T2 - Curs (13.03.2022)

③ $(\mathcal{M}_2^s(\mathbb{R}), +, \cdot)$, $R_0 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ rep. canonic

$R' = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$

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a) R' reper în $\mathcal{M}_2^s(\mathbb{R})$

b) $R_0 \xrightarrow{A} R'$, $R' \xrightarrow{B} R_0$

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a) Fie $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (a_{11} \; a_{12} \; a_{21} \; a_{22}) \in \mathbb{R}^4$

$R' = \{ \underbrace{(1,1,1,3)}_{u}, \underbrace{(0,1,1,0)}_{v}, \underbrace{(1,0,0,0)}_{w} \}$

R' SLI $\iff au + bv + cw = 0$, $a, b, c \in \mathbb{R}$

$(a+c, a+b, a+b, 3a) = 0 \Rightarrow a = b = c = 0 \Rightarrow R'$ SLI ①

$M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{pmatrix}$ $\quad rg M = 3$

$\mathcal{M}_2^s(\mathbb{R}) = \begin{pmatrix} x & y \\ y & w \end{pmatrix} = (x \; y \; y \; w) \in \mathbb{R}^4$

Card $R' = 3 = \dim_{\mathbb{R}} \mathcal{M}_2^s(\mathbb{R}) \Rightarrow R'$ SG ②

①, ② $\Rightarrow R'$ reper în $\mathcal{M}_2^s(\mathbb{R})$

b) $\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow a = 1, \; b = 3, \; c = 1$

analog pentru $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ și $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = a \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 1 \end{cases}$

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = a \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} a = 1/3 \\ b = -1/3 \\ c = -1/3 \end{cases}$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = a \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} a = 0 \\ b = 1 \\ c = 0 \end{cases}$

$B = \begin{pmatrix} 0 & 1/3 & 0 \\ 0 & -1/3 & 1 \\ 1 & -1/3 & 0 \end{pmatrix}$ $\quad$ Obs. că $A = B^{-1}$.

② a) $V' = \{(x, y, z) \in \mathbb{R}^3 \mid tg\, y = 0\} \subset \mathbb{R}^3$

b) $V'' = \{f \in \mathcal{F}(\mathbb{R}) \mid f \text{ bijectivă}\} \subset \mathcal{F}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ funcție}\}$

c) $V''' = \{P \in \mathbb{R}_3[X] \mid \text{grad } P = 2\} \subset \mathbb{R}_2[X]$

subspații vectoriale

a) fie $u, v \in V'$, $a, b \in \mathbb{R}$ $\overset{?}{\Rightarrow} au + bv \in V'$

$au + bv = (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$; $tg\, y_1 = 0$, $tg\, y_2 = 0$, $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$

$tg\, y_1 = 0 \Rightarrow y_1 = k_1\pi$, $k_1 \in \mathbb{Z}$

$tg\, y_2 = 0 \Rightarrow y_2 = k_2\pi$, $k_2 \in \mathbb{Z}$

$\Rightarrow ay_1 + by_2 = ak_1\pi + bk_2\pi$

fie $a = 0$, $b = \frac{1}{2} \Rightarrow tg(ay_1 + by_2) = tg\left(\frac{k_2\pi}{2}\right) \neq 0$, $k_2 \in \mathbb{Z}$. ⚡

$\Rightarrow V'$ nu e subspațiu vectorial.

b) fie $u, v \in V''$, $a, b \in \mathbb{R}$ $\overset{?}{\Rightarrow} a \cdot u + bv \in V''$

fie $u(x) = x$, u bijectivă

$v(x) = -x$, v bijectivă

$u(x) + v(x) = 0$, nu e bijecție $\Rightarrow u(x) + v(x) \notin V''$ ⚡

$\Rightarrow V''$ nu e subspațiu vectorial

c) fie $u, v \in V'''$, $a, b \in \mathbb{R}$ $\overset{?}{\Rightarrow} au + bv \in V'''$

$au + bv = a(\alpha' X^2 + \alpha'' X + \alpha''') + b(\beta' X^2 + \beta'' X + \beta''')$, $\alpha', \beta' \in \mathbb{R}^*$, $\alpha'', \beta'', \alpha''', \beta''' \in \mathbb{R}$

$= (a\alpha' + b\beta')X^2 + (a\alpha'' + b\beta'')X + (a\alpha''' + b\beta''')$

dacă $a = b = 0 \Rightarrow au + bv$ nu are grad 2 $\Rightarrow$ ⚡

$\Rightarrow V'''$ nu e subspațiu vectorial

1 a) $M_n(\mathbb{R}) = M_n(\mathbb{R})^s \oplus M_n(\mathbb{R})^a$

b) $\dim_{\mathbb{R}} M_n^s(\mathbb{R})$, $\dim_{\mathbb{R}} M_n^a(\mathbb{R})$

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b) $$M_n(\mathbb{R})^s = \left\langle \underbrace{\left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \right\}}_{R_1} \right\rangle$$

$$\dim_{\mathbb{R}} M_n^s(\mathbb{R}) = \frac{n(n+1)}{2}$$

$$M_n(\mathbb{R})^a = \left\langle \underbrace{\left\{ \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & -1 & 0 \end{pmatrix} \right\}}_{R_2} \right\rangle$$

$$\dim_{\mathbb{R}} M_n^a(\mathbb{R}) = \frac{n(n-1)}{2}$$

Astfel, $\dim_{\mathbb{R}} M_n^s(\mathbb{R}) = \frac{n^2+n}{2}$; $\dim_{\mathbb{R}} M_n^a(\mathbb{R}) = \frac{n^2-n}{2}$

a) Conform T. Grassmann:

$$\dim (M_n(\mathbb{R})^s + M_n(\mathbb{R})^a) = \dim M_n(\mathbb{R})^s + \dim M_n(\mathbb{R})^a - \dim (M_n(\mathbb{R})^s \cap M_n(\mathbb{R})^a)$$

$$\dim (M_n(\mathbb{R})^s + M_n(\mathbb{R})^a) = \frac{n(n+1)}{2} + \frac{n(n-1)}{2} - 0 = n^2 = \dim M_n(\mathbb{R})$$

$M_n(\mathbb{R})^s, M_n(\mathbb{R})^a \subset M_n(\mathbb{R})$ (subspații vectoriale) $\Rightarrow$

$$\Rightarrow M_n(\mathbb{R}) = M_n(\mathbb{R})^s \oplus M_n(\mathbb{R})^a$$