T3 – Seminar

(3) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_1 + x_3, 0, 2x_1 + x_2)$.

a) $[f]_{R_0 R_0}$

b) $\ker f$, $\mathrm{Im} f$, dim, reper

a) $R_0 = \{e_1, e_2, e_3\} \xrightarrow{f} R_0 = \{e_1, e_2, e_3\}$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad A = [f]_{R_0 R_0} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

b) $\ker f = \{x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3}\} = S(A)$, $\dim \ker f = 3 - \mathrm{rg} A = 1$

$$A = \left(\begin{array}{ccc|c} \boxed{1} & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \\ \boxed{2} & 1 & \boxed{0} & 0 \end{array} \right) \qquad \begin{cases} x_1 + x_3 = 0 \\ 0 = 0 \\ 2x_1 + x_2 = 0 \end{cases}$$

$\mathrm{rg} A = 2 \Rightarrow$ x_2 – nec. sec.; x_1, x_3 – nec. pr.

$$\begin{cases} x_1 + x_3 = 0 \Rightarrow -x_1 = +x_3 \\ -2x_1 = x_2 \end{cases}$$

$\ker f = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \begin{cases} x_1 + x_3 = 0 \\ 0 = 0 \\ 2x_1 + x_2 = 0 \end{cases} \} = \{(x_1, -2x_1, -x_1) \mid x_1 \in \mathbb{R}\} = \langle\{(1,-2,-1)\}\rangle$

$\dim \ker f = 1.$

Conform T. dimensiunii: $\dim \mathbb{R}^3 = \dim \ker f + \dim \operatorname{Im} f$

$\Rightarrow \dim \operatorname{Im} f = 2$

$\operatorname{Im} f = \{y \in \mathbb{R}^3 \mid \exists x \in \mathbb{R}^3 \text{ a.î. } f(x) = y\} = \{y \in \mathbb{R}^3 \mid y_2 = 0\}$

$= \{y \in \mathbb{R}^3 \mid$

$$\begin{cases} x_1 + x_3 = y_1 \\ 0 = y_2 \\ 2x_1 + x_2 = y_3 \end{cases} \qquad A = \left(\begin{array}{ccc|c} 1 & 0 & 1 & y_1 \\ 0 & 0 & 0 & y_2 \\ 2 & 1 & 0 & y_3 \end{array}\right)$$

$\det A = 0$

$\operatorname{rg} A = 2 \Rightarrow \operatorname{rg} \bar{A} = 2 \Rightarrow \Delta_c = 0$

$$\Delta_c = 0 \Rightarrow 0 = \begin{vmatrix} 1 & 1 & y_1 \\ 2 & 0 & y_3 \\ 0 & 0 & y_2 \end{vmatrix} \Rightarrow y_2 (-1)^6 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 0 \Rightarrow y_2 = 0$$

$\operatorname{Im} f = \{(y_1, 0, y_3) \mid y_1, y_3 \in \mathbb{R}\} = \langle\{(1,0,0),(0,0,1)\}\rangle$

$R_1 = \{(1,2,-1)\}$ reper în $\ker f$

$R_2 = \{e_1, e_3\}$ reper în $\operatorname{Im} f$

① $f \in \operatorname{End}(\mathbb{R}^3)$, $A = [f]_{R_0 R_0} = \begin{pmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix}$

Val. proprii, subsp. proprii, reper

$P(\lambda) = \det(A - \lambda I_n) = 0$

$$\begin{vmatrix} -3-\lambda & -7 & -5 \\ 2 & 4-\lambda & 3 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0 \Rightarrow 3\lambda^2 - \lambda^3 - 3\lambda + 1 = -(\lambda - 1)^3 = 0$$

$\Rightarrow \lambda = 1.$

$\lambda_1 = 1, \; m_1 = 1.$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_1 x = x\}$

$$\begin{cases} -3x_1 - 7x_2 - 5x_3 = x_1 \\ 2x_1 + 4x_2 + 3x_3 = x_2 \\ x_1 + 2x_2 + 2x_3 = x_3 \end{cases} \Leftrightarrow \begin{cases} -4x_1 - 7x_2 - 5x_3 = 0 \\ 2x_1 + 3x_2 + 3x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \end{cases}$$

$$B = \begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \qquad \operatorname{rg} B = 2$$

$$\begin{cases} -4x_1 - 7x_2 = 5x_3 \\ 2x_1 + 3x_2 = -3x_3 \end{cases}$$

$-x_2 = -x_3 \Rightarrow x_2 = x_3.$

$\Rightarrow x_1 = \frac{-3x_3 - 3x_3}{2} = -3x_3$

$V_{2} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = -3x_3,\ x_2 = x_3\} = \{(-3x_3, x_3, x_3) \mid x_3 \in \mathbb{R}\} = \langle \underbrace{\{(-3,1,1)\}}_{R_1} \rangle.$

$R_1 = \{(-3,1,1)\}$ SG, $R_1 \subset \mathbb{R}^3$ SLI $\Rightarrow R_1$ reper în $\mathbb{R}_3$

② $f: \mathbb{R}^2 \to \mathbb{R}^2$ liniară a.î. $f(1,1) = (3,5)$, $f(-1,2) = (0,1)$

a) $f(x) = ?$
b) izo $- ?$

a) f liniară $\Leftrightarrow f(ax + by) = af(x) + bf(y)$
$f((a-b, a+2b)) = (3a, 5a+b)$

$$\begin{cases} a - b = x_1 \\ a + 2b = x_2 \end{cases} \Rightarrow 3b = x_2 - x_1 \Rightarrow b = \frac{x_2 - x_1}{3} \Rightarrow a = \frac{x_2 + 2x_1}{3}$$

$f(x) = \left(\frac{1}{3}(2x_1 + x_2), \frac{1}{3}(-x_1 + x_2)\right).$

b) $f(x) = y \Leftrightarrow y = AX$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{vmatrix} = 1/9 \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = \frac{1}{9}(2+1) = \frac{1}{3} \neq 0 \Rightarrow \exists A^{-1} \Rightarrow$$

$\Rightarrow f$ bijectivă $\Rightarrow f$ izomorfism.

– Bonus seminar 6 –

② $f:\mathbb{R}^3\to\mathbb{R}^3$, $f(x)=(x_1+x_2-x_3,\ -x_1-x_2+x_3,\ x_1+x_2+x_3)$.

a) $[f]_{R,R}$, $R=\{e_1'=e_1+e_2+e_3,\ e_2'=e_1+e_3,\ e_3'=e_1+e_2\}$.

b) $\mathbb{R}^3=\operatorname{Im}f\oplus W$.

$s:\mathbb{R}^3\to\mathbb{R}^3$ sim faţă de W, $s(0,1,1)=?$

c) $\mathbb{R}^3=f(V')\oplus U$.

$V'=\{x\in\mathbb{R}^3 \mid \begin{cases}x_1+2x_2+x_3=0\\ -x_1+x_2+2x_3=0\end{cases}\}$.

$p:\mathbb{R}^3\to\mathbb{R}^3$ pr. pe $f(V')$

$p(2,-1,3)=?$

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a) $R=\{e_1'=e_1+e_2+e_3,\ e_2'=e_1+e_3,\ e_3'=e_1+e_2\}\xrightarrow{A} R=\{e_1',e_2',e_3'\}$.

$f(e_1')=(1,-1,3)=a(1,1,1)+b(1,0,1)+c(1,1,0)\Rightarrow\begin{cases}a=1\\ b=2\\ c=-2\end{cases}$

$f(e_2')=(0,0,2)=a(1,1,1)+b(1,0,1)+c(1,1,0)\Rightarrow\begin{cases}a=2\\ b=0\\ c=-2\end{cases}$

$f(e_3')=(2,-2,2)=a(1,1,1)+b(1,0,1)+c(1,1,0)\Rightarrow\begin{cases}a=-2\\ b=4\\ c=0.\end{cases}$

$$A=\begin{pmatrix}1&2&-2\\2&0&4\\-2&-2&0\end{pmatrix}$$

b) $\operatorname{Im}f=\{y\in\mathbb{R}^3 \mid \exists x\in\mathbb{R}^3 \text{ a.î. } f(x)=y\}$

$\begin{cases}x_1+x_2-x_3=y_1\\ -x_1-x_2+x_3=y_2\\ x_1+x_2+x_3=y_3\end{cases}$ $\quad A=\left(\begin{array}{ccc|c}1&1&-1&y_1\\-1&-1&1&y_2\\1&1&1&y_3.\end{array}\right)$

$\det A=0$, $\operatorname{rg}A=2\Rightarrow \Delta_c=0\Rightarrow\begin{vmatrix}-1&1&y_2\\1&1&y_3\\1&-1&y_1\end{vmatrix}=0\Rightarrow -2y_1-2y_2=0$.

$y_1=-y_2$

$\operatorname{Im}f=\{(y_1,y_1,y_3) \mid y_1,y_3\in\mathbb{R}\}=\langle\{(1,1,0),(0,0,1)\}\rangle$.

$\operatorname{rg}\begin{pmatrix}1&0&1\\1&0&0\\0&1&0\end{pmatrix}=3\Rightarrow R=\operatorname{Im}f\cup\{e_1\}\Rightarrow W=\{e_1\}$, R reper în $\mathbb{R}^3$.

$p(0,1,1)=\underbrace{a(1,1,0)+b(0,0,1)}_{\in \operatorname{Im}f}+\underbrace{c(1,0,0)}_{\in W}$

$\begin{cases}a=1\\ b=1\\ c=-1\end{cases}$ $\quad v_1=(1,1,1)$, $v_2=(-1,0,0)$. $\quad p(0,1,1)=(-1,0,0)$.

$s(0,1,1)=2p-\mathrm{id}_{\mathbb{R}^3}=(-2,0,0)-(0,1,1)=(-2,-1,-1)$.

c)

$V' = \{x \in \mathbb{R}^3 \mid \begin{cases} x_1 + 2x_2 + x_3 = 0 \\ -x_1 + x_2 + 2x_3 = 0 \end{cases}\} = S(A)$ $\quad \dim V' = 3 - rg A = 1.$

$A = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right)$

$rg A = 2$

$\begin{cases} x_1 + 2x_2 = -x_3 \\ -x_1 + x_2 = -2x_3 \end{cases}$

$3x_2 = -3x_3$

$x_2 = -x_3, \quad x_1 = x_2 + 2x_3 = x_3.$

$V' = \{(x_3, -x_3, x_3) \mid x_3 \in \mathbb{R}\} = \langle\{(1,-1,1)\}\rangle$

$f((1,-1,1)) = (-1, 1, 1) \Rightarrow f(V') = \langle\{(-1,1,1)\}\rangle$

$rg\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = 3 \Rightarrow R = f(V') \cup U, \; U = \{e_1, e_2\} \rightarrow R$ reper în $\mathbb{R}^3$.

$(2,-1,3) = \underbrace{a(-1,1,1)}_{\in f(V')} + \underbrace{b(1,0,0) + c(0,1,0)}_{\in U}$

$\Rightarrow \begin{cases} a = 3 \\ b = 5 \\ c = -4 \end{cases}$ $\qquad v_1 = (-3, 3, 3)$, $\; v_2 = (5, -4, 0)$

$P(2,-1,3) = (-3, 3, 3)$

6) $f: \mathbb{R}^4 \to \mathbb{R}^4$ liniară $\quad A = [f]_{R_0R_0} = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix}$

a) val. pr., subsp. pr. coresp.

b) $U = \langle\{e_1 + 2e_2, \; e_2 + e_3 + 2e_4\}\rangle$

U subsp. invariant al lui f i.e. $f(U) \subset U$.

a) $P(\lambda) = \det(A - \lambda I_4) = 0.$

$\begin{vmatrix} 1-\lambda & 0 & 2 & -1 \\ 0 & 1-\lambda & 4 & -2 \\ 2 & -1 & -\lambda & 1 \\ 2 & -1 & -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 2 & -1 \\ 2-2\lambda & 1-\lambda & 0 & 0 \\ 2 & -1 & -\lambda & 1 \\ 2 & -1 & -1 & 2-\lambda \end{vmatrix} =$

$= (2-2\lambda)(-1)^3 \cdot \begin{vmatrix} 0 & 2 & -1 \\ -1 & -\lambda & 1 \\ -1 & -1 & 2-\lambda \end{vmatrix} + (1-\lambda)\cdot(-1)^4 \begin{vmatrix} 1-\lambda & 2 & -1 \\ 2 & -\lambda & 1 \\ 2 & -1 & 2-\lambda \end{vmatrix} =$

$= (2-2\lambda)(1-\lambda) + (1-\lambda)\cdot(1-\lambda)(\lambda^2 - 2\lambda - 1)$
$= (1-\lambda)(2-2\lambda + (1-\lambda)(\lambda^2-2\lambda-1)) = 0$

1) $1-\lambda = 0 \Rightarrow \lambda = 1$.
2) $2(1-\lambda) + (1-\lambda)(\lambda^2-2\lambda-1) = 0$
$(1-\lambda)(2+\lambda^2-2\lambda-1) = 0$.
$(1-\lambda)^3 = 0 \Rightarrow \lambda = 1$.

$\lambda_1 = 1, \ m_1 = 4$

$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid f(x) = \lambda_1 x = x \}$.

$\begin{cases} 2x_3 - x_4 = 0 \\ 4x_3 - 2x_4 = 0 \\ 2x_1 - x_2 - x_3 + x_4 = 0 \\ 2x_1 - x_2 - x_3 + x_4 = 0. \end{cases} \Rightarrow \begin{cases} 2x_3 = x_4 \\ 2x_1 - x_2 = -x_3 \end{cases} \to \begin{cases} x_4 = 2x_2 - 4x_1 \\ x_3 = x_2 - 2x_1 \end{cases}$

$V_{\lambda_1} = \{ (x_1, x_2, x_2 - 2x_1, 2x_2 - 4x_1) \mid x_1, x_2 \in \mathbb{R} \} = \langle \{ (1, 0, -2, -4), (0, 1, 1, 2) \} \rangle$

b) $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \Rightarrow f(e_1') = (1, 2, 0, 0)$.

$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} \Rightarrow f(e_2'') = (0, 1, 1, 2)$

$f(U) = \langle \{ (1,2,0,0), (0,1,1,2) \} \rangle = \langle \{ e_1 + 2e_2, e_2 + e_3 + 2e_4 \} \rangle \Rightarrow f(U) \subset U$

$\Rightarrow U$. subsp. invariant al lui f.

⑦ $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_2, x_3, 2x_1 - 5x_2 + 4x_3)$.

f. nu e endomorfism diagonalizabil.

$[f]_{R_0 R_0} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$

$P(\lambda) = \det(A - \lambda I_3) = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{vmatrix} = 0 \Rightarrow -(\lambda-1)^2(\lambda - 2) = 0$.

$\lambda_1 = 1, \ m_1 = 2$.
$\lambda_2 = 2, \ m_2 = 1$

$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid f(x) = \lambda_1 x = x \}$.

$\begin{cases} x_2 = x_1 \\ x_3 = x_2 \\ 2x_1 - 5x_2 + 3x_3 = 0 \end{cases} \Rightarrow x_1 = x_2 = x_3 \Rightarrow V_{\lambda_1} = \langle \{ (1,1,1) \} \rangle$

$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_2 x = 2x\}$.

$$\begin{cases} x_2 = 2x_1 \\ x_3 = 2x_2 = 4x_1 \\ 2x_1 - 10x_1 + 8x_1 = 0. \end{cases} \Rightarrow x_3 = 2x_2 = 4x_1.$$

$V_{\lambda_2} = \langle\{(4, 2, 1)\}\rangle$

$$\left.\begin{array}{l} \dim V_{\lambda_1} \neq m_1 \\ \dim V_{\lambda_2} = m_2 \end{array}\right| \Rightarrow f \text{ nu e endomorfism diagonalizabil}$$

— Bonus seminar 5 —

④ $f : \mathbb{R}_3[x] \to \mathbb{R}_2[x]$, $f(p) = p'$

a) $[f]_{R_0, R_0'} = A = ?$

b) dim ker f, dim Im f.

a) $R_0 = \{1, x, x^2, x^3\} \xrightarrow{A} R_0' = \{1, x, x^2\}$.

$f(1) = a1 + bx + cx^2 = 0 \Rightarrow a = b = c = 0$

$f(x) = a1 + bx + cx^2 = 1 \Rightarrow a = 1, b = c = 0.$

$f(x^2) = a1 + bx + cx^2 = 2x \Rightarrow a = c = 0, b = 2$

$f(x^3) = a1 + bx + cx^2 = 3x^2 \Rightarrow a = b = 0, c = 3.$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

b) $\dim \ker f = 4 - \operatorname{rg} A = 4 - 3 = 1$

T. dimensiunii: $\dim \mathbb{R}_3[x] = \dim \ker f + \dim \operatorname{Im} f$

$4 = 1 + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 3$

⑤ $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1 + 2x_2 + x_3, -x_1 - 2x_2 - x_3, x_1 + x_2 + x_3)$

a) $[f]_{R_0 R_0} = A = ?$

b) dim ker f, dim Im f

c) $U' = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \{x_1 - x_2 + x_3 = 0,\ x_1 + 2x_2 - x_3 = 0\}\}$

$f(U') = ?$

a) $f = A \cdot X$, $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

b) $\dim \ker f = 3 - \operatorname{rg} A = 3 - 2 = 1.$

$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

T. dimensiunii: $\dim \mathbb{R}^3 = \dim \ker f + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 2.$

c)

$B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$

$\operatorname{rg} B = 2 \Rightarrow \begin{cases} x_1 - x_2 = -x_3 \\ x_1 + 2x_2 = x_3 \end{cases}$

$3x_2 = 2x_3$

$x_2 = \frac{2}{3}x_3 \rightarrow x_1 = -\frac{1}{3}x_3.$

$V' = \{(-\frac{1}{3}x_3, \frac{2}{3}x_3, x_3) \mid x_3 \in \mathbb{R}\} = \langle \{(-1, 2, 3)\} \rangle.$

$f(-1,2,3) = (6, -6, 4) \Rightarrow f(V') = \langle \{(6, -6, 4)\} \rangle.$

(8) $f: \mathbb{R}_1[x] \to \mathbb{R}^3$, $f(ax+b) = (a, b, a+b)$.

$R_1 = \{2x-1, -x+1\}$, $R' = \{(1,1,1), (1,1,0), (1,0,0)\}$ repere.

a) f liniară

b) $[f]_{R_1,R'} = A = ?$

c) $\ker f$, $\operatorname{Im} f$

a) f liniară $\Leftrightarrow f(c_1(a_1x+b_1) + c_2(a_2x+b_2)) = c_1 f(x) + c_2 f(y)$ $\forall a_1x+b_1, a_2x+b \in \mathbb{R}_1[x]$, $\forall c_1, c_2 \in \mathbb{R}$.

$f(c_1(a_1x+b_1) + c_2(a_2x+b_2)) =$

$= f(a_1c_1x + b_1c_1 + a_2c_2x + c_2b_2) = (a_1c_1 + a_2c_2, b_1c_1 + b_2c_2, a_1c_1 + a_2c_2 + b_1c_1 + b_2c_2)$

$= (a_1c_1, b_1c_1, a_1c_1 + b_1c_1) + (a_2c_2, b_2c_2, a_2c_2 + b_2c_2) =$

$= c_1(a_1, b_1, a_1+b_1) + c_2(a_2, b_2, a_2+b_2)$

$= c_1 f(a_1x+b_1) + c_2 f(a_2x+b_2).$

$\Rightarrow f$ liniară.

b) $R_1 = \{2x-1, -x+1\} \xrightarrow{A} R' = \{(1,1,1), (1,1,0), (1,0,0)\}$

$f(2x-1) = a(1,1,1) + b(1,1,0) + c(1,0,0) = (2,-1,1) \Rightarrow a = 1$ $b = -2$ $c = 3$

$f(-x+1) = a(1,1,1) + b(1,1,0) + c(1,0,0) = (-1,1,0) \rightarrow a = 0, b = 1, c = -2.$

$A = [f]_{R_1,R'} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 3 & -2 \end{pmatrix}$

c) $\ker f = \{ x \in \mathbb{R}_1[X] \mid f(x) = 0_{\mathbb{R}^3} \}$

$f(x) = 0_{\mathbb{R}^3} \Rightarrow f(ax+b) = (a, b, a+b) = (0,0,0) \Rightarrow a = b = 0.$

$\ker f = \{ 0_V \}$

$\left\{ \dim \ker f = 0. \right.$

$\operatorname{Im} f = \{ y \in \mathbb{R}^3 \mid f(x) = y \}$

T. dimensiunii: $\dim \mathbb{R}_1[X] = \dim \operatorname{Im} f + \dim \ker f \Rightarrow \dim \operatorname{Im} f = 2.$

$\begin{cases} x = y_1 \\ -2x + y = y_2 \\ 3x - 2y = y_3 \end{cases}$ $\quad A = \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 3 & -2 \end{pmatrix} \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix}$

$\operatorname{rg} A = \operatorname{rg} \bar{A} = 2 \Rightarrow \begin{vmatrix} 1 & 0 & y_1 \\ -2 & 1 & y_2 \\ 3 & -2 & y_3 \end{vmatrix} = 0. \Rightarrow y_3 + 4y_1 - 3y_1 + 2y_2 = 0$

$y_1 + 2y_2 + y_3 = 0 \Rightarrow y_1 = -2y_2 - y_3$

$\operatorname{Im} f = \langle \{ (-2, 1, 0), (-1, 0, 1) \} \rangle$

(9) $g : \mathbb{R}^3 \to \mathbb{R}^3$ liniară, $g(v_i) = u_i$, $i = \overline{1,3}$

$v_1 = (-1, 1, 1)$, $v_2 = (1, 1, 1)$, $v_3 = (0, 2, 1)$.

$u_1 = 2v_1 + 3v_2 - v_3$, $u_2 = v_1 + 3v_2 + v_3$, $u_3 = v_3$.

a) $g((-1,1,1)) = 2v_1 + 3v_2 - v_3 = (1, 3, 4) = e_1 + 3e_2 + 4e_3$

$g((1,1,1)) = v_1 + 3v_2 + v_3 = (2, 6, 5) = 2e_1 + 6e_2 + 5e_3$

$g((0,2,1)) = v_3 = (0, 2, 1) = 2e_2 + e_3$

$g(x,y,z) = (e_1 + 3e_2 + 4e_3, 2e_1 + 6e_2 + 5e_3, 2e_2 + e_3)$

b) $[g]_{R_0 R_0} = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 5 \\ 0 & 2 & 1 \end{pmatrix}$

c) $\ker f = \{ x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3} \} = \{ \}$

$\dim \ker f = 3 - \operatorname{rg} A = 3 - 3 = 0.$

$\operatorname{rg} A = \operatorname{rg} \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 5 \\ 0 & 2 & 1 \end{pmatrix} = 3$

$\operatorname{Im} f = \{ y \in \mathbb{R}^3 \mid f(x) = y \} = \mathbb{R}^3$

T. dim. $\dim \mathbb{R}^3 = \dim \ker f + \dim \operatorname{Im} f = 3$

(10) $f : \mathbb{R}_1[X] \to \mathbb{R}_2[X]$, $f(ax+b) = ax^2 + (a+2b)X + a - b$

$R = \{1, X\}$, $R' = \{X^2, X^2+X+1, X^2+X\}$ repere

a) f liniară
b) $[f]_{R,R'}$
c) $\ker f$, $\operatorname{Im} f$

a) f liniară $\iff f(c_1(a_1x+b_1) + c_2(a_2x+b_2)) = c_1 f(a_1x+b_1) + c_2 f(a_2x+b_2)$
$\forall c_1, a_1, b_1, c_2, a_2, b_2 \in \mathbb{R}$

$f(c_1(a_1x+b_1) + c_2(a_2x+b_2)) = f((a_1c_1+a_2c_2)x + b_1c_1 + b_2c_2) =$

$= (a_1c_1+a_2c_2)x^2 + (a_1c_1+a_2c_2+2b_1c_1+2b_2c_2)x + a_1c_1 + a_2c_2 - b_1c_1 - b_2c_2 =$

$= c_1(a_1x^2 + (a_1+2b_1)x + a_1 - b_1) + c_2(a_2x^2 + (a_2+2b_2)x + a_2 - b_2) =$

$= c_1 f(a_1x+b_1) + c_2 f(a_2x+b_2) \Rightarrow$ f liniară

b) $f(1) = 2X - 1 = aX^2 + b(X^2+X+1) + c(X^2+X) \Rightarrow b = -1, c = 3, a = -2.$

$f(X) = X^2 + X + 1 = aX^2 + b(X^2+X+1) + c(X^2+X) \Rightarrow b = 1, a = c = 0.$

$[f]_{R,R'} = \begin{pmatrix} -2 & 0 \\ -1 & 1 \\ 3 & 0 \end{pmatrix}$

c) $\ker f = \{ x \in \mathbb{R}_1[X] \mid f(x) = 0 \} = \{0\}$. =

$f(ax+b) = 0 \Rightarrow a = 0 = b$

T. dim: $\dim \mathbb{R}_1[X] = \underbrace{\dim \ker f}_{0} + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 2$

$\operatorname{Im} f = \{ y \in \mathbb{R}_2[X] \mid f(x) = y \}$

$$A = \left(\begin{array}{cc|c} -2 & 0 & y_1 \\ -1 & 1 & y_2 \\ 3 & 0 & y_3 \end{array}\right)$$

$rg\, A = rg\, \overline{A} \Rightarrow \det \overline{A} = 0 \Rightarrow \begin{vmatrix} -2 & 0 & y_1 \\ -1 & 1 & y_2 \\ 3 & 0 & y_3 \end{vmatrix} = -2y_3 - 3y_1 = 0 \Rightarrow 3y_1 = -2y_3$

$\operatorname{Im} f = \langle \{ (0,1,0), (1,0,-\frac{3}{2}) \} \rangle$

⑪ $f : \mathbb{R}_2[X] \to \mathbb{R}_1[X]$ liniară

$f(X+2) = X-1$

$f(-X^2+3) = 2X+3$

$f(2X+5) = -X+1$

$f = ?$

Fie $g_i(x) = a_i X^2 + b_i X + c_i$, $a_i, b_i, c_i \in \mathbb{R}$, $g_1 = a_1X^2 + b_1X + c_1$ + $\alpha_2 g_2(a_2X^2 + b_2X + c_2)$

f liniară $\Leftrightarrow f(\alpha_1 g_1(x) + \alpha_2 g_2(x) + \alpha_3 g_3(x)) = \alpha_1 f(g_1(x)) + \alpha_2 f(g_2(x)) + \alpha_3 f(g_3(x))$

$f(-\alpha_2 X^2 + (\alpha_1 + 2\alpha_3)X + 2\alpha_1 + 3\alpha_2 + 5\alpha_3) = (\alpha_1 + 2\alpha_2 - \alpha_3)X + (-\alpha_1 + 3\alpha_2 + \alpha_3)$

$$\begin{cases} -\alpha_2 = a \\ \alpha_1 + 2\alpha_3 = b \\ 2\alpha_1 + 3\alpha_2 + 5\alpha_3 = c \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_1 = -6a + 5b - 2c \\ \alpha_2 = -a \\ \alpha_3 = 3a - 2b + c \end{cases}$$

$f(aX^2 + bX + c) = (-11a + 7b - 3c)X + (-6a + 3b - c)$

⑫ $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1 + x_2 + 3x_3, x_1 + x_2 + a, x_1 + x_3 + a)$, $a \in \mathbb{R}$

f apl. liniară

$a = ?$

f apl. liniară $\Leftrightarrow f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

$\Rightarrow (\alpha x_1 + \alpha x_2 + 3\alpha x_3 + \beta y_1 + \beta y_2 + 3\beta y_3, \alpha x_1 + \alpha x_2 + \beta y_1 + \beta y_2 + a, \alpha x_1 + \alpha x_3 + \beta y_1 + \beta y_3 + a)$

$= (\alpha x_1 + \alpha x_2 + 3\alpha x_3 + \beta y_1 + \beta y_2 + 3\beta y_3, \alpha x_1 + \alpha x_2 + \beta y_1 + \beta y_2 + 2a, \alpha x_1 + \alpha x_3 + \beta y_1 + \beta y_3 + 2a)$

$\Rightarrow a = 0$

(13) $f: \mathbb{R}_1[x] \to \mathbb{R}_1[x]$ liniară

a) $[f]_{R_0 R_0} = A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

b) $[f]_{RR} = A' = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$, $R = \{x-1, 2x+2\}$

a) $\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \Rightarrow \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + 2b_1 \\ 3a_1 + 4b_1 \end{pmatrix}$

$f(ax+b) = (a+2b)x + (3a+4b)$

b) $\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \Rightarrow \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + 2b_1 \\ -b_1 \end{pmatrix}$

$f(ax+b) = (a+2b)x - b$

(15) $f_m: \mathbb{R}^3 \to \mathbb{R}^3$, $[f_m]_{R_0 R_0} = \begin{pmatrix} 1 & -1 & m \\ 2 & 3 & m \\ m & 1 & 0 \end{pmatrix}$
apl. liniară

$f_m \in Aut(\mathbb{R}^3)$

$m = ?$

$f_m \in Aut(\mathbb{R}^3) \Rightarrow f$ bij $\Rightarrow \det([f_m]_{R_0 R_0}) = 0 \Rightarrow \begin{vmatrix} 1 & -1 & m \\ 2 & 3 & m \\ m & 1 & 0 \end{vmatrix} = 0$

$\overset{L_1 - L_2}{\Longrightarrow} \begin{vmatrix} -1 & -4 & 0 \\ 2 & 3 & m \\ m & 1 & 0 \end{vmatrix} = 0 \Rightarrow m \cdot (-1)^{3+2} \cdot \begin{vmatrix} -1 & -4 \\ m & 1 \end{vmatrix} = 0 \Rightarrow$

$\Rightarrow -m(-1+4m) = 0 \Rightarrow \begin{cases} m = 0 \\ m = 1/4 \end{cases}$ $S = \{0, 1/4\}$

(16) $f: \mathbb{R}^2 \to \mathbb{R}^3$ liniară, $[f]_{R_0, R_0'} = \begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & 0 \end{pmatrix}^T$

$\ker f$, $\operatorname{Im} f$

$\ker f = \{x \in \mathbb{R} \mid f(x) = 0_{\mathbb{R}^3}\}$ $\dim \ker f = 2 - 2 = 0$

$rg \begin{pmatrix} 1 & -1 \\ -2 & -1 \\ 1 & 0 \end{pmatrix} = 2$ $\begin{cases} -2x_1 - y_2 = 0 \\ x_1 = 0 \end{cases}$ $\ker f = \{0_{\mathbb{R}^2}\}$

$-2y_2 = -y_3 \Rightarrow y_2 = y_3 = 0$

$-y_2 = 0$

$R_1 = \{(1,0,0)\}$ ... în $\ker f$

T. dimensiunii: $\dim \mathbb{R}^2 = \dim \ker f + \dim \operatorname{Im} f$
$\Rightarrow \dim \operatorname{Im} f = 2$

$\operatorname{Im} f = \{ y \in \mathbb{R}^3 \mid f(x) = y \}$

$$A = \left(\begin{array}{cc|c} 1 & -1 & y_1 \\ -2 & -1 & y_2 \\ 1 & 0 & y_3 \end{array}\right)$$

$rg\, A = rg\, \overline{A} = 2 \Rightarrow \begin{vmatrix} 1 & -1 & y_1 \\ -2 & -1 & y_2 \\ 1 & 0 & y_3 \end{vmatrix} = 0 \Rightarrow y_1 - y_2 - 3y_3 = 0$

$y_1 = y_2 + 3y_3$

$\operatorname{Im} f = <\{(1,1,0), (3,0,1)\}>$

(14) $f : \mathbb{R}^3 \to \mathbb{R}^3$

$f(x) = (x_1 - 2x_2 + 3x_3, m x_1 + 3x_2 - x_3, x_2 - 3x_3)$

a) $m = ?$ f inj.
b) $m = 1$, $\operatorname{Im} f = ?$
c) $m = -\frac{8}{3}$, $\dim \operatorname{Im} f = ?$

a) $y = AX$, $A = \begin{pmatrix} 1 & -2 & 3 \\ m & 3 & -1 \\ 0 & 1 & -3 \end{pmatrix} = [f]_{R_0, R_0}$

$\ker f = \{ x \in \mathbb{R} \mid f(x) = 0_{\mathbb{R}^3} \}$

f injectivă $\Leftrightarrow \ker f = \{0_{\mathbb{R}^3}\} \Rightarrow \dim \ker f = 0 \Rightarrow rg\, A = 3 \Rightarrow \det A \neq 0$

$\det A = \begin{vmatrix} 1 & -2 & 3 \\ m & 3 & -1 \\ 0 & 1 & -3 \end{vmatrix} = -3m - 8$

$\det A \neq 0 \Rightarrow m \in \mathbb{R} \setminus \{-\frac{8}{3}\}$ a.î. f injectivă

b) $m = 1 \Rightarrow A = \begin{pmatrix} 1 & -2 & 3 \\ 1 & 3 & -1 \\ 0 & 1 & -3 \end{pmatrix}$

$\det A = 3 \Rightarrow \dim \ker f = 0$

T. dimensiunii: $\dim \mathbb{R}^3 = \dim \ker f + \dim \operatorname{Im} f$
$\Rightarrow \dim \operatorname{Im} f = \dim \mathbb{R}^3 = 3$

$\left.\begin{array}{l} \operatorname{Im} f \subseteq \mathbb{R}^3 \\ \dim \operatorname{Im} f = \dim \mathbb{R}^3 \end{array}\right| \Rightarrow \operatorname{Im} f = \mathbb{R}^3.$

c) $m = -\frac{8}{3} \Rightarrow \det A = 0 \Rightarrow rg\, A = 2$

$\dim \ker f = 3 - rg\, A = 1.$

T. Dimensioni: $\dim \mathbb{R}^3 = \dim \ker f + \dim \operatorname{Im} f.$

$\Rightarrow \dim \operatorname{Im} f = 3 - 1 = 2.$