T2 – Seminar (13.03.2022)

① $(\mathbb{R}^4,+,\cdot)_{|\mathbb{R}}$ R_0 – rep. canonic

$S = \{(1,0,-1,2), (1,1,1,1), (2,1,0,3), (3,2,1,4)\}$ — u, v, w, t

a) S este SLD
b) S' SLI max, extindere la reper R în $\mathbb{R}^4$
c) $R_0 \xrightarrow{A} R$, $A = ?$
d) coord. ale $x = (1,2,3,-4)$ în rap. cu R

a) fie $a,b,c,d \in \mathbb{R}$ a.î. $au + bv + c\cdot w + d\cdot t = 0$

$(a+b+2c+3d,\ b+c+2d,\ -a+b+d,\ 2a+b+3c+4d) = 0$

$$(*)\ \begin{cases} a+b+2c+3d = 0 \\ b+c+2d = 0 \\ -a+b+d = 0 \\ 2a+b+3c+4c = 0 \end{cases}$$

avem SLD $\Leftrightarrow$ (*) admite sol. nenule

$$A = \left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 4 & 0 \end{array}\right)$$

$$\det A = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ -1 & 1 & 0 & 1 \\ 2 & 1 & 3 & 4 \end{vmatrix} \xrightarrow[L_4 - 2L_1]{L_3 + L_1} \begin{vmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & -1 & -1 & -2 \end{vmatrix} = 1\cdot(-1)^2 \cdot \begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ -1 & -1 & -2 \end{vmatrix} = 0 \quad (L_1 = \tfrac{1}{2}L_2)$$

$\left.\begin{array}{l}\det A = 0 \\ (*) \text{ SLD}\end{array}\right| \Rightarrow$ (*) admite sol. nenule $\Rightarrow$ SLD

b) $t = v + w$
$u = w - v$ (era evident SLD (!!))

$\mathrm{rg}\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \\ 1 & 3 \end{pmatrix} = 2$, $S' = \{v, w\}$

$\det\begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} \neq 0 \Rightarrow \underbrace{S' \cup \{e_1, e_4\}}_{R}$ SLI

card $R = 4 = $ card $\mathbb{R}^4 \Rightarrow R'$ SG

$\Rightarrow R$ reper $\mathbb{R}^4$

c) $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix}$

d) $(1, 2, 3, -4) = x_1' e_1' + x_2' e_2' + x_3' e_3' + x_4' e_4'$

$= (x_1' + 2x_2' + 3x_3',\ x_1' + x_2',\ x_1',\ x_1' + 3x_2' + x_4')$

$$\begin{cases} x_1' + 2x_2' + 3x_3' = 1 \\ x_1' + x_2' = 2 \\ x_1' = 3 \\ x_1' + 3x_2' + x_4' = -4 \end{cases}$$

$x_1' = 3 \Rightarrow x_2' = -1 \Rightarrow x_3' = \frac{1-3+2}{3} = 0 \Rightarrow x_4' = -4$

$(x_1', x_2', x_3', x_4') = (3, -1, 0, -4)$

③ $V' = \langle \{(1,1,1,1), (3,2,1,3), (2,1,0,2)\} \rangle$

sist. de ecuații cu mul. sol. este $V' \subset \mathbb{R}^4$

$x = a(1,1,1,1) + b(3,2,1,3) + c(2,1,0,2) \in V'$

$(x_1, x_2, x_3, x_4) = (a + 3b + 2c,\ a + 2b + c,\ a + b,\ a + 3b + 2c)$

$$\begin{cases} a + 3b + 2c = x_1 \\ a + 2b + c = x_2 \\ a + b = x_3 \\ a + 3b + 2c = x_4 \end{cases}$$

$$A = \left(\begin{array}{ccc|c} 1 & 3 & 2 & x_1 \\ 1 & 2 & 1 & x_2 \\ 1 & 1 & 0 & x_3 \\ 1 & 3 & 2 & x_4 \end{array}\right)$$

$\text{rg} A = \text{rg} \bar{A} = 2$

$d_1 = \begin{vmatrix} 1 & 3 & x_1 \\ 1 & 2 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} = 2x_2 - x_3 - x_1 = 0$

$d_2 = \begin{vmatrix} 1 & 3 & x_1 \\ 1 & 2 & x_2 \\ 1 & 3 & x_4 \end{vmatrix} = -x_4 + x_1 = 0$

$V' = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \begin{cases} -x_1 + 2x_2 - x_3 = 0 \\ x_1 - x_4 = 0 \end{cases} \}$

② $(\mathcal{M}_2(\mathbb{R}), +, \cdot)_{/\mathbb{R}}$

$V' = \{A = \begin{pmatrix} u & -u-x \\ 0 & x \end{pmatrix} \mid u, x \in \mathbb{R}\}$ sp. vect.

a) baza în V'

b) V'' subspațiu complementar lui V'

a) $A=\begin{pmatrix} u & -u-x \\ 0 & x \end{pmatrix}=\begin{pmatrix} u & -u \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & -x \\ 0 & x \end{pmatrix}=u\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}+x\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$

$A\in\langle\underbrace{\left\{\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}\right\}}_{S}\rangle$

$S=\left\{\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}\right\}$ SG.

$S'=\{(1\ -1\ 0\ 0),(0\ -1\ 0\ 1)\}\subset\mathbb{R}^4$

$M=\begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad rg\,M=2\ (max)\xRightarrow[L.i.]{Crit.}$ S' SLi, S' SG $\Rightarrow\left\{\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}\right\}$ bază

b) $M'=\begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

$\det M'=\begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}=1\cdot(-1)^{4+2}\cdot\begin{vmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}=1\cdot(-1)^{3+2}\cdot\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix}=(-1)\neq 0$

$rg\,M'=4$

$\mathbb{R}^4=S'\oplus S''$, $S''=\{e_3,e_1\}$

– bonus seminar 3 –

① $+:\mathbb{R}^2\times\mathbb{R}^2\longrightarrow\mathbb{R}^2$
$\cdot:\mathbb{R}\times\mathbb{R}^2\longrightarrow\mathbb{R}^2$

a) $(x,y)+(x',y')=(x+x',0)$
$\alpha(x,y)=(\alpha x,\alpha y)$

1) evident $(\mathbb{R}^2,+)$ grup abelian.
2) $a(b(x,y))=a(bx,by)=(abx,aby)$
$(ab)(x,y)=(abx,aby)$
3) $(a+b)(x,y)=((a+b)x,(a+b)y)$
$a(x,y)+b(x,y)=(ax,ay)+(bx,by)=((a+b)x,0)$ $\Rightarrow$ NU avem un sp. vect.

b) $(x,y)+(x',y')=(x+x',y')$
$\alpha(x,y)=(\alpha x,\alpha y)$

NU avem un spațiu vectorial deoarece "+" nu e comutativă.

c) $(x,y)+(x',y')=(x+x',y+y')$
$\alpha(x,y)=(0,\alpha y)$

NU avem un sp. vectorial deoarece $1(x,y)=(0,y)\neq(x,y)$

(5) $(\mathcal{K}=\{a_0, a_1, a_2, a_3\}, +)$ grupul lui Klein.

$\cdot : \mathbb{Z}_2 \times \mathcal{K} \to \mathcal{K}$

$\hat{0} \cdot a_i = a_0 \quad \forall i = \overline{0,3}$

$\hat{1} \cdot a_i = a_i$

$(\mathcal{K}, +, \cdot)_{|\mathbb{Z}_2}$ sp. vect.

$\mathcal{K} = \{(\hat{0},\hat{0}), (\hat{1},\hat{0}), (\hat{0},\hat{1}), (\hat{1},\hat{1})\}$.

1) $(\mathcal{K}, +)$ grup comutativ

2) $a(b(\hat{0},\hat{0})) = (ab)(\hat{0},\hat{0}) = (\hat{0},\hat{0}) \pmod 2$

$a(b(\hat{1},\hat{1})) = (ab)(\hat{1},\hat{1}) = (\widehat{ab}, \widehat{ab}) \pmod 2$

$a(b(\hat{1},\hat{0})) = (ab)(\hat{1},\hat{0}) = (\widehat{ab}, \hat{0}) \pmod 2$

$a(b(\hat{0},\hat{1})) = (ab)(\hat{0},\hat{1}) = (\hat{0}, \widehat{ab}) \pmod 2$

3) $(a+b)(\hat{0},\hat{0}) = a(\hat{0},\hat{0}) + b(\hat{0},\hat{0}) = (\hat{0},\hat{0})$

$(a+b)(\hat{0},\hat{1}) = a(\hat{0},\hat{1}) + b(\hat{0},\hat{1}) = (\hat{0}, \widehat{a+b})$

$(a+b)(\hat{1},\hat{0}) = a(\hat{1},\hat{0}) + b(\hat{1},\hat{0}) = (\widehat{a+b}, \hat{0})$

$(a+b)(\hat{1},\hat{1}) = a(\hat{1},\hat{1}) + b(\hat{1},\hat{1}) = (\widehat{a+b}, \widehat{a+b})$

4) fie $k_i \in \{\hat{0}, \hat{1}\}$, $i = \overline{0,3}$

$b((k_{i_1}, k_{i_2}) + (k_{i_3} + k_{i_4})) = b(k_{i_1} + k_{i_3}, k_{i_2} + k_{i_4}) =$

$= (bk_{i_1} + bk_{i_3}, bk_{i_2} + bk_{i_4}) = b(k_{i_1}, k_{i_2}) + b(k_{i_3}, k_{i_4})$

5) $\hat{1} \cdot a_i = a_i$

$1,2,3,4,5 \Longrightarrow (\mathcal{K}, +, \cdot)_{|\mathbb{Z}_2}$ sp. vectorial.

(13) $\dim_{\mathbb{C}}(\mathbb{C}) = 1$

---

fie $a \in \mathbb{C}$, $x \in \mathbb{C}$.

putem scrie $x = a \cdot 1 \Rightarrow x \in \langle\{1\}\rangle \Rightarrow \dim_{\mathbb{C}} \mathbb{C} = 1$.

$\dim_{\mathbb{R}} \mathbb{C} = 2$

---

fie $a \in \mathbb{R}$, $b \in \mathbb{R}$, $x \in \mathbb{C}$

putem scrie $x = a \cdot 1 + b \cdot i \Rightarrow x \in \langle\{1, i\}\rangle \Rightarrow \dim_{\mathbb{R}} \mathbb{C} = 2$

$\dim_{\mathbb{C}}(\mathbb{C}^n) = n$

---

fie $a_i \in \mathbb{C}$, $x \in \mathbb{C}^n$, $i = \overline{1,n}$

putem scrie $x = a_1(1,0,\dots,0) + \dots + a_n(0,\dots,0,1) \Rightarrow x \in \langle\{(1,0,\dots,0), \dots, (0,\dots,0,1)\}\rangle$

$\Rightarrow \dim_{\mathbb{C}} \mathbb{C}^n = n$.

$\dim_{\mathbb{R}} \mathbb{C}^n = 2n$

Fie $a_i, b_i \in \mathbb{R}$, $x \in \mathbb{C}^n$, $i=\overline{1,n}$
putem scrie $x = a_1(1,0,\dots,0)+\dots+a_n(0,\dots,0,1)+b_1(i,0,\dots,0)+\dots+b_n(0,\dots,0,i)$

$\Rightarrow x \in \langle\{(1,0,\dots,0),\dots,(0,\dots,0,1);(i,0,\dots,0),\dots,(0,\dots,0,i)\}\rangle$

$\Rightarrow \dim_{\mathbb{R}} \mathbb{C}^n = 2n.$

(14) $(\mathbb{R}^3,+,\cdot)_{/\mathbb{R}}$

$S=\{u_1=(1,5,3),\ u_2=(2,0,6)\}$
$S'=\{w_1=(-1,7,-3),\ w_2=(4,5,12)\}$

$\langle S\rangle = \langle S'\rangle$

$rg\begin{pmatrix}1&2\\5&0\\3&6\end{pmatrix}=2$ (maxim) $\xRightarrow[L.I]{\text{crit.}}$ S este SLI $\Rightarrow \dim\langle S\rangle=2$

$rg\begin{pmatrix}-1&4\\7&5\\-3&12\end{pmatrix}=2$ (maxim) $\xRightarrow[L.I]{\text{crit.}}$ S' este SLI $\Rightarrow \dim\langle S'\rangle=2$

$u_1 \in S' \Leftrightarrow \exists\, a_1,b_1 \in \mathbb{R}$ ai $u_1 = a_1w_1+b_1w_2$.

$$\begin{cases}-a_1+4b_1=1\\7a_1+5b_1=5\\-3a_1+12b_1=3\end{cases}$$

$$A_1=\left(\begin{array}{cc|c}-1&4&1\\7&5&5\\-3&12&3\end{array}\right)$$

$\left.\begin{array}{l}rg\,A_1=2\\rg\,\bar{A}_1=2\end{array}\right\} \Rightarrow$ SCD

$$\Delta_c=\begin{vmatrix}-1&4&1\\7&5&5\\-3&12&3\end{vmatrix}=0\ (3L_1=L_3)$$

$u_2 \in S' \Leftrightarrow \exists\, a_2,b_2 \in \mathbb{R}$ ai $u_2 = a_2w_1+b_2w_2$.

$$\begin{cases}-a_2+4b_2=2\\7a_2+5b_2=0\\-3a_2+12b_2=6\end{cases}\qquad A_2=\left(\begin{array}{cc|c}-1&4&2\\7&5&0\\-3&12&6\end{array}\right)$$

$\left.\begin{array}{l}rg\,A_2=2\\rg\,\bar{A}_2=2\end{array}\right\} \Rightarrow$ SCA

$$\Delta_c=\begin{vmatrix}-1&4&2\\7&5&0\\-3&12&6\end{vmatrix}=0\ (3\cdot L_1=L_3)$$

$\left.\begin{array}{l}\langle S\rangle \subset \langle S'\rangle\\ \dim\langle S\rangle=\dim\langle S'\rangle=2\end{array}\right\} \Rightarrow \langle S\rangle=\langle S'\rangle$

(13) $(V_1,+,\cdot)_{/\mathbb{K}}$ sp. vect, $B_1=\{e_1,\dots,e_n\}$ bază

$(V_2,+,\cdot)_{/\mathbb{K}}$ sp. vect, $B_2=\{f_1,\dots,f_m\}$ bază

$(V_1\times V_2,+,\cdot)_{/\mathbb{K}}$ are baza $B=\{(e_1,0_{V_2}),\dots,(e_n,0_{V_2}),(0_{V_1},f_1),\dots,(0_{V_1},f_m)\}$

$\dim(V_1\times V_2)=\dim V_1+\dim V_2=n+m$

$(V_1,+,\cdot)_{/\mathbb{K}}$ sp. vect. cu $B_1=\{e_1,\dots,e_n\}$ bază $\Rightarrow$

$\Rightarrow v_1=a_1e_1+\dots+a_ne_n,\ v_1\in V_1,\ a_i\in\mathbb{K},\ i=\overline{1,n}$

$v_2=b_1f_1+\dots+b_mf_m,\ v_2\in V_2,\ b_j\in\mathbb{K},\ j=\overline{1,m}$

$v_3\in V_1\times V_2\Rightarrow v_3=(v_1,v_2)=(v_1,0_{V_2})+(0_{V_1},v_2)\Rightarrow$

$\Rightarrow v_3=a_1(e_1,0_{V_2})+\dots+a_n(e_n,0_{V_2})+b_1(0_{V_1},f_1)+\dots+b_m(0_{V_1},f_m)$

$\Rightarrow B=\{(e_1,0_{V_2}),\dots,(e_n,0_{V_2}),(0_{V_1},f_1),\dots,(0_{V_1},f_m)\}$ bază în $V_1\times V_2$

având baza $B\Rightarrow\dim(V_1\times V_2)=\dim<\{(e_1,0_{V_2}),\dots,(e_n,0_{V_2}),(0_{V_1},f_1),\dots,(0_{V_1},f_m)\}>$

$=\dim V_1+\dim V_2=n+m.$

(12) sp. vect $(\mathbb{R}^n,+,\cdot)_{\mathbb{R}}$, $B=\{f_1,\dots,f_n\}$ bază

$(\mathbb{C}^n,+,\cdot)_{/\mathbb{R}}$ are $B'=\{f_1,if_1,\dots,f_n,if_n\}$ bază

---

fie $x\in\mathbb{R}^n$, B bază $\Rightarrow x=f_1a_1+\dots+f_n\cdot a_n,\ a_i\in\mathbb{R},\ i=\overline{1,n}$

fie $x'\in\mathbb{C}^n$

putem scrie $x'=f_1a_1+\dots+f_na_n+f_1b_1i+\dots+f_nib_n,\ a_i,b_i\in\mathbb{R},\ i=\overline{1,n}$

$\Rightarrow x'=<\{f_1,\dots,f_n,f_1i,\dots,f_ni\}>$ SG

$\{f_1,\dots,f_n\}$ bază $\Rightarrow\{f_1,\dots,f_n\}$ SLI $\rightarrow\{f_1i,\dots,f_ni\}$ SLI

$\rightarrow\{f_1,\dots,f_n,f_1i,\dots,f_ni\}$ SLI

$\Rightarrow B'=\{f_1,if_1,\dots,f_n,if_n\}$ bază pentru $(\mathbb{C}^n,+,\cdot)_{/\mathbb{R}}$

— Bonus seminar 4 —

(8) $(\mathbb{R}^4,+,\cdot)_{|\mathbb{R}}$, $V' = \langle\{(1,2,-1,0), (1,0,0,3)\}\rangle$

a) V' printr-un sist. ec. lin

b) $\mathbb{R}^4 = V' \oplus V''$, $V'' = ?$, V'' printr-un sist. de ec liniare

a) $x = (x_1, x_2, x_3, x_4) \in V' \Leftrightarrow \exists\, a, b \in \mathbb{R}$ a.i.

$$rg\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 0 \\ 0 & 3 \end{pmatrix} = 2$$

$x = a(1,2,-1,0) + b(1,0,0,3)$

$(x_1, x_2, x_3, x_4) = (a+b, 2a, -a, 3b)$

$$\begin{cases} a+b = x_1 \\ 2a = x_2 \\ -a = x_3 \\ 3b = x_4 \end{cases} \quad \bar{A} = \left(\begin{array}{cc|c} 1 & 1 & x_1 \\ 2 & 0 & x_2 \\ -1 & 0 & x_3 \\ 0 & 3 & x_4 \end{array}\right)$$

$\Delta_p = \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2 \neq 0 \quad rg\, A = 2$

$rg\, A = rg\, \bar{A} \Rightarrow$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & x_1 \\ 2 & 0 & x_2 \\ -1 & 0 & x_3 \end{vmatrix} = 1\cdot(-1)^3 \cdot \begin{vmatrix} 2 & x_2 \\ -1 & x_3 \end{vmatrix} = -(2x_3 + x_2) = 0$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & x_1 \\ 2 & 0 & x_2 \\ 0 & 3 & x_4 \end{vmatrix} = 6x_1 - 3x_2 - 2x_4 = 0.$$

$V' = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \begin{cases} x_2 + 2x_3 = 0 \\ 6x_1 - 3x_2 - 2x_4 = 0 \end{cases}\} = S(B)$, $B = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 6 & -3 & 0 & -2 \end{pmatrix}$

b) $R' = \{(1,2,-1,0), (1,0,0,3)\}$ reper în V'

extindem la reper în $\mathbb{R}^4$

$$\det\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \end{pmatrix} = 1\cdot(-1)^{3+4} \cdot \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 3 & 0 \end{vmatrix} = (-1)\cdot(-3) = 3 \neq 0$$

$R'' = \{e_2, e_3\}$ reper în V''

$R = R' \cup R''$ reper în $\mathbb{R}^4$.

$V'' = \langle R' \rangle$.

$x = (x_1, x_2, x_3, x_4) \in V'' \Leftrightarrow \exists\, a, b \in \mathbb{R}$ a.i.

$x = a e_2 + b e_3$

$$\begin{cases} a = x_2 \\ b = x_3 \end{cases} \quad A = \left(\begin{array}{cc|c} 0 & 0 & x_1 \\ 1 & 0 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & x_4 \end{array}\right)$$

$\Delta_p = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow rg\, A = 2$

$rg\, A = rg\, \bar{A} = 2. \Rightarrow$

$$\Delta_1 = \begin{vmatrix} 1 & 0 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & x_4 \end{vmatrix} = x_4 = 0$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & x_1 \end{vmatrix} = x_1 = 0$$

$V'' = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \{ x_1 = 0, x_4 = 0 \} = S(B'')$, $B'' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

9) $(\mathbb{R}^4, +, \cdot)_{\mathbb{R}}$, $V' = \langle \{u, v, w\} \rangle$, $V'' = \langle \{u', v', w'\} \rangle$

$u = (2, 3, 11, 5)$, $v = (1, 1, 5, 2)$, $w = (0, 1, 1, 1)$
$u' = (2, 1, 3, 2)$, $v' = (1, 1, 3, 4)$, $w' = (5, 2, 6, 2)$

a) $V' \oplus V'' = \mathbb{R}^4$
b) V', V'' sist. ec. liniare.

b) $x \in V' \Rightarrow \exists a, b, c \in \mathbb{R}$ ai $x = (2a+b, 3a+b+c, 11a+5b+c, 5a+2b+c)$

$$\begin{cases} 2a+b = x_1 \\ 3a+b+c = x_2 \\ 11a+5b+c = x_3 \\ 5a+2b+c = x_4 \end{cases} \qquad \bar{A'} = \left(\begin{array}{ccc|c} 2 & 1 & 0 & x_1 \\ 3 & 1 & 1 & x_2 \\ 11 & 5 & 1 & x_3 \\ 5 & 2 & 1 & x_4 \end{array}\right)$$

$\mathrm{rg}\, A' = 2 = \mathrm{rg}\, \bar{A'} \Rightarrow$ $\Delta_1 = \begin{vmatrix} 2 & 1 & x_1 \\ 3 & 1 & x_2 \\ 11 & 5 & x_3 \end{vmatrix} = 0 \Rightarrow 4x_1 + x_2 - x_3 = 0$

$\Delta_2 = \begin{vmatrix} 2 & 1 & x_1 \\ 3 & 1 & x_2 \\ 5 & 2 & x_4 \end{vmatrix} = 0 \Rightarrow x_1 + x_2 - x_4 = 0$

$V' = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \{ 4x_1 + x_2 - x_3 = 0, \; x_1 + x_2 - x_4 = 0 \} = S(B')$, $B' = \begin{pmatrix} 4 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}$

$x \in V'' \Rightarrow \exists a, b, c \in \mathbb{R}$ ai $x = (2a+b+5c, a+b+2c, 3a+3b+6c, 2a+4b+2c)$

$$\begin{cases} 2a+b+5c = x_1 \\ a+b+2c = x_2 \\ 3a+3b+6c = x_3 \\ 2a+4b+2c = x_4 \end{cases} \qquad \bar{A''} = \left(\begin{array}{ccc|c} 2 & 1 & 5 & x_1 \\ 1 & 1 & 2 & x_2 \\ 3 & 3 & 6 & x_3 \\ 2 & 4 & 2 & x_4 \end{array}\right)$$

$\mathrm{rg}\, A'' = 2 = \mathrm{rg}\, \bar{A''} \Rightarrow$ $\Delta_1 = \begin{vmatrix} 2 & 1 & x_1 \\ 1 & 1 & x_2 \\ 3 & 3 & x_3 \end{vmatrix} = 0 \Rightarrow -3x_2 + x_3 = 0$

$\Delta_2 = \begin{vmatrix} 2 & 1 & x_1 \\ 1 & 1 & x_2 \\ 2 & 4 & x_4 \end{vmatrix} = 0 \Rightarrow 2x_1 - 6x_2 + x_4 = 0.$

$V'' = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \{ -3x_2 + x_3 = 0, \; 2x_1 - 6x_2 + x_4 = 0 \} = S(B'')$, $B'' = \begin{pmatrix} 0 & -3 & 1 & 0 \\ 2 & -6 & 0 & 1 \end{pmatrix}$

a) $\mathrm{rg}\begin{pmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ 11 & 5 & 1 \\ 5 & 2 & 1 \end{pmatrix} = 2 \Rightarrow V' = \langle \{u, v\} \rangle \Rightarrow \dim V' = 2$

$\mathrm{rg}\begin{pmatrix} 2 & 1 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \\ 2 & 4 & 2 \end{pmatrix} = 2 \Rightarrow V'' = \langle \{u', v'\} \rangle \Rightarrow \dim V'' = 2$

$$V' \cap V'' = \left\{ (x_1, x_2, x_3, x_4) \;\middle|\; \begin{cases} 4x_1 + x_2 - x_3 = 0 \\ x_1 + x_2 - x_4 = 0 \\ -3x_2 + x_3 = 0 \\ 2x_1 - 6x_2 + x_4 = 0 \end{cases} \right\}$$

$$A=\begin{pmatrix} 4 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & -3 & 1 & 0 \\ 2 & -6 & 0 & 1 \end{pmatrix} \quad \det A = \begin{vmatrix} 4 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & -3 & 1 & 0 \\ 2 & -6 & 0 & 1 \end{vmatrix} \overset{L_4+L_2}{=\!=} \begin{vmatrix} 4 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & -3 & 1 & 0 \\ 3 & -5 & 0 & 0 \end{vmatrix} =$$

$$=(-1)\cdot(-1)^{2+4}\cdot\begin{vmatrix} 4 & 1 & -1 \\ 0 & -3 & 1 \\ 3 & -5 & 0 \end{vmatrix} = (-1)(3-9+20) = -14 \neq 0 \Rightarrow rg A = 4.$$

Conform T. Grassmann:

$\dim(V_1+V_2) = \dim V_1 + \dim V_2 - \dim V_1 \cap V_2 = 2+2-(4-rg A) = rg A = 4.$

$\dim V_1 + V_2 = 4 = \dim_{\mathbb{R}} \mathbb{R}^4$
$V_1 \cup V_2 \subset \mathbb{R}^4$ (subspații vect.) $\Rightarrow V_1 \oplus V_2 = \mathbb{R}^4$
$\dim V_1 \cap V_2 = 0$

(10) $(\mathbb{R}^4, +, \cdot)_{/\mathbb{R}}$ $\quad A=\begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & 3 & 2 & -1 \\ 3 & 2 & 4 & 2 \end{pmatrix}$, $V' = \{x \in \mathbb{R}^4 \mid Ax = 0\}$

a) $\dim_{\mathbb{R}} V' = ?$ bază în V'
b) $\mathbb{R}^4 = V' \oplus V^4$
c) $x=(1,2,1,2)$ suma vectori V', V^4

a) $\dim_{\mathbb{R}} V' = 4 - rg A = 4-2 = 2.$

$rg A = rg \begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & 3 & 2 & -1 \\ 3 & 2 & 4 & 2 \end{pmatrix}$:

$d_2 = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5 \neq 0 \Rightarrow rg A \geq 2.$

$d_3' = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & 2 \\ 3 & 2 & 4 \end{vmatrix} = 0$

$d_3'' = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & -1 \\ 3 & 2 & 2 \end{vmatrix} = 0$

$\Rightarrow rg A = 2$

$$\begin{cases} x_1 - x_2 = -2x_3 - 3x_4 \quad /\cdot 3 \\ 2x_1 + 3x_2 = -2x_3 + x_4 \end{cases} \Leftrightarrow \begin{cases} 3x_1 - 3x_2 = -6x_3 - 9x_4 \\ 2x_1 + 3x_2 = -2x_3 + x_4 \end{cases}$$

$5x_1 = -8x_3 - 8x_4.$

$x_1 = -\frac{8}{5}(x_3+x_4)$

$x_2 = x_1 + 2x_3 + 3x_4 = \frac{2}{5}x_3 + \frac{7}{5}x_4$

$V' = \{(-\frac{8}{5}x_3 - \frac{8}{5}x_4, \frac{2}{5}x_3 + \frac{7}{5}x_4, x_3, x_4) \mid x_3, x_4 \in \mathbb{R}\} =$

$= \{(-\frac{8}{5}x_3, \frac{2}{5}x_3, x_3, 0) + (-\frac{8}{5}x_4, \frac{7}{5}x_4, 0, x_4) \mid x_3, x_4 \in \mathbb{R}\} =$

$= \langle\{(-\frac{8}{5}, \frac{2}{5}, 1, 0), (-\frac{8}{5}, \frac{7}{5}, 0, 1)\}\rangle$ SG. $\Rightarrow$ Bază R' în V'

$R' = \{(-\frac{8}{5}, \frac{2}{5}, 1, 0), (-\frac{8}{5}, \frac{7}{5}, 0, 1)\}$ SLI

b) $\det \begin{vmatrix} -\frac{8}{5} & -\frac{8}{5} & 1 & 0 \\ \frac{2}{5} & \frac{7}{5} & 0 & \boxed{1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 1 \cdot (-1)^{2+4} \cdot \begin{vmatrix} -\frac{8}{5} & -\frac{8}{5} & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \cdot (-1)^{1+3} = 1 \neq 0$

$\Rightarrow rg\, A = 4 \quad \Rightarrow V'' = \langle \{e_1, e_2\} \rangle$

$R = \{(-\frac{8}{5}, \frac{2}{5}, 1, 0), (-\frac{8}{5}, \frac{7}{5}, 0, 1), (1, 0, 0, 0), (0, 1, 0, 0)\}$

c) $x = (1,2,1,2) = \underbrace{a(-\frac{8}{5}, \frac{2}{5}, 1, 0) + b(-\frac{8}{5}, \frac{7}{5}, 0, 1)}_{u \in V'} + \underbrace{c(1,0,0,0) + d(0,1,0,0)}_{v \in V''}$

$\Rightarrow \begin{cases} b = 2 \\ a = 1 \\ -\frac{8}{5}(a+b) + c = 1 \Rightarrow c = 1 + \frac{24}{5} = \frac{29}{5} \\ \frac{2}{5}a + \frac{7}{5}b + d = 2 \Rightarrow d = 2 - \frac{2}{5} - \frac{14}{5} = \frac{-6}{5} \end{cases}$

$u = (-\frac{24}{5}, \frac{16}{5}, 1, 2)$

$v = (\frac{29}{5}, -\frac{6}{5}, 0, 0) \in V''$

$x = u + v = (\frac{5}{5}, \frac{10}{5}, 1, 2) = (1, 2, 1, 2)$