# Chapter 1

# Signal and system representations

## 1.1 Introduction

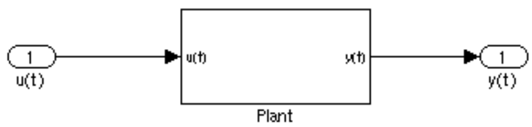


Figure 1.1: Diagram of a dynamical system

Traditionally, in the context of autonomous systems and related fields such as control engineering, picturing a dynamical system $\Sigma$ can be roughly summarized in the figure above (see fig 1.1). Variable $u(t)$ is a signal/stimuli whose variations have some effect on the behavior of system $\Sigma$ . Signal $u(t)$ is called the control input. Variable $y(t)$ is a signal representing a selected part of the dynamics of the system. It typically represents what is measured.

**Example : A car as a dynamical system?**

- The speed of the car is what is usually measured, ie it is therefore $y(t)$.
- Let us take the gas pedal position as the control input $u(t)$.

Most systems that will be considered in this course are related to robotics and whose physics will often be modelled by ordinary differential equations (ODEs for short), although this is not always necessarily the case. Let us see a few examples.

## 1.2 A few systems

Let us start with a very simple if not trivial example of a dynamical system, where the time variable $t$ is discrete. In this case, we talked about difference equations instead of differential equations. as we will see, quite a few such discrete-time representations are used in different techniques used in autonomous systems.

**Example: Bank Account with Interest Rate**

Consider a bank account with an interest rate of 3% per year. We have an initial deposit of $y_0 = 10000\,\mathrm{DKK}$ on the account. The model of this relatively simple system can be given as

$$y(t+1) = (1+0.03)y(t), \qquad y(0) = y_0 = 10000\,\mathrm{DKK} \tag{1.1}$$

where the unit of $t$ ($t$ is here a discrete unit of time, ie $t \in \mathbb{Z}$ or $\mathbb{N}$) is the year, and $y(t)$ represents the amount of money on the bank account at year $t$.
In the present situation, the dynamical behavior of the system depends only on the initial condition y(0).

Let us now add an input to this system to represent an influx of money (add or remove money from the account). In this case, the new model with an input can be represented with the following difference equation:

$$y(t+1) = 1.03y(t) + u(t), \qquad y(0) = y_0 = 10000\,\mathrm{DKK} \tag{1.2}$$

□

Coming back to ODEs and with a time variable belonging to the set of real numbers, a first well-known system, quite important for its illustrative but also for its applicative virtues, is the Mass-Spring-Damper system.

**Example: Mass-Spring-Damper system**

A second order linear differential equation will mathematically represent the system shown in figure 1.2:

$$m\ddot{y}(t) + d\dot{y}(t) + ky(t) = u(t) \tag{1.3}$$

where $y(0)$ is the initial position of mass and $\dot{y}(0)$ is the initial velocity of mass. □

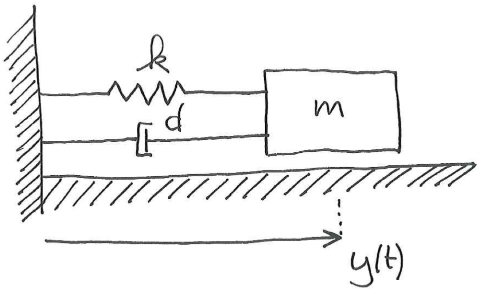


Figure 1.2: Mass-Spring-Damper System

Other systems can be modeled by a *combination* of ODEs.

**Example: Pharmacokinetics using compartment model**

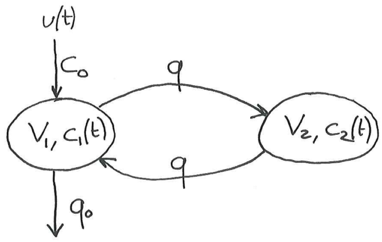


Figure 1.3: Compartment Model

As is often the case when taking medecine, a drug for a part of the body will be injected from another part (think aspirine for the blood injested into the stomach through the mouth for example). Hence the drug will propagate from place to place until it reaches the right concentration in the “target organ”.

To model this, the compartment model diagram is used (see figure 1.3), where $V_1$ and $V_2$ are the constant volumes for each compartment, and $c_1$, $c_2$ are their respective time-varying drug concentrations. A drug of concentration $c_0$ is injected in compartment 1 at a volume flow rate $u(t)$ (control input). Constant parameters $q_0$ and $q$ are outflow rates resulting in a decrease of concentration over time.
The set of differential equations representing this systems is given below.

$$\begin{cases} V_1\dot{c_1}(t) = q\left(c_2(t) - c_1(t)\right) - q_0 c_1(t) + c_0 u(t) \\ V_2\dot{c_2}(t) = q\left(c_1(t) - c_2(t)\right)) \end{cases} \tag{1.4}$$

with

$$y(t) = c_2(t) \tag{1.5}$$

since what we interested in is to monitor $c_2(t)$. □

The two previous examples were linear, and as we will see, a wide arsenal of techniques (Laplace transform, Fourier, etc) are available for them. However, other systems are inherently *nonlinear* in their nature. In this case, Laplace transforms do not apply, and their dynamical properties are generally studied based primarily on their differential equation representation.

**Example: Pendulum**

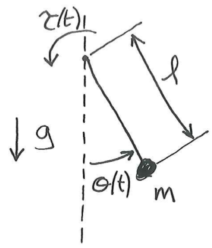


Figure 1.4: Pendulum

Consider the following simple model of a pendulum without friction and subject to a gravity field

$$ml^2\ddot{\theta}(t) + mgl\sin\theta(t) = 0 \tag{1.6}$$

where the measurement and the initial conditions are given by

$$y(t) = \theta(t), \qquad \theta(0) = \theta_0, \qquad \dot{\theta}(0) = \omega_0$$

and the parameters $m$, $l$ and $g$ represent the mass of the pendulum "tip", the length of the pendulum and the gravity constant, respectively.
Adding rotational damping to the system gives

$$ml^2\ddot{\theta}(t) + d_\theta l^2\dot{\theta}(t) + mgl\sin\theta(t) = 0 \tag{1.7}$$

(with $d_\theta$ the damping coefficient), while adding an external torque $\tau(t)$ to the pendulum results in

$$ml^2\ddot{\theta}(t) + d_\theta l^2\dot{\theta}(t) + mgl\sin\theta(t) = \tau(t) \tag{1.8}$$

where the control input is therefore

$$u(t) = \tau(t)$$

□

Another well-known dynamical system combining linear and nonlinear dynamics is the so-called Lorenz system. This system is at the origin of Chaos Theory (the butterfly effect, have you heard about that?).

**Example: Lorenz System**

One of the different versions of the Lorenz system is given by the 3 following coupled differential equations

$$\begin{cases} \dot{x}_1(t) = -px_1(t) + px_2(t) \\ \dot{x}_2(t) = -x_1(t)x_3(t) - x_2(t) \\ \dot{x}_3(t) = x_1(t)x_2(t) - x_3(t) \end{cases} \tag{1.9}$$

where $p$ is a constant. □

To finish with this set of relatively simple examples, let us mention the field of mobile robotics, which is rich with systems that are nonlinear and combine several inputs and outputs.

**Example: The 2-Wheeled Robot**

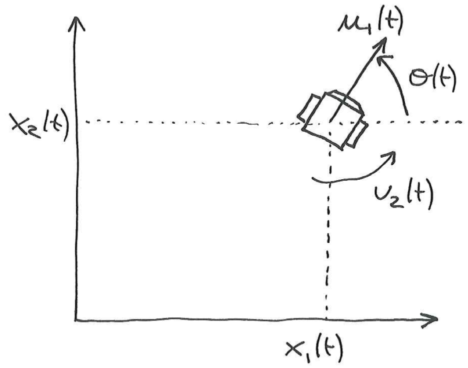


Figure 1.5: 2-Wheeled Robot

Considering a 2-wheeled robot as pictured in figure 1.5, a mathematical description of it is given by the following equations:

$$\begin{cases} \dot{x}_1(t) = u_1(t)\cos\theta(t) \\ \dot{x}_2(t) = u_1(t)\sin\theta(t) \\ \dot{\theta}(t) = u_2(t) \end{cases} \tag{1.10}$$

where $x_1(t)$, $x_2(t)$ represent the position of the robot in the plane, and $\theta(t)$ the orientation of the robot. The robot is steered by the longitudinal velocity $u_1(t)$ and the rotational velocity $u_2(t)$.

Hence if we would like to monitor all 3 of the dynamic variables, we have

$$y_1(t) = x_1(t), \qquad y_2(t) = x_2(t), \qquad y_3(t) = \theta(t)$$

while we clearly also have the 2 control inputs $u_1(t)$ and $u_2(t)$. □

The so-called car-like robot is obtained by only slightly modifying the equations of the 2-wheeled robot.

**Example: Car-like Robot**

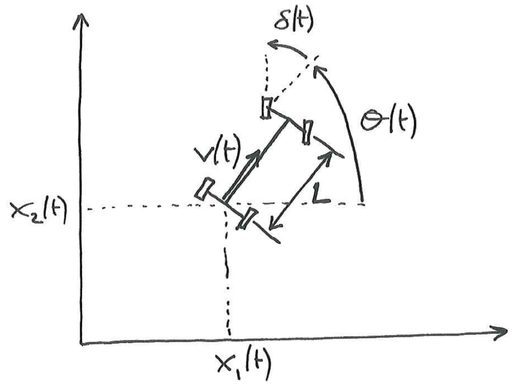


Figure 1.6: Car-like Robot

Consider the following mathematical representation

$$\begin{cases} \dot{x}_1(t) = v(t)\cos\theta(t) \\ \dot{x}_2(t) = v(t)\sin\theta(t) \\ \dot{\theta}(t) = \frac{v(t)}{L}\tan\delta(t) \end{cases} \tag{1.11}$$

with the following inputs

$$u_1(t) = v(t), \qquad u_2(t) = \delta(t)$$

where $\delta(t)$ represents the steering angle of the front wheels of the car, and $L$ is the constant length between the rear axis and the front axis. □

## 1.3 Block Diagrams and Matlab/Simulink

To simulate systems, Matlab/Simulink uses a graphical representation of systems called "block diagrams". This kind of representation is very much in use in autonomous systems and control engineering.
To build a block diagram, one typically combines a number of basic blocks by linking them together. See below a few of these basic blocks.

- Summator

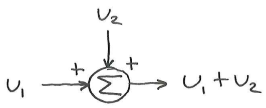


Figure 1.7: Summator

- Gain

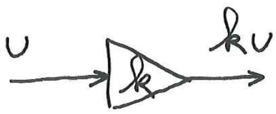


Figure 1.8: Gain

- Integrator

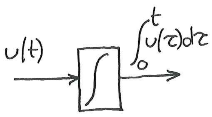


Figure 1.9: Integrator

- Differentiator

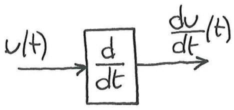


Figure 1.10: Differentiator

- Multiplier

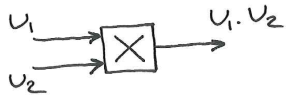


Figure 1.11: Multiplier

The next step consists in linking them to implement/realize a system, often represented mathematically as a differential equation. Let us see that through a simple example.

**Example: Block diagrams for a linear first-order differential equation**

Consider the following first-order linear differential equation given by

$$\frac{d}{dt}y(t) = -3y(t), \qquad y(0) = 6 \tag{1.12}$$

A first simple block diagram implementation consists in connecting a differentiator block with a gain as in figure 1.12.

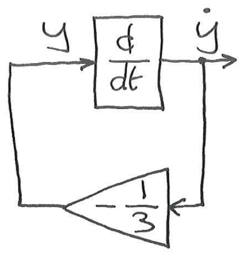


Figure 1.12: Block diagram of a first-order system: the differentiator solution

First note that the initial condition is not present and therefore not accounted for. This is not difficult to solve, and can be done by simply adding a constant input to signal $y(t)$.
However, the problem is that this solution is typically quite sensitive to noise or discretization errors, mostly due to the effect that differentiation has on high-frequency signals (ie it amplifies them).

Hence another possible solution consists in using a integrator. Indeed, note that

$$\int_0^t \dot{y}(\tau)d\tau = y(t) - y(0) \tag{1.13}$$

which obviously gives

$$y(t) = \int_0^t \dot{y}(\tau)d\tau + y(0) \tag{1.14}$$

or, in block diagram dialect

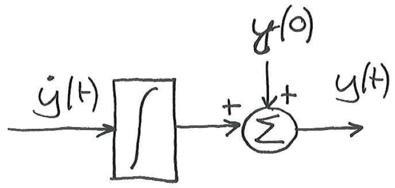


Figure 1.13: Integration of a time-derivative

The above solution is now quite robust to noise thanks to the smoothing properties of the integrator.

Hence, system (1.12) can be implemented/represented by the block diagram hereafter

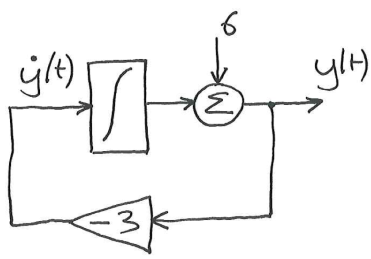


Figure 1.14: Block diagram of a first-order system: the integrator solution

or, using another notation to incorporate the initial condition into the integrator

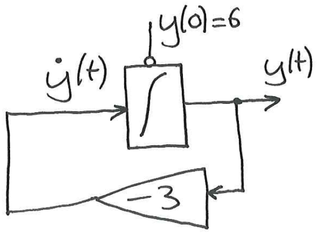


Figure 1.15: A different notation for the integrator solution

**Exercises**

Make a block diagram for each system described in section (1.2).

Simulating systems represented by difference equations where the time is discrete is not more difficult. In this case, instead of an integrator, we just use the following unit delay block

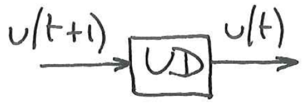


Figure 1.16: The unit delay block

With this block, we can now go back to our bank account example, which is represented by

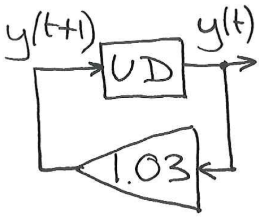


Figure 1.17: Block account for the bank account example

And similarly for multi-dimensional, nonlinear, time-varying systems or a combination thereof.