T4 - Seminar

① $f:\mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1 - 2x_2, -2x_1 + 2x_2 - 2x_3, -2x_2 + 3x_3)$.

$\xrightarrow{\text{endom.}}$ diag.

$$A = [f]_{R_0, R_0} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}.$$

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & 2-\lambda & -2 \\ 0 & -2 & 3-\lambda \end{vmatrix} = 0 \to (\lambda+1)(\lambda-2)(\lambda-5) = 0$$

$\lambda_1 = -1$, $m_1 = 1$

$\lambda_2 = 2$, $m_2 = 1$

$\lambda_3 = 5$, $m_3 = 1$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_1 x = -x\}$.

$rg\begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 4 \end{pmatrix} = 2 \Rightarrow \dim V_{\lambda_1} = 1 = m_1$ (1)

$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_2 x = 2x\}$.

$rg\begin{pmatrix} -1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & 1 \end{pmatrix} = 2 \Rightarrow \dim V_{\lambda_2} = 1 = m_2$ (2).

$V_{\lambda_3} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_3 x = 5x\}$.

$rg\begin{pmatrix} -4 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -2 \end{pmatrix} = 2 \Rightarrow \dim V_{\lambda_3} = 1 = m_3$ (3)

(1), (2), (3) $\Rightarrow f$ endomorfism diagonalizabil.

(2) $G = \begin{pmatrix} 3 & -2 & -4 \\ -2 & 6 & 2 \\ -4 & 2 & 3 \end{pmatrix}$ mat. as. f. pătratice $Q: \mathbb{R}^3 \to \mathbb{R}$ rap. la R_0.

a) $Q = ?$ $g = ?$ (f polară), $\ker g = ?$

b) Q la formă canonică + reper

a) $Q = 3x_1^2 + 6x_2^2 + 3x_3^2 - 4x_1x_2 - 8x_1x_3 + 4x_2x_3$

$g(x,y) = 3x_1y_1 - 2x_1y_2 - 4x_1y_3 - 2x_2y_1 + 6x_2y_2 + 2x_2y_3 - 4x_3y_1 + 2x_3y_2 + 3x_3y_3$

$\ker g = \{x \in V \mid g(x,y) = 0 \ \forall y \in V\}$.

$\det G = \begin{vmatrix} 3 & -2 & -4 \\ -2 & 6 & 2 \\ -4 & 2 & 3 \end{vmatrix} = -34 \neq 0 \Rightarrow$

$\Rightarrow$ g nedegenerată $\Rightarrow \ker g = \{0_V\}$.

b) Metoda Gauss

$Q = 3x_1^2 + 6x_2^2 + 3x_3^2 - 4x_1x_2 - 8x_1x_3 + 4x_2x_3$

$= 3(x_1^2 - \frac{4}{3}x_1x_2 - \frac{8}{3}x_1x_3) + 6x_2^2 + 3x_3^2 + 4x_2x_3$

$= 3(x_1 - \frac{2}{3}x_2 - \frac{4}{3}x_3)^2 - \frac{4}{3}x_2^2 - \frac{16}{3}x_3^2 - \frac{16}{3}x_2x_3 + 6x_2^2 + 3x_3^2 + 4x_2x_3$

$= 3(x_1 - \frac{2}{3}x_2 - \frac{4}{3}x_3)^2 + \frac{14}{3}x_2^2 - \frac{7}{3}x_3^2 - \frac{4}{3}x_2x_3$

$$= 3\left(x_1 - \frac{2}{3}x_2 - \frac{4}{3}x_3\right)^2 + \frac{14}{3}\left(x_2^2 - \frac{4}{14}x_2x_3\right) - \frac{7}{3}x_3^2$$

$$= 3\left(x_1 - \frac{2}{3}x_2 - \frac{4}{3}x_3\right)^2 + \frac{14}{3}\left(x_2 - \frac{1}{7}x_3\right)^2 - \frac{2}{21}x_3^2 - \frac{7}{3}x_3^2$$

$$= 3\left(x_1 - \frac{2}{3}x_2 - \frac{4}{3}x_3\right)^2 + \frac{14}{3}\left(x_2 - \frac{1}{7}x_3\right)^2 - \frac{17}{7}x_3^2$$

$$\begin{cases} x_1' = x_1 - \frac{2}{3}x_2 - \frac{4}{3}x_3 \\ x_2' = x_2 - \frac{1}{7}x_3 \\ x_3' = x_3 \end{cases} \Rightarrow Q(x) = 3x_1'^2 + \frac{14}{3}x_2'^2 - \frac{17}{7}x_3'^2$$

$(2,1)$ – signatura

$$\Rightarrow \begin{cases} x_1 = x_1' - \frac{2}{3}x_2' - \frac{10}{7}x_3' \\ x_2 = x_2' + \frac{1}{7}x_3' \\ x_3 = x_3' \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{2}{3} & -\frac{10}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \Rightarrow \mathcal{R}' = \left\{(1,0,0), \left(-\frac{2}{3},1,0\right), \left(-\frac{10}{7},\frac{1}{7},1\right)\right\}$$

reper.

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② b) $[f]_{R_0,R_0} = A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$ $f \in End(\mathbb{R}^3)$

$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} -\lambda & 1 & 0 \\ -4 & 4-\lambda & 0 \\ -2 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -\lambda & 1 \\ -4 & 4-\lambda \end{vmatrix} =$

$= (2-\lambda)(\lambda^2 - 4\lambda + 4) = (2-\lambda)(\lambda-2)^2 = -(\lambda-2)^3 = 0 \Rightarrow \lambda_1 = 2, m_1 = 3$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_1 x = 2x\}$

$\begin{cases} x_2 = 2x_1 \\ -4x_1 + 4x_2 = 2x_2 \\ -2x_1 + x_2 + 2x_3 = 2x_3 \end{cases} \Leftrightarrow \begin{cases} 2x_1 = x_2 \\ 2x_1 = x_2 \\ 2x_1 = x_2 \end{cases}$ $V_{\lambda_1} = \langle \{(1,2,0), (0,0,1)\} \rangle$

$\dim V_{\lambda_1} \neq m_1 \Rightarrow \nexists$ reper R in $\mathbb{R}^3$ ai $[f]_{R_0,R_0}$ mat. diagonală

c) $[f]_{R_0R_0} = A = \begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}$, $f \in End(\mathbb{R}^3)$

$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & -3 & 4 \\ 4 & -7-\lambda & 8 \\ 6 & -7 & 7-\lambda \end{vmatrix} = 0$

$\Rightarrow -\lambda^3 + \lambda^2 + 5\lambda + 3 = 0$

$(\lambda+1)^2(\lambda-3) = 0.$ $\lambda_1 = -1, m_1 = 2$

$\lambda_2 = 3, m_2 = 3$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_1 x = -x\}$

$rg \begin{vmatrix} 2 & -3 & 4 \\ 4 & -6 & 8 \\ 6 & -7 & 8 \end{vmatrix} = 2 \rightarrow \dim V_{\lambda_1} = 3 - 2 = 1 \neq m_1$

$\Rightarrow \nexists$ reper R in $\mathbb{R}^3$ ai $[f]_{R_0R_0}$ mat diag.

③ $f \in End(\mathbb{R}^3)$ $[f]_{R_0 R_0} = \begin{pmatrix} -1 & 0 & -3 \\ 3 & 2 & 3 \\ -3 & 0 & -1 \end{pmatrix} = A$

a) val pr, subsp pr

b) R reper in $\mathbb{R}^3$ ai $[f]_{RR} = A'$ = diag.

c) $R_0 \to R$, det. C

d) A^n

a) $P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} -1-\lambda & 0 & -3 \\ 3 & 2-\lambda & 3 \\ -3 & 0 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)(-1)^4 \begin{vmatrix} -1-\lambda & -3 \\ -3 & -1-\lambda \end{vmatrix} = (2-\lambda)\left((-1-\lambda)^2 - 9\right) =$

$= (2-\lambda)(\lambda^2 + 2\lambda - 8) = (2-\lambda)(\lambda-2)(\lambda+4) = -(2-\lambda)^2(\lambda+4)$

$\lambda_1 = 2, m_1 = 2$, $\lambda_2 = -4, m_2 = 1$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_1 x = 2x\} = \langle\{(0,1,0), (1,0,-1)\}\rangle$

$\begin{cases} -3x_1 - 3x_3 = 0 \\ 3x_1 + 3x_3 = 0 \\ -3x_1 + 3x_3 = 0 \end{cases} \Rightarrow x_1 = -x_3$

$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_2 x = -4x\} = \langle\{(1, -\frac{6}{5}, 1)\}\rangle$

$\begin{cases} 3x_1 - 3x_3 = 0 \\ 3x_1 + 5x_2 + 3x_3 = 0 \\ -3x_1 + 3x_3 = 0 \end{cases} \Rightarrow x_1 = x_3$

$5x_2 = -6x_3 \Rightarrow x_2 = -\frac{6}{5}x_3$

b) $\dim V_1 = m_1$, $\dim V_2 = m_2 \Rightarrow A$ diagonalizabilă

$R = \{(0,1,0), (1,0,-1), (1,-\frac{6}{5},1)\}$ reper în $\mathbb{R}^3$ ai $[f]_{RR} = A'$,

$A' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix}$ diagonală

c) $C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -\frac{6}{5} \\ 0 & -1 & 1 \end{pmatrix}$, $\det C = -1 - 1 = -2$.

d) $A' = C^{-1}AC$

$A = CA'C^{-1} \Rightarrow A^n = CA'^n C^{-1}$

$C^{-1} = \begin{pmatrix} 3 & 1 & 3 \\ 1/2 & 0 & -1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}$

$A^n = C A'^n C^{-1}$

$$A^n = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -6 \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & (-4)^n \end{pmatrix} \begin{pmatrix} 3 & 1 & 3 \\ 1/2 & 0 & -1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 2^n & (-4)^n \\ 2^n & 0 & -6\cdot(-4)^n \\ 0 & (-2)^n & (-4)^n \end{pmatrix} \begin{pmatrix} 3 & 1 & 3 \\ 1/2 & 0 & -1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix} =$$

$= (\text{calcule})$

⑥ $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (4x_1 + x_2 + x_3,\ x_1 + 4x_2 + x_3,\ x_1 + x_2 + 4x_3)$.

$\exists$ rep. diag

$f = Ax \Rightarrow A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$.

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} =$$

$$\overset{L_2-L_1}{\underset{L_3-L_1}{=}} (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (6-\lambda)(3-\lambda)^2$$

$\lambda_1 = 6,\ m_1 = 1$

$\lambda_2 = 3,\ m_2 = 2$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_1 x = 6x\}$

$$\text{rg} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = 2 \Rightarrow \dim V_{\lambda_1} = 1 = m_1 \quad ①.$$

$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_2 x = 3x\}$

$$\text{rg} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 1 \Rightarrow \dim V_{\lambda_2} = 2 = m_2 \quad ②$$

①, ② $\Rightarrow \exists$ reper în rap. cu care $[f]_{RR}$ diagonală.

④ $f_0=0, f_1=1, f_{n+2}=f_{n+1}+f_n$

a) $A=?$ a.î. $\begin{pmatrix} f_{n+2} \\ f_{n+1} \end{pmatrix} = A \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$ b) Diag + A^n c) $f_n=?$

a) $\begin{pmatrix} f_{n+2} \\ f_{n+1} \end{pmatrix} = A \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} \Rightarrow A \in M_2(\mathbb{R})$

evident $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ convine

verificare $\begin{pmatrix} f_{n+2} \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} f_{n+1} + f_n \\ f_{n+1} \end{pmatrix}$ Adevărat

b) $P(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} =$

$= -\lambda + \lambda^2 - 1 = \lambda^2 - \lambda - 1$

$\lambda_1 = \frac{1+\sqrt{5}}{2}$, $m_1 = 1$

$\lambda_2 = \frac{1-\sqrt{5}}{2}$ $m_2 = 1$

$V_{\lambda_1} = \{x \in \mathbb{R}^2 \mid f(x) = \lambda_1 x = \frac{1+\sqrt{5}}{2} x\}$

$\begin{cases} x_1 + x_2 = \frac{1+\sqrt{5}}{2} x_1 \\ x_1 = \frac{1+\sqrt{5}}{2} x_2 \end{cases} \Rightarrow \left(\frac{1+\sqrt{5}}{2} + 1\right) x_2 = \left(\frac{1+\sqrt{5}}{2}\right) x_1$

$x_1 = \frac{3+\sqrt{5}}{1+\sqrt{5}} x_2 =$

$V_{\lambda_1} = \langle \{ (\frac{3+\sqrt{5}}{1+\sqrt{5}}, 1) \} \rangle$ $\dim V_{\lambda_1} = m_1 = 1$. ①

$V_{\lambda_2} = \{x \in \mathbb{R}^2 \mid f(x) = \lambda_2 x = \frac{1-\sqrt{5}}{2} x\} = \langle \{ (\frac{3-\sqrt{5}}{1-\sqrt{5}}, 1) \} \rangle$

$\dim V_{\lambda_2} = m_2 = 1$ ②

①, ② $\Rightarrow \exists R$ reper a.î. A diagonală

$R = \{ (\frac{3+\sqrt{5}}{1+\sqrt{5}}, 1), (\frac{3-\sqrt{5}}{1-\sqrt{5}}, 1) \}$

$A' = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} =$

$A^n = C A'^n C^{-1}$

$R = \{ (\frac{1+\sqrt{5}}{2}, 1), (\frac{1-\sqrt{5}}{2}, 1) \}$.

Putem scrie. C-ul ca $C = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}$.

$$C^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ -1 & \frac{\sqrt{5}+1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{\sqrt{5}-1}{2\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{\sqrt{5}+1}{2\sqrt{5}} \end{pmatrix}$$

$$A^n = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\frac{1+\sqrt{5}}{2})^n & 0 \\ 0 & (\frac{1-\sqrt{5}}{2})^n \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{\sqrt{5}-1}{2\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{\sqrt{5}+1}{2\sqrt{5}} \end{pmatrix}.$$

c) Observ că $\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = A^n \begin{pmatrix} f_1 \\ f_0 \end{pmatrix} \Rightarrow f_n = \frac{-1}{\sqrt{5}} f_1 + \frac{\sqrt{5}+1}{}$

$$\Rightarrow f_n = \left(\frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n \cancel{f_1} \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n \right) f_1 + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right) \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n f_0$$

(9) $f \in End(\mathbb{R}^3)$, $g \in L(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

$g_f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g_f(x, y) = g(f(x), y)$ $\forall x, y \in \mathbb{R}^3$

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a) $g_f \in L(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

b) $G = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ -2 & -1 & -1 \end{pmatrix}$, $A = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$. mat. as. lui f și g.

$\tilde{G} = ?$ (mat. as. lui g_f.

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a) $g_f \in L(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R}) \Leftrightarrow g_f(x, y) = \sum_{i,j=1}^{n} g_{ij} x_i y_j$

$\Leftrightarrow g(f(x), y) = \sum_{i,j=1}^{n} g_{ij} f(x_i) y_j$

(10) $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g(x,y) = x_1y_1 + x_1y_3 + 3x_2y_1 + x_2y_2 + 2x_2y_3$
$+ 2x_3y_1 - x_3y_2 + x_3y_3$

$G^s = \frac{1}{2}(G + G^T)$, $G^a = \frac{1}{2}(G - G^T)$

$g^s: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$
$g^a: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ ai G^s, G^a sunt mat. as. în rap. cu $\mathcal{R}_0$.

$g = \underset{\in \, L^s(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})}{g^s} + \underset{\in \, L^a(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})}{g^a}$

$$G = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix} \quad G^T = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$G^s = \begin{pmatrix} 1 & 3/2 & 3/2 \\ 3/2 & 1 & 1/2 \\ 3/2 & 1/2 & 1 \end{pmatrix} \quad G^a = \begin{pmatrix} 0 & -3/2 & -1/2 \\ 3/2 & 0 & 3/2 \\ 1/2 & -3/2 & 0 \end{pmatrix}$$

$g^s(x,y) = x_1y_1 + \frac{3}{2}x_1y_2 + \frac{3}{2}x_1y_3 + \frac{3}{2}x_2y_1 + x_2y_2 + \frac{1}{2}x_2y_3 + \frac{3}{2}x_3y_1 + \frac{1}{2}x_3y_2 + x_3y_3$

$g^a(x,y) = -\frac{3}{2}x_1y_2 - \frac{1}{2}x_1y_3 + \frac{3}{2}x_2y_1 + \frac{3}{2}x_2y_3 + \frac{1}{2}x_3y_1 - \frac{3}{2}x_3y_2$

$g(x,y) = x_1y_1 + x_1y_3 + 3x_2y_1 + x_2y_2 + 2x_2y_3 + 2x_3y_1 - x_3y_2 + x_3y_3$

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(3) $Q:\mathbb{R}^4\to\mathbb{R}$, $Q(x)=x_1^2+x_2^2+x_3^2-2x_4^2-2x_1x_2+2x_1x_3-2x_1x_4$
$+2x_2x_3-4x_2x_4$

f. canonică

$$G=\begin{pmatrix}1&-1&1&-1\\-1&1&1&-2\\1&1&1&0\\-1&-2&0&-2\end{pmatrix}$$

$Q(x)=(x_1^2-2x_1x_2+2x_1x_3-2x_1x_4)+x_2^2+x_3^2-2x_4^2+2x_2x_3-4x_2x_4$

$=(x_1-x_2-x_3-x_4)^2-x_2^2-x_3^2-x_4^2-\cancel{2x_2x_3}-2x_2x_4-2x_3x_4+$
$+x_2^2+x_3^2-2x_4^2+\cancel{2x_2x_3}-4x_2x_4$

$=(x_1-x_2-x_3-x_4)^2-3x_4^2-6x_2x_4-2x_3x_4$

$=(x_1-x_2-x_3-x_4)^2-\frac{1}{3}(9x_4^2-18x_2x_4-6x_3x_4)$

$=(x_1-x_2-x_3-x_4)^2-\frac{1}{3}(3x_4-3x_2-x_3)^2+3x_2^2+\frac{1}{3}x_3^2+2x_2x_3$

$=(x_1-x_2-x_3-x_4)^2-\frac{1}{3}(3x_4-3x_2-x_3)^2+\frac{1}{3}(9x_2^2+6x_2x_3)+\frac{1}{3}x_3^2$

$=(x_1-x_2-x_3-x_4)^2-\frac{1}{3}(3x_4-3x_2-x_3)^2+\frac{1}{3}(3x_2+x_3)^2-\frac{1}{3}x_3^2+\frac{1}{3}x_3^2$

$=(x_1-x_2-x_3-x_4)^2-\frac{1}{3}(3x_4-3x_2-x_3)^2+\frac{1}{3}(3x_2+x_3)^2$

(4) $Q:\mathbb{R}^3\to\mathbb{R}$ formă pătratică, $G=\begin{pmatrix}1&2&1\\2&3&2\\1&2&1\end{pmatrix}$ în rap. cu $\mathcal{B}_0=\{e_1,e_2,e_3\}$

diag. Q

$\operatorname{rg} G=2$

$Q(x)=x_1^2+3x_2^2+x_3^2+4x_1x_2+2x_1x_3+4x_2x_3$

$=(x_1^2+4x_1x_2+x_1x_3)+3x_2^2+x_3^2+4x_2x_3=$

$=(x_1+2x_2+x_3)^2-x_3^2-4x_2^2-4x_2x_3+4x_2x_3=$

$=(x_1+2x_2+x_3)^2-x_2^2$

$$G'=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$$

⑤ $Q: \mathbb{R}^3 \to \mathbb{R}$, f. polar, $G' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ mat. as. în raport

cu $R' = \{e_1' = (1,1,1),\ e_2' = (0,1,0),\ e_3' = (1,0,0)\}$

Q f. can

$R^0 \xrightarrow{C} R'$

$C = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\qquad C = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$G = C G' C^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$Q(x) = 2x_1^2 + 2x_2^2 + 6x_1x_2 + 2x_1x_3 + 2x_2x_3$$
$$= \frac{1}{2}(4x_1^2 + 12x_1x_2 + 4x_1x_3) + 2x_2^2 + 2x_2x_3$$

$$= \frac{1}{2}(2x_1 + 3x_2 + x_3)^2 - 3x_2x_3 - \frac{9}{2}x_2^2 - \frac{1}{2}x_3^2 + 2x_2^2 + 2x_2x_3$$

$$= \frac{1}{2}(2x_1 + 3x_2 + x_3)^2 - \frac{5}{2}x_2^2 - x_2x_3 + \frac{1}{2}x_3^2 -$$

$$= \frac{1}{2}(2x_1 + 3x_2 + x_3)^2 - \frac{5}{2}\left(\frac{25}{4}x_2^2 + \frac{2}{5}x_2x_3\right) - \frac{1}{2}x_3^2 =$$

$$= \frac{1}{2}(2x_1 + 3x_2 + x_3)^2 - \frac{5}{2}\left(\frac{5}{2}x_2 + \frac{2}{25}x_3\right)^2 + \frac{2}{125}x_3^2 - \frac{1}{2}x_3^2 =$$

$$= \frac{1}{2}(2x_1 + 3x_2 + x_3)^2 - \frac{5}{2}\left(\frac{5}{2}x_2 + \frac{2}{25}x_3\right)^2 - \frac{121}{250}x_3^2$$

(Probabil calculele sunt greșite)

Schimbarea reperului

$$\begin{cases} x_1' = 2x_1 + 3x_2 + x_3 \\ x_2' = \frac{5}{2}x_2 + \frac{2}{25}x_3 \\ x_3' = x_3 \end{cases} \qquad Q(x) = \frac{1}{2}x_1' - \frac{5}{2}x_2' - \frac{121}{250}x_3^2$$

⑥ $Q: \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = 2x_1^2 + 5x_2^2 + 2x_3^2 - 4x_1x_2 - 2x_1x_3 + 4x_2x_3$

a) $G = ?$ rap. R_0

b) $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ f. pol

c) f. can. prin Jacobi + Gauss + Th. inerției Sylvester

a) $G = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 5 & 2 \\ -1 & 2 & 2 \end{pmatrix}$

b) $g(x,y) = 2x_1y_1 - 2x_1y_2 - x_1y_3 - 2x_2y_1 + 5x_2y_2 + 2x_2y_3 - x_3y_1 + 2x_3y_2 + 2x_3y_3$

c) $\Delta_1 = 2$
$\Delta_2 = 7 - 4 = 3 \Rightarrow \exists R'$ a.î. $Q(x) = \frac{1}{2}x_1'^2 + \frac{2}{3}x_2'^2 + \frac{3}{7}x_3'^2$
$\Delta_3 = 7$

signatura $(3, 0)$

$Q(x) = \frac{1}{2}(4x_1^2 - 8x_1x_2 - 4x_1x_3) + 5x_2^2 + 2x_3^2 + 4x_2x_3$

$= \frac{1}{2}(2x_1 - 2x_2 - x_3)^2 - 2x_2^2 - \frac{1}{2}x_3^2 - 2x_2x_3 + 5x_2^2 + 2x_3^2 + 4x_2x_3$

$= \frac{1}{2}(2x_1 - 2x_2 - x_3)^2 + 3x_2^2 + \frac{3}{2}x_3^2 + 2x_2x_3$

$= \frac{1}{2}(2x_1 - 2x_2 - x_3)^2 + \frac{1}{3}(9x_2^2 + 6x_2x_3) + \frac{3}{2}x_3^2$

$= \frac{1}{2}(2x_1 - 2x_2 - x_3)^2 + \frac{1}{3}(3x_2 + x_3)^2 - \frac{1}{3}x_3^2 + \frac{3}{2}x_3^2$

$= \frac{1}{2}(2x_1 - 2x_2 - x_3)^2 + \frac{1}{3}(3x_2 + x_3)^2 - \frac{7}{2}x_3^2$

$$\begin{cases} x_1' = 2x_1 - 2x_2 - x_3 \\ x_2' = 3x_2 + x_3 \\ x_3' = x_3 \end{cases} \Longleftrightarrow \begin{cases} x_1 = (x_1' + 2x_2' + x_3')/2 \\ x_2 = (x_2' - x_3')/3 \\ x_3 = x_3' \end{cases}$$

$Q(x) = \frac{1}{2}x_1'^2 + \frac{1}{3}x_2'^2 - \frac{7}{2}x_3'^2$

$R' = \{(1/2, 0, 0), (1, 1/3, 0), (1/2, -1/3, 1)\}$ reper.

(3,0) signatura

În ambele metode avem (3,0) signatura $\xrightarrow{\text{Th. inerție}}$ Sylvester

(8) $g: M_2(\mathbb{R}) \times M_2(\mathbb{R}) \to \mathbb{R}$

$g(x, y) = 2\,Tr(xy) - Tr(x)\,Tr(y), \ \forall\, x, y \in M_2(\mathbb{R})$.

a) $g \in L^s(M_2(\mathbb{R}), M_2(\mathbb{R}); \mathbb{R})$

b) $G =$ mat. rep. cu $\mathcal{R}_0 = \{E_{ij}\}_{i,j=\overline{1,2}}$

c) expr. analitica $Q: M_2(\mathbb{R}) \to \mathbb{R}$ + f. polar.

d) f. canonică Q.

a) Fie $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, $y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$.

$g(x,y) = 2(x_1y_1 + x_2y_3 + x_3y_2 + x_4y_4) - (x_1 + x_4)(y_1 + y_4) =$

$= x_1y_1 - x_1y_4 + 2x_2y_3 + 2x_3y_2 + x_4y_4 - x_4y_1$

$$G = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$G = G^T$
$g = x^T G y$ $\Bigg| \Rightarrow g \in L^s(M_2(\mathbb{R}), M_2(\mathbb{R}), \mathbb{R})$

b) $$G = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

c) $Q(x) = x_1^2 + x_4^2 - 2x_1x_4 + 4x_2x_3$

d) $Q(x) = (x_1^2 - 2x_1x_4 + x_4^2) + 4x_2x_3 = (x_1 - x_4)^2 + 4x_2x_3$.

(9) $g \in L^0(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$ $G = \begin{pmatrix} 2 & 1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix}$ mat. rep. cu $\mathcal{R}_0$

$Q: \mathbb{R}^3 \to \mathbb{R}$ f. pătr. as. lui $g^s \in L^s(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

$G^s = \frac{1}{2}(G + G^T)$

Q formă canonică

$$G^{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

$$g^{D}(x,y) = 2x_1y_1 - 1x_2y_2 - 1x_2y_3 - 1x_3y_2 - 2x_3y_3$$

$$Q(x) = 2x_1^2 - x_2^2 - 2x_3^2 - 2x_2x_3$$

$$= 2x_1^2 - (x_2 + x_3)^2 - x_3^2$$

$$Q(x) = 2x_1'^2 - x_2'^2 - x_3'^2$$

$$\begin{cases} x_1' = x_1 \\ x_2' = x_2 + x_3 \\ x_3' = x_3 \end{cases} \Longleftrightarrow \begin{cases} x_1 = x_1' \\ x_2 = x_2' - x_3' \\ x_3 = x_3' \end{cases}$$

$D' = \{(1,0,0), (0,1,-1), (0,0,1)\}$ reper