Maxwell Lagrangian: A Derivation

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Here we perform an exercise in Classical Field Theory by deriving a Lagrangian for Classical Electromagnetism by considering a Lagrangian that gives rise to the familiar Maxwell's Equations in free space. Note that we are working in natural units ($c = \hbar = \epsilon_0 = \mu_0 = 1$). Thus,

$$\nabla \cdot \vec{E} = \rho \tag{1}$$

$$\nabla \cdot \vec{B} = 0 \tag{2}$$

$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t} \tag{3}$$

$$\nabla \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}. \tag{4}$$

The plan is to now condense these four equations into a single equation of motion that is generated by the Maxwell Lagrangian. This is done by considering the potential formalism of classical electrodynamic which asserts that

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \tag{5}$$

$$\vec{B} = \nabla \times \vec{A}. \tag{6}$$

These potential terms prove very useful due to their automatic satisfaction of Eq. (2) and Eq. (3). This allows these equations to be discarded and replaced with the definitions of the potential. The remaining equations, Eq. (1) and Eq. (4), in terms of the potentials, read:

$$-\nabla^2\Phi - \frac{\partial}{\partial t}\left(\nabla \cdot \vec{A}\right) = \rho \tag{7}$$

$$\nabla \times \left(\nabla \times \vec{A}\right) = \vec{J} - \frac{\partial}{\partial t}\left(\nabla\Phi + \frac{\partial \vec{A}}{\partial t}\right). \tag{8}$$

This brings us from four equations to two, one step closer to our desired single equation of motion. Before going any further, let us simplify the two equations into:

$$\nabla^2\Phi + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{A}\right) = -\rho \tag{9}$$

$$-\nabla^2\vec{A} + \nabla\left(\nabla \cdot \vec{A}\right) = \vec{J} - \frac{\partial}{\partial t}\left(\nabla\Phi\right) - \frac{\partial^2\Phi}{\partial t^2}, \tag{10}$$

where we have used the identity $\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2\vec{A}$ to obtain Eq. (10). This identity is derived via the use of Levi-Civita Tensors in Appendix A.

Now, we take a step back and note a key difference between the field and potential formalism of EM: electromagnetic fields are uniquely determined whereas potentials can take on one of infinitely many forms so long as they satisfy $\nabla \times \vec{A} = \vec{B}$ and $-\nabla\Phi - \frac{\partial \vec{A}}{\partial t} = \vec{E}$. Therefore, through a process known as "Gauge-Fixing," we may chose a condition for the potential that will be convenient for us. In this case, we will use

$\nabla \cdot \vec{A} = -\frac{\partial \Phi}{\partial t}$ so as to cancel out terms in Eq. (10). This is known as the Lorenz Gauge, and it reduces the equations to:

$$\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = \rho \tag{11}$$

$$\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \vec{J} \tag{12}$$

These two equations can now be reduced into one equation expressed in the language of four-vectors and index notation as

$$\partial_\mu \partial^\mu A^\nu = J^\nu, \tag{13}$$

where we have defined the four-vectors $A^\nu = (\Phi, -\vec{A})$ and $J^\nu = (\rho, -\vec{J})$. The resulting Lagrangian is given by $\mathcal{L} = \frac{1}{2}\left(\partial_\mu A^\nu\right)^2 + J^\mu A_\mu$.

But not so fast! A key symmetry that we impose on the Lagrangian is gauge symmetry, which means that the form of the Lagrangian is invariant under gauge transformations. In this case, these gauge transformations take the form $A_\mu \to A_\mu - \partial_\mu \alpha$, for some function α. Intuitively, what this means is that our arbitrary choice of gauge should not effect the results

In this Lagrangian, however, it is evident that this symmetry is not respected. Therefore, we need to make some changes to fix this. The only way that we can do this is by introducing $\partial_\mu A^\mu$-terms because we have set it to zero in the Lorentz Gauge ($\frac{\partial \Phi}{\partial t} + \nabla \cdot \vec{A} = \partial_\mu A^\mu = 0$).

If we alter the Lagrangian to

$$\mathcal{L} = \frac{1}{2}(\partial_\mu A^\nu)^2 - \frac{1}{2}(\partial_\mu A^\mu)^2 + J^\mu A_\mu, \tag{14}$$

it is gauge invariant because the first two term transforms as

$$\begin{aligned}\mathcal{L}'_{\text{first terms}} = \frac{1}{2}\left[\partial_\mu A^\nu - \partial_\mu \partial_\nu \alpha\right]^2 - \frac{1}{2}\left[\partial_\mu A^\mu - \partial_\mu \partial^\mu \alpha\right]^2 = \mathcal{L}_{\text{original}} + \frac{1}{2}(\partial_\mu \partial_\nu A^\nu)(\partial^\mu \partial^\sigma A_\sigma) - \frac{1}{2}(\partial_\mu \partial^\mu \alpha)(\partial_\nu \partial^\nu \alpha) \\ + (\partial_\mu A^\mu)(\partial_\sigma \partial^\sigma \alpha) - (\partial_\mu A^\nu)(\partial_\mu \partial_\nu \alpha).\end{aligned} \tag{15}$$

Using integration by parts, all the terms on the far right cancel each other, leaving just the original Lagrangian, meaning that first two terms in the Lagrangian are gauge invariant. The third term, however, seems to not be gauge invariant at first because

$$\mathcal{L}'_{\text{current term}} = J^\mu (A_\mu - \partial_\mu \alpha) = J^\mu A_\mu - J^\mu \partial_\mu \alpha. \tag{16}$$

However, integrating the second term by parts and using the continuity equation from electromagnetism ($\partial_\mu J^\mu = 0$), we see that it is gauge invariant. We have now shown that the entire Lagrangian is gauge invariant.

The Lagrangian can be rewritten from Eq. (14) to

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + J^\mu A_\mu,} \tag{17}$$

using the fact that the electromagnetic-field-strength tensor is given by $F_{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A^\mu$. We are now done with the derivation, but I want to include a clarifying summary as well as a few clarifications below.

Summary of the derivation:

1. We transformed Maxwell's equations into a set of two coupled differential equations in terms of the electric and magnetic potentials.

2. We picked the Lorenz Gauge to very nicely simplify the equations by setting $\partial_\mu A^\mu = 0$.

3. We noticed that the Lagrangian did not respect gauge symmetry.

4. We introduced $\partial_\mu A^\mu$ terms because they can make the Lagrangian gauge invariant without altering it because they're equal to zero.

5. This yielded our desired gauge invariant Lagrangian.

Clarifications:

1. Of course we could have never imposed a gauge at all and have written a single indexed equation for Eq. (9) and Eq. (10), but it is far from evident that it will take the form $\Box A_\nu - \partial_\mu \partial_\nu A^\mu = -J_\nu$. We instead opted to use the gauge as a means of simplifying the mathematics.

2. The astute reader may have noticed that we have not included higher order terms in $\partial_\mu A^\mu$. There are many correct explanations for why. I will use this one involving dimensional analysis: if we were to include higher order terms, they would need to be scaled by a dimensionful constant. Because we are building these new terms out of only powers of $\partial_\mu A^\mu$, there would be no other terms proportional to that constant to cancel terms and make the Lagrangian gauge invariant.