TS - Seminar

① $(\mathbb{R}^3, g_0)$, $u = (0,1,-1)$, $\measuredangle \varphi = \pi$, axa $<\{u\}>$

trans. ort. sp. 1.

$x_2 - x_3 = 0 \Rightarrow x_2 = x_3$

$<\{u\}>^\perp = <\{(1,0,0), (0,1,1)\}>$.

$R = \{\frac{1}{\sqrt2}(0,1,-1), (1,0,0), \frac{1}{\sqrt2}(0,1,1)\}$

$$C = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \end{pmatrix} \quad R_0 = \{e_1^0, e_2^0, e_3^0\} \xrightarrow{C} R = \{e_1, e_2, e_3\}.$$

$$[f]_{RR} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = A'$$

$$A = CA'C^T = \begin{pmatrix} 0 & 1 & 0 \\ 1/\sqrt2 & 0 & 1/\sqrt2 \\ -1/\sqrt2 & 0 & 1/\sqrt2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1/\sqrt2 & -1/\sqrt2 \\ 1 & 0 & 0 \\ 0 & 1/\sqrt2 & 1/\sqrt2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$f(x) = (-x_1, -x_3, -x_2)$, $f: \mathbb{R}^3 \to \mathbb{R}^3$

② $(\mathbb{R}_2[X], g_0)$, $g_0(P, Q) = \sum_{i=0}^{2} a_i b_i$, $P = a_0 + a_1 X + a_2 X^2$
$Q = b_0 + b_1 X + b_2 X^2$.

ortonormeze $R = \{x, x - x^2, 1 + x + x^2\}$ resp. cu g_0

$e_1 = f_1 = X$

$$e_2 = f_2 - \frac{<f_2, e_1>}{<e_1, e_1>} e_1 = X - X^2 - X = -X^2$$

$<f_2, e_1> = X$
$<e_1, e_1> = X$

$$e_3 = f_3 - \frac{<f_3, e_1>}{<e_1, e_1>} e_1 - \frac{<f_3, e_2>}{<e_2, e_2>} e_2 = 1 + X + X^2 - X + (-X^2) - 1.$$

$<f_3, e_1> = X$
$<e_1, e_1> = X$
$<f_3, e_2> = -X^2$
$<e_2, e_2> = X^2$

$$e_1' = \frac{1}{\|e_1\|} e_1 = x$$

$$e_2' = \frac{1}{\|e_2\|} e_2 = -x^2$$

$$e_3' = \frac{1}{\|e_3\|} e_3 = 1$$

$\mathcal{R}' = \{e_1' = x, e_2' = -x^2, e_3' = 1\}$ reper ortonormat.

(3) $(\mathbb{R}^3, g_0)$, $U = \{x \in \mathbb{R}^3 \mid \begin{cases} x_1 + x_3 = 0 \\ 3x_2 + x_3 = 0 \end{cases}\}$

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a) $U^\perp$

b) $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ rep. ortonormat $\mathbb{R}^3$ a.î. $\mathcal{R}_1$ rep. ort. în U,

$\mathcal{R}_2$ rep. ort. în $U^\perp$

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a) $\begin{cases} x_1 + x_3 = 0 \\ 3x_2 + x_3 = 0 \end{cases} \Rightarrow$

$\Rightarrow 3x_2 - x_1 = 0 \quad \Rightarrow \quad x_1 = 3x_2$

$x_3 = -3x_2$

$U = \langle\{(3, 1, -3)\}\rangle$

$3x_1 + x_2 - 3x_3 = 0 \Rightarrow x_2 = 3x_3 - 3x_1$

$U^\perp = \langle\{(1, -3, 0), (0, 3, 1)\}\rangle$

b) $\mathcal{R}_1 = \{\frac{1}{\sqrt{19}}(3, 1, -3)\}$

$e_2 = f_2 = (1, -3, 0) \quad e_2' = \frac{1}{\sqrt{10}}(1, -3, 0)$

$e_3 = f_3 - \frac{\langle f_3, e_2\rangle}{\langle e_2, e_2\rangle} e_2 = (0, 3, 1) - \frac{-9}{10}(1, -3, 0) = \left(\frac{9}{10}, \frac{3}{10}, 1\right)$

$= \frac{1}{10}(9, 3, 10)$

$\langle f_3, e_2\rangle = -9 \qquad e_3' = \frac{1}{10\sqrt{190}}(9, 3, 10)$

$\langle e_2, e_2\rangle = 10$

$\mathcal{R}_2 = \{\frac{1}{\sqrt{10}}(1, 0, 3), \frac{1}{10\sqrt{190}}(9, 3, 10)\}$

$\mathcal{R} = \{e_1' = \frac{1}{\sqrt{19}}(3, 1, -3), e_2' = \frac{1}{\sqrt{10}}(1, -3, 0), e_3' = \frac{1}{10\sqrt{190}}(9, 3, 10)\}$

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⑥ $(\mathbb{R}^3, g)$, $U = \langle \{(1,0,1), (1,1,2)\} \rangle$

a) $U^\perp$

b) $R = R_1 \cup R_2$ ort. în $\mathbb{R}^3$ (R_1 r.o. U, R_2 r.o. $U^\perp$)

a) $u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} = e_2 + e_3 - e_1 - 2e_2 = (-1, -1, 1)$

$U^\perp = \langle \{(-1,-1,1)\} \rangle$

b) $e_1 = f_1 = (1,0,1)$ $\quad e_1' = \frac{1}{\sqrt{2}}(1,0,1)$

$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 = (1,1,2) - \frac{3}{2}(1,0,1) = \left(-\frac{1}{2}, 1, \frac{1}{2}\right) = \frac{1}{2}(-1,2,1)$

$\langle f_2, e_1 \rangle = 3$ $\quad e_2' = \frac{1}{2\sqrt{6}}(-1,2,1)$

$\langle e_1, e_1 \rangle = 2$

$R_1 = \{e_1' = \frac{1}{\sqrt{2}}(1,0,1), e_2' = \frac{1}{2\sqrt{6}}(-1,2,1)\}$ r.o. în U

$R_2 = \{e_3' = \frac{1}{\sqrt{3}}(-1,-1,1)\}$ r.o. în $U^\perp$

$R = R_1 \cup R_2$ r.o. în $\mathbb{R}^3$

⑦ $(\mathbb{R}_2[X], +, \cdot)_{/\mathbb{R}}$, $g : \mathbb{R}_2[X] \times \mathbb{R}_2[X] \to \mathbb{R}$

$g(P,Q) = \sum_{k=0}^{2} a_k b_k$, $P = a_0 + a_1 X + a_2 X^2$

$Q = b_0 + b_1 X + b_2 X^2$

ortonormeze $\{2, 3-2x, 1-2x+x^2\}$

$e_1 = f_1 = 2$

$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 = 3 - 2x - \left(\frac{3}{2} \cdot 2\right) = -2x$

$\langle f_2, e_1 \rangle = 6$

$\langle e_1, e_1 \rangle = 4$

$e_3 = f_3 - \frac{\langle f_3, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 - \frac{\langle f_3, e_2 \rangle}{\langle e_2, e_2 \rangle} e_2 = 1 - 2x + x^2 - \frac{1}{\cancel{2}} \cdot \cancel{2} + 2x = x^2$

$\langle f_3, e_1 \rangle = 2$

$\langle e_1, e_1 \rangle = 4$

$\langle f_3, e_2 \rangle = 4x$

$\langle f_2, e_2 \rangle = 4x$

$e_1' = 1$ $\quad e_2' = -x$ $\quad e_3' = x^2$

$R = \{1, -x, x^2\}$ reper ortonormat

8) $(\mathbb{R}^3, g_0)$, $U = \{x \in \mathbb{R}^3 \mid \begin{cases} x_1 - x_3 = 0 \\ 2x_2 - x_3 = 0 \end{cases}\}$

a) $U^\perp$

b) $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ r.o. în $\mathbb{R}^3$ ai $\mathcal{R}_1$ r.o. U, $\mathcal{R}_2$ r.o. $U^\perp$

a) $\begin{matrix} x_1 = x_3 \\ 2x_2 = x_3 = x_1 \end{matrix} \Rightarrow U = \langle\{(2,1,2)\}\rangle$.

$2x_1 + x_2 + 2x_3 = 0 \Rightarrow x_2 = -2x_1 - 2x_3$

$U^\perp = \langle\{(1,-2,0), (0,-2,1)\}\rangle$

b) $e_1 = f_1 = (2,1,2) \Rightarrow e_1' = \frac{1}{3}(2,1,2)$

$e_2 = f_2 = (1,-2,0) \Rightarrow e_2' = \frac{1}{\sqrt{5}}(1,-2,0)$.

$e_3 = f_3 - \frac{\langle f_3, e_2\rangle}{\langle e_2, e_2\rangle} e_2 = (0,-2,1) - \frac{4}{5}(1,-2,0) = \left(\frac{-4}{5}, \frac{-2}{5}, 1\right) = \frac{1}{5}(-4,-$

$\langle f_3, e_2\rangle = 4$

$\langle e_2, e_2\rangle = 5$.

$\Rightarrow e_3' = \frac{1}{\sqrt{5}\sqrt{5}}(-4,-2,5)$.

$\mathcal{R}_1 = \{e_1' = \frac{1}{3}(2,1,2)\}$ r.o. în U

$\mathcal{R}_2 = \{e_2' = \frac{1}{\sqrt{5}}(1,-2,0),\ e_3' = \frac{1}{\sqrt{5}\sqrt{5}}(-4,-2,5)\}$

9) $(\mathbb{R}^3, g_0)$, $f \in \mathrm{End}(\mathbb{R}^3)$, $[f]_{\mathcal{R}_0\mathcal{R}_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 1/2 \\ 0 & -1/2 & \frac{\sqrt{3}}{2} \end{pmatrix}$

$f \in O(\mathbb{R}^3) \Leftrightarrow \mathcal{R}_0 \xrightarrow{A} \mathcal{R}' = \{e_1', e_2', e_3'\}$

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(12) $(\mathbb{R}^4, g_0)$ $\mathcal{R} = \{f_1=(-1,2,2,1),\ f_2=(-1,1,-5,-3),\ f_3=(-3,2,8,7),\ f_4=(0,-1,1,0)\}$

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$e_1 = f_1 = (-1,2,2,1)$.

$e_2 = f_2 - \frac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1 = (-1,1,-5,-3) - \frac{-10}{10}(-1,2,2,1) = (-2,3,-3,-2)$

$\langle f_2, e_1\rangle = 1+2-10-3 = -10$

$\langle e_1, e_1\rangle = 1+4+4+1 = 10$.

$e_3 = f_3 - \frac{\langle f_3, e_1\rangle}{\langle e_1, e_1\rangle} e_1 - \frac{\langle f_3, e_2\rangle}{\langle e_2, e_2\rangle} e_2 = f_3 - 3e_1 + e_2 = (-2,-1,-1,1)$

$\langle f_3, e_1\rangle = 3+4+16+7 = 30$.

$\langle e_1, e_1\rangle = 10$.

$\langle f_3, e_2\rangle = 6+6-24-14 = -26$

$\langle e_2, e_2\rangle = 4+9+9+4 = 26$

$e_4 = f_4 - \frac{\langle f_4, e_1\rangle}{\langle e_1, e_1\rangle} e_1 - \frac{\langle f_4, e_2\rangle}{\langle e_2, e_2\rangle} e_2 - \frac{\langle f_4, e_3\rangle}{\langle e_3, e_3\rangle} e_3 = f_4 - \frac{3}{13} e_2 = \left(\frac{6}{13}, \frac{-10}{13}, \frac{16}{13}, \frac{-3}{13}\right)$

$= \frac{1}{13}(6,-10,16,-3)$

$\langle f_4, e_1\rangle = 0$

$\langle f_4, e_2\rangle = -6$.

$\langle e_2, e_2\rangle = -26$.

$\langle f_4, e_3\rangle = 0$

$\mathcal{R}_1 = \{e_1' = \frac{1}{\sqrt{10}}(-1,2,2,1),\ e_2' = \frac{1}{\sqrt{26}}(-2,3,-3,-2),\ e_3' = \frac{1}{\sqrt{7}}(-2,-1,-1,1),\ e_4' = \frac{1}{13\sqrt{401}}(6,-10,16,-3)\}$

reper ortonormat.

(14) $(\mathcal{M}_2(\mathbb{R}), g)$, $g(A,B) = Tr(A^T \cdot B)$, $\forall A,B \in \mathcal{M}_2(\mathbb{R})$

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a) g - pr. sc.

b) $\mathcal{R} = \left\{\begin{pmatrix}1&0\\2&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}1&2\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$.

II

a) $A = \begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}$, $B = \begin{pmatrix}y_1 & y_2\\ y_3 & y_4\end{pmatrix}$ $y_i, x_i \in \mathbb{R}$

$g(A,B) = Tr\begin{pmatrix}x_1 & x_3\\ x_2 & x_4\end{pmatrix}\begin{pmatrix}y_1 & y_2\\ y_3 & y_4\end{pmatrix} = Tr\begin{pmatrix}x_1y_1+x_3y_3 & x_1y_2+x_3y_4\\ x_2y_1+x_4y_3 & x_2y_2+x_4y_4\end{pmatrix}$

$= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 = g_0$ evident Produs scalar

b) fie $\mathcal{R} = \{(1,0,2,1), (0,1,1,0), (1,2,1,0), (0,0,0,1)\}$

$e_1 = f_1 = (1,0,2,1)$

$e_2 = f_2 - \frac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1 = (0,-1,1,0) - \frac{1}{3}(1,0,2,1) = \left(\frac{-1}{3}, -1, \frac{1}{3}, -\frac{1}{3}\right)$

$= \frac{1}{3}(-1,-3,1,-1)$

$\langle f_2, e_1\rangle = 2$

$\langle e_1, e_1\rangle = 1+4+1 = 6$

$e_3 = f_3 - \frac{\langle f_3, e_1\rangle}{\langle e_1, e_1\rangle} e_1 - \frac{\langle f_3, e_2\rangle}{\langle e_2, e_2\rangle} e_2 = (1,2,1,0) - \frac{1}{2}(1,0,2,1) - \frac{1}{2}(-1,-3,1,-1)$

$\langle f_3, e_1\rangle = 3$

$\langle e_1, e_1\rangle = 6$

$\langle f_3, e_2\rangle = \frac{1}{3}(-6) = -2$

$\langle e_2, e_2\rangle = \frac{1}{9} \cdot 12 = \frac{4}{3}$

$= \left(1, \frac{7}{2}, \frac{-1}{2}, 0\right) = \frac{1}{2}(2,7,-1,0)$

$e_4 = f_4 - \frac{\langle f_4, e_1\rangle}{\langle e_1, e_1\rangle} e_1 - \frac{\langle f_4, e_2\rangle}{\langle e_2, e_2\rangle} e_2 - \frac{\langle f_4, e_3\rangle}{\langle e_3, e_3\rangle} e_3 = f_4 - \frac{1}{6} e_1 + \frac{1}{4} e_2$

$\langle f_4, e_1\rangle = 1$

$\langle e_1, e_1\rangle = 6$

$\langle f_4, e_2\rangle = -1/3$

$\langle e_2, e_2\rangle = 4/3$

$\langle f_4, e_3\rangle = 0$

$\langle e_3, e_3\rangle = -$

$= (0,0,0,1) - \frac{2}{12}(1,0,2,1) + \frac{1}{12}(-1,-3,1,-1)$

$= \left(\frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{3}{4}\right) = \frac{1}{4}(-1,-1,-1,3)$

$\mathcal{R}_1 = \{e_1' = \frac{1}{\sqrt{6}}(1,0,2,1),\ e_2' = \frac{1}{6\sqrt{3}}(-1,-3,1,-1),\ e_3' = \frac{1}{6\sqrt{6}}(2,7,-1,0),\ e_4' = \frac{1}{8\sqrt{3}}(-1,-1,-1,3)\}$

reper ortonormat

(11) $C([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ cont}\}$

$h(f,g) = \int_a^b f(t) g(t)\, dt \quad \forall f,g \in C([a,b])$

$(C([a,b]), h)$ s.v.e

$h(f,g) = h(g,f)$

$h(af+bg, u) = a h(f,u) + b\, h(g,u)$

① $h(f,g) = \int_a^b f(t)\, g(t)\, dt = \int_a^b g(t) f(t)\, dt = h(g,f) \quad \forall f,g \in C([a,b])$

② $h(af+bg, u) = \int_a^b (af(t) + bg(t)) u(t)\, dt =$

$= \int_a^b (af(t)u(t) + bg(t)u(t))\, dt = a\int_a^b f(t)u(t)\, dt + b\int_a^b g(t)u(t)\, dt$

$= a h(f, \mu) + b h(g, \mu)$ $\Rightarrow$

$1, 2 \Longrightarrow \left(C([a,b]), h\right)$ S.V.E.