T1 – Curs (27.02.2022)

① $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix}$

a) A^{-1} (Th. H–C)
b) dacă $B = A^6 + A^5 + A^4 + A + I_3$, atunci $a, b, c \in \mathbb{R}$ = ? ai $B = aA^2 + bA + cI_3$

a) Conform Teoremei Hamilton Cayley

$$A^3 - \sigma_1 A^2 + \sigma_2 A - \sigma_3 I_3 = O_3$$

$\sigma_1 = 6$

$\sigma_2 = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = -4 + 3 + 7 = 6$

$\sigma_3 = 6 - 3 - (-1 + 18) = 3 - 17 = -14$

$$A^3 - 6A^2 + 6A + 14I_3 = O_3 \quad | \cdot A^{-1}$$

$$A^2 - 6A + 6I_3 + 14A^{-1} = O_3$$

$$A^{-1} = -\frac{1}{14}(A^2 - 6A + 6I_3)$$

$$A^2 = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 3 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 5 & -6 \\ 9 & 9 & -8 \\ 3 & 5 & 8 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{14}\begin{pmatrix} 7 & -7 & 0 \\ -9 & 3 & -2 \\ 3 & -1 & -4 \end{pmatrix}$$

b) $A^6 = (6A^2 + 6A - 14I_3)^2 = 36A^4 + 36A^2 + 196I_3 + 72A^3 - 168A^2 - 168A =$
$= 36A^4 + 72A^3 - 132A^2 - 168A + 196I_3$

$A^5 = A^2(6A^2 + 6A - 14I_3) = 6A^4 + 6A^3 - 14A^2 = 274A^2 + 168A - 168I_3$

$A^4 = A(6A^2 + 6A - 14I_3) = 6A^3 + 6A^2 - 14A = 42A^2 + 22A - 14I_3$

$A^3 = 6A^2 + 6A - 14I_3$

$B = 36(42A^2 + 22A - 14I_3) + 72(6A^2 + 6A - 14I_3) - 132A^2 - 168A + 196I_3 +$
$+ 274A^2 + 168A + 196I_3 + 42A^2 + 22A - 14I_3 + A + I_3$

$= 2128A^2 + 1247A - 1133I_3$

$(a, b, c) = (2128, 1247, -1133)$

② $A \in \mathcal{M}_2(\mathbb{R})$

a) dacă $tr(A) = 0$, at. $A^2B = BA^2 \quad \forall B \in \mathcal{M}_2(\mathbb{R})$
b) dacă $tr(A) \neq 0$ și $A^2B = BA^2$, at $AB = BA$

a) $tr(A) = 0 \Rightarrow A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$

$$A^2 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a^2 + bc & 0 \\ 0 & a^2 + bc \end{pmatrix} = (a^2 + bc)I_2$$

$$\left.\begin{array}{l}A^2B=(a^2+bc)B\\BA^2=(a^2+bc)B\end{array}\right| \Rightarrow A^2B=BA^2 \; \forall B\in M_2(\mathbb{R}),\ trA=0$$

b) Știm din Th. H-C că $A^2-tr(A)\cdot A+det(A)\,I_2=O_2$

$$A^2=tr(A)\cdot A-det(A)\,I_2$$

$$A^2B=BA^2$$

$$(tr(A)\cdot A-det(A)\,I_2)B=B(tr(A)A-det(A)\,I_2)$$

$$tr(A)\cdot(AB)-det(A)\cdot B=tr(A)(BA)-det(A)B$$

$$tr(A)(AB)=tr(A)(BA) \quad ,\ tr(A)\neq 0.$$

$$AB=BA$$

Q.E.D.

③ $\begin{cases}\frac{1}{2}x=ax+by+cz\\ \frac{1}{2}y=cx+ay+bz\\ \frac{1}{2}z=bx+cy+az\end{cases}$, $a,b,c\in\mathbb{Z}$

Sol. unică

Rescriem sistemul $\Rightarrow$ $\begin{cases}(2a-1)x+2by+2cz=0\\ 2cx+(2a-1)y+2bz=0\\ 2bx+2cy+(2a-1)z=0\end{cases}$

$$A=\left(\begin{array}{ccc|c}2a-1&2b&2c&0\\2c&2a-1&2b&0\\2b&2c&2a-1&0\end{array}\right) \quad (2a+2b+2c-1)\begin{vmatrix}1&2b&2c\\1&&\\1&&\end{vmatrix}$$

$$\det A=\begin{vmatrix}2a-1&2b&2c\\2c&2a-1&2b\\2b&2c&2a-1\end{vmatrix}=(2a+2b+2c-1)\begin{vmatrix}1&2b&2c\\1&2a-1&2b\\1&2c&2a-1\end{vmatrix}$$

$$=-\frac{1}{2}(2a+2b+2c-1)\left[(2a-2b-1)^2+(2a-2c-1)^2+(2b-2c)^2\right]$$

$$=-\left(a+b+c-\frac{1}{2}\right)\left[(2b-2a+1)^2+(2c-2a+1)^2+(2b-2c)^2\right]$$

PPRA $\det A=0$

I. $\left.\begin{array}{l}a+b+c=-\frac{1}{2}\\ \text{dar } a,b,c\in\mathbb{Z}\end{array}\right| \Rightarrow$ ↯

II. $\left.\begin{cases}b-a=\frac{1}{2}\\ c-a=\frac{1}{2}\\ b=c\end{cases}\right| \Rightarrow$ ↯

dar $a,b,c\in\mathbb{Z}$

Așadar $\det A\neq 0 \Rightarrow$

$\Rightarrow$ Sistemul are o soluție unică, soluția $(0,0,0)$