A. Demonstrații și Definiții

1. Spațiu vectorial

$(V,+,\cdot)_{\mathbb{K}}$ sp. vect $\longrightarrow$ $\left| \begin{array}{ll} + : V \times V \to V & \text{leg. int.} \\ \cdot : \mathbb{K} \times V \to V & \text{leg. ext} \end{array} \right.$

1. $(V,+)$ grup. abelian.
2. $a(bx) = (ab)x$
3. $(a+b)x = ax+bx$
4. $a(x+y) = ax+ay$.
5. $1_{\mathbb{K}} \cdot x = x$

2. Subspațiu vectorial

Combinație liniară: $\forall a, b \in \mathbb{K}, \forall x, y \in V' \Rightarrow ax+by \in V'$.

3. Liniară

$f(ax+by) = af(x) + bf(y)$.

4. Izo(inj + surj

1. Injectivă: $\ker f = \{0_V\}$, $\dim V_1 = \dim \operatorname{Im} f$.
2. Surjectivă: $\dim \operatorname{Im} f = \dim V_2$, $\dim V_1 = \dim \ker f + \dim V_2$.
3. Bijectivă: $\dim V_1 = \dim V_2 = \operatorname{rg} A$.
4. Izomorfism $\Leftrightarrow$ Bijecție. aka. $\det A \neq 0$. $(\exists y = AX, A^{-1})$.

5. Sumă directă

$V_1 \oplus V_2 \Leftrightarrow V_1 \cap V_2 = \{0_V\}$.

$\circledast$ Se folosește T. Grassmann.

B. Teoreme

1. T. Schimbului

Fie $(V,+,\cdot)_{\mathbb{K}}$ un ~~sp~~ sp. vect. finit generat.

Fie $\left.\begin{array}{l} \{x_1, \dots x_n\} \text{ SG} \\ \{y_1, \dots y_n\} \text{ SLI} \end{array}\right| \Rightarrow \{y_1, \dots y_n\}$ SG.

2. T. Crit. de liniar independență

S SLI $\Leftrightarrow$ rangul matricei componentelor vectorilor din S în rap. cu orice reper e maxim

3. T. Grassmann

$\dim_{\mathbb{K}} (V_1 + V_2) = \dim_{\mathbb{K}} V_1 + \dim_{\mathbb{K}} V_2 - \dim_{\mathbb{K}} V_1 \cap V_2$

4. T. Dimensiunii

$\dim V_1 = \dim \ker f + \dim \operatorname{Im} f$.

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① $f:\mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1 - x_2 + x_3,\ x_1 - x_2 + x_3,\ x_3)$

a) $A = [f]_{R_0, R_0}$

$y = Ax$, $A = [f]_{R_0,R_0} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

b) val. pr, subsp. pr, repere, diag.

$P(\lambda) = \det(A - \lambda I_3) = 0$

$\begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & -1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - 1)(\lambda)^2 = 0 \Rightarrow \lambda_1 = 1,\ m_1 = 1$; $\lambda_2 = 0,\ m_2 = 2$.

$V_{\lambda_1} = \{x \in \mathbb{R} \mid f(x) = \lambda_1 x = x\} = \langle\{(1,1,1)\}\rangle$.

$\begin{cases} x_1 - x_2 + x_3 = x_1 \\ x_1 - x_2 + x_3 = x_2 \\ x_3 = x_3 \end{cases} \Rightarrow \begin{cases} x_2 = x_3 \\ x_1 = x_2 \\ x_3 = x_3 \end{cases} \Rightarrow x_1 = x_2 = x_3$

$R_1 = \{(1,1,1)\}$ SG + SLi, din $V_{\lambda_1} = m_1$ ①.
$\to$ reper in V_{λ_1}

$V_{\lambda_2} = \{x \in \mathbb{R} \mid f(x) = \lambda_2 x = 0\} = \ker f = \langle\{(1,1,0)\}\rangle$

$\begin{cases} x_1 - x_2 + x_3 = 0 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_2 \\ x_3 = 0 \end{cases}$

$R_2 = \{(1,1,0)\}$ SG + SLi, din $V_{\lambda_2} \neq m_2$ ②
$\to$ reper în V_{λ_2}

①, ② $\Rightarrow f$ nu e endom. diagonalizabil.

c). $\ker f$, $\operatorname{Im} f$, repere.

$\ker f = \{x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3}\} = S(A)$, din $\ker f = 3 - \operatorname{rg} A = 3 - 2 = 1$.

$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & \boxed{0} & 1 \end{pmatrix}$ $\operatorname{rg} A = 2 \Rightarrow \ker f = \{(x_1, x_1, 0) \mid x_1 \in \mathbb{R}\} = \langle\{(1,1,0)\}\rangle$

Teorema dimensiunii: $\dim \mathbb{R}^3 = \dim \ker f + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 2$.

$\operatorname{Im} f = \{y \in \mathbb{R}^3 \mid f(x) = y\}$

$\begin{cases} x_1 - x_2 + x_3 = y_1 \\ x_1 - x_2 + x_3 = y_2 \\ x_3 = y_3. \end{cases}$ $\quad A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$\operatorname{rg} A = \operatorname{rg} \bar{A} \Rightarrow y_1 = y_2$

$\operatorname{Im} f = \langle\{(1,1,0), (0,0,1)\}\rangle$

$R_1 = \{(1,1,0)\}$ reper în $\ker f$ (SG + SLi)

$R_2 = \{(1,1,0), (0,0,1)\}$ reper în $\operatorname{Im} f$ (SG + SLi (rg = 2 maxim)).

$rg\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3 \Rightarrow$ $R = \ker f \cup \{e_1, e_3\}$ reper în $\mathbb{R}^3$

$W = \{e_1, e_3\}$.

$(1,0,3) = \underbrace{a(1,1,0)}_{v_1 \in \ker f} + \underbrace{b(1,0,0) + c(0,0,1)}_{v_2 \in W} = (a+b, a, c)$

Calcule $\Rightarrow a = 0,\ b = 1,\ c = 3.$

$v_1 \in \ker f,\ v_1 = (0,0,0)$

$v_2 \in W,\ v_2 = (1,0,3)$ $\qquad p(1,0,3) = (0,0,0).$

$s = 2p - id_{\mathbb{R}^3} \Rightarrow s(1,0,3) = (0,0,0) - (1,0,3) = (-1,0,-3).$

(2)

$$\begin{tikzcd} R \arrow[r, "A'"] & R \arrow[d, "C^{-1}"] \\ R_0 \arrow[u, "C"] \arrow[r, "A"] & R_0 \end{tikzcd}$$

$A = [f]_{R_0 R_0}$

$C = (e^1, e^2, e^3) \Rightarrow C^{-1}$ calcule

$A' =$ matricea diagonală

(3)

$g : \mathbb{R}^3 \to \mathbb{R}^3,\ g(x,y) = x_1y_1 - x_2y_2 - x_1y_3 - x_3y_1 + 2x_2y_3 + 2x_3y_2$

a) $g \in L^s(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

b) $G = ?$

c) $\ker g = ?$ g nedegenerată

d) $G' = ?$ (rap. cu $e_1' = (1,1,1),\ e_2' = (1,2,1),\ e_3' = (0,0,1)$)

a) b) $G = \begin{array}{c|c} & \begin{matrix} y_1 & y_2 & y_3 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} & \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix} \end{array}$

formă liniară $\boxed{X^T G Y} \Rightarrow g$ biliniară

$G = G^T$

$\Rightarrow g \in L^s(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

$$\boxed{\begin{gathered} g_{ij} = g(e_i, e_j) \\ g(x_1e_1 + x_2e_2, y_1e_1 + y_2e_2) \end{gathered}}$$

c) $\ker g = \{x \in \mathbb{R}^3 \mid g(x,y) = 0\ \forall\, y \in \mathbb{R}^3\}.$

$g(x, e_1) = 0 \Rightarrow x_1 - x_3 = 0$

$g(x, e_2) = 0 \Rightarrow -x_2 + 2x_3 = 0$

$g(x, e_3) = 0 \Rightarrow -x_1 + 2x_2 = 0$

$B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix}$

$\det G = 1 - 4 = -3 \neq 0$

g nedeg. $\Leftrightarrow \ker g = \{0_{\mathbb{R}^3}\} \Leftrightarrow \det G \neq 0$

d) $R_0 \xrightarrow{C} R'$

$C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ $\qquad G' = C^T G C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 3 & 3 \\ 1 & 3 & 0 \end{pmatrix}$

...Gauss, Jacobi, transf. ortog. + F. canonica ... $(x_1-a)^2+(x_2-b)^2=r^2$

(C) ④ $f \in End(\mathbb{R}^3)$ $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1.$
$v_1 = (-3, 2, 1), v_2 = (2, 1, 0), v_3 = (-6, 3, 1)$

$f(v_1) = \lambda_1 v_1$
$f(v_2) = \lambda_2 v_2$ $\quad R = \{v_1, v_2, v_3\}$ SLI
$f(v_3) = \lambda_3 v_3$ $\quad |R| = 3 = \dim_{\mathbb{R}} \mathbb{R}^3.$

$$A' = [f]_{R,R} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_0 \xrightarrow{C} R.$$

$A' = C^{-1}AC. \Rightarrow A = CA'C^{-1}$
$v_1 = -3e_1 + 2e_2 + e_3$
$v_2 = 2e_1 + e_2$
$v_3 = -6e_1 + 3e_2 + e_3$

$$C = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix} \qquad C^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & -3 \\ -1 & -2 & 1 \end{pmatrix}$$

$A = CA'C^{-1}$ (calcule)

⑤ $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (2x_1 + 2x_2, x_1 + x_3, x_1 + 3x_2 - 2x_3)$

a) $f|_{V'}: V' \to V''$ izo $\quad$, $V' = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\}$, $V'' = \{(\sim) \in \mathbb{R}^3 \mid 3x_1 - 4x_2 - 2x_3 = 0\}$
b) $f(V' \cap V'')$

a) $V' = \langle\{(1,0,1), (0,1,1)\}\rangle$ SG + SLi $\Rightarrow R' = \{(1,0,1), (0,1,1)\}$ - reper în V'
↳ rg. maxim (SLI)

$f(1,0,1) = (2,2,-1) \in V''$
$f(0,1,1) = (2,1,1) \in V''$
$R'' = \{(2,2,-1), (2,1,1)\}$ reper în V''

b) $V' \cap V'' = \{x \in \mathbb{R}^3 \mid \begin{cases} x_1 + x_2 - x_3 = 0 \\ 3x_1 - 4x_2 - 2x_3 = 0 \end{cases}\}$

$$B = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 3 & -4 & -2 & 0 \end{array}\right) \quad \text{rg}B = 2 \qquad \dim V' \cap V'' = 3 - 2 = 1.$$

$V' \cap V'' = \langle\{6, 1, 7\}\rangle$ SG + SLi (un vector)

$f(V' \cap V'') = \langle\{f(6,1,7)\}\rangle = \langle\{(14, 13, -5)\}\rangle$

⑥ $(\mathbb{R}^3, +, \cdot)_{/\mathbb{R}}$ $\quad S = \{(1,2,3), (-1,1,5)\}$
$S' = \{(1,5,11), (2,1,-2), (3,6,9)\}$

a) $\langle S \rangle = \langle S' \rangle = V'$
b) V' printr-un SEC.
c) V'' ai $\mathbb{R}^3 = V' \oplus V''$

a) $\text{rg}\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 5 \end{pmatrix} = 2$ (maxim) $\xRightarrow{\text{astfel}}$ S SLi $\Rightarrow \dim \langle S \rangle = 2.$

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 5 & 1 & 6 \\ 11 & -2 & 9 \end{pmatrix} = 0$$

$S'' = \{(1,5,11), (2,1,-2)\}$ SLi maximal în S'
$\langle S' \rangle = \langle S'' \rangle \Rightarrow \dim \langle S' \rangle = 2.$

$(1,2,3)\ v=(-1,1,5)$
$=(1,5,11),\ v'=(2,1,-2)$

$u\in S' \Leftrightarrow \exists\, a,b\in\mathbb{R}$ a.î. $u=au'+bv'$

$$\begin{cases} a+2b=1\\ 5a+b=2\\ 11a-2b=3\end{cases}$$

$rg\,A = rg\,\bar{A}=2 \Rightarrow SCD.$

$v\in S' \Leftrightarrow \exists\, a',b'\in\mathbb{R}$ a.î. $v=a'u'+b'v'$

$$\begin{cases} a'+2b'=1\\ 5a'+b'=1\\ 11a'-2b'=5\end{cases}$$

$rg\,A = rg\,\bar{A}=2 \Rightarrow SCD$

$\langle S\rangle \subset \langle S'\rangle$
$\dim S = \dim S' = 2$ $\Big| \Rightarrow \langle S\rangle = \langle S'\rangle$.

b) $V'=\langle\{(1,5,11),(2,1,-2)\}\rangle=\{(x,y,z)\in\mathbb{R}^3 \mid \exists\, a,b\in\mathbb{R}$ a.î. $(x,y,z)=a(1,5,11)+b(2,1,-2)\}$

$$\begin{cases} a+2b=x\\ 5a+b=y\\ 11a-2b=z.\end{cases}$$

$rg\,A = rg\,\bar{A}=2 \Rightarrow \Delta_c=0 \Rightarrow \begin{vmatrix}1&2&x\\5&1&y\\11&-2&z\end{vmatrix} = -21x+24y-9z=0.$

$V'=\{(x,y,z)\in\mathbb{R}^3 \mid -7x+8y-3z=0\}.$

c) $R'=\{(1,5,11),(2,1,-2)\}$ rep. în V'

$rg\begin{pmatrix}1&2&1\\5&1&0\\11&-2&0\end{pmatrix}=3$ $\quad R'\cup\{e_1\}$ reper în $\mathbb{R}^3$
$V''=\langle\{e_1\}\rangle.$

7) $R_0=\{e_1=1, e_2=x, e_3=x^2\}$, $R'=\{-1+2x+3x^2,\ x-x^2,\ x-2x^2\}$.
$(\mathbb{R}_2[x],+,\cdot)/\mathbb{R}$

a) R' reper în $\mathbb{R}_2[x]$, $A=?$ $(R_0 \xrightarrow{A} R')$.
b) coord. lui $P=3-x+x^2$ rap. cu R'

a) $A=\begin{pmatrix}-1&0&0\\2&1&1\\3&-1&-2\end{pmatrix}$ $\quad \det A = 1\neq 0 \Rightarrow rg\,A=3$ (max) $\xRightarrow{C.N.S.L.I}$ R' SLI
$\dim_{\mathbb{R}} \mathbb{R}_2[x]=3=|R'| \Rightarrow R'$ SG $\Bigg| \Rightarrow R'$ reper în $\mathbb{R}_2[x]$

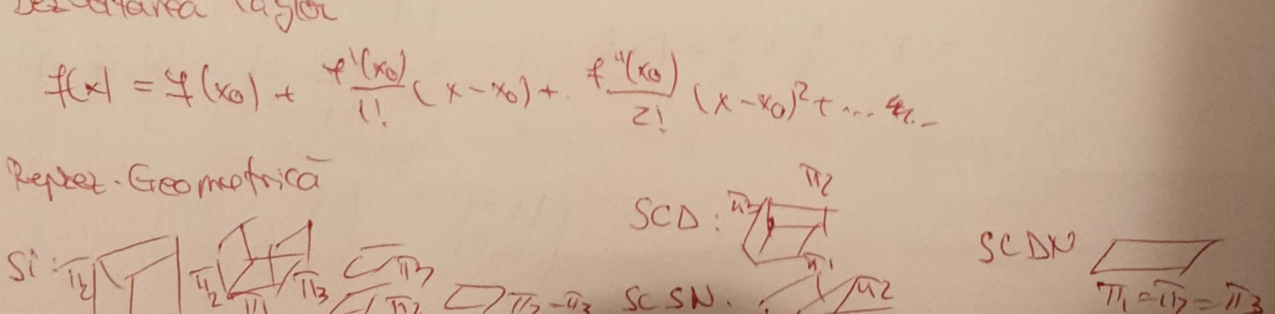
b) $P=3-x+x^2$
$3-x+x^2 = x_1'e_1'+x_2'e_2'+x_3'e_3'$
$= x_1'(-1+2x+3x^2)+x_2'(x-x^2)+x_3'(x-2x^2)$

$$\begin{cases} -x_1'=3\\ 2x_1'+x_2'+x_3'=-1\\ 3x_1'-x_2'-2x_3'=1\end{cases} \Rightarrow (x_1',x_2',x_3')=(-3,20,-15)$$ coord. în rap. cu R'

Dezvoltarea Taylor

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\dots$$

Reprez. Geometrică



OBSERVAȚII

- SCI – admite doar $(0,0,0\ldots)$
- SLD – admite sol. nenule
- Th. H–C: $A^3 - \underset{\text{tr.}}{\sigma_1} A + \underset{S}{\sigma_2} A - \underset{\text{det.}}{\sigma_3} I_3 = O_3$

- Laplace. $(\times 2)$.

$$\det A = \begin{vmatrix} a & -b & -a & b \\ a & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{vmatrix} = \begin{vmatrix} a & -b \\ b & a \end{vmatrix} (-1)^{1+2+1+2} \cdot \begin{vmatrix} c & -d \\ d & c \end{vmatrix} + \begin{vmatrix} a & -a \\ b & -b \end{vmatrix} (-1)^{1+2+1+3} \cdot \begin{vmatrix} -d & -d \\ c & c \end{vmatrix}$$

$$+ \ldots$$

- $\ker f = \{0_V\} \Rightarrow \dim \ker f = 0$.

$$\begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1 & 2 & 1 \\ x_2 & 2 & 3 & 2 \\ x_3 & 1 & 2 & 1 \end{array}$$

$$g = x_1 y_1 + 2x_1 y_2 + x_1 y_3 + x_2 y_1 + 3x_2 y_2 + \ldots$$

Lucrare I (14.05.2020).

① $(\mathbb{R}^3, +, \cdot)_{/\mathbb{R}}$, $S = \{(1,2,3), (0,1,1), (4,8,12), (0,-1,-1)\}$

a) S' SLI maximal din S

b) extindere la $\mathbb{R}^3$

a) $rg\begin{pmatrix} 1 & 0 & 4 & 0 \\ 2 & 1 & 8 & -1 \\ 3 & 1 & 12 & -1 \end{pmatrix} = 2. \Rightarrow S' = \{(1,2,3),(0,1,1)\}$ SLI maximal în S

$d_2 = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1.$

$d_3' = \begin{vmatrix} 1 & 0 & 4 \\ 2 & 1 & 8 \\ 3 & 1 & 12 \end{vmatrix} = 20 - 20 = 0.$

$d_3'' = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{vmatrix} = 0 \quad (C_2' = (-1)C_3)$

b) $\det\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & 0 \end{vmatrix} = 2 - 3 = -1 \neq 0 \Rightarrow S' \cup \{e_1\} = SLI$

$R = \{(1,2,3), (0,1,1), (1,0,0)\}$ reper în $\mathbb{R}^3$

$\left.\begin{array}{l} \text{card } R = \dim_{\mathbb{R}} \mathbb{R}^3 = 3 \\ R \text{ SLI} \end{array}\right| \Rightarrow R$ reper

② $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1, x_1, x_1 + x_2 + x_3)$

a) $\ker f = ?$

b) $\operatorname{Im} f = ?$

a) $\ker f = \{x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3}\} = S(A)$, $\dim \ker f = 3 - rg A = 1$

$\begin{cases} x_1 = 0 \\ x_1 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases}$ $A = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right)$

$\det A = 0.$

$\begin{cases} x_1 = 0 \\ x_1 + x_2 = -x_3 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = x_3 \end{cases}$

$\ker f = \{(0, -x_3, x_3) \mid x_3 \in \mathbb{R}\} = \langle\{0,-1,1\}\rangle$

b) T. dimensiunii: $\dim \mathbb{R}^3 = \dim \ker f + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 2.$

$\operatorname{Im} f = \{y \in \mathbb{R}^3 / \exists x \in \mathbb{R}^3 \text{ a.î. } f(x) = y\}. = \{y \in \mathbb{R}^3 / y_1 + y_2 + y_3 = 0\}$

$\begin{cases} x_1 = y_1 \\ x_1 = y_2 \\ x_1 + x_2 + x_3 = y_3 \end{cases}$ $A = \left(\begin{array}{ccc|c} 1 & 0 & 0 & y_1 \\ 1 & 0 & 0 & y_2 \\ 1 & 1 & 1 & y_3 \end{array}\right)$

$\det A = 0$

$rg A = 2 \Rightarrow rg \bar{A} = 2 \Rightarrow \Delta_c = 0. \Rightarrow \begin{vmatrix} 1 & 0 & y_2 \\ 1 & 1 & y_3 \\ 1 & 0 & y_1 \end{vmatrix} = 0 \Rightarrow y_1 - y_2 = 0.$

$\mathrm{Im} f = \{(y_1, y_1, y_3)\} = \{(y_1, y_1, 0) + (0, 0, y_3)\} =$
$= \{y_1(1,1,0) + y_3(0,0,1) \mid y_1, y_2 \in \mathbb{R}\} = <\{(1,1,0), (0,0,1)\}$.

④ $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x) = (x_1 + x_2, x_1 - x_2)$

$[f]_{R_1, R_1}$, $R_1 = \{(1,1), (0,2)\}$

$f(1,1) = (2,1) = a(1,1) + b(0,2) \Rightarrow a = 2$, $b = -1/2$

$f(0,2) = (2,-2) = a(1,1) + b(0,2) \Rightarrow a = 2, b = -2$.

$$[f]_{R_1, R_1} = \begin{pmatrix} 2 & 2 \\ -\frac{1}{2} & -2 \end{pmatrix}$$

⑤ $f: \mathbb{R}_2[X] \to \mathbb{R}_1[X]$, $f(P) = P'$

$[f]_{R_1, R_2}$ $R_1 = \{1, x, x^2\}$ rep. in $\mathbb{R}_2[X]$, $R_2 = \{1, x\}$

$f(1) = 0 = 0 \cdot 1 + 0 \cdot X$
$f(x) = 1 = 1 \cdot 1 + 0 \cdot X$
$f(x^2) = 2x = 0 \cdot 1 + 2 \cdot X$

$$[f]_{R_1, R_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

⑥ $(\mathbb{R}^3, +, \cdot)_{/\mathbb{R}}$, $U = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$.

a) reper în U.
b) subsp. $W \subset \mathbb{R}^3$ a.î. $\mathbb{R}^3 = U \oplus W$.

a) $x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3$

$U = \{(-x_2 - x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\} = <\{(-1,1,0), (-1,0,1)\}>$

$R = \{(-1,1,0), (-1,0,1)\}$ SG

$\mathrm{rg}\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$ (maxim) $\Rightarrow R$ SLI

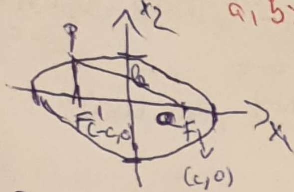
$\Rightarrow R$ reper în U.

b) $\mathrm{rg}\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 3 \Rightarrow R' = \{(-1,1,0), (-1,0,1), (1,0,0)\} =$
$= U \oplus W$

R' reper în $\mathbb{R}^3$, $W = \{(1,0,0)\} = \{e_1\}$

④ Conice

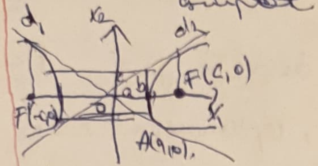
ELIPSA $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1$, $a,b>0$.



$PF+PF'=2a$

HIPERBOLA $\frac{x_1^2}{a^2}-\frac{x_2^2}{b^2}=1$, $a,b>0$.

$d_1 \cup d_2 : x_2=\pm\frac{b}{a}x_1$ asimptote

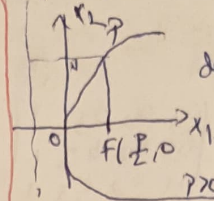


$|PF-PF'|=2a$, $a>0$

PARABOLA $x_2^2=2px_1$

d – dr. directoare

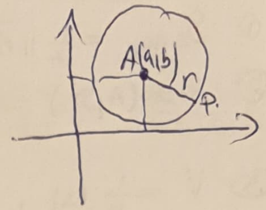
$d: x_1=-\frac{p}{2}$



$dist(P,d)=PF'$

$p>0$.

CERCUL $(x_1-a)^2+(x_2-b)^2=r^2$



$PA=r$.

exemplu

F.G: $f(x_1,x_2)=a_{11}x_1^2+2a_{12}x_1x_2+a_{22}x_2^2+2b_1x_1+2b_2x_2+c=0$.

$e=\frac{c}{a}$

Hip: $c^2=a^2+b^2$

Elip: $c^2=a^2-b^2$

$\Gamma: f(x_1,x_2)=5x_1^2+8x_1x_2+5x_2^2-18x_1-18x_2+9=0$

$\mathcal{R}\{0;e_1,e_2\} \xrightarrow[\text{translație}]{\theta} \mathcal{R}'=\{P_0;e_1,e_2\} \xrightarrow[\text{rotație}]{\tau} \mathcal{R}''=\{P_0;e_1',e_2'\}$

$A=\begin{pmatrix}a_{11}&a_{12}\\a_{12}&a_{22}\end{pmatrix}=\begin{pmatrix}5&4\\4&5\end{pmatrix}$ $\delta=\det A=9 \neq 0 \Rightarrow$ Centru unic

P_0 centru $\begin{cases}10x_1+8x_2-18=0\\8x_1+10x_2-18=0\end{cases}$ ← deriv parțiale.

$\Rightarrow x_1=x_2=1 \Rightarrow P_0(1,1)$.

$\tilde{A}=\begin{pmatrix}A&B^T\\B&c\end{pmatrix}=\begin{pmatrix}5&4&-9\\4&5&-9\\-9&-9&9\end{pmatrix}$ $\Delta=\det\tilde{A}=-81$.

$\theta: x=x'+x_0 \Leftrightarrow \begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}x_1'\\x_2'\end{pmatrix}+\begin{pmatrix}x_1^0\\x_2^0\end{pmatrix}=\begin{pmatrix}x_1'\\x_2'\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}$.

$\theta(\Gamma): 5x_1'^2+8x_1'x_2'+5x_2'^2+\underbrace{\frac{\Delta}{\delta}}_{=-9}=0$, ← aducem la f. canonică

$Q:\mathbb{R}^2\to\mathbb{R}^2$, $Q(x)=5x_1'^2+8x_1'x_2'+5x_2'^2$

$P(\lambda)=\det(A-\lambda I_2) \Rightarrow \lambda_1=9$, $\lambda_1=1$. ← dacă ajut la 7l ⇒ Gram-Schmidt (dim=2)

– calcule – $\Rightarrow V_{\lambda_1}=\{x\in\mathbb{R}^2 \mid Ax=9x\}=\langle\{(1,1)\}\rangle \Rightarrow e_1'=\frac{1}{\sqrt2}(1,1)$

$V_{\lambda_2}=\{x\in\mathbb{R}^2 \mid Ax=x\}=\langle\{(-1,1)\}\rangle \Rightarrow e_2'=\frac{1}{\sqrt2}(-1,1)$

fie τ rotație, $\tau: x'=Rx''$, $R=\begin{pmatrix}\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{pmatrix}\in SO(2)$

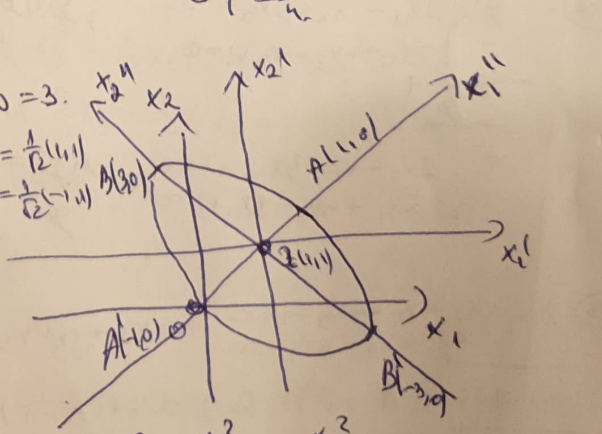
$\Rightarrow \varphi=\frac{\pi}{4}$.

$\tau(\theta(\Gamma)): 9x_1''^2+x_2''^2-9=0 \Rightarrow$ ELIPSĂ

$x_1''^2+\frac{x_2''^2}{9}=1$.

avem o elipsă cu $a=1$, $b=3$.

desenăm în sensul $(P_0 \to e_i)$ $e_1'=\frac{1}{\sqrt2}(1,1)$ $e_2'=\frac{1}{\sqrt2}(-1,1)$



$\mathcal{E}: \frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1$.

COLȚUL DE TEORIE

$A\in O(3)$, $\det A=1$ (SPEȚA 1) (rotație)

$\det A=-1$ (SPEȚA 2) (+translație) (simetrie)

SPEȚA 1	SPEȚA 2
a) $\lambda=1$. ~~[illegible]~~ $A=\begin{pmatrix}1&0&0\\0&\cos&-\sin\\0&\sin&\cos\end{pmatrix}$.	a) $\lambda=1$. (val. proprie) $A=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$
b) $\lambda=-1$. $A=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	b) $\lambda=-1$. $A=\begin{pmatrix}-1&0&0\\0&\cos&-\sin\\0&\sin&\cos\end{pmatrix}$
$TrA=1+2\cos\varphi$ Axa: $f(x)=x$	$TrA=-1+2\cos\varphi$ Axa: $f(x)=-x$

$A\in O(2)$

a) $\det A=1$ rotație

$A=\begin{pmatrix}\cos&-\sin\\\sin&\cos\end{pmatrix}$

b) $\det A=-1$.

$A=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ transl.

$A\in O(1)$

$f\in\{id_E, -id_E\}$

$f(x)=f(x\cdot e)=\pm x$

(B) Arii, volume, distanțe, unghiuri

① $S_{\Delta ABC} = \frac{1}{2}\|\vec{AB}\times\vec{AC}\|$

② $dist(A,\mathcal{D}) = \frac{\|\vec{AB}\times\vec{AC}\|}{\|\vec{BC}\|}$

③ $V = \frac{1}{6}|\Delta|$, $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix}$, coplanare $\Leftrightarrow \Delta = 0$.

④ $dist(A,\pi) = \frac{|aa_1 + ba_2 + ca_3 + d|}{\sqrt{a^2+b^2+c^2}}$, $\pi: ax_1 + bx_2 + cx_3 + d = 0$.

⑤ $dist(\mathcal{D}_1,\mathcal{D}_2) = \frac{|\langle \vec{AB}, N\rangle|}{\|N\|}$, $N = u\times v$.

⑥ $\measuredangle(\mathcal{D}_1,\mathcal{D}_2) = \measuredangle(u_1,u_2) = \measuredangle\varphi \in [0,\pi]$, $\cos\varphi = \frac{\langle u_1, u_2\rangle}{\|u_1\|\|u_2\|}$

⑦ $\measuredangle(\pi_1,\pi_2) = \measuredangle(N_1,N_2) = \measuredangle\varphi$, $\cos\varphi = \frac{\langle N_1,N_2\rangle}{\|N_1\|\|N_2\|}$

⑧ $\measuredangle(\mathcal{D},\pi) = \measuredangle(\mathcal{D},\mathcal{D}') = \measuredangle\varphi$
$= \frac{\pi}{2} - \measuredangle(\mathcal{D}, N_\pi)$.
$\rightarrow$ $\mathcal{D}$ – dreaptă orientată de u
π – plan orientat de N
$\mathcal{D}'$ – proiecția pe π a lui $\mathcal{D}$

⑨ $h = \frac{|\Delta|}{\|N\|} = \frac{|\vec{AB} \wedge u \wedge v|}{\|N\|} = \frac{|\langle\vec{AB}, N\rangle|}{\|N\|}$

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⑩ Poz. a 2 drepte.

$C = \begin{pmatrix} v_1 & -v_1' \\ v_2 & -v_2' \\ \vdots & \vdots \\ v_n & -v_n' \end{pmatrix} \left| \begin{matrix} b_1 - a_1 \\ b_2 - a_2 \\ \vdots \\ b_n - a_n \end{matrix} \right.$

$rg\, C = rg\, \bar{C} = 2 \Rightarrow$ concurente
$rg\, C = rg\, \bar{C} = 1 \Rightarrow \mathcal{D}_1 = \mathcal{D}_2$
$rg\, C = 2,\ rg\, \bar{C} = 3 \Rightarrow$ necoplanare
$rg\, C = 1,\ rg\, \bar{C} = 2 \Rightarrow \mathcal{D}_1 \parallel \mathcal{D}_2$

Ex1 $A(3,-1,3)$, $B(5,1,-1)$, $u = (-3,5,-6)$

a) $\mathcal{D}$ a.î. $A\in\mathcal{D}$, $V_{\mathcal{D}} = \langle u\rangle$, b) ec. AB c) $\mathcal{D}\cap$ pl. de coord.

a) $M(x_1,x_2,x_3)\in\mathcal{D} \Rightarrow \exists t\in\mathbb{R}$ a.î. $\vec{AM} = tu$.
$(x_1-3, x_2+1, x_3-3) = t(-3,5,-6)$
$\mathcal{D}: \begin{cases} x_1 = -3t+3 \\ x_2 = 5t-1 \\ x_3 = -6t+3 \end{cases}$, $t\in\mathbb{R}$, – sau – $\mathcal{D}: \frac{x_1-3}{-3} = \frac{x_2+1}{5} = \frac{x_3-3}{-6} = t$

b) $\vec{AB} = (2,2,-4) = 2(1,1,-2)$.
$AB: \frac{x_1-3}{1} = \frac{x_2+1}{1} = \frac{x_3-3}{-2} = t$ – sau – $AB: \begin{cases} x_1 = t+3 \\ x_2 = t-1 \\ x_3 = -2t+3 \end{cases}$, $t\in\mathbb{R}$

c) $\mathcal{D}\cap Ox = \{P(\frac{3}{2}, \frac{3}{2}, 0)\}$ (analog).
$x_3 = 0 \Rightarrow t = \frac{1}{2}$

Ex2. $\mathcal{D}': \begin{cases} 2x_1 - x_2 + 3x_3 = -1. \\ 5x_1 + 4x_2 - x_3 + 1 = 0. \end{cases}$, $\mathcal{D}\parallel\mathcal{D}'$, $A(2,-5,3)$, $A\in\mathcal{D}$.

$\pi_1: 2x_1 - x_2 + 3x_3 + 1 = 0$
$\pi_2: 5x_1 + 4x_2 - 3x_3 + 1 = 0$

$u_{\mathcal{D}'} = N_1\times N_2 = -4e_1 + 17e_2 + 13e_3$

$\mathcal{D}: \frac{x_1-2}{-4} = \frac{x_2+5}{17} = \frac{x_3-3}{13} = t$, $\mathcal{D}: \begin{cases} x_1 = -4t+2 \\ x_2 = 17t-5 \\ x_3 = 13t+3 \end{cases}$ $t\in\mathbb{R}$

Ex3 $\mathcal{D}: \begin{cases} x_1 + x_2 - x_3 + 1 = 0 \\ 2x_1 + x_2 - 3x_3 + 2 = 0 \end{cases}$ $\pi: x_1 + x_2 + x_3 = 0$, $A(1,1,1)$

a) $dist(A,\mathcal{D})$ b) $dist(A,\pi)$

a) $dist(A,\mathcal{D}) = \frac{\|\vec{AB}\times\vec{AC}\|}{\|\vec{BC}\|} = x$ $B(-1,0,0)$, $C(1,-1,1)$

$x_3 = t \Rightarrow \begin{cases} x_1 = 2t-1 \\ x_2 = -t \end{cases}$

$\vec{AB} = (-2,-1,-1)$
$\vec{AC} = (0,-2,0)$
$\vec{BC} = (2,-1,1)$
$\vec{AB}\times\vec{AC} = (-2,0,4)$

$\mathcal{D}: \begin{cases} x_1 = 2t-1 \\ x_2 = -t \\ x_3 = t \end{cases}$, $t\in\mathbb{R}$

$x = \frac{\sqrt{20}}{\sqrt{6}} = \frac{\sqrt{120}}{6} = \frac{\sqrt{30}}{3}$

b) $dist(A,\pi) = \frac{|1\cdot1 + 1\cdot1 + 1\cdot1 + 0|}{\sqrt{1^2+1^2+1^2}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$

... $|1-\lambda \quad 0 \quad 1| - (1-\lambda)(\ldots-\lambda)(\ldots-\lambda) = 0$ ...

(ex3) $\pi_1: x_1 - 2x_2 - 1 = 0$ $\quad d_1: \begin{cases} x_1 - x_2 - 2 = 0 \\ x_1 + x_3 - 3 = 0 \end{cases}$ $\quad d_2: \frac{x_1 - 1}{3} = \frac{x_2 - 1}{0} = \frac{x_3 - 1}{-1}$

$\pi_2: 2x_2 + x_3 - 2 = 0$

$\pi_3: x_2 - x_3 - 1 = 0$ $\quad d: \frac{x_1 - 1}{-1} = \frac{x_2}{2} = \frac{x_3 + 1}{-5}$

a) $\measuredangle(d_1, d_2)$ b) $\measuredangle(d, \pi)$, c) $\measuredangle(\pi_1, \pi_2)$, d) dist (d_1, d_2)

a) $u_{d_1} = N_1 \times N_2 = (1, -1, 0) \times (1, 0, 1) = (-1, -1, 1)$ $\quad u_{d_2} = (3, 0, -1)$

$\measuredangle(d_1, d_2) = \measuredangle(u_1, u_2) = \measuredangle \varphi \in [0, \pi]$, $\cos\varphi = \frac{\langle u_1, u_2 \rangle}{\|u_1\| \|u_2\|} = \frac{-4}{\sqrt{30}} = \frac{-2\sqrt{30}}{15}$

$\varphi = \pi - \arccos\left(\frac{2\sqrt{30}}{15}\right)$

b) $\measuredangle(d, \pi_3) = \measuredangle \varphi \in [0, \pi]$,

$\cos\varphi = \frac{\langle u_d, N_{\pi_3} \rangle}{\|N_{\pi_3}\| \|u_d\|} = \frac{7}{\sqrt{2}\sqrt{30}} = \frac{7\sqrt{15}}{30}$ $\Rightarrow \beta = \frac{\pi}{2} - \arccos\frac{7\sqrt{15}}{30}$

fie $\beta = \measuredangle(d, \pi)$

c) $\measuredangle(\pi_1, \pi_2) = \measuredangle(N_1, N_2) = \measuredangle \varphi \in [0, \pi]$, $\cos\varphi = \frac{\langle N_1, N_2 \rangle}{\|N_1\| \|N_2\|} = \frac{-4}{5} \Rightarrow \varphi = \pi - \arccos\frac{-4}{5}$

d) $\text{dist}(d_1, d_2) = \frac{|\langle \vec{AB}, N \rangle|}{\|N\|} = \frac{3}{\sqrt{14}}$

$N = u \times v = (3, 0, -1) \times (-1, -1, 1) = (-1, -2, -3) = -(1, 2, 3)$

$A(0, -2, 3) \in d_1$
$B(1, -1, 1) \in d_2$ $\Rightarrow \vec{AB} = (1, 1, -2)$

(ex4) $d_1: \frac{x_1 - 1}{1} = \frac{x_2 - 2}{-1} = \frac{x_3 + 2}{2} = t$, $\quad d_2: \begin{cases} 2x_1 - x_3 - 1 = 0 \\ 2x_2 + x_3 + 3 = 0 \end{cases}$

a) d_1, d_2 coplanare, b) ec. pl. d_1, d_2, c) dist(d_1, d_2)

a) $u_{d_2} = N_1 \times N_2 = (2, 0, -1) \times (0, 2, 1) = 2(1, -1, 2)$.

$\pi: \begin{vmatrix} x_1 - 1 & 1 & 0 \\ x_2 - 2 & -1 & -4 \\ x_3 + 2 & 2 & 3 \end{vmatrix} = 0$ sau $\begin{vmatrix} x_1 & x_2 & x_3 & 1 \\ 1 & 2 & -2 & 1 \\ 1 & -2 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{vmatrix} = 0$ sau $N = \vec{A_1A_2} \times u_1 = (a, b, c)$,

$\langle N, \vec{AM} \rangle = 0$

$a(x_1 - 1) + b(x_2 - 2) + c(x_3 + 2) = 0$

$A_1(1, 2, -2) \in d_1$, $A_2(1, -2, 1) \in d_2$, $A_3(2, 1, 0) \in d_2$ $\Rightarrow$ $u_{d_2} = 2u_{d_1} \Rightarrow d_1 \parallel d_2 \Rightarrow$ coplanare

sau: $\Delta = 0$.

(ex5) $d_1: \begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 - 1 = 0 \end{cases}$, $\quad d_2: \begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases}$

a) d_1, d_2 necopl. b) ec. $\perp$ d_1, d_2, c) dist(d_1, d_2)

b) $x_3 = t \Rightarrow d_1: \begin{cases} x_1 = -t \\ x_2 = 1 + t \\ x_3 = t \end{cases}$, $P_1 \in d_1$

$d_2: \begin{cases} x_1 = s \\ x_2 = 0 \\ x_3 = 0 \end{cases}$, $P_2 \in d_2$

$u_1 = (-1, 1, 1)$, $u_2 (1, 0, 0)$. $\vec{P_1P_2} = (s + t, -1 - t, -t)$

$\begin{cases} \langle \vec{P_1P_2}, u_1 \rangle = 0 \\ \langle \vec{P_1P_2}, u_2 \rangle = 0 \end{cases} \Rightarrow s = \frac{1}{2}, t = -\frac{1}{2}$

$P_1\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), P_2\left(\frac{1}{2}, 0, 0\right)$.

$\vec{P_1P_2} = \left(0, -\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(0, -1, 1)$

$d: \frac{x_1 - \frac{1}{2}}{0} = \frac{x_2}{-1} = \frac{x_3}{1}$, $\quad d: \begin{cases} x_1 = \frac{1}{2} \\ x_2 = -x_3 \end{cases}$

c) $d(d_1, d_2) = d(P_1, P_2) = \|\vec{P_1P_2}\| = \frac{\sqrt{2}}{2}$

a) $\left(\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{array}\right)$

$\left.\begin{array}{l} \text{rg } C = 2 \\ \text{rg } \bar{C} = 3 \end{array}\right| \Rightarrow d_1, d_2$ necoplanare

(C) Gauss, Jacobi, Transf. ortog. + f. canonică, Gram-Schmidt,

① Sver: a) $g \in C^0(\mathbb{R}^3, \mathbb{R}^3, \mathbb{R})$, Q poz. definită

b) $g(x,y) = g(y,x)$, $g(ax+by, z) = ag(x,z) + bg(y,z)$.

② Proc. Gram-Schmidt: $e_1 = f_1$, $e_2 = f_2 - \frac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1$, $e_3 = f_3 - \frac{\langle f_3, e_1\rangle}{\langle e_1, e_1\rangle} e_1 - \frac{\langle f_3, e_2\rangle}{\langle e_2, e_2\rangle} e_2 \dots$

$e_1' = \frac{e_1}{\|e_1\|}$, $e_2' = \frac{e_2}{\|e_2\|}$, ...

③ Prod. scalar: Sver / Q poz. def.

④ g - prod. scalar, Q_1 - f. pătratică / analoturat, Q_2 - f. canonică

(ex1) $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g(x,y) = x_2 y_1 + x_1 y_2 + 2x_3 y_1 + 2x_1 y_3 = X^T G Y$.

a) f. pătratică

b) f. canonică, poz. def.

a) $G = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$ $Q(x) = 2x_1x_2 + 4x_1x_3$.

Obs: dacă G' alt reper folosit

$G = C G' C^{-1}$

Sylvester păstrează semnul.

Sch. repr: $\begin{cases} y_1 = x_1 + x_2 \\ y_2 = x_1 - x_2 \\ y_3 = x_3 \end{cases} \Leftrightarrow \begin{cases} x_1 = \frac{1}{2}(y_1 + y_2) \\ x_2 = \frac{1}{2}(y_1 - y_2) \\ x_3 = y_3 \end{cases}$

$Q(x) = 2\frac{1}{4}(y_1^2 - y_2^2) + 4\frac{1}{2}y_3(y_1 + y_2) = 2(\frac{1}{2}y_1 + y_3)^2 - 2(\frac{1}{2}y_2 - y_3)^2$.

Sch. repr. $\begin{cases} z_1 = \sqrt{2}(\frac{1}{2}y_1 + y_3) \\ z_2 = \sqrt{2}(\frac{1}{2}y_2 - y_3) \\ z_3 = y_3 \end{cases}$ $Q(x) = z_1^2 - z_2^2$

(1,1) Q nu e poz. def.

(ex2) $Q: \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$.

a) G, b) $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, f. pol., c) f. can. (Gauss, Jac), gen.

a) $G = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$ b) $g(x,y) = x_1y_1 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_1y_3 + \frac{1}{2}x_2y_1 + x_2y_2 + \frac{1}{2}x_2y_3 + \frac{1}{2}x_3y_1 + \frac{1}{2}x_3y_2 + x_3y_3$

c) (MG) $Q(x) = (x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3)^2 - \frac{1}{4}x_2^2 - \frac{1}{4}x_3^2 - \frac{1}{2}x_2x_3 + x_2^2 + x_3^2 + x_2x_3$

$= (x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3)^2 + \frac{3}{4}x_2^2 + \frac{3}{4}x_3^2 + \frac{1}{2}x_2x_3 =$

$= (x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3)^2 + \frac{3}{4}(x_2 + \frac{1}{3}x_3)^2 + \frac{2}{3}x_3^2$

Sch. reperul: $\begin{cases} x_1' = x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ x_2' = x_2 + \frac{1}{3}x_3 \\ x_3' = x_3 \end{cases}$ $\Rightarrow Q(x) = x_1'^2 + \frac{3}{4}x_2'^2 + \frac{2}{3}x_3'^2$

Sch. reperul: $\begin{cases} x_1'' = x_1' \\ x_2'' = \frac{\sqrt{3}}{2}x_2' \\ x_3'' = \sqrt{\frac{2}{3}}x_3' \end{cases}$ $\Rightarrow Q(x) = x_1''^2 + x_2''^2 + x_3''^2$ → f. normală

(3, 0) sign

(MJ) $\Delta_1 = 1 \neq 0$

$\Delta_2 = \frac{3}{4} \neq 0$ $\Rightarrow \exists$ R' ai $Q(x) = \frac{1}{\Delta_1}x_1'^2 + \frac{\Delta_1}{\Delta_2}x_2'^2 + \frac{\Delta_2}{\Delta_3}x_3'^2$

$\Delta_3 = \frac{1}{2} \neq 0$ $= x_1'^2 + \frac{4}{3}x_2'^2 + \frac{3}{2}x_3'^2$. (poz. def.)

Generalizare: $Q(x) = \sum_{i=1}^{n} x_i^2 + \sum_{i<j} x_i x_j$ poz. def.

(Ex3) $P(\lambda)=\det(A-\lambda I_3)=\begin{vmatrix}1-\lambda & 0 & 1\\ 0 & 1-\lambda & 0\\ 1 & 0 & 1-\lambda\end{vmatrix}=(1-\lambda)(2-\lambda)(-\lambda)=0$ $\quad A=\begin{pmatrix}1&0&1\\0&1&0\\1&0&1\end{pmatrix}$

calcule $\Rightarrow V_{\lambda_1}=<\{(0,1,0)\}>$, $V_{\lambda_2}=<\{(1,0,1)\}$, $V_{\lambda_3}=<\{(1,0,-1)\}>$.

$R=\{e_1=(0,1,0),\ e_2=\frac{1}{\sqrt2}(1,0,1),\ e_3=\frac{1}{\sqrt2}(1,0,-1)\}$

$Q(x)=x_1'^2+2x_2'^2$ $\quad(2,0)$ - sign. $\Rightarrow$ nu e p.d.

$R_0 \longrightarrow R$ $\quad h:\mathbb{R}^3\to\mathbb{R}^3$

$C=\begin{pmatrix}0 & 1/\sqrt2 & 1/\sqrt2\\ 1 & 0 & 0\\ 0 & 1/\sqrt2 & -1/\sqrt2\end{pmatrix}=[f]_{R_0R}$ $\quad h(x)=\left(\frac{1}{\sqrt2}x_2+\frac{1}{\sqrt2}x_3,\ x_1,\ \frac{1}{\sqrt2}x_2-\frac{1}{\sqrt2}x_3\right)\in O(\mathbb{R}^3)$

(Ex3) $(\mathbb{R}^3,g_0)$ s.e.v.e, $u=(1,1,0)$

a) $<\{u\}>^\perp$ rep. orton.

b) f rot. sp 1. (rotatie), $\varphi=\frac{\pi}{2}$, axa $=<\{u\}>$ $\quad(f\in SO(\mathbb{R}^3))$

a) $<\{u\}>^\perp=\{x\in\mathbb{R}^3\mid g(x,u)=0\}=<\{(1,-1,0),(0,0,1)\}>$
$\qquad\qquad\qquad\qquad\qquad\qquad f_2 \qquad f_3$

$\{f_2,f_3\}$ rep. ortogonal.

b) $\{e_1=\frac{1}{\sqrt2}(1,1,0),\ e_2=\frac{1}{\sqrt2}(1,-1,0),\ e_3=(0,0,1)\}$ reper ortonormat

$[f]_{RR}=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}=A'$

$C=\frac{1}{\sqrt2}\begin{pmatrix}1&1&0\\1&-1&0\\0&0&\sqrt2\end{pmatrix}$ $\quad A'=C^{-1}AC$, C^{-1} ortogonal.

$A=CA'C^T=\frac12\begin{pmatrix}-1&1&0\\-1&-1&0\\0&0&\sqrt2\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}\begin{pmatrix}-1&-1&0\\1&-1&0\\0&0&\sqrt2\end{pmatrix}$

$=\frac12\begin{pmatrix}1&1&-\sqrt2\\1&1&\sqrt2\\\sqrt2&-\sqrt2&0\end{pmatrix}$

$f:\mathbb{R}^3\to\mathbb{R}^3$, $f(x)=\frac12(x_1+x_2-\sqrt2x_3,\ x_1+x_2+\sqrt2x_3,\ \sqrt2x_1-\sqrt2x_2)$.

(Ex4) $A=[f]_{R_0R_0}=\frac19\begin{pmatrix}8&1&-4\\1&8&4\\-4&4&-7\end{pmatrix}$

a) $f\in O(\mathbb{R}^3)$ sp. 2. $\quad, f=$ o o R_φ.

b) φ rot, axa lui

c) $R=\{e_1,e_2,e_3\}$ orton. $[f]_{RR}=\begin{pmatrix}-1&0&0\\0&\cos&-\sin\\0&\sin&\cos\end{pmatrix}$

Obs: $e_3=\frac{1}{10}(\cancel{9}\,3,10)$.

$\|e_3\|=\sqrt{150}\Rightarrow e_3'=\frac{1}{\sqrt{150}}(9,3,10)$

(*) dupa ce ... $\frac{1}{10}$.

a) $AA^T=I_3$, $\det A=-1\Rightarrow f\in O(\mathbb{R}^3)$ spe. 2

b) $\operatorname{tr} A=1=-1+2\cos\varphi\Rightarrow\cos\varphi=1\Rightarrow\varphi=0$, axa: $f(x)=-x$.

c) $f:\mathbb{R}^3\to\mathbb{R}$, $f(x)=\frac19(8x_1+x_2-4x_3,\ x_1+8x_2+4x_3,\ -4x_1+4x_2-7x_3)$.

$\det\begin{pmatrix}17&1&-4\\1&17&4\\-4&4&2\end{pmatrix}=0.$ $\Rightarrow <\{u\}>^\perp=\{(x_1,-x_1+4x_3,x_3)\mid x_1,x_3\in\mathbb{R}\}$

$u=(1,-1,4)$ $\quad=<\{(1,1,0),(0,4,1)\}>$
$\qquad\qquad\qquad\qquad\qquad f_2 \qquad f_3$

$\{f_2,f_3\}$ rep. in $<\{u\}>^\perp$

Aplicam Gram-Schmidt.

$e_2'=f_2=(1,1,0)$ $\quad e_2=\frac{1}{\sqrt2}(1,1,0)$

$e_3'=f_3-\dfrac{g_0(f_3,e_2')}{g_0(e_2',e_2')}e_2'=(0,4,1)-2(1,1,0)=(-2,2,1)\Rightarrow e_3=\frac13(-2,2,1)$

$e_1=\frac{1}{3\sqrt2}(1,-1,4)$. $\quad R_0=\{e_1^0,e_2^0,e_3^0\}\xrightarrow{C}R=\{e_1,e_2,e_3\}$ sch. de rep. ortonom. la fel orientate

$C=\begin{pmatrix}\frac{1}{3\sqrt2}&\frac{1}{\sqrt2}&-\frac23\\-\frac{1}{3\sqrt2}&\frac{1}{\sqrt2}&\frac23\\\frac{4}{3\sqrt2}&0&\frac13\end{pmatrix}$ $\det C=1>0\Rightarrow$ la fel orientat

$A'=C^TAC$

$[f]_{RR}=\begin{pmatrix}-1&0&0\\0&\cos0&-\sin0\\0&\sin0&\cos0\end{pmatrix}=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$