T3 – Curs

① $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1 - x_2 + x_3,\ x_1 - x_2 + x_3,\ x_3)$.

a) $A = [f]_{R_0 R_0}$

b) val. proprii, subsp. proprii; repere în fiecare

c) $\ker f$, $\operatorname{Im} f$, repere

d) $\mathbb{R}^3 = \ker f \oplus W$, $W = ?$

$P(1,0,3)$, $pr(1,0,3) = ?$ (față de $\ker f$).

a) $R_0 = \{e_1, e_2, e_3\} \xrightarrow{A} R_0 = \{e_1, e_2, e_3\}$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad A = [f]_{R_0 R_0} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

b) $P(\lambda) = \det(A - \lambda I_3) = 0.$

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & -1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-1)^6 \cdot \begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\left[(1-\lambda)(-1-\lambda) + 1\right] = 0.$$

I. $1-\lambda = 0 \Rightarrow \lambda = 1.$

II. $(1-\lambda)(-1-\lambda) = -1.$

$(1-\lambda)(1+\lambda) = 1. \Rightarrow 1-\lambda^2 = 1 \Rightarrow \lambda = 0$

$\lambda_1 = 1, \; m_1 = 1$

$\lambda_2 = 0, \; m_2 = 2.$

$V_{\lambda_1} = \{x \in \mathbb{R} \mid f(x) = \lambda_1 x = x\} = \{(x_1, x_1, x_1) \mid x_1 \in \mathbb{R}\} = \langle\{(1,1,1)\}\rangle.$

$$\begin{cases} x_1 - x_2 + x_3 = x_1 \\ x_1 - x_2 + x_3 = x_2 \\ x_3 = x_3 \end{cases} \Leftrightarrow \begin{cases} x_2 = x_3 \\ x_1 = x_2 \\ x_3 = x_3 \end{cases} \Rightarrow x_1 = x_2 = x_3.$$

$R_1 = \{(1,1,1)\}$. SG + SLi ($R_1 \subset \mathbb{R}^3$ subsp. vect.) $\Rightarrow R_1$ reper din $\mathbb{R}^3$.

$V_{\lambda_2} = \{x \in \mathbb{R} \mid f(x) = \lambda_2 x = 0\} = \{(x_1, x_1, 0) \mid x_1 \in \mathbb{R}\} = \langle\{(1,1,0)\}\rangle$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_3 = 0. \end{cases} \Rightarrow \begin{cases} x_1 = x_2 \\ x_3 = 0. \end{cases}$$

$R_2 = \{(1,1,0)\}$

c) $\ker f = \{x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3}\} = S(A)$, $\dim \ker f = 3 - \operatorname{rg} A = 1$.

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \operatorname{rg} A = 2.$$

$$\begin{cases} -x_2 + x_3 = -x_1 \\ x_3 = 0 \end{cases} \Rightarrow x_1 = x_2.$$

$\ker f = \{(x_1, x_1, 0) \mid x_1 \in \mathbb{R}\} = \langle \{(1,1,0)\} \rangle$.

$R_1 = \{(1,1,0)\}$ reper în $\ker f$

T. dimensiunii: $\dim \mathbb{R}^3 = \dim \ker f + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 2$.

$\operatorname{Im} f = \{y \in \mathbb{R}^3 \mid \exists\, x \in \mathbb{R}^3$ a.î. $f(x) = y\}$.

$$\begin{cases} x_1 - x_2 + x_3 = y_1 \\ x_1 - x_2 + x_3 = y_2 \\ x_3 = y_3 \end{cases} \qquad \bar{A} = \left(\begin{array}{ccc|c} 1 & -1 & 1 & y_1 \\ 1 & -1 & 1 & y_2 \\ 0 & 0 & 1 & y_3 \end{array}\right)$$

$$\operatorname{rg} A = \operatorname{rg} \bar{A} = 2 \Rightarrow \Delta_c = 0 \Rightarrow \begin{vmatrix} -1 & 1 & y_2 \\ 0 & 1 & y_3 \\ -1 & 1 & y_1 \end{vmatrix} = 0 \Rightarrow -y_1 - y_3 + y_2 + y_3 = 0$$

$$y_1 = y_2$$

$\operatorname{Im} f = \{(y_1, y_1, y_3) \mid y_1, y_3 \in \mathbb{R}\} = \langle \{(1,1,0), (0,0,1)\} \rangle$.

$R_2 = \{(1,1,0), (0,0,1)\}$ reper în $\operatorname{Im} f$

d). $\operatorname{rg} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3 \Rightarrow \mathbb{R}^3 = \ker f \cup \{e_1, e_3\} \Rightarrow W = \{e_1, e_3\}$

$$(1,0,3) = \underbrace{a(1,1,0)}_{\in \ker f} + \underbrace{b(1,0,0) + c(0,0,1)}_{\in W} = (a+b, a, c).$$

$$\begin{cases} a + b = 1 \\ a = 0 \\ c = 3 \end{cases} \Leftrightarrow \begin{cases} b = 1 \\ a = 0 \\ c = 3 \end{cases}$$

$v_1 \in \ker f$, $v_1 = (0,0,0)$. $\qquad p(1,0,3) = (0,0,0)$.
$v_2 \in W$, $v_2 = (1,0,3)$.

$s = 2p - id_{\mathbb{R}^3} \Rightarrow s(1,0,3) = (0,0,0) - (1,0,3) = (-1,0,-3)$

③ $f: \mathbb{R}^2 \to \mathbb{R}^2$, $A = \begin{pmatrix} \alpha & 1-\alpha \\ 1 & 2 \end{pmatrix} = [f]_{R_0, R_0}$

$\alpha = ?$ a.î. 1) $\lambda = 1$ val. proprie
2) $\lambda = -1$ — // —
3) $0 \notin \sigma(f)$

$\lambda = 1$ val. proprie $\Rightarrow \det(A - I_2) = 0$

$\begin{vmatrix} \alpha-1 & 1-\alpha \\ 1 & 1 \end{vmatrix} = 0$

$\alpha - 1 - 1 + \alpha = 0$

$\alpha = 1$

$\lambda = -1$ val. proprie $\Rightarrow \det(A + I_2) = 0$

$\begin{vmatrix} \alpha+1 & 1-\alpha \\ 1 & 3 \end{vmatrix} = 0$

$3\alpha + 3 - 1 + \alpha = 0$

$4\alpha = 2$

$\alpha = 1/2$

$0 \notin \sigma(f) \Rightarrow \det(A) \neq 0$

$\begin{vmatrix} \alpha & 1-\alpha \\ 1 & 2 \end{vmatrix} \neq 0 \Rightarrow 2\alpha - 1 + \alpha \neq 0$

$3\alpha \neq 1$

$\alpha \neq 1/3$

② $S: V \to W$ liniară

$S^*: W^* \to V^*$, $S^*(f) = f \circ S$, $\forall f \in W^*$ (pull-back).

a) S^* liniară

b) S surj $\Rightarrow S^*$ inj

a) fie $f, g \in W^*$, $a, b \in K$.

$S^*(af + bg) = (af + bg) \circ S = (af) \circ S + (bg) \circ S = a S^*(f) + b S^*(g)$

$\Rightarrow S^*$ liniară

b) $\ker S^* = \{ f \in W^* \mid S^*(f) = 0_{V^*} \}$

$S^*(f) = f \circ S = f(S(x)) =$

S surj $\Rightarrow S(x) = W \Rightarrow f(W) = 0_K \Rightarrow \ker S^* = \{0_{V^*}\}$

astfel, dacă S surjectivă $\Rightarrow S^*$ injectivă