Exercise 32

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The language L $\{a^n \mid n = 2^m \text{ for some natural number } m\}$ on input alphabet $\{a\}$ can be decided by the following algorithm A:

1. Reject if the tape is blank.
2. Scan right along the tape until a blank is reached, crossing out every other zero and keeping track of parity of whether the number of 0's is equal to 1 and whether it is odd. Accept if 1, otherwise reject if odd.
3. Return to step 2.

I will prove A decides the L by mathematical induction. A rejects a^0, and accepts a^1. A accepts inputs in L and rejects inputs not in L for all inputs with length $< 2^1$. Assume as an inductive hypothesis that for some $n \geq 1$, A accepts inputs in L and rejects inputs not in L for all inputs with length $< 2^n$. All inputs with $2^n \leq$ length $< 2^{n+1}$ have either an even or an odd length. A should and does reject all odd inputs. A accepts an even input a^{2n} iff A accepts a^n, and rejects an even input a^{2n} iff A rejects a^n. For all even inputs a^{2n} with length $< 2^{n+1}$, a^n has length $< 2^n$, and by the inductive hypothesis, A accepts such inputs in L and rejects such inputs not in L. Therefore, A accepts inputs in L and rejects inputs not in L for all inputs with $2^n \leq$ length $< 2^{n+1}$, and along with the inductive hypothesis, this means that A accepts inputs in L and rejects inputs not in L for all inputs with length $< 2^{n+1}$. By mathematical induction, A accepts all inputs in L and rejects all inputs not in L. Therefore, A decides L, and L is decidable.

I will prove that L is not context free using the contrapositive of the pumping lemma for context-free languages. For all numbers p, there exists a string $s = a^{2^{\lfloor log_2(x) \rfloor + 1}}$ which is in L and has length larger than p. For all partitions of s into five pieces $s = uvxyz$ satisfying the conditions $|vy| > 0$

and $|vxy| \leq p$, $0 < |vy| < |s|$, and so uv^2xy^2z is longer than s, but less than twice as long as s and is therefore not in L. Therefore, L is not context free.

Pumping lemma for context free languages from Michael Sipser's *Introduction to the theory of computation 3rd edition.*