Module 7
Differential Fluid Flow

7.1 Introduction to Differential Analysis

7.1.1 Initial and Boundary Conditions

The system or control volume analysis covered in previous lessons depended on a finite control volume, and fluid properties such as velocity and pressure would be studied as they flowed through the control surfaces. While this analysis is sufficient for simplified problems, the assumptions required by them make it inapplicable for most complex problems. Therefore, an alternative approach is made where a differential element (usually an infinitesimally small control volume) is analyzed and the fluid properties for the element are determined by solving the partial differential equations that result from applying the laws of conservation.

Definitions:

- field: a dependent variable that is a function of more than one independent variable
- initial conditions: known conditions that depend on time
- boundary conditions: known conditions that depend on a space coordinate

Examples of initial conditions:

- Letting go of a body at a height z so that it drops in free fall. The initial velocity (at the instant it is let go) is zero.

Examples of boundary conditions:

- No-slip condition, where the velocity of a fluid at the boundary (wall) is zero.
- Open channel (free-surface) flow, where the pressure at the surface is atmospheric.

If an equation has solutions for a property along a two- or three-dimensional plane, the plot of these solutions showing its slope is called a direction field. Note that in order to have a direction field for a given interval in the xy plane or xyz space, solutions for the property of interest must exist at any point at that interval. If this is the case, we can say that the solutions to the equation are smooth. Mathematically, a function F has smooth solutions on an interval if the following differential equation is satisfied for that interval:

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

7.1.2 The Del Operator

The gradient of a function is the change of a differentiable function with respect to space. Mathematically, we can define the gradient operator (also called del or nabla) as $\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$. Let's keep in mind that del is an operator, not a vector. Now, we can apply the operator to a scalar function in order to obtain the gradient of the function:

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

We can also apply the gradient operator to a vector valued function to create a scalar function by:

$$\nabla \cdot \vec{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

Or we can create an operator to map a vector valued function to a scalar by:

$$\vec{f} \cdot \nabla = f_x \frac{\partial}{\partial x} + f_y \frac{\partial}{\partial y} + f_z \frac{\partial}{\partial z}$$

Mathematically, we can relate the material derivative of the velocity vector to the del operator as follows. First let's form an operator $\vec{v} \cdot \nabla$:

$$\vec{v} \cdot \nabla = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$$

Applying this operator to the velocity field:

$$(\vec{v} \cdot \nabla)\vec{v} = u \frac{\partial \vec{v}}{\partial x} + v \frac{\partial \vec{v}}{\partial y} + w \frac{\partial \vec{v}}{\partial z}$$

Recall that the material derivative of the velocity is the sum of the local and convective acceleration:

$$\frac{D\vec{v}}{Dt} = u \frac{\partial \vec{v}}{\partial x} + v \frac{\partial \vec{v}}{\partial y} + w \frac{\partial \vec{v}}{\partial z} + \frac{\partial \vec{v}}{\partial t}$$

So, we can substitute with our operator and rearrange:

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}$$

Note that this operator can also be expressed as $\vec{v} \cdot \nabla \vec{v}$, so the material derivative can be expressed in an alternate form:

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}$$

7.1.3 Divergence

A common combination of partial differential equations in fluid mechanics is the divergence of the vector field. If we define a vector $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$, then the divergence is found by:

$$\text{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The divergence can also be written in terms of the del operator:

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left(P\hat{\imath} + Q\hat{\jmath} + R\hat{k} \right) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Let's recall the velocity of a fluid in vector form:

$$\vec{v} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$$

The divergence of the velocity vector is:

$$\text{div}(\vec{v}) = \nabla \cdot \vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

7.1.4 Curl

The curl of a vector field $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$ is the vector field found by:

$$\text{curl}(\vec{F}) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

The curl (sometimes called rotation) can be written in terms of the del operator:

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

If we apply the operator $\left(\vec{f} \cdot \nabla\right)$ to its vector field, we get the following vector identity:

$$(\vec{f} \cdot \nabla)\vec{f} = \frac{1}{2}\nabla(f^2) - \vec{f} \times (\nabla \times \vec{f})$$

Where f is the magnitude of $\vec{f}$.

7.2 Differential Conservation of Mass

7.2.1 Continuity Equation – Differential Form

Let's consider an infinitesimally small control volume with dimensions, dx, dy, dz. The law of conservation of mass tells us that the change in mass in the control volume is equal to the difference between the mass fluxes entering and leaving the control volume:

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm}{dt}$$

The mass of a differential element is $m = \rho dV = \rho dx dy dz$ where $\rho = f(x, y, z, t)$. The mass flux entering the control volume, in the x-direction, is then:

$$\dot{m}_{in_x} = \rho(x)\frac{dx}{dt}(x)dy\,dz = \rho(x)u(x)dydz$$

The mass flux leaving the control volume, in the x-direction is:

$$\dot{m}_{out_x} = \rho(x + dx)\frac{dx}{dt}(x + dx)dy\,dz = \rho(x + dx)u(x + dx)dydz$$

This formulation can be followed for the mass fluxes entering and leaving the control volume in the y- and z-directions to yield:

$$\dot{m}_{in} - \dot{m}_{out} = \rho(x)u(x)dydz + \rho(y)v(y)dxdz + \rho(z)w(z)dxdy - \rho(x + dx)u(x + dx)dydz$$
$$- \rho(y + dy)v(y + dy)dxdz - \rho(z + dz)w(z + dz)dxdy$$

The change in mass can be expressed as:

$$\frac{dm}{dt} = \frac{d(\rho dxdydz)}{dt} = \frac{\partial \rho}{\partial t} \, dx \, dy \, dz$$

Note that the change in density with respect to time has to be taken as a partial derivative since density is a function of both time and space. This allows us to express the continuity equation as:

$$\rho(x)u(x)dydz + \rho(y)v(y)dxdz + \rho(z)w(z)dxdy - \rho(x + dx)u(x + dx)dydz - \rho(y + dy)v(y + dy)dxdz$$
$$- \rho(z + dz)w(z + dz)dxdy = \frac{\partial \rho}{\partial t} \, dx \, dy \, dz$$

Dividing both sides by $dxdydz$:

$$\frac{\rho(x)u(x)}{dx} + \frac{\rho(y)v(y)}{dy} + \frac{\rho(z)w(z)}{dz} - \frac{\rho(x + dx)u(x + dx)}{dx} - \frac{\rho(y + dy)v(y + dy)}{dy} - \frac{\rho(z + dz)w(z + dz)}{dz} = \frac{\partial \rho}{\partial t}$$

And rearranging:

$$\frac{\rho(x)u(x) - \rho(x + dx)u(x + dx)}{dx} + \frac{\rho(y)v(y) - \rho(y + dy)v(y + dy)}{dy} + \frac{\rho(z)w(z) - \rho(z + dz)w(z + dz)}{dz} = \frac{d\rho}{dt}$$

From Calculus, we know that the derivative of u with respect to x is $\frac{du}{dx} = \frac{u(x+dx)-u(x)}{dx}$, so we can express the continuity equation as:

$$-\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z} = \frac{\partial \rho}{\partial t}$$

Rearranging:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

We can express the continuity equation in terms of the del operator as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

Or, using a different notation:

$$\rho_t + \text{div}(\rho \vec{v}) = 0$$

This is the continuity equation in its differential form. We can arrive at a different form of the continuity equation by starting with the partial differential equation $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$ and differentiating the products:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + \rho\frac{\partial w}{\partial z} = 0$$

Which is then simplified to:

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \vec{v} = 0$$

The former form of the continuity equation is more common in mathematics and physics, whereas the latter is more common in engineering.

7.2.2 Simplifications of the Continuity Equation

Some of the simplifications we have looked at in the class can now be applied to the differential form of the continuity equation. For example, steady flow requires no change with respect to time, so $\frac{\partial \rho}{\partial t} = 0$ and the continuity equation becomes:

$$\text{div}(\rho\vec{v}) = 0$$

Incompressible flow requires no change in density, which means that we can treat ρ as a constant and the continuity equation becomes:

$$\text{div}(\vec{v}) = 0$$

Try to think about what the above equation means mathematically for the velocity vector, the divergence is:

$$\text{div}(\vec{v}) = \nabla \cdot \vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

This means that knowing the velocity vector, we can determine if our flow is incompressible or not.

7.3 Differential Conservation of Momentum

7.3.1 The Stress Tensor

In 3D-space, there are nine stress components that act on a cube-shaped fluid element. These components form the stress tensor. Consider a cube-shaped fluid element with faces in the x-, y-, and z-directions. The stress tensor for this fluid element is:

$$\tau_{ij} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

Here, the first subscript refers to the face on which the component acts, and the second subscript denotes the direction of the component. Stress components that act normal to a face are called normal stresses, and are denoted with the variable σ. Stress components that act tangential to a face are called shear stresses, and are denoted with the variable τ. For an incompressible fluid, if moments are taken about the axes passing through the centers of the faces of the element, then we will get that:

$$\tau_{yx} = \tau_{xy}$$
$$\tau_{zx} = \tau_{xz}$$
$$\tau_{zy} = \tau_{yz}$$

7.3.2 Equations of Motion

Consider a differential fluid element with dimensions dx, dy, and dz. Applying Newton's 2nd Law, we can express the sum of external forces in terms of the stress tensor, and we can consider gravity as acting in the z direction only. This gives us three scalar equations. In the x-direction, the equation is:

$$\sum F_x = ma_x$$

$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx\right) dydz - \sigma_{xx} dydz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dxdz - \tau_{yx} dxdz + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) dxdy - \tau_{zx} dxdy$$

$$= \rho dxdydz \frac{Du}{Dt}$$

Solving:

$$\frac{\partial \sigma_{xx}}{\partial x} dxdydz + \frac{\partial \tau_{yx}}{\partial y} dydxdz + \frac{\partial \tau_{zx}}{\partial z} dzdxdy = \rho dxdydz \frac{Du}{Dt}$$

Dividing by $dxdydz$:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \frac{Du}{Dt}$$

Repeating the process for the y-direction:

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} = \rho \frac{Dv}{Dt}$$

In the z-direction, we also have to include gravitational force: $F_W = mg = \rho dxdydzg$:

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} - \rho g = \rho \frac{Dw}{Dt}$$

These three equations form the general formulation of the momentum equation, also known as the equations of motion for fluid particles. The acceleration terms on the right side of the equation can be expanded following the definition of the material derivative, giving a more commonly used form of the equations:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right)$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} = \rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}\right)$$

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} - \rho g = \rho \left(u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}\right)$$

7.3.3 Differential Form of the Euler Equations

We can use the general formulation of the momentum equation to derive a form of Euler's equations. Let's recall that Euler derived his equations following an inviscid flow assumption. Since viscous forces are neglected in an ideal fluid, the stress tensor becomes:

$$\tau_{ij} = \begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$

As there is no viscosity component for an inviscid flow, the normal stress component is equal to the pressure acting on each face. For an infinitesimally small fluid element, we have already determined that the pressure is the same in all directions. Applying this tensor to the momentum equation yields:

$$-\frac{\partial P}{\partial x} = \rho \frac{Du}{Dt}$$

$$-\frac{\partial P}{\partial y} = \rho \frac{Dv}{Dt}$$

$$-\frac{\partial P}{\partial z} - \rho g = \rho \frac{Dw}{Dt}$$

We can rearrange and write this in vector form (expressing gravitational acceleration as a vector: $\vec{g} = -g\hat{k}$):

$$\rho \frac{D}{Dt}(u\hat{\imath} + v\hat{\jmath} + w\hat{k}) = -\left(\frac{\partial P}{\partial x}\hat{\imath} + \frac{\partial P}{\partial y}\hat{\jmath} + \frac{\partial P}{\partial z}\hat{k}\right) + \rho\vec{g}$$

$$\rho \frac{D\vec{v}}{Dt} = -\nabla P + \rho\vec{g}$$

This is Euler's equation in differential form. In thermodynamics, the equation may be taken in unit-mass terms by dividing over density: $\frac{D\vec{v}}{Dt} = -\nabla w + \vec{g}$ where w is the specific thermodynamic work found by dividing pressure over density.

7.3.4 Differential Form of the Bernoulli Equation

Let's express the material derivative in the Euler equation in terms of the del operator:

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}\right) = -\nabla P + \rho\vec{g}$$

We can use this form of Euler's equation to derive the Bernoulli equation by assuming steady flow ($\frac{\partial \vec{v}}{\partial t} = 0$).

$$\rho(\vec{v} \cdot \nabla)\vec{v} = -\nabla P + \rho\vec{g}$$

Applying the vector identity $(\vec{f} \cdot \nabla)\vec{f} = \frac{1}{2}\nabla(f^2) - \vec{f} \times (\nabla \times \vec{f})$:

$$\rho\left(\frac{1}{2}\nabla(v^2) - \vec{v} \times (\nabla \times \vec{v})\right) = -\nabla P + \rho\vec{g}$$

Dividing by density:

$$\frac{1}{2}\nabla(v^2) - \vec{v} \times (\nabla \times \vec{v}) = -\frac{\nabla P}{\rho} + \vec{g}$$

Recall that the gravity vector is $\vec{g} = -g\hat{k}$. We can express the unit vector $\hat{k}$ in terms of the del operator:

$$\nabla z = \frac{\partial z}{\partial x}\hat{\imath} + \frac{\partial z}{\partial y}\hat{\jmath} + \frac{\partial z}{\partial z}\hat{k} = 0\hat{\imath} + 0\hat{\jmath} + 1\hat{k} = \hat{k}$$

Substituting back into the Euler equation and rearranging gives:

$$\frac{\nabla P}{\rho} + \frac{1}{2}\nabla(v^2) + g\nabla z = \vec{v} \times (\nabla \times \vec{v})$$

Since our flow travels along a stream line, let's define a differential length vector $\overrightarrow{ds}$ that travels in the direction of the streamline. Taking the dot product of $\overrightarrow{ds}$ with the terms in the equation above yields:

$$\frac{\nabla P}{\rho} \cdot \overrightarrow{ds} + \frac{1}{2}\nabla(v^2) \cdot \overrightarrow{ds} + g\nabla z \cdot \overrightarrow{ds} = \vec{v} \times (\nabla \times \vec{v}) \cdot \overrightarrow{ds}$$

Remember that the cross product of two vectors yields a vector normal to both vectors. Therefore $\vec{v} \times (\nabla \times \vec{v})$ is normal to the velocity vector, which makes it normal to the stream vector $\overrightarrow{ds}$. The dot product of two orthogonal vectors is zero, therefore $\vec{v} \times (\nabla \times \vec{v}) \cdot \overrightarrow{ds} = 0$.

Note also that for a scalar function f and a differential length vector $\overrightarrow{ds}$, $\nabla f \cdot \overrightarrow{ds} = df$. Applying this relationship to the equation above yields:

$$\frac{dP}{\rho} + \frac{1}{2}dv^2 + gdz = 0$$

Note that this already looks similar to the derivation we performed in a previous lesson (though the velocity term is slightly different). Integrating this equation along a streamline and dividing by gravitational acceleration yields the Bernoulli equation we are already familiar with.

$$\frac{P}{\gamma} + \frac{v^2}{2g} + z = \text{const.}$$

7.3.5 Navier-Stokes Equations for Newtonian Fluids

Recall the general formulation of the equations of motion:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right)$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}\right)$$

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} - \rho g = \rho\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}\right)$$

When deriving these equations, we assumed that the z-axis was parallel to the gravitational force vector. In order for our equations to apply to axes set in any direction, let's take the gravitational force and express it as a vector with components in the x-, y- and z-directions. Thus, the equations take the form:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x = \rho \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \right)$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y = \rho \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \right)$$

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \rho g_z = \rho \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \right)$$

In this formulation, if the z-axis were aligned with the gravitational force, then $g_x = g_y = 0$ and $g_z = -g$ where g is the gravitational acceleration.

For incompressible Newtonian fluids, the relationships between the stress and strain rate are given by empirical relationships called constitutive equations:

$$\sigma_{xx} = -P + 2\mu\frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -P + 2\mu\frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -P + 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$$

Combining these equations and the conservation of momentum, we arrive at the Navier-Stokes equations, the most significant set of equations in fluid mechanics. Let's start by rearranging and substituting the relevant constitutive equations to the left side of the x-direction equation:

$$\frac{\partial}{\partial x}\left(-P + 2\mu\frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right) + \rho g_x = \rho \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \right)$$

Now we can manipulate algebraically:

$$\rho g_x - \frac{\partial P}{\partial x} + 2\mu\frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \mu\frac{\partial}{\partial y}\frac{\partial u}{\partial y} + \mu\frac{\partial}{\partial y}\frac{\partial v}{\partial x} + \mu\frac{\partial}{\partial z}\frac{\partial w}{\partial x} + \mu\frac{\partial}{\partial z}\frac{\partial u}{\partial z} = \rho \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \right)$$

$$\rho g_x - \frac{\partial P}{\partial x} + \mu\frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \mu\frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \mu\frac{\partial}{\partial y}\frac{\partial u}{\partial y} + \mu\frac{\partial}{\partial y}\frac{\partial v}{\partial x} + \mu\frac{\partial}{\partial z}\frac{\partial w}{\partial x} + \mu\frac{\partial}{\partial z}\frac{\partial u}{\partial z} = \rho \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \right)$$

…and rearrange:

$$\rho g_x - \frac{\partial P}{\partial x} + \mu \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \mu \frac{\partial}{\partial y}\frac{\partial u}{\partial y} + \mu \frac{\partial}{\partial z}\frac{\partial u}{\partial z} + \mu \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \mu \frac{\partial}{\partial y}\frac{\partial v}{\partial x} + \mu \frac{\partial}{\partial z}\frac{\partial w}{\partial x} = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right)$$

In mathematics, a function is symmetrical if the second-order partial derivatives with respect to two different variables is the same regardless of the order of derivation. In other words: $\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$. Most physical functions are symmetric, as is the case for our equations. Therefore, let's change the order of operation for the second-order partial derivatives:

$$\rho g_x - \frac{\partial P}{\partial x} + \mu \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \mu \frac{\partial}{\partial y}\frac{\partial u}{\partial y} + \mu \frac{\partial}{\partial z}\frac{\partial u}{\partial z} + \mu \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \mu \frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \mu \frac{\partial}{\partial x}\frac{\partial w}{\partial z} = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right)$$

Factorize:

$$\rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial}{\partial z}\frac{\partial u}{\partial z}\right) + \mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right)$$

$$\rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right)$$

We can substitute the divergence of the velocity vector: $\nabla \cdot \vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$:

$$\rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\frac{\partial}{\partial x}(\nabla \cdot \vec{v}) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

Now we can repeat the same process in the y- and z-directions:

$$\rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \mu\frac{\partial}{\partial y}(\nabla \cdot \vec{v}) = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \mu\frac{\partial}{\partial z}(\nabla \cdot \vec{v}) = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Adding all three terms gives:

$$\rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\frac{\partial}{\partial x}(\nabla \cdot \vec{v}) + \rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \mu\frac{\partial}{\partial y}(\nabla \cdot \vec{v}) + \rho g_z$$

$$- \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \mu\frac{\partial}{\partial z}(\nabla \cdot \vec{v})$$

$$= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$+ \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Rearranging:

$$\rho g_x + \rho g_y + \rho g_z - \frac{\partial P}{\partial x} - \frac{\partial P}{\partial y} - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$$+ \mu\frac{\partial}{\partial x}(\nabla \cdot \vec{v}) + \mu\frac{\partial}{\partial y}(\nabla \cdot \vec{v}) + \mu\frac{\partial}{\partial z}(\nabla \cdot \vec{v})$$

$$= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$+ \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Factorizing:

$$\rho(g_x + g_y + g_z) - \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)P + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$$+ \mu\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)(\nabla \cdot \vec{v})$$

$$= \rho\left(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} + \frac{\partial w}{\partial t} + u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + v\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) + w\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} + \frac{\partial w}{\partial z}\right)\right)$$

Writing in vector form:

$$\rho\vec{g} - \nabla P + \mu\left(\frac{\partial^2 \vec{v}}{\partial x^2} + \frac{\partial^2 \vec{v}}{\partial y^2} + \frac{\partial^2 \vec{v}}{\partial z^2}\right) + \mu\nabla(\nabla \cdot \vec{v}) = \rho\left(\frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z}\right)$$

Substituting $u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z} = \vec{v} \cdot \nabla\vec{v}$:

$$\rho\vec{g} - \nabla P + \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{v} + \mu\nabla(\nabla \cdot \vec{v}) = \rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla\vec{v}\right)$$

Substituting $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla \cdot \nabla = \nabla^2$:

$$\rho\vec{g} - \nabla P + \mu\nabla^2\vec{v} + \mu\nabla(\nabla \cdot \vec{v}) = \rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla\vec{v}\right)$$

Substituting $\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla\vec{v}$:

$$\rho\frac{D\vec{v}}{Dt} = \rho\vec{g} - \nabla P + \mu\nabla^2\vec{v} + \mu\nabla(\nabla \cdot \vec{v})$$

Remember that the constitutive equations we used apply for incompressible flow, where the divergence is $\nabla \cdot \vec{v} = 0$:

$$\rho\frac{D\vec{v}}{Dt} = -\nabla P + \rho\vec{g} + \mu\nabla^2\vec{v}$$

This is the Navier-Stokes equation for incompressible flow of a Newtonian fluid. For compressible fluids, the term $\lambda\nabla \cdot \vec{v}$ is added to the right side of the constitutive equations for normal stresses, where λ is the second-coefficient of viscosity. Furthermore, note that we have treated viscosity as a constant since we are assuming our fluid is Newtonian.

7.4 Potential Flow Hydrodynamics

7.4 Vorticity Equation

Recall that the material derivative can be expressed in terms of the del operator. This means that acceleration can take the form:

$$\vec{a} = \frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}$$

Recall that vorticity is calculated as twice the angular velocity. We can now use the definition of the del operator to express vorticity as the curl of the velocity vector:

$$\vec{\omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\imath} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = \nabla \times \vec{v}$$

We use this definition to study vorticity using the Navier-Stokes equations by taking their curl:

$$\nabla \times \rho \frac{D\vec{v}}{Dt} = \nabla \times (-\nabla P + \rho \vec{g} + \mu \nabla^2 \vec{v})$$

$$\nabla \times \rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}\right) = -\nabla \times \nabla P + \rho \nabla \times \vec{g} + \mu \nabla \times \nabla^2 \vec{v}$$

The curl of the gradient of a scalar function is zero: $-\nabla \times \nabla P = 0$. The curl of a constant is zero: $\rho \nabla \times \vec{g}$. We can also apply the following relationships:

$$\nabla \times \frac{\partial \vec{v}}{\partial t} = \frac{\partial}{\partial t}(\nabla \times \vec{v}) = \frac{\partial \vec{\omega}}{\partial t}$$

$$\nabla \times \nabla^2 \vec{v} = \nabla^2(\nabla \times \vec{v}) = \nabla^2 \vec{\omega}$$

$$\nabla \times \left((\vec{v} \cdot \nabla)\vec{v}\right) = (\vec{v} \cdot \nabla)\vec{\omega} - (\vec{\omega} \cdot \nabla)\vec{v}$$

If we treat our fluid as incompressible and Newtonian, then the curl of the Navier-Stokes equations takes the form:

$$\rho \left(\frac{\partial \vec{\omega}}{\partial t} + (\vec{v} \cdot \nabla)\vec{\omega} - (\vec{\omega} \cdot \nabla)\vec{v}\right) = \mu \nabla^2 \vec{\omega}$$

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla)\vec{v} + \nu \nabla^2 \vec{\omega}$$

Where ν is the kinematic viscosity. This equation is known as the vorticity equation. Notice that the vorticity equation does not consider pressure nor gravitational forces.

7.5 Differential Conservation of Energy
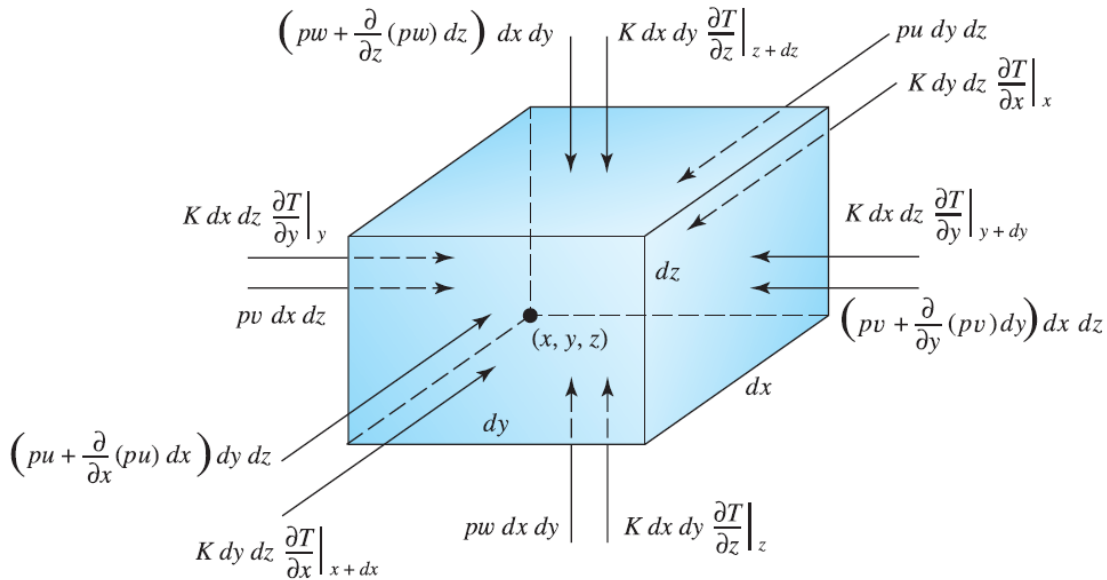
7.5.1 Differential Energy Equation

The heat transfer rate through an area A is calculated with Fourier's law of heat transfer:

$$\dot{Q} = -KA\frac{\partial T}{\partial n}$$

where K is the thermal conductivity (assumed to be a constant), T is the temperature, and n is the direction normal to the area. The work rate through A due to pressure forces is:

$$\dot{W} = PAv$$

If we study an infinitesimal fluid element that is inviscid, we will have the following heat transfer and work terms:

We can write the first law of thermodynamics considering the terms in the infinitesimal element:

$$\dot{Q} - \dot{W} = \frac{DE}{Dt}$$

$$Kdydz\left(\frac{\partial T}{\partial x}\bigg|_{x+dx} - \frac{\partial T}{\partial x}\bigg|_{x}\right) - \frac{\partial}{\partial x}(Pu)dxdydz + Kdxdz\left(\frac{\partial T}{\partial y}\bigg|_{y+dy} - \frac{\partial T}{\partial y}\bigg|_{y}\right) - \frac{\partial}{\partial y}(Pv)dxdydz$$

$$+ Kdxdy\left(\frac{\partial T}{\partial z}\bigg|_{z+dz} - \frac{\partial T}{\partial z}\bigg|_{z}\right) - \frac{\partial}{\partial z}(Pw)dxdydz = \rho dxdydz\frac{D}{Dt}\left(\frac{u^2 + v^2 + w^2}{2} + gz + \tilde{u}\right)$$

Dividing by $dxdydz$, knowing that $\frac{\left(\frac{\partial T}{\partial x}\big|_{x+dx} - \frac{\partial T}{\partial x}\big|_{x}\right)}{dx} = \frac{\partial^2 T}{\partial x^2}$, and rearranging:

$$K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) - P\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - u\frac{\partial P}{\partial x} - v\frac{\partial P}{\partial y} - w\frac{\partial P}{\partial z}$$

$$= \rho u \frac{Du}{Dt} + \rho v \frac{Dv}{Dt} + \rho w \frac{Dw}{Dt} + \rho g \frac{Dz}{Dt} + \rho \frac{D\tilde{u}}{Dt}$$

Since we are studying an inviscid element, we can apply Euler's scalar equations so that

$$-u\frac{\partial P}{\partial x} = \rho u \frac{Du}{Dt}$$

$$-v\frac{\partial P}{\partial y} = \rho v \frac{Dv}{Dt}$$

$$-w\frac{\partial P}{\partial z} - \rho g w = \rho w \frac{Dw}{Dt}$$

where $\frac{Dz}{Dt} = w$. This allows us to reduce our equation to:

$$\rho\frac{D\tilde{u}}{Dt} = K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) - P\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

Which can be then simplified by using the del operator:

$$\rho\frac{D\tilde{u}}{Dt} = K\nabla^2 T - P\nabla \cdot \vec{v}$$

This is the differential form of the energy equation.

7.5.2 Energy in Liquids

For liquids, we use $\nabla \cdot \vec{v} = 0$ and $\tilde{u} = c_P T$ where c_P is the specific heat. The energy equation simplifies to:

$$\rho c_P \frac{DT}{Dt} = K\nabla^2 T$$

Define thermal diffusivity as:

$$\alpha = \frac{K}{\rho c_P}$$

which in turn allows us to express the energy equation for incompressible liquids as:

$$\frac{DT}{Dt} = \alpha \nabla^2 T$$

7.5.3 Energy in Gases

For gases, we can write the energy equation in terms of enthalpy by using $\tilde{u} = h - \frac{P}{\rho}$ and rearranging

$$\rho \frac{D}{Dt}\left(h - \frac{P}{\rho}\right) + P\nabla \cdot \vec{v} = K\nabla^2 T$$

$$\rho\left(\frac{Dh}{Dt} - \frac{1}{\rho}\frac{DP}{Dt} + \frac{P}{\rho^2}\frac{D\rho}{Dt}\right) + P\nabla \cdot \vec{v} = K\nabla^2 T$$

$$\rho\frac{Dh}{Dt} - \frac{DP}{Dt} + \frac{P}{\rho}\frac{D\rho}{Dt} + P\nabla \cdot \vec{v} = K\nabla^2 T$$

From the continuity equation (the form that uses the material derivative) we get that $\nabla \cdot \vec{v} = -\frac{1}{\rho}\frac{D\rho}{Dt}$. Inserting:

$$\rho\frac{Dh}{Dt} - \frac{DP}{Dt} + \frac{P}{\rho}\frac{D\rho}{Dt} - \frac{P}{\rho}\frac{D\rho}{Dt} = K\nabla^2 T$$

$$\rho\frac{Dh}{Dt} - \frac{DP}{Dt} = K\nabla^2 T$$

This allows the energy equation to take the special form:
$$\rho\frac{Dh}{Dt} = K\nabla^2 T + \frac{DP}{Dt}$$

For gases, $\left|\frac{DP}{Dt}\right| \ll |P\nabla \cdot \vec{v}|$, so the equation can simplify to:

$$\rho\frac{Dh}{Dt} = K\nabla^2 T$$

For ideal gases, $dh = c_P dT$ and so:

$$\rho c_P \frac{DT}{Dt} = k\nabla^2 T$$

7.5.4 Energy in Viscous Flow

So far, the equations derived in this section have assumed inviscid flow. If we have to consider viscous forces, then the energy equation will take the form

$$\rho\frac{D\tilde{u}}{Dt} = K\nabla^2 T - P\nabla \cdot \vec{v} + \vec{\Phi}$$

where

$$\vec{\Phi} = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right]$$