Exercise 47

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I conjecture the following classifications, but provide full proof for only (a) and (e):

(a) i. Proof follows.

(b) iii for typical encodings. Easily decidable. Can be proved non context-free for some encodings via the pumping lemma.

(c) iii. Easily decidable. For pumping length p, the string $\mathrm{a}^p\mathrm{b}^{2p}\mathrm{a}^p$ cannot be partitioned into $uvxyz$ and pumped due to the requirement that $|vxy| \leq p$.

(d) iv for typical encodings. Recognized by running in parallel M on all possible inputs, increasing allowable run time and input length simultaneously. If it were decidable, then E_{TM} would be decidable by checking if every state is accessible for every subset of states, and accepting iff the accept state is not in the largest accepted subset.

(e) ii Proof follows.

The set $L_1 = \{u \in \{0,1,2\}^n \mid 7 \text{ does not divide } \sum_{i=0}^{n} u_i 3^{n-i}\}$ is recognized by deterministic finite automata $M_1 = (\mathbb{Z}_7,\ \mathbb{Z}_3,\ \delta(\sigma, x) \equiv 3\sigma + x \bmod 7,\ 0,\ \mathbb{Z}_7 \setminus \{0\})$ and is, therefore, regular. At each point in computation M_1's state is equivalent to the numeral processed that far mod 7. M_1 accepts if and only if the whole numeral is not divisible by 7. The existence of M_1 demonstrates that L_1 is regular.

Let L_2 be the set such that $u \in L_2$ if and only if $u = \emptyset$ or there exist $s, t \in L_2$ such that $u = [s]t$. (adapted from *Mathematics for Computer Science*, by Eric Lehman, F. Thomson Leighton, and Albert R. Meyer, revision of June

6, 2018, page 222.) Consider the push down automata:

$$M_2 = (\{q_0, q_1, q_2\},\ \{[,]\},\ \{\#, [\},\ \delta(x) = \begin{cases} \{(q_1, \#)\} & x = (q_0, \epsilon, \epsilon) \\ \{(q_1, [)\} & x = (q_1, [, \epsilon) \\ \{(q_1, \epsilon)\} & x = (q_1,], [) \\ \{(q_2, \epsilon)\} & x = (q_1, \epsilon, \#) \\ \emptyset & \text{otherwise} \end{cases},\ q_0, \{q_2\})$$

M_2 accepts $\emptyset$. Assuming M_2 properly handles all inputs smaller than u, M_2 accepts u if there exist $s, t \in L_2$ such that $u = [s]t$ by adding # and [to the stack, processing s, resulting in an unchanged stack, removing the [from the stack, resulting in a stack consisting of a single #, and processing t on that stack, which accepts. Further, M_2 rejects all strings that start with a], and only accepts a string u that start with a [by pushing [to the stack, processing an accepting string s, resulting in an unchanged stack, processing a], and then processing an accepting string t. This means that M_2 will only accept a string accept a string u if there exist $s, t \in L_2$ such that $u = [s]t$. By structural induction, M_2 recognizes L_2. The existence of M_2 demonstrates that L_2 is context-free.

Any deterministic finite automaton which recognizes L_2 must be in a distinct state after processing each of {[, [[, [[[, ...} because each corresponds to a different set of possible accepting continuations. This necessitates an infinite number of states. L_2 is not regular.

L_2 is context-free but not regular.