Electron-Positron to Muon-Antimuon Scattering

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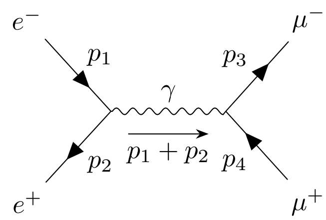
The only Feynman diagram that contributes to electron-positron to muon-antimuon scattering at tree level is the s-channel. The u and t channels do not appear because they contain vertices involving electrons and muons. This is disallowed because the QED Lagrangian, which determines particle interactions, does not contain interaction terms involving different types of species of fermions. Below is the Feynman diagram of the s-channel.



(1)

Applying the Feynman rules to the above diagram yields,

$$i\mathcal{M} = \bar{v}(p_2)(-ie\gamma^\mu)u(p_1)\frac{-i\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]}{s}\bar{u}(p_3)(-ie\gamma^\nu)v(p_4) \tag{2}$$

The $\frac{p_\mu p_\nu}{p^2}$ of course vanishes due to Gauge Invariance, but for pedagogical reasons, lets take a look at how this occurs.

$$\bar{v}(p_2)\gamma^\mu u(p_1)k_\mu = \bar{v}(p_2)k_\mu\gamma^\mu u(p_1) = \bar{v}(p_2)\not{p}_1\gamma^\mu u(p_1) + \bar{v}(p_2)\not{p}_2\gamma^\mu u(p_1) \tag{3}$$

Seeing as the spinors are on shell, they obey the Dirac Equation, meaning that $\not{p}_1 u(p_1) = mu(p_1)$ and $\not{p}_1\bar{v}(p_2) = -m\bar{v}(p_2)$. As a result, the expression in equation (3) cancels

$$m\bar{v}(p_2)u(p_1) - m\bar{v}(p_2)u(p_1) = 0 \tag{4}$$

Therefore, we do not include the gauge term. The amplitude can now be written as

$$\mathcal{M} = \frac{e^2}{s}\left[\bar{v}(p_2)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_3)\gamma_\mu v(p_4)\right] \tag{5}$$

where contractions between spinors and gamma matrices have been grouped with brackets. The factors have been grouped this way because the bracketed factors yield scalars and thus commute with the other brackets.

Since we want to obtain the $|\mathcal{M}|^2$, our next step will be to determine $\mathcal{M}^\dagger$. To do that, we must obtain a rule for taking the adjoint of the bracketed factors.

$$\left(\bar{\psi}_1\gamma^\mu\psi_2\right)^\dagger = \left(\psi_1^\dagger\gamma_0\gamma^\mu\psi_2\right)^\dagger = \psi_2^\dagger\left(\gamma^\mu\right)^\dagger\gamma_0\psi_1 = \psi_2^\dagger\gamma_0\gamma^\mu\psi_1 = \bar{\psi}_2\gamma^\mu\psi_1 \tag{6}$$

where we have made use of $\gamma_0^\dagger = \gamma_0$, $\left(\gamma^\mu\right)^\dagger\gamma_0 = \gamma_0\gamma^\mu$, and the definition of psi bar $\bar{\psi} = \psi^\dagger\gamma_0$. Making use of this,

$$\mathcal{M}^\dagger = \frac{e^2}{s}\left[\bar{v}(p_4)\gamma_\nu u(p_3)\right]\left[\bar{u}(p_1)\gamma^\nu v(p_2)\right] \tag{7}$$

where we have switched the μ dummy index to a ν dummy index so that there is no confusion when we multiply them to get $|\mathcal{M}|^2$

$$|\mathcal{M}|^2 = \frac{e^4}{s^2}\left[\bar{v}(p_2)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_3)\gamma_\mu v(p_4)\right]\left[\bar{v}(p_4)\gamma^\nu u(p_3)\right]\left[\bar{u}(p_1)\gamma_\nu v(p_2)\right] \tag{8}$$

Now we wish to average over final spin states as we are calculating the unpolarized cross section. Unsupressing spin and component indices and using, $\Sigma \bar{v}^s_\beta(p_4)v^s_\alpha(p_4) = (\not{p}_4 - m_\mu \mathbb{1})_{\alpha\beta}$ and $\Sigma \bar{u}^s_\alpha(p_3)u^s_\beta(p_3) = (\not{p}_3 + m_\mu \mathbb{1})_{\alpha\beta}$ we obtain

$$\sum_{s,s'}\left[\bar{u}^s_\alpha(p_3)\gamma^{\alpha\beta}_\mu v^{s'}_\beta(p_4)\right]\left[\bar{v}^\sigma_{s'}(p_4)\gamma^\nu_{\sigma\delta}u^\delta_s(p_3)\right] = \sum_s \bar{u}^s_\alpha(p_3)\gamma^{\alpha\beta}_\mu(\not{p}_4 - m_\mu\mathbb{1})^\sigma_\beta\gamma^\nu_{\sigma\delta}u^\delta_s(p_3) \tag{9}$$

$$= (\not{p}_3 + m_\mu\mathbb{1})^\delta_\alpha\gamma^{\alpha\beta}_\mu(\not{p}_4 - m_\mu\mathbb{1})^\sigma_\beta\gamma^\nu_{\sigma\delta} \tag{10}$$

$$= \mathrm{Tr}\left[(\not{p}_3 + m_\mu\mathbb{1})\gamma_\mu(\not{p}_4 - m_\mu\mathbb{1})\gamma^\nu\right] \tag{11}$$

where we are using m_μ to denote the muon mass and m_e to denote the electron mass.

Let us assume that we do not know the polarization of the initial states. Thus, if we perform the calculation numerous times, we will get an average in the form of a contraction scaled by $\frac{1}{4}$ because there are four possible initial spin states. Thus we need

$$\frac{1}{4}\sum_{s,s'}\left[\bar{v}(p_2)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_1)\gamma_\nu v(p_2)\right] = \frac{1}{4}\,\mathrm{Tr}\left[(\not{p}_1 + m_e\mathbb{1})\gamma_\nu(\not{p}_2 - m_e\mathbb{1}\gamma^\mu)\right] \tag{12}$$

Therefore,

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4s^2}\,\mathrm{Tr}\left[(\not{p}_1 + m_e\mathbb{1})\gamma_\nu(\not{p}_2 - m_e\mathbb{1})\gamma^\mu\right]\mathrm{Tr}\left[(\not{p}_3 + m_\mu\mathbb{1})\gamma_\mu(\not{p}_4 - m_\mu\mathbb{1})\gamma^\nu\right] \tag{13}$$

Now for the time-consuming part: evaluating the traces. To do this note that

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right] = 4g^{\mu\nu} \tag{14}$$

and

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right] = 4\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}\right) \tag{15}$$

Therefore, the first trace factor yields

$$p_1^\alpha p_2^\beta\,\mathrm{Tr}\left[\gamma_\alpha\gamma_\nu\gamma_\beta\gamma^\mu\right] - m_e^2\,\mathrm{Tr}\left[\gamma_v\gamma^\mu\right] = 4p_1^\alpha p_2^\beta\left(g_{\alpha\nu}g^\mu_\beta - g_{\alpha\beta}g^\mu_\nu + g^\mu_\alpha g_{\nu\beta}\right) - 4m_e^2 g^\mu_\nu \tag{16}$$

$$= 4(p_1)_\nu p_2^\mu - 4(p_1\cdot p_2)g^\mu_\nu + 4p_1^\mu(p_2)_\nu - 4m^2 g^\mu_\nu \tag{17}$$

$$= 4((p_1)_\nu p_2^\mu + p_1^\mu(p_2)_\nu) - 4(p_1\cdot p_2 + m^2)g^\mu_\nu \tag{18}$$

In a similar fashion the second trace factor yields

$$\mathrm{Tr}\left[(\not{p}_3 + m_\mu\mathbb{1})\gamma_\mu(\not{p}_4 - m_\mu\mathbb{1})\gamma^\nu\right] = 4((p_3)_\mu p_4^\nu + p_3^\nu(p_4)_\mu) - 4(p_3\cdot p_4 + m^2)g^\nu_\mu \tag{19}$$

Abbreviating $p_n \cdot p_m$ as p_{nm} and expanding out we get

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{4e^4}{s^2}\left[2p_{13}p_{24} + 2p_{14}p_{23} - 2p_{12}(p_{34} + m_\mu^2) - 2p_{34}(p_{12} + m_e^2) + 4(p_{12} + m_e^2)(p_{34} + m_\mu^2)\right] \tag{20}$$

$$= \frac{8e^4}{s^2}\left[p_{13}p_{24} + p_{14}p_{23} + m_\mu^2 p_{12} + m_e^2 p_{34} + 2m_e^2 m_\mu^2\right] \tag{21}$$

Now we express this in terms of the Lorentz invariant Mandelstam variables

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = 2m_e^2 + 2p_{12} = 2m_\mu^2 + 2p_{34} \tag{22}$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 = m_e^2 + m_\mu^2 - 2p_{13} = m_e^2 + m_\mu^2 - 2p_{24} \tag{23}$$

$$(p_1 - p_4)^2 = (p_2 - p_3)^2 = m_e^2 + m_\mu^2 - 2p_{13} = m_e^2 + m_\mu^2 - 2p_{23} \tag{24}$$

These can be used to isolate the four momentum contractions

$$p_{12} = \frac{s - 2m_e^2}{2} \tag{25}$$

$$p_{34} = \frac{s - 2m_\mu^2}{2} \tag{26}$$

$$p_{13} = p_{24} = \frac{m_e^2 + m_\mu^2 - t}{2} \tag{27}$$

$$p_{14} = p_{23} = \frac{m_e^2 + m_\mu^2 - u}{2} \tag{28}$$

After a lot of algebra, the averaged amplitude can be written as

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{2e^4}{s^2}\left[t^2 + u^2 + 4s(m_e^2 + m_\mu^2) - 2(m_e^2 + m_\mu^2)^2\right] \tag{29}$$

We're almost done. We just need to calculate the cross section using this amplitude. To do that we need to make a choice of four momenta that satisfies energy-momentum conservation. The easiest way to do this is through the center-of-mass frame. In this frame, all particles have equal energies, and the incoming and outgoing particles have equal and opposite momenta.

$$p_1 = \left(E, \vec{k}\right),\ \ p_2 = \left(E, -\vec{k}\right),\ \ p_3 = (E,\ \vec{p}),\ \ p_4 = (E, -\vec{p}) \tag{30}$$

Expressing the Mandelstam variables in terms of the center-of-mass frame momenta we get

$$s = (p_1 + p_2)^2 = 4E^2 = E_{\text{CM}}^2 \tag{31}$$

$$t = (p_1 - p_3)^2 = m_e^2 + m_\mu^2 - 2E^2 + 2\vec{k}\cdot\vec{p} \tag{32}$$

$$u = (p_1 - p_4)^2 = m_e^2 + m_\mu^2 - 2E^2 - 2\vec{k}\cdot\vec{p} \tag{33}$$

Combining the above relations with the 2 to 2 differential scattering cross section formula

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{1}{64\pi^2 E_{\text{CM}}^2}\frac{|\vec{p}_f|}{|\vec{p}_i|}\left(\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\right) \tag{34}$$

we get

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{32\pi^2 E_{\text{CM}}^2 s^2}\frac{|\vec{p}|}{|\vec{k}|}\left[t^2 + u^2 + 4s(m_e^2 + m_\mu^2) - 2(m_e^4 + 2m_e^2 m_\mu^2 + m_\mu^4)\right] \tag{35}$$

$$= \frac{\alpha^2}{16E^6}\frac{|\vec{p}|}{|\vec{k}|}\left(E^4 + (\vec{k}\cdot\vec{p})^2 + E^2(m_e^2 + m_\mu^2)^2\right) \tag{36}$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant. Noting that $\vec{k}\cdot\vec{p} = |\vec{k}||\vec{p}|\cos(\theta)$, $|\vec{k}| = \sqrt{E^2 - m_e^2}$, and $|\vec{p}| = \sqrt{E^2 - m_\mu^2}$, we find obtain the generalized form of the tree-level differential scattering cross section for electron-positron to muon-antimuon scattering,

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{\text{CM}}^2}\sqrt{\frac{1-\frac{m_\mu^2}{E^2}}{1-\frac{m_e^2}{E^2}}}\left[1+\left(1-\frac{m_e^2}{E^2}\right)\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2(\theta)+\frac{m_e^2+m_\mu^2}{E^2}\right]} \tag{37}$$

In the high energy limit, $\frac{m_e^2}{E^2}$ can be taken to be zero, giving

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{\text{CM}}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{m_\mu^2}{E^2}+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2(\theta)\right) \tag{38}$$

In the ultra-high energy limit, $\frac{m_\mu^2}{E^2}$ can be taken to be zero, giving

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{\text{CM}}^2}\left(1+\cos^2(\theta)\right) \tag{39}$$