Neat Integral: Two Different Ways

David Pevzner

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1 The Integral

The Integral we will be dealing with is

$$I = \int_0^\infty \frac{\sin^2(x)}{x^2\,(x^2+1)}\,\mathrm{dx}\,. \tag{1}$$

We will evaluate this integral two different ways: first via pure Feynman integration and second via a combination of Feynman integration and complex analysis.

2 Method 1: Pure Feynman Integration

As is always the first step of Feynman integration, we insert a parameter into the the integral. In this case, it is clear that we should define

$$I(\alpha) = \int_0^\infty \frac{\sin^2(\alpha x)}{x^2\,(x^2+1)}\,\mathrm{dx}\,. \tag{2}$$

Then we differentiate with respect to the parameter α with the goal of being able to write a differential equation relating the various derivatives of $I(\alpha)$.

$$\frac{dI}{d\alpha} = \int_0^\infty \frac{\sin(2\alpha x)}{x(x^2+1)}\,\mathrm{dx}\,. \tag{3}$$

And we differentiate again!

$$\frac{d^2I}{d\alpha^2} = 2\int_0^\infty \frac{\cos(2\alpha x)}{x^2+1}\,\mathrm{dx}\,. \tag{4}$$

And again!

$$\frac{d^3I}{d\alpha^3} = -4\int_0^\infty \frac{x\sin(2\alpha x)}{x^2+1}. \tag{5}$$

With some keen observation, it can be seen that

$$\frac{dI}{d\alpha} - \frac{1}{4}\frac{d^3I}{d\alpha^3} = \int_0^\infty \frac{\sin(2\alpha x)}{x}\,\mathrm{dx}\,. \tag{6}$$

The integral on the right side is related to the Dirichlet integral by a simple u-substitution and thus evaluates to $\pi/2$. If the reader is unfamiliar with the Dirichlet integral, they can consult Appendix A below. Thus,

$$\frac{d^3I}{d\alpha^3} - 4\frac{dI}{d\alpha} = -2\pi. \tag{7}$$

Now we solve the differential equation for $I(\alpha)$. The first step will be to make the substitution $\Psi = \frac{dI}{d\alpha}$. This yields

$$\frac{d^2\Psi}{dx^2} - 4\Psi = -2\pi \tag{8}$$

This can be solved with the understanding that the solution for Ψ is just the sum of the solution of the homogeneous differential equation and the particular solution of the ODE. Thus,

$$\Psi(\alpha) = Ae^{2\alpha} + Be^{-2\alpha} + \frac{\pi}{2}, \tag{9}$$

where A and B are unknown constants.

Now Ψ can be integrated over α to give $I(\alpha)$ as

$$I(\alpha) = \frac{1}{2}Ae^{2\alpha} - \frac{1}{2}Be^{-2\alpha} + \frac{\pi}{2}\alpha + C \tag{10}$$

in terms of the three as of yet unknown constants: A, B, C.

Since we have three unknown constants, we need three boundary conditions to eliminate all three constants. These bounds will be

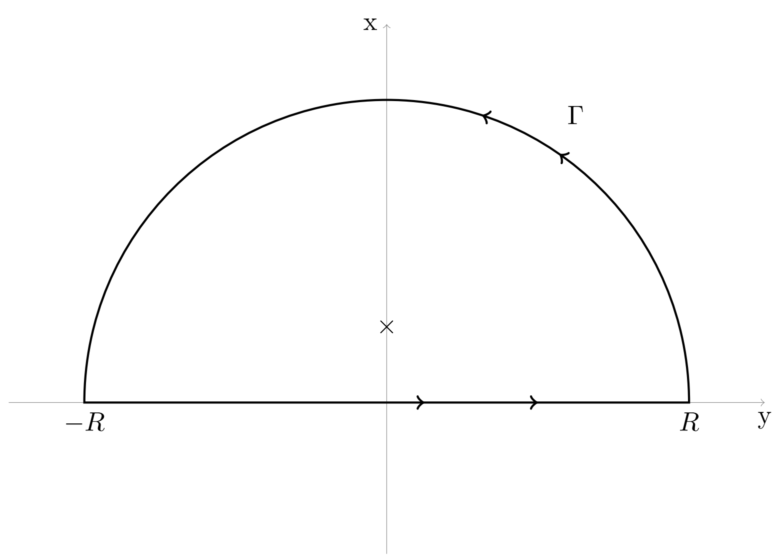
$$\begin{cases} I(0) = 0 & (11) \\ I'(0) = 0 & (12) \\ I''(0) = 0. & (13) \end{cases}$$

Applying these bounds, we find that $A = 0$, $B = -\frac{\pi}{2}$, $C = -\frac{\pi}{4}$. Now the only thing left to do is calculate the original integral by finding the value of $I(\alpha)$ at $\alpha = 1$. Doing this, we get

$$\boxed{I = \frac{\pi}{4}\left(1 + \frac{1}{e^2}\right).} \tag{14}$$

3 Method 2: Feynman Integration + Complex Analysis

In this method, which is my personal favorite, rather than continuing to evaluate higher and higher order derivatives with respect to α, we stop at second order [Eq. (4)]. Then, we evaluate this integral with complex analysis and integrate to obtain $I(\alpha)$, just as we would have using method 1. For the contour integration, we choose $f(z) = \frac{e^{2i\alpha z}}{z^2+1}$ because of its close relation to $\frac{\cos(2\alpha x)}{x^2+1}$ by Euler's Formula, and a simple semicircular contour shown below.



Using the contour, we have

$$\oint_C f(z)\,\mathrm{dz} = \int_\Gamma f(z)\,\mathrm{dz} \; + \int_{-\infty}^{\infty} f(x)\,\mathrm{dx}\,. \tag{15}$$

The contour integral can be evaluated via the residue theorem to give $2\pi i\left(\frac{e^{-2\alpha}}{2i}\right) = \frac{\pi}{e^{2\alpha}}$.

The integral over Γ is parameterized by the curve $z = Re^{i\phi}$. Substituting this into $f(z)$ and integrating as R goes to infinity yields

$$\int_\Gamma f(z)\,\mathrm{dz} = \lim_{R\to\infty}\int_0^\pi \frac{iRe^{i\phi}e^{2i\alpha Re^{i\phi}}}{R^2e^{2i\phi}+1}d\phi = i\lim_{R\to\infty}\int_0^\pi \frac{e^{i\alpha Re^{i\phi}}}{Re^{i\phi}}d\phi = 0. \tag{16}$$

Alternatively, this integral can be done by applying/proving Jordan's Lemma with various inequalities. Using Eq. (15), along with the calculated values for the contour and gamma integral, we get that

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,\mathrm{dx} = \frac{\pi}{e^{2\alpha}}. \tag{17}$$

Using Euler's formula, noting that the imaginary part of the integral vanishes, and using $\cos(x)$'s symmetry to simplify the integral, we get that

$$\int_0^{\infty} \frac{\cos(x)}{x^2+1}\,\mathrm{dx} = \frac{\pi}{2e^{2\alpha}}. \tag{18}$$

Therefore, Eq. (4) implies that

$$\frac{d^2I}{d\alpha^2} = \pi e^{-2\alpha}. \tag{19}$$

Therefore,

$$\frac{dI}{d\alpha} = -\frac{\pi}{2}e^{-2\alpha} + C, \tag{20}$$

and

$$I(\alpha) = \frac{\pi}{4}e^{-2\alpha} + C\alpha + D, \tag{21}$$

where C and D are as of yet unknown constants.

Applying the boundary conditions, $I(0) = 0$ and $I'(0) = 0$ shows that $C = \frac{\pi}{2}$ and $D = -\frac{\pi}{4}$. Therefore,

$$I = I(1) = \frac{\pi}{4}\left(1 + \frac{1}{e^2}\right) \tag{22}$$

4 Appendix A: Dirichlet Integral

In this appendix we will evaluate the famous integral

$$I = \int_0^{\infty} \frac{\sin(x)}{x}\,\mathrm{dx}\,. \tag{23}$$

To do this, we can use Feynman Integration by introducing the parameter β as follows

$$I(\beta) = \int_0^{\infty} \frac{\sin(x)e^{-\beta x}}{x}\,\mathrm{dx}\,. \tag{24}$$

Differentiating with respect to β yields

$$\frac{dI}{d\beta} = \int_0^{\infty} \sin(x)e^{-\beta x}\,\mathrm{dx}\,. \tag{25}$$

This can be trivially evaluated via integration by parts to show that

$$\frac{dI}{d\beta} = -\frac{1}{\beta^2 + 1}. \tag{26}$$

Therefore,

$$I(\beta) = -\arctan(\beta) + C, \tag{27}$$

where C is some unknown constant. Considering that in the limit that β goes to infinity, the integral goes to zero, we can tell that $C = \frac{\pi}{2}$. As a result,

$$I(\beta) = \frac{\pi}{2} - \arctan(\beta) \tag{28}$$

Substituting in zero for β gives the result:

$$\boxed{\int_0^\infty \frac{\sin(x)}{x}\,\mathrm{dx} = \frac{\pi}{2}} \tag{29}$$