Bhabha Scattering $[e^+e^- \to e^+e^-]$

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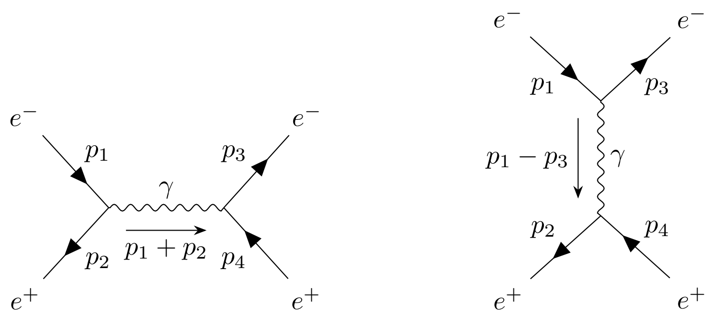
The two Feynman diagrams that contribute to Bhabha Scattering are the s and t channels. The u-channel does not contribute because it is unphysical for an electron to radiate off a photon and then become a positron. In such a case, the electron would just transition to a lower energy state. The appropriate Feynman diagrams are shown below.



(1)

The total amplitude will be the difference of the amplitudes corresponding to these two diagrams due to the relative minus sign between them. For more information see Appendix A.

$$\mathcal{M} = \mathcal{M}_s - \mathcal{M}_t \tag{2}$$

Therefore,

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_s|^2 + \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_t|^2 - \frac{1}{4}\sum_{\text{spins}} \left(\mathcal{M}_s\mathcal{M}_t^\dagger + \mathcal{M}_s^\dagger\mathcal{M}_t\right). \tag{3}$$

Notice that $\frac{1}{4}\sum |\mathcal{M}_s|^2$ is just the squared amplitude for electron-positron to muon-antimuon scattering but with the muon mass set equal to the electron mass. Making use of this crossing and our previous calculation of the squared amplitude for this event we obtain

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_s|^2 = \frac{2e^4}{s^2}\left(t^2 + u^2\right). \tag{4}$$

Note that we have taken the electron mass relative to the energy to be zero for convenience. The brave reader may attempt to calculate the generalized cross section at their own risk.

Now we must consider the s and t amplitudes so that we can calculate the other cross sections.

$$\mathcal{M}_s = \frac{e^2}{s}\left[\bar{v}(p_2)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_3)\gamma_\mu v(p_4)\right] \tag{5}$$

$$\mathcal{M}_t = \frac{e^2}{t}\left[\bar{u}(p_3)\gamma^\mu u(p_1)\right]\left[\bar{v}(p_2)\gamma_\mu v(p_4)\right] \tag{6}$$

Meaning,

$$\mathcal{M}_t^\dagger = \frac{e^2}{t} \left[\bar{v}(p_4)\gamma^\nu v(p_2)\right] \left[\bar{u}(p_1)\gamma_\nu u(p_3)\right], \tag{7}$$

and

$$\mathcal{M}_s^\dagger = \frac{e^2}{s} \left[\bar{v}(p_4)\gamma_\mu u(p_3)\right] \left[\bar{u}(p_1)\gamma^\mu v(p_2)\right]. \tag{8}$$

Thus,

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_t|^2 = \frac{e^4}{4t^2} \operatorname{Tr}\left[\not{p}_2\gamma_\mu\not{p}_4\gamma^\nu\right] \operatorname{Tr}\left[\not{p}_1\gamma_\nu\not{p}_3\gamma^\mu\right]. \tag{9}$$

Now let us employ some trickery to make our lives easier. Remember that

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_s|^2 = \frac{e^4}{4s^2} \operatorname{Tr}\left[\not{p}_1\gamma_\nu\not{p}_2\gamma^\mu\right] \operatorname{Tr}\left[\not{p}_3\gamma_\mu\not{p}_4\gamma^\nu\right] = \frac{2e^4}{s^2}\left(t^2+u^2\right). \tag{10}$$

Notice how Eq. (9) is related to this by the crossing $p_1 \to p_1$, $p_2 \to -p_3$, $p_3 \to -p_2$, $p_4 \to p_4$. We put the negatives to ensure s goes to t. In terms of Mandelstam variables the crossing can be written $s \to t$, $t \to$, $u \to u$.

Now the magic: we can calculate $\frac{1}{4}\sum |\mathcal{M}_t|^2$ by applying this crossing symmetry to $\frac{1}{4}\sum |\mathcal{M}_s|^2$ to get

$$\frac{1}{4}\sum |\mathcal{M}_t|^2 = \frac{2e^4}{t^2}\left(s^2+u^2\right). \tag{11}$$

The interference terms are given by

$$\frac{1}{4}\sum_{\text{spins}} \mathcal{M}_t^\dagger \mathcal{M}_s = \frac{e^4}{st} \operatorname{Tr}\left[\not{p}_4\gamma^\nu\not{p}_2\gamma^\mu\not{p}_1\gamma_\nu\not{p}_3\gamma_\mu\right], \tag{12}$$

and

$$\frac{1}{4}\sum_{\text{spins}} \mathcal{M}_s^\dagger \mathcal{M}_t = \frac{e^4}{st} \operatorname{Tr}\left[\not{p}_2\gamma_\nu\not{p}_4\gamma_\mu\not{p}_3\gamma^\nu\not{p}_1\gamma^\mu\right]. \tag{13}$$

The lengthy and tedious calculation of these traces is left as an exercise for the reader ;). It can be shown that the first one yields

$$\frac{1}{4}\sum_{\text{spins}} \mathcal{M}_t^\dagger \mathcal{M}_s = \frac{8e^4}{st}\left(p_2\cdot p_3\right)\left[\left(p_1\cdot p_3\right)-\left(p_1\cdot p_2\right)\right] = \frac{2e^4 u}{st}\left(s+t\right) = -\frac{2e^4u^2}{st}. \tag{14}$$

The two interference terms are related by the crossing $s \to s$, $t \to t$, $u \to u$, meaning that the two interference terms are equal.

Using all the terms contributing to the squared amplitude that we have calculated, we find that the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left[\frac{t^2+u^2}{s^2}+\frac{s^2+u^2}{t^2}+\frac{2u^2}{st}\right] = \frac{\alpha^2}{2s}\left[\left(\frac{s}{t}\right)^2+\left(\frac{t}{s}\right)^2+u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2\right]. \tag{15}$$

We could in principle leave the differential scattering cross section like that, but lets express it in terms of an angle θ. After a lot of tedious algebra we get

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2E_{\text{CM}}^2}\left[\frac{1}{2}\left(1+\cos^2(\theta)\right)+\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}-\frac{2\cos^4(\theta/2)}{\sin^2(\theta/2)}\right]} \tag{16}$$

Appendix A: Relative Minus Signs

work in progress