

# Neat Integral: Two Different Ways

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## 1 The Integral

The Integral we will be dealing with is

$$I = \int_0^\infty \frac{\sin^2(x)}{x^2(x^2+1)} dx. \quad (1)$$

We will evaluate this integral two different ways: first via pure Feynman integration and second via a combination of Feynman integration and complex analysis.

## 2 Method 1: Pure Feynman Integration

As is always the first step of Feynman integration, we insert a parameter into the the integral. In this case, it is clear that we should define

$$I(\alpha) = \int_0^\infty \frac{\sin^2(\alpha x)}{x^2(x^2+1)} dx. \quad (2)$$

Then we differentiate with respect to the parameter  $\alpha$  with the goal of being able to write a differential equation relating the various derivatives of  $I(\alpha)$ .

$$\frac{dI}{d\alpha} = \int_0^\infty \frac{\sin(2\alpha x)}{x(x^2+1)} dx. \quad (3)$$

And we differentiate again!

$$\frac{d^2 I}{d\alpha^2} = 2 \int_0^\infty \frac{\cos(2\alpha x)}{x^2+1} dx. \quad (4)$$

And again!

$$\frac{d^3 I}{d\alpha^3} = -4 \int_0^\infty \frac{x \sin(2\alpha x)}{x^2+1} dx. \quad (5)$$

With some keen observation, it can be seen that

$$\frac{dI}{d\alpha} - \frac{1}{4} \frac{d^3 I}{d\alpha^3} = \int_0^\infty \frac{\sin(2\alpha x)}{x} dx. \quad (6)$$

The integral on the right side is related to the Dirichlet integral by a simple u-substitution and thus evaluates to  $\pi/2$ . If the reader is unfamiliar with the Dirichlet integral, they can consult Appendix A below. Thus,

$$\frac{d^3 I}{d\alpha^3} - 4 \frac{dI}{d\alpha} = -2\pi. \quad (7)$$

Now we solve the differential equation for  $I(\alpha)$ . The first step will be to make the substitution  $\Psi = \frac{dI}{d\alpha}$ . This yields

$$\frac{d^2\Psi}{dx^2} - 4\Psi = -2\pi \quad (8)$$

This can be solved with the understanding that the solution for  $\Psi$  is just the sum of the solution of the homogeneous differential equation and the particular solution of the ODE. Thus,

$$\Psi(\alpha) = Ae^{2\alpha} + Be^{-2\alpha} + \frac{\pi}{2}, \quad (9)$$

where  $A$  and  $B$  are unknown constants.

Now  $\Psi$  can be integrated over  $\alpha$  to give  $I(\alpha)$  as

$$I(\alpha) = \frac{1}{2}Ae^{2\alpha} - \frac{1}{2}Be^{-2\alpha} + \frac{\pi}{2}\alpha + C \quad (10)$$

in terms of the three as of yet unknown constants:  $A$ ,  $B$ ,  $C$ .

Since we have three unknown constants, we need three boundary conditions to eliminate all three constants. These bounds will be

$$\begin{cases} I(0) = 0 \\ I'(0) = 0 \\ I''(0) = 0. \end{cases} \quad (11)$$

$$\begin{cases} I'(0) = 0 \\ I''(0) = 0. \end{cases} \quad (12)$$

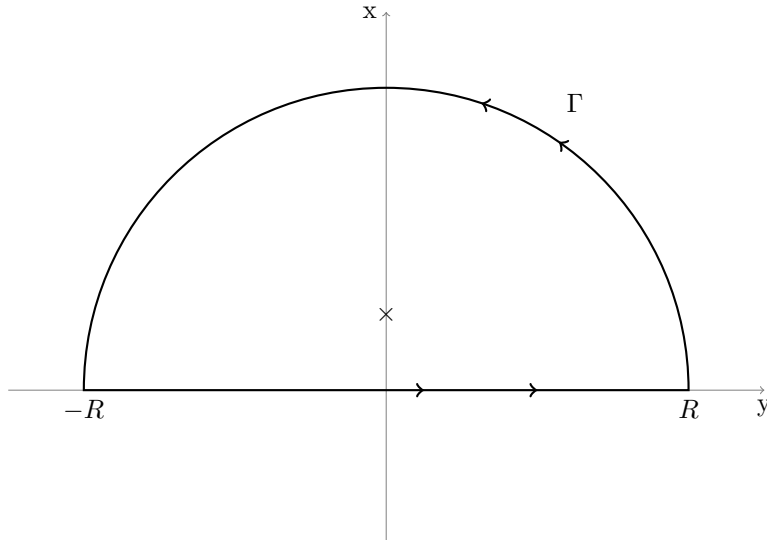
$$\begin{cases} I''(0) = 0. \end{cases} \quad (13)$$

Applying these bounds, we find that  $A = 0$ ,  $B = -\frac{\pi}{2}$ ,  $C = -\frac{\pi}{4}$ . Now the only thing left to do is calculate the original integral by finding the value of  $I(\alpha)$  at  $\alpha = 1$ . Doing this, we get

$$\boxed{I = \frac{\pi}{4} \left( 1 + \frac{1}{e^2} \right)}. \quad (14)$$

### 3 Method 2: Feynman Integration + Complex Analysis

In this method, which is my personal favorite, rather than continuing to evaluate higher and higher order derivatives with respect to  $\alpha$ , we stop at second order [ Eq. (4)]. Then, we evaluate this integral with complex analysis and integrate to obtain  $I(\alpha)$ , just as we would have using method 1. For the contour integration, we choose  $f(z) = \frac{e^{2i\alpha z}}{z^2+1}$  because of its close relation to  $\frac{\cos(2\alpha x)}{x^2+1}$  by Euler's Formula, and a simple semicircular contour shown below.



Using the contour, we have

$$\oint_C f(z) dz = \int_{\Gamma} f(z) dz + \int_{-\infty}^{\infty} f(x) dx. \quad (15)$$

The contour integral can be evaluated via the residue theorem to give  $2\pi i \left( \frac{e^{-2\alpha}}{2i} \right) = \frac{\pi}{e^{2\alpha}}$ .

The integral over  $\Gamma$  is parameterized by the curve  $z = Re^{i\phi}$ . Substituting this into  $f(z)$  and integrating as  $R$  goes to infinity yields

$$\int_{\Gamma} f(z) dz = \lim_{R \rightarrow \infty} \int_0^{\pi} \frac{iRe^{i\phi} e^{2i\alpha Re^{i\phi}}}{R^2 e^{2i\phi} + 1} d\phi = i \lim_{R \rightarrow \infty} \int_0^{\pi} \frac{e^{i\alpha Re^{i\phi}}}{Re^{i\phi}} d\phi = 0. \quad (16)$$

Alternatively, this integral can be done by applying/proving Jordan's Lemma with various inequalities. Using Eq. (15), along with the calculated values for the contour and gamma integral, we get that

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + 1} dx = \frac{\pi}{e^{2\alpha}}. \quad (17)$$

Using Euler's formula, noting that the imaginary part of the integral vanishes, and using  $\cos(x)$ 's symmetry to simplify the integral, we get that

$$\int_0^{\infty} \frac{\cos(x)}{x^2 + 1} dx = \frac{\pi}{2e^{2\alpha}}. \quad (18)$$

Therefore, Eq. (4) implies that

$$\frac{d^2 I}{d\alpha^2} = \pi e^{-2\alpha}. \quad (19)$$

Therefore,

$$\frac{dI}{d\alpha} = -\frac{\pi}{2} e^{-2\alpha} + C, \quad (20)$$

and

$$I(\alpha) = \frac{\pi}{4} e^{-2\alpha} + C\alpha + D, \quad (21)$$

where  $C$  and  $D$  are as of yet unknown constants.

Applying the boundary conditions,  $I(0) = 0$  and  $I'(0) = 0$  shows that  $C = \frac{\pi}{2}$  and  $D = -\frac{\pi}{4}$ . Therefore,

$$I = I(1) = \frac{\pi}{4} \left( 1 + \frac{1}{e^2} \right) \quad (22)$$

## 4 Appendix A: Dirichlet Integral

In this appendix we will evaluate the famous integral

$$I = \int_0^{\infty} \frac{\sin(x)}{x} dx. \quad (23)$$

To do this, we can use Feynman Integration by introducing the parameter  $\beta$  as follows

$$I(\beta) = \int_0^{\infty} \frac{\sin(x) e^{-\beta x}}{x} dx. \quad (24)$$

Differentiating with respect to  $\beta$  yields

$$\frac{dI}{d\beta} = \int_0^{\infty} \sin(x) e^{-\beta x} dx. \quad (25)$$

This can be trivially evaluated via integration by parts to show that

$$\frac{dI}{d\beta} = -\frac{1}{\beta^2 + 1}. \quad (26)$$

Therefore,

$$I(\beta) = -\arctan(\beta) + C, \quad (27)$$

where C is some unknown constant. Considering that in the limit that  $\beta$  goes to infinity, the integral goes to zero, we can tell that  $C = \frac{\pi}{2}$ . As a result,

$$I(\beta) = \frac{\pi}{2} - \arctan(\beta) \quad (28)$$

Substituting in zero for  $\beta$  gives the result:

$$\boxed{\int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}} \quad (29)$$