T1 - Seminar (24.02.2022)

(1) $\begin{cases} x+y+az-t=0 \\ 2x+y-z+t=0 \\ 3x-y-z-t=0 \\ ax-2y-2z-2t=0 \end{cases}$

$$A=\left(\begin{array}{cccc|c} 1 & 1 & a & -1 & 0 \\ 2 & 1 & -1 & 1 & 0 \\ 3 & -1 & -1 & -1 & 0 \\ a & -2 & -2 & -2 & 0 \end{array}\right)$$

$$\det(A)=\begin{vmatrix} 1 & 1 & a & -1 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & -1 \\ a & -2 & -2 & -2 \end{vmatrix} \overset{L_4-2L_3}{=} \begin{vmatrix} 1 & 1 & a & -1 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & -1 \\ a-6 & 0 & 0 & 0 \end{vmatrix} = (a-6)(-1)^{1+4}\cdot\begin{vmatrix} 1 & a & -1 \\ 1 & -1 & 1 \\ -1 & -1 & -1 \end{vmatrix} \overset{C_1-C_3}{=}$$

$$=(6-a)\cdot\begin{vmatrix} 2 & a & -1 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{vmatrix} = (6-a)\cdot 2\cdot(-1)^{1+1}\cdot\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} = (6-a)\cdot 2\cdot 2 = 4(6-a)$$

(1) $\det(A)\neq 0 \Rightarrow a\in\mathbb{R}\setminus\{6\} \Rightarrow$ SCD $\Rightarrow S=\{(0,0,0,0)\}$.

(2) $\det(A)=0 \Rightarrow a=6$.

$$A=\left(\begin{array}{cccc|c} 1 & 1 & 6 & -1 & 0 \\ 2 & 1 & -1 & 1 & 0 \\ 3 & -1 & -1 & -1 & 0 \\ 6 & -2 & -2 & -2 & 0 \end{array}\right)$$

$$\Delta_p=\begin{vmatrix} 1 & 1 & 6 \\ 2 & 1 & -1 \\ 3 & -1 & -1 \end{vmatrix} \underset{L_2+L_3}{\overset{L_1+L_3}{=}} \begin{vmatrix} 4 & 0 & 5 \\ 5 & 0 & -2 \\ 3 & -1 & -1 \end{vmatrix} = (-1)\cdot(-1)^{2+3}\cdot\begin{vmatrix} 4 & 5 \\ 5 & -2 \end{vmatrix} = -8-25=-33\neq 0$$

$\Rightarrow$ rang $A=3=$ rang $\overline{A}$ $\Rightarrow$ SC simplu N, cu $\begin{cases} t=\alpha \text{ - nec. sec.} \\ x,y,z \text{ - nec. prin} \end{cases}$

$$\begin{cases} x+y+6z=\alpha \\ 2x+y-z=-\alpha \\ 3x-y-z=\alpha \end{cases}$$

$\Delta=-33\neq 0 \Rightarrow$ SCD $\Rightarrow$ Aplicăm Cramer: $x=\frac{\Delta x}{\Delta}, y=\frac{\Delta y}{\Delta}, z=\frac{\Delta z}{\Delta}$

$$\Delta_x=\begin{vmatrix} \alpha & 1 & 6 \\ -\alpha & 1 & -1 \\ \alpha & -1 & -1 \end{vmatrix} \underset{L_2+L_3}{\overset{L_1+L_3}{=}} \begin{vmatrix} 2\alpha & 0 & 5 \\ 0 & 0 & -2 \\ \alpha & -1 & -1 \end{vmatrix} = 0-(4\alpha) = -4\alpha \Rightarrow x=\frac{4\alpha}{33}$$

$$\Delta_y=\begin{vmatrix} 1 & \alpha & 6 \\ 2 & -\alpha & -1 \\ 3 & \alpha & -1 \end{vmatrix} \underset{L_3+L_2}{\overset{L_1+L_2}{=}} \begin{vmatrix} 3 & 0 & 5 \\ 2 & -\alpha & -1 \\ 5 & 0 & -2 \end{vmatrix} = (-\alpha)\cdot(-1)^{2+2}\cdot\begin{vmatrix} 3 & 5 \\ 5 & -2 \end{vmatrix} = 31\alpha \Rightarrow y=\frac{-31\alpha}{33}$$

$$\Delta_z=\begin{vmatrix} 1 & 1 & \alpha \\ 2 & 1 & -\alpha \\ 3 & -1 & \alpha \end{vmatrix} \underset{L_3+L_2}{\overset{L_1+L_2}{=}} \begin{vmatrix} 3 & 2 & 0 \\ 2 & 1 & -\alpha \\ 5 & 0 & 0 \end{vmatrix} = 5\cdot(-1)^{3+1}\cdot\begin{vmatrix} 2 & 0 \\ 1 & -\alpha \end{vmatrix} = -10\alpha \Rightarrow z=\frac{10\alpha}{33}$$

$$S=\left\{\left(\frac{4\alpha}{33}, \frac{-31\alpha}{33}, \frac{10\alpha}{33}, \alpha\right) \mid \alpha\in\mathbb{R}\right\}$$

② $A = \begin{pmatrix} a & -b & -a & b \\ b & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{pmatrix}$

$\det A = 4(a^2+b^2)(c^2+d^2)$ + Laplace

$\det(A) = \begin{vmatrix} a & -b & -a & b \\ b & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{vmatrix} = \begin{vmatrix} a & -b \\ b & a \end{vmatrix} \cdot (-1)^{1+2+1+2} \cdot \begin{vmatrix} c & -d \\ d & c \end{vmatrix} + \begin{vmatrix} a & -a \\ b & -b \end{vmatrix} \cdot (-1)^{1+2+1+3} \cdot \underbrace{\begin{vmatrix} -d & -d \\ c & c \end{vmatrix}}_{\substack{0 \\ (C_1 = C_2)}}$

$+ \begin{vmatrix} a & b \\ b & -a \end{vmatrix} \cdot (-1)^{1+2+1+4} \cdot \begin{vmatrix} -d & c \\ c & d \end{vmatrix} + \begin{vmatrix} -b & -a \\ a & -b \end{vmatrix} \cdot (-1)^{1+2+2+3} \cdot \begin{vmatrix} c & -d \\ d & c \end{vmatrix} +$

$+ \begin{vmatrix} -b & b \\ a & -a \end{vmatrix} \cdot (-1)^{1+2+2+4} \cdot \underbrace{\begin{vmatrix} c & c \\ d & d \end{vmatrix}}_{\substack{0 \\ (C_1 = C_2)}} + \begin{vmatrix} -a & b \\ -b & -a \end{vmatrix} \cdot (-1)^{1+2+3+4} \cdot \begin{vmatrix} c & -d \\ d & c \end{vmatrix} =$

$= (a^2+b^2)(c^2+d^2) + 0 + (-a^2-b^2)(-d^2-c^2) + (b^2+a^2)(c^2+d^2) + 0 + (a^2+b^2)(c^2+d^2)$

$= 4(a^2+b^2)(c^2+d^2)$

③ $A \in M_2(\mathbb{R}), A^2 = O_2$

$P_A = \det(A - xI_2)$ pol caract.

$P_A(1) + \dots + P_A(22)$

$P_A(A) = A^2 - trA \cdot A + \det A \cdot I_2 = O_2$

$trA \cdot A - \det A \, I_2 = O_2$

$A = \frac{\det A}{trA} I_2$

$P_A = \det(A - xI_2) = \det\left(\left(\frac{\det A}{trA} - x\right) I_2\right) = \left(\frac{\det A}{trA} - x\right)^2$.

$A^2 = O_2 \Rightarrow (\det A)^2 = 0 \Rightarrow \det A = 0$

$P_A(1) + \dots + P_A(22) = 1^2 + \dots + 22^2 = \frac{22 \cdot 23 \cdot 45}{6} = 11 \cdot 23 \cdot 15 =$

$= 3795$

– bonus: exerciții nerezolvate la seminar –

(18) $\sum_{j=1}^{4} a_{ij} x_j = 4^{i-1}$, $\forall i = \overline{1,4}$, $a_{ij} = j^{i-1}$ $\forall i,j = \overline{1,4}$

$i=1 \Rightarrow a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = 1 \Rightarrow x_1 + x_2 + x_3 + x_4 = 1$

$i=2 \Rightarrow a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = 4 \Rightarrow x_1 + 2x_2 + 3x_3 + 4x_4 = 4$

$i=3 \Rightarrow a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = 16 \Rightarrow x_1 + 4x_2 + 9x_3 + 16x_4 = 16$

$i=4 \Rightarrow a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = 64 \Rightarrow x_1 + 8x_2 + 27x_4 + 64x_4 = 64$

$$A = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 4 & 9 & 16 & 16 \\ 1 & 8 & 27 & 64 & 64 \end{array}\right)$$

$$\det A = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{vmatrix} \xrightarrow[\substack{C_3 - C_1 \\ C_2 - C_1}]{C_4 - C_1} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ 1 & 3 & 8 & 15 \\ 1 & 7 & 26 & 63 \end{vmatrix} = 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 1 & 2 & 3 \\ 3 & 8 & 15 \\ 7 & 26 & 63 \end{vmatrix} \xrightarrow[C_2 - 2C_1]{C_3 - 3C_1}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 3 & 2 & 6 \\ 7 & 12 & 42 \end{vmatrix} = 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 6 \\ 12 & 42 \end{vmatrix} = 84 - 42 = 12 \neq 0 \Rightarrow SCD$$

Aplicăm Cramer $\Rightarrow x_1 = \frac{\Delta x_1}{\Delta}$, $x_2 = \frac{\Delta x_2}{\Delta}$, $x_3 = \frac{\Delta x_3}{\Delta}$, $x_4 = \frac{\Delta x_4}{\Delta}$.

$$\Delta x_1 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 4 & 2 & 3 & 4 \\ 16 & 4 & 9 & 16 \\ 64 & 8 & 27 & 64 \end{vmatrix} \underset{=}{C_1 = C_4} 0 \Rightarrow x_1 = 0$$

analog $\Rightarrow x_2 = x_3 = 0$.

$\Delta x_4 = \Delta \Rightarrow x_4 = 1$.

$$S = \{(0, 0, 0, 1)\}.$$

(15) $\begin{cases} x + y + mz - t = 0 \\ 2x + y - z + t = 0 \\ 3x - y - z - t = 0 \\ mx - 2y - 2t = 0 \end{cases}$, $m \in \mathbb{R}$

$m = 2$ ai sol. nenule

$$A = \left(\begin{array}{cccc|c} 1 & 1 & m & -1 & 0 \\ 2 & 1 & -1 & 1 & 0 \\ 3 & -1 & -1 & -1 & 0 \\ m & -2 & 0 & -2 & 0 \end{array}\right)$$

sistemul admite sol. nenule $\Rightarrow \det A = 0$

$$\det A = \begin{vmatrix} 1 & 1 & m & -1 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & -1 \\ m & -2 & 0 & -2 \end{vmatrix} \underline{\underline{C_4 - C_2}} \begin{vmatrix} 1 & 1 & m & -2 \\ 2 & 1 & -1 & 0 \\ 3 & -1 & -1 & 0 \\ m & -2 & 0 & 0 \end{vmatrix} = (-2)(-1)^{1+4} \cdot \begin{vmatrix} 2 & 1 & -1 \\ 3 & -1 & -1 \\ m & -2 & 0 \end{vmatrix}$$

$$= 2 \cdot \left(6 - m - (m+4)\right) = 2(2-2m) = 4(1-m).$$

$\det A = 0 \Rightarrow m = 1.$

(14) $\begin{cases} x + m(y + z + t) = a \\ y + m(x + z + t) = b \\ z + m(x + y + t) = c \\ t + m(x + y + z) = d \end{cases}$, $m, a, b, c \in \mathbb{R}$

discutie

$$A = \left(\begin{array}{cccc|c} 1 & m & m & m & a \\ m & 1 & m & m & b \\ m & m & 1 & m & c \\ m & m & m & 1 & d \end{array}\right)$$

$$\det A = \begin{vmatrix} 1 & m & m & m \\ m & 1 & m & m \\ m & m & 1 & m \\ m & m & m & 1 \end{vmatrix} = (3m+1) \begin{vmatrix} 1 & m & m & m \\ 1 & 1 & m & m \\ 1 & m & 1 & m \\ 1 & m & m & 1 \end{vmatrix} \begin{array}{l} \underline{\underline{L_3 - L_1}} \\ L_2 - L_1 \\ L_4 - L_1 \end{array}$$

$$= (3m+1) \begin{vmatrix} 1 & m & m & m \\ 0 & 1-m & 0 & 0 \\ 0 & 0 & 1-m & 0 \\ 0 & 0 & 0 & 1-m \end{vmatrix} = (3m+1) \cdot 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 1-m & 0 & 0 \\ 0 & 1-m & 0 \\ 0 & 0 & 1-m \end{vmatrix} =$$

$= (3m+1)(1-m)^3$ (determinant triunghiular)

(1) $\det A \neq 0 \Rightarrow m \in \mathbb{R} \setminus \{-\frac{1}{3}, 1\} \Rightarrow$ SCD $\Rightarrow$ Aplicăm Cramer

$x = \frac{\Delta x}{\Delta}$, $y = \frac{\Delta y}{\Delta}$, $z = \frac{\Delta z}{\Delta}$, $t = \frac{\Delta t}{\Delta}$.

$$\Delta_x = \begin{vmatrix} a & m & m & m \\ b & 1 & m & m \\ c & m & 1 & m \\ d & m & m & 1 \end{vmatrix} = \ldots$$

$$\Delta_y = \begin{vmatrix} 1 & a & m & m \\ m & b & m & m \\ m & c & 1 & m \\ m & d & m & 1 \end{vmatrix} = \ldots$$

$$\Delta_z = \begin{vmatrix} 1 & m & a & m \\ m & 1 & b & m \\ m & m & c & m \\ m & m & d & 1 \end{vmatrix} = \ldots$$

$$\Delta_t = \begin{vmatrix} 1 & m & m & a \\ m & 1 & m & b \\ m & m & 1 & c \\ m & m & m & d \end{vmatrix} = \ldots$$

$S = \left\{ \left(\frac{\Delta x}{\Delta}, \frac{\Delta y}{\Delta}, \frac{\Delta z}{\Delta}, \frac{\Delta t}{\Delta} \right) \right\}$

② $\det A = 0$, $m = 1 \Rightarrow rg\, A = 1$

① $a = b = c = d = 1 \Rightarrow rg\, \bar{A} = 1 = rg\, A \Rightarrow$ SC triplu det.

$$S = \{(1-\alpha-\beta-\gamma, \alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in \mathbb{R}\}$$

② dacă nu avem $a = b = c = d = 1 \Rightarrow rg\, \bar{A} \neq 1 \Rightarrow$ SCI

$S = \emptyset$.

③ $\det A = 0$, $m = -\frac{1}{3}$.

$$A = \begin{pmatrix} 1 & -1/3 & -1/3 & -1/3 \\ -1/3 & 1 & -1/3 & -1/3 \\ -1/3 & -1/3 & 1 & -1/3 \\ -1/3 & -1/3 & -1/3 & 1 \end{pmatrix} \left| \begin{matrix} a \\ b \\ c \\ d \end{matrix} \right.$$

$$\Delta_p = \begin{vmatrix} -1/3 & 1 & -1/3 \\ -1/3 & -1/3 & 1 \\ -1/3 & -1/3 & -1/3 \end{vmatrix} = \frac{-16}{27} \neq 0 \Rightarrow rg\, A = 3$$

$$\Delta_c = \begin{vmatrix} 1 & -1/3 & -1/3 & a \\ -1/3 & 1 & -1/3 & b \\ -1/3 & -1/3 & 1 & c \\ -1/3 & -1/3 & -1/3 & d \end{vmatrix} \xlongequal{L_1 - L_4} \begin{vmatrix} \frac{4}{3} & 0 & 0 & a-d \\ -1/3 & 1 & -1/3 & b \\ -1/3 & -1/3 & 1 & c \\ -1/3 & -1/3 & -1/3 & d \end{vmatrix} =$$

$$= (a-d)(-1)^5 \cdot \begin{vmatrix} -1/3 & 1 & -1/3 \\ -1/3 & -1/3 & 1 \\ -1/3 & -1/3 & -1/3 \end{vmatrix} + \frac{4}{3} \cdot (-1)^2 \cdot \begin{vmatrix} 1 & -1/3 & b \\ -1/3 & 1 & c \\ -1/3 & -1/3 & d \end{vmatrix} =$$

$$= \frac{16}{27}(a-d) + \frac{4}{9}(b+c+2d) = \frac{4}{9}\left(\frac{4}{3}a + b + c + \frac{2}{3}d\right)$$

$\Delta_c \neq 0 \Rightarrow \frac{4}{3}a + b + c + \frac{2}{3} \neq 0 \Rightarrow rg\, \bar{A} = 4$.

$rg\, A \neq rg\, \bar{A} \Rightarrow$ SI $\qquad S = \emptyset$

$\Delta_c = 0 \Rightarrow \frac{4}{3}a + b + c + \frac{2}{3} = 0 \Rightarrow$ SC simplu n.

$\{x, y, z$ – nec. principale
$t = \alpha$ – nec. sec

$$\begin{cases} -x + 3y - z = 3b + \alpha & 1. \\ -x - y + 3z = 3c + \alpha & 2. \\ -x - y - z = 3d - 3\alpha & 3. \end{cases}$$

$1, 3 \Rightarrow 4y = 3b - 3d + 4\alpha$.

$y = \frac{1}{4}(3b - 3d + 4\alpha) = \frac{3}{4}(b-d) + \alpha$

$2, 3 \Rightarrow 4z = 3c - 3d + 4\alpha$

$z = \frac{1}{4}(3c - 3d + 4\alpha) = \frac{3}{4}(c-d) + \alpha$

$x = 3\alpha - 3d - y - z$

$x = 3\alpha - 3d - \frac{3}{4}(b-d) - \alpha - \frac{3}{4}(c-d) - \alpha$

$x = \alpha - 3d + \frac{3}{2}d - \frac{3}{4}b - \frac{3}{4}c$

$x = \alpha - \frac{3}{4}b - \frac{3}{4}c - \frac{3}{2}d$.

$$S = \left\{\left(\alpha - \frac{3}{4}b - \frac{3}{4}c - \frac{3}{2}d, \frac{3}{4}(b-d) + \alpha, \frac{3}{4}(c-d) + \alpha, \alpha\right) \mid \alpha \in \mathbb{R}\right\}$$

(8) Fie $A \in \mathcal{M}_n(\mathbb{R})$ o matrice $rg A = 1$

$\exists \alpha \in \mathbb{R}$ ai $A^2 = \alpha A$

$$rg A = 1 \Rightarrow A = \begin{pmatrix} \alpha_1 x_1 & \dots & \alpha_1 x_n \\ \alpha_2 x_1 & \dots & \alpha_2 x_n \\ \dots & & \dots \\ \alpha_n x_1 & \dots & \alpha_n x_n \end{pmatrix}$$

$$A^2 = \begin{pmatrix} \alpha_1 x_1 & \dots & \alpha_1 x_n \\ \alpha_2 x_1 & \dots & \alpha_2 x_n \\ \dots & & \dots \\ \alpha_n x_1 & \dots & \alpha_n x_n \end{pmatrix} \begin{pmatrix} \alpha_1 x_1 & \dots & \alpha_1 x_n \\ \alpha_2 x_1 & \dots & \alpha_2 x_n \\ \dots & & \dots \\ \alpha_n x_1 & \dots & \alpha_n x_n \end{pmatrix} = \begin{pmatrix} \alpha_1 x_1(\alpha_1 x_1 + \dots + \alpha_n x_n) & \dots & \alpha_1 x_n(\alpha_1 x_1 + \dots + \alpha_n x_n) \\ \alpha_2 x_1(\alpha_1 x_1 + \dots + \alpha_n x_n) & \dots & \alpha_2 x_n(\alpha_1 x_1 + \dots + \alpha_n x_n) \\ \dots & & \dots \\ \alpha_n x_1(\alpha_1 x_1 + \dots + \alpha_n x_n) & \dots & \alpha_n x_n(\alpha_1 x_1 + \dots + \alpha_n x_n) \end{pmatrix}$$

$$= (\alpha_1 x_1 + \dots + \alpha_n x_n) \begin{pmatrix} \alpha_1 x_1 & \dots & \alpha_1 x_n \\ \alpha_2 x_1 & \dots & \alpha_2 x_n \\ \dots & & \dots \\ \alpha_n x_1 & \dots & \alpha_n x_n \end{pmatrix} = \alpha A, \quad \alpha = (\alpha_1 x_1 + \dots + \alpha_n x_n)$$

(9) $A \in \mathcal{M}_2(\mathbb{C})$

a) ex. ai $rg A \neq rg(A^2)$

b) dacă $rg A = rg A^2$, atunci $rg A^n = rg A \;\forall n \in \mathbb{N}$.

a) $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, $rg A = 1$

$$A^2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow rg A^2 = 0 \Rightarrow rg A \neq rg A^2$$

b) caz 1: $rg A = rg A^2 = 0 \Rightarrow A = O_2$

evident $rg(A^n) = 0 = rg(A) \;\forall n \in \mathbb{N}$

caz 2: $rg A = rg A^2 = 1 \Rightarrow A^2 = \alpha A$ (ex. 8)

$A^n = \alpha^{n-1} \cdot A \Rightarrow$

$\Rightarrow rg A^n = rg A \;\forall n \in \mathbb{N}$

caz 3: $rg A = rg A^2 = 2$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det A^k = (\det A)^k$

$rg A = 2 \Rightarrow \det A \neq 0 \Rightarrow (\det A)^n \neq 0 \Rightarrow \det A^n \neq 0$

$\Rightarrow rg A^n = 2 = rg A \;\forall n \in \mathbb{N}$

$\Rightarrow$ Dacă $rg A = rg A^2$, atunci $rg A^n = rg A \;\forall n \in \mathbb{N}$