2.7 Linear state-feedback for nonlinear systems

The interest of linear state-feedback control lies essentially in its simplicity: take the current state as information, and multiply it by a few gains to compute the control input which is then applied back to the plant. As we have seen, the computation of the gains can be done in a quite simple way.
Interestingly, applying linear state-feedback to systems which are nonlinear is also possible. The basic idea consists in choosing a point of interest, where one wishes to stabilize the system, ie an equilibrium point, and then apply the right linear state-feedback controller based on the linear approximation of the system around the chosen equilibrium point.

More concretely, start with nonlinear system described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{2.62}$$

and compute a linear approximation of (2.62) around $(\mathbf{x}^*, \mathbf{u}^*)$ to obtain

$$\delta\dot{\mathbf{x}} = \mathbf{A}\delta\mathbf{x} + \mathbf{B}\delta\mathbf{u} \tag{2.63}$$

If this linear system is not stable, then it can be stabilized with a linear state-feedback law given by

$$\delta\mathbf{u}(t) = -\mathbf{K}\delta\mathbf{x}(t) \tag{2.64}$$

where we know how to compute the values of the gain row vector $\mathbf{K}$ so that we have the desired poles in closed-loop. However, remember that $\delta\mathbf{u}(t)$ represents only small deviations around a nominal $\mathbf{u}^*$, and therefore that $\delta\mathbf{u}(t)$ should not be applied directly to the nonlinear system. Indeed, the control input signal is and will always be $\mathbf{u}(t)$. Hence, using the way $\delta\mathbf{u}(t)$ is defined, the control input applied to the nonlinear system is given by

$$\mathbf{u}(t) = \mathbf{u}^* + \delta\mathbf{u}(t) \tag{2.65}$$

Similarly, the variable $\delta\mathbf{x}(t)$ used for feedback in (2.64) still represents small deviations around $\mathbf{x}^*$, not the state $\mathbf{x}(t)$ as measured from the nonlinear system itself. As a consequence, what enters our feedback controller (2.64) is

$$\delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}^* \tag{2.66}$$

Thus, combining (2.64), (2.65) and (2.66), a way to implement our linear state-feedback stabilizer of a nonlinear system would be given by the following expression

$$\mathbf{u}(t) = -\mathbf{K}(\mathbf{x}(t) - \mathbf{x}^*) + \mathbf{u}^* \tag{2.67}$$

This linear controller and its nonlinear plant are represented together in the block diagram below (see figure 2.6).

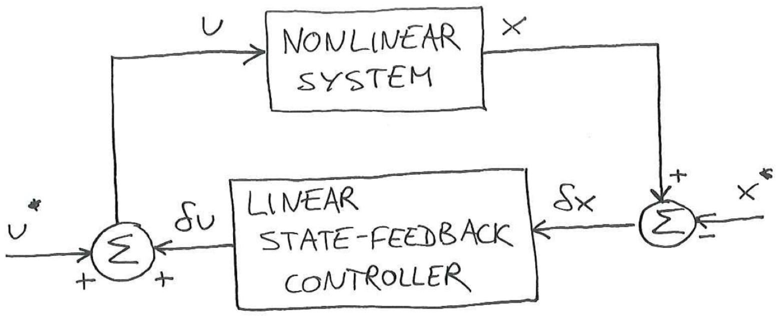

Figure 2.6: Linear state-feedback controller for a nonlinear system

2.8 Tracking control for linear systems

One can also extend the result of section 2.6 for a system to *track* or follow a particular trajectory of reference $\mathbf{x}_d(t)$. In this case, the feedback controller will have to compensate for errors *around* this trajectory.

First, we need to make sure it is actually possible at all for the system to follow this trajectory, especially when there is no disturbance or difference in initial conditions. Hence, we will call *feasible* trajectory for the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ a desired trajectory $\mathbf{x}_d(t)$ if an open-loop control signal $\mathbf{u}_d(t)$ can be found such that $\mathbf{x}_d(t)$ is a solution for the system with $\mathbf{u}_d(t)$ as input. That is we have $\mathbf{x}_d(t)$ and $\mathbf{u}_d(t)$ verifying

$$\dot{\mathbf{x}}_d(t) = \mathbf{A}\mathbf{x}_d(t) + \mathbf{B}\mathbf{u}_d(t) \tag{2.68}$$

Note that this expression can be seen as an extension of the equilibrium relation (2.46) we saw in section 2.6. Indeed, if $\mathbf{x}_d(t)$ is constant, then $\dot{\mathbf{x}}_d(t) = 0$ and expression (2.68) reduces to $0 = \mathbf{A}\mathbf{x}_d + \mathbf{B}\mathbf{u}_d$, which is nothing else than (2.46).

Once the feasible trajectory $\mathbf{x}_d(t)$ and its associated $\mathbf{u}_d(t)$ are defined, we can introduce, similarly again to section 2.6 the "delta" variables given by

$$\Delta\mathbf{x}(t) := \mathbf{x}(t) - \mathbf{x}_d(t) \quad \text{and} \quad \Delta\mathbf{u}(t) := \mathbf{u}(t) - \mathbf{u}_d(t) \tag{2.69}$$

which again leads to the error dynamics around $(\mathbf{x}_d(t), \mathbf{u}_d(t))$

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{A}\Delta\mathbf{x}(t) + \mathbf{B}\Delta\mathbf{u}(t) \tag{2.70}$$

from which we obtain the *tracking controller* expression

$$\mathbf{u}(t) = \mathbf{K}(\mathbf{x}(t) - \mathbf{x}_d(t)) + \mathbf{u}_d(t) \tag{2.71}$$

Interestingly, note that tracking controller (2.71) thus combines an open-loop/feedforward controller with a standard linear state-feedback controller, as represented in figure 2.7.

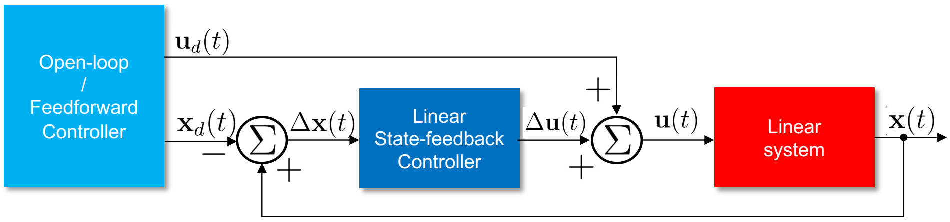

Figure 2.7: OL + Linear State-FB = tracking controller.

2.9 Integration using linear state-feedback

2.9.1 Basic principle

Through our simulations/experiments, we have seen that linear state-feedback is quite good at coping with transitory disturbances, such as high-frequency noise or differences in initial conditions.

However, other more persistent disturbances can perturb the system. For example, a constant wind will influence the behavior of a plane, a heavy passenger in the car might push the driver to increase the action on the throttle to maintain speed.

To see how persistent disturbances affect systems, let us start with the simple linear scalar system represented by

$$\dot{x}(t) = ax(t) + bu(t) \tag{2.72}$$

where a and b are the constant parameters of the system. Applying the state-feedback controller $u(t) = -kx(t)$, we will get the closed-loop dynamics

$$\dot{x}(t) = (a - bk)\, x(t) \tag{2.73}$$

which will be stable around equilibrium point $x^* = 0$ provided $a - bk < 0$. Assume now that a constant disturbance d affects the system additively, ie we have

$$\dot{x}(t) = ax(t) + bu(t) + d \tag{2.74}$$

Using the same feedback law $u(t) = -kx(t)$ means that, because of the disturbance d, closed-loop dynamics (2.73) become

$$\dot{x}(t) = (a - bk)\, x(t) + d \tag{2.75}$$

for which we have a new equilibrium point given by

$$x^* = \frac{-1}{a - bk} d \tag{2.76}$$

Expression (2.76) means that, unless we set the gain of our controller to $k = \infty$ (both unrealistic and unreasonable), the equilibrium cannot be $x^* = 0$ anymore.

Hence, the simple feedback law $u(t) = -kx(t)$ cannnot completely compensate for constant disturbances.

Let us now introduce a state-feedback law where an *integral term* is added:

$$u(t) = -kx(t) - k_I \int_0^t x(\tau)d\tau \tag{2.77}$$

with k_I a constant scalar gain tuning the influence/importance of the integral term. Putting (2.77) into disturbed system (2.74), we get the closed-loop dynamics

$$\dot{x} = ax - bkx - bk_I \int_0^t x(\tau)d\tau + d \tag{2.78}$$

Expression (2.78), combining both integral and differential terms, is simply called an *integro-differential equation.* In order to obtain a differential equation, differentiate (2.78) to get

$$\ddot{x} = (a - bk)\dot{x} - bk_I x \tag{2.79}$$

where it is important to notice that the constant disturbance term has disappeared since $\dot{d} = 0$. Converting this second-order system into a state-space representation by defining the state coordinates $[x_1, x_2]^T := [x, \dot{x}]^T$, we get

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -bk_I & a - bk \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.80}$$

which has equilibrium point

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} x^* \\ \dot{x}^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.81}$$

We know that the whole state $[x_1(t), x_2(t)]^T$ and hence $x_1(t) = x(t)$, will converge to the origin of the state-space provided each eigenvalue of the matrix

$$\begin{bmatrix} 0 & 1 \\ -bk_I & a - bk \end{bmatrix} \tag{2.82}$$

has a strictly negative real part. The result is quite important, because it means that provided that k and k_I are chosen properly, ie so that the dynamics (2.80) are stable, controller (2.77) can stabilize our scalar system around the origin, and this despite the presence of an additive constant disturbance!

More generally, and starting from a disturbed system described by the following state-space representation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{u} + \mathbf{d}), \tag{2.83}$$

we can design the following controller

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) - \mathbf{K}_I \int_0^t \mathbf{C}\mathbf{x}(\tau)d\tau = -\mathbf{K}\mathbf{x}(t) - \mathbf{K}_I \int_0^t \mathbf{y}(\tau)d\tau. \tag{2.84}$$

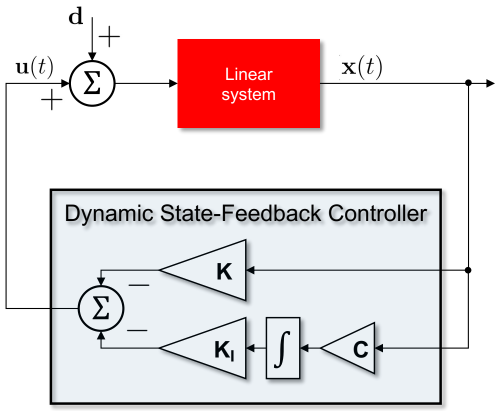

Figure 2.8: State-feedback control with integration

2.9.2 Stabilization around a constant reference

Let us come back to our scalar system

$$\dot{x} = ax + bu \tag{2.85}$$

and assume this time that we want to stabilize it around an equilibrium point *not* at the origin, ie we have $x^* \neq 0$. For convenience, let us rewrite this equilibrium point as $r := x^*$ (often called "reference"). As an alternative to what we saw in section 2.6, ie a combined feedforward/state-feedback control strategy, consider instead the following dynamic state-feedback controller (with integral term)

$$u(t) = kx(t) + k_I \int_0^t (x(\tau) - r)\, d\tau \tag{2.86}$$

This gives the closed-loop dynamics, after differentiation,

$$\ddot{x} = (a + bk)\,\dot{x} + bk_I\,(x - r) \tag{2.87}$$

whose state-space representation is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ bk_I & a + bk \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -bk_I \end{bmatrix} r \tag{2.88}$$

or, in component form

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = (a + bk)x_2 + bk_I(x_1 - r) \end{cases} \tag{2.89}$$

Let us now look at the equilibrium points of system (2.88) or (2.89). Setting the derivative of the state components to 0, we get

$$\left.\begin{array}{l} 0 = x_2^* \\ 0 = (a + bk)x_2^* + bk_I(x_1^* - r) \end{array}\right\} \Rightarrow \begin{cases} x_1^* = r \\ x_2^* = 0 \end{cases} \tag{2.90}$$

which means that we are free to choose any r we want! The important implication of this is that, provided of course that k and k_I are chosen so that the system is stabilized, using integral controller means that the system will be stabilized around the desired reference $r = x^*$.
This particular technique is quite interesting because, contrary to the method that we have seen in section 2.6 for stabilizing around an equilibrium point x^*, it does not require the computation of u^*. It is implicitely computed by the dynamic controller (2.86).

The above discussion can be generalized to higher-order systems. In this case, we simply replace (2.84) with

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) - \mathbf{K}_I \int_0^t (\mathbf{y}(\tau) - \mathbf{r}(\tau))d\tau \tag{2.91}$$

2.9.3 Polynomial disturbances

More complex disturbances than the simple constant signals seen in section 2.9.1 can be considered using the same basic integral idea. Indeed, consider again our scalar example but affected this time with a constantly increasing disturbance

$$\dot{x} = ax + bu + d.t \tag{2.92}$$

where the constant term d is multiplied by the time variable t. To compensate for this first-order polynomial disturbance, one will then use the following linear state-feedback controller with double integrator

$$u(t) = kx(t) + k_I \int_0^t x(\tau)\, d\tau + k_{II} \int_0^t \int_0^\tau x(\sigma) d\sigma d\tau \tag{2.93}$$

and similarly for higher-order polynomial disturbances where we will simply need more integrators.

2.9.4 A link with PID control

Linear state-feedback controller with integration can be related to PID controller, which we have briefly talked about at the beginning of this chapter (see again figure 2.9 for reference). Recall that, mathematically, a PID controller is given by

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau + k_d \frac{de}{dt}(t). \tag{2.94}$$

In order to compare a PID controller with linear state-feedback techniques, let us first assume, for simplicity, that the reference signal is such that $r(t) = 0$ (i.e. we want to stabilize the system output around the origin). In this case, controller (2.94) simplifies into

$$u(t) = -k_p y(t) - k_i \int_0^t y(\tau) d\tau - k_d \dot{y}(t) \tag{2.95}$$

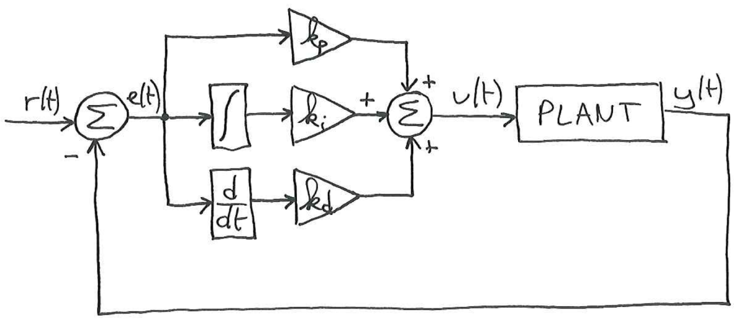

Figure 2.9: PID Controller

Furthermore, we will also assume that the plant is a second-order system represented by the following state-space represenation (in component form)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a_0 x_1 - a_1 x_2 + bu \\ y = x_1 \end{cases} . \tag{2.96}$$

As we have seen, in this case, a linear state-feedback controller with integral term takes the form

$$u(t) = -k_1 x_1(t) - k_2 x_2(t) - k_I \int_0^t x_1(\tau) d\tau \tag{2.97}$$

which, because we have a controllability canonical form, we also have $x_1 = y$ and $x_2 = \dot{y}$, and this implies

$$u(t) = -k_1 y(t) - k_2 \dot{y}(t) - k_I \int_0^t y(\tau) d\tau \tag{2.98}$$

Note the striking resemblance between the above expression and equation (2.95). Indeed, we have exactly $k_p = k_1$, $k_d = k_2$ and $k_i = k_I$, ie k_1 plays the same role as the proportional gain k_p, k_2 the same as derivative gain k_d and k_I the same as k_i.

A quite similar analysis can be carried out for second-order systems stabilized around a non-zero reference input signal. However, for higher-order systems, a state-feedback controller will have more gains, allowing to control more complex dynamics (roughly corresponding to adding higher derivative terms in the PID control frameworks), while the PID controller will basically remain the same. However, PID control turns out to be quite effective and sufficiently performant while remaining quite simple in many cases, hence its popularity.