COMS W4701: Artificial Intelligence

Lecture 20: Exact Inference in Bayes Nets

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Today

- Exact inference in Bayes nets
- Inference by enumeration
- Factors and variable elimination

Inference in Bayes Nets

- General task: Find the *posterior* distribution of a set of **query** variables X given a set of observed **evidence** e
- There may also be **hidden** variables Y interacting with X and E

- *Enumeration* strategy: Construct joint distributions via “simplified” chain rule and remove hidden variables via marginalization

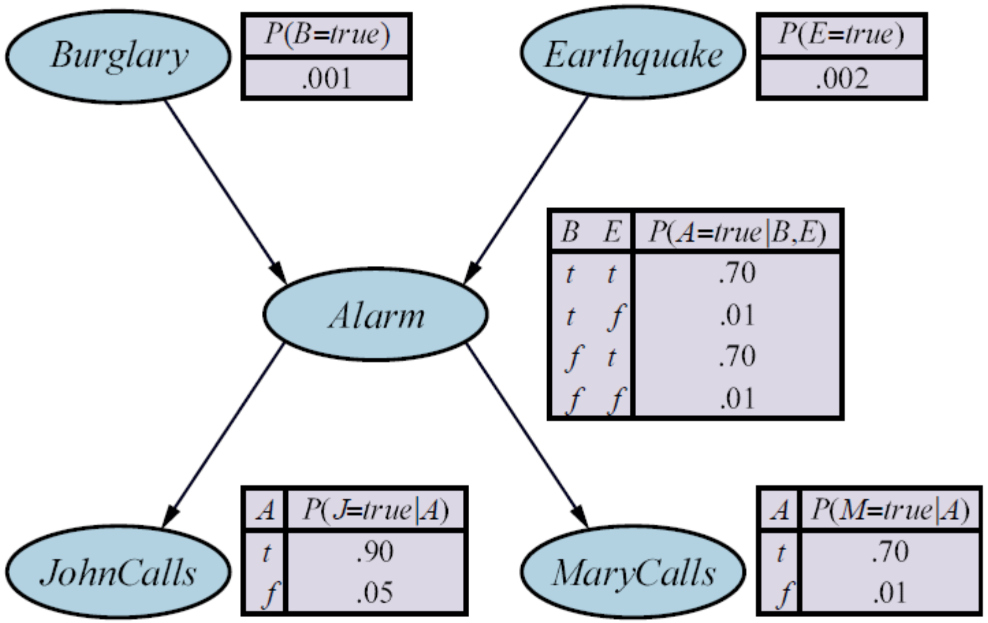
$$\boxed{P(\boldsymbol{X} \mid \boldsymbol{e}) \propto P(\boldsymbol{X}, \boldsymbol{e}) = \sum_{\boldsymbol{y}} P(\boldsymbol{X}, \boldsymbol{y}, \boldsymbol{e})}$$

- Y will generally include *ancestors* of X and E but not descendants

Example: Alarm Network

- Y will generally include ancestors of X and E but not descendants

$$P(+b,-e,+a) = P(+b)P(-e)P(+a|+b,-e)$$
$$= (.001)(.998)(.01) = 9.98 \times 10^{-6}$$



$$P(+a) = \sum_{b,e} P(b,e,+a) = \sum_{b,e} P(b)P(e)P(+a|b,e)$$
$$= (.001)(.002)(.7) + (.001)(.998)(.01)$$
$$+ (.999)(.002)(.7) + (.999)(.998)(.01) = .01138$$

$$P(-b|+a) = \frac{\sum_e P(-b,e,+a)}{P(+a)} = \frac{\sum_e P(-b)P(e)P(+a|-b,e)}{P(+a)}$$
$$= \big((.999)(.002)(.7) + (.999)(.998)(.01)\big)/.01138 = .999$$

Example: Alarm Network

- Local independence properties can help simplify expressions before computation

$$P(+j|-a,+e,+b,+m) = P(+j|-a) = 0.05$$

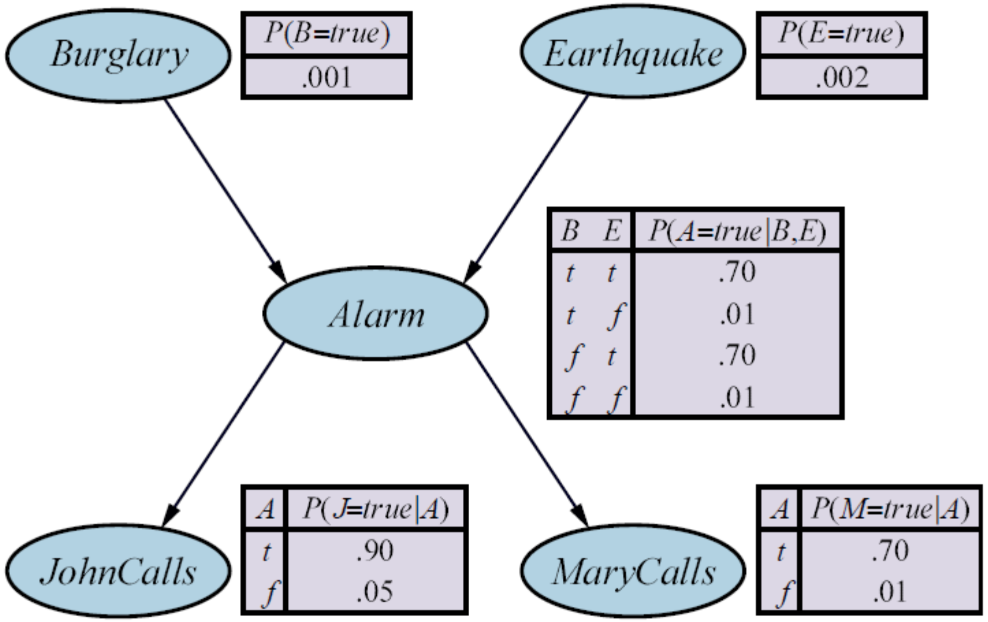
$$P(+j,+m|-a,+e,+b) = P(+j,+m|-a)$$

$$= P(+j|-a)P(+m|-a) = (0.05)(0.01) = 0.0005$$

$$P(+j,+e|-a,+b,+m) = P(+j|-a)P(+e|-a,+b)$$

$$= P(+j|-a)P(+e,-a,+b)/P(-a,+b)$$

$$= \frac{P(+j|-a)P(+b)P(+e)P(-a|+b,+e)}{\sum_e P(+b)P(e)P(-a|+b,e)} = \frac{(.05)(.001)(.002)(.3)}{(.001)(.002)(.3)+(.001)(.998)(.99)} = 3.05 \times 10^{-5}$$



P(B=true)
.001

P(E=true)
.002

B	E	P(A=true|B,E)
t	t	.70
t	f	.01
f	t	.70
f	f	.01

A	P(J=true|A)
t	.90
f	.05

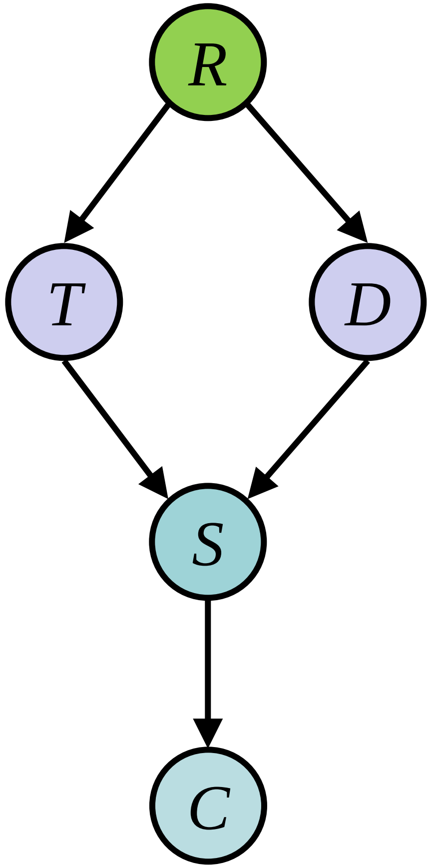
A	P(M=true|A)
t	.70
f	.01

Querying Distributions

- We can also query entire distributions all at once
- Computational complexity will generally be exponential in number of query and hidden variables
- Ex: $P(R|+s) \propto P(R,+s) = \sum_{t,d} P(R,t,d,+s)$

$$= \sum_{t,d} P(R)P(t|R)P(d|R)P(+s|t,d)$$

- Compute joint probabilities over all relevant variables by *multiplying* CPTs, and sum out the hidden variables



Factor Representation

- CPTs may represent marginal distributions, conditional distributions, or neither
- In any case, they are just tables or **factors** over which we are multiplying or adding
- Each factor f_i is a CPT indexed by the values of its input variables

$$P(R|+s) \propto \sum_{t,d} P(R)P(t|R)P(d|R)P(+s|t,d) = \sum_{t,d} f_1(R)f_2(R,t)f_3(R,d)f_4(t,d)$$

- *Multiplying* factors: *Pointwise multiplication* over the common variables, new factor depends on the *union* of dependencies

$$f_1(X,Y) \times f_2(Y,Z) = f_3(X,Y,Z)$$

- *Summing* over a factor: Same as marginalization of a joint distribution

$$\sum_y f_3(X,y,Z) = f_4(X,Z)$$

Example

$$P(R|+s) \propto P(R,+s) = \sum_{t,d} P(R)P(t|R)P(d|R)P(+s|t,d)$$

R	T	D	P(R,T,D,+s)
+r	+t	+d	(0.5)(0.7)(0.7)(0.1)
+r	+t	-d	(0.5)(0.7)(0.3)(0.4)
+r	-t	+d	(0.5)(0.3)(0.7)(0.2)
+r	-t	-d	(0.5)(0.3)(0.3)(0.9)
-r	+t	+d	(0.5)(0.6)(0.6)(0.1)
-r	+t	-d	(0.5)(0.6)(0.4)(0.4)
-r	-t	+d	(0.5)(0.4)(0.6)(0.2)
-r	-t	-d	(0.5)(0.4)(0.4)(0.9)

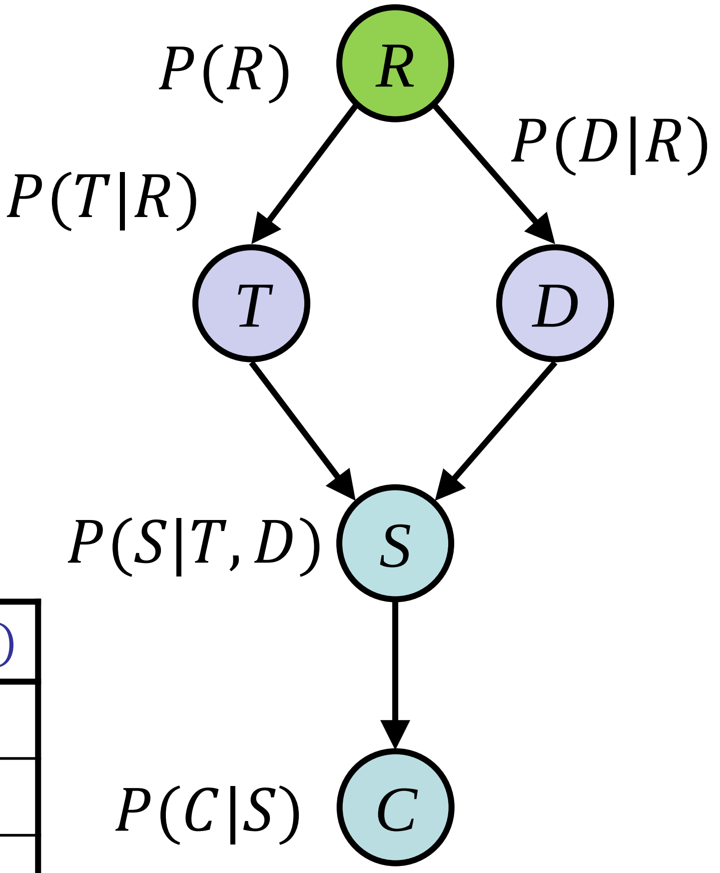
Joint distribution size: $2^3 = 8$ rows

R	$f_1(R)$
+r	0.5
-r	0.5

T	R	$f_2(T,R)$
+t	+r	0.7
+t	-r	0.6
-t	+r	0.3
-t	-r	0.4

D	R	$f_3(D,R)$
+d	+r	0.7
+d	-r	0.6
-d	+r	0.3
-d	-r	0.4

T	D	$f_4(T,D)$
+t	+d	0.1
+t	-d	0.4
-t	+d	0.2
-t	-d	0.9

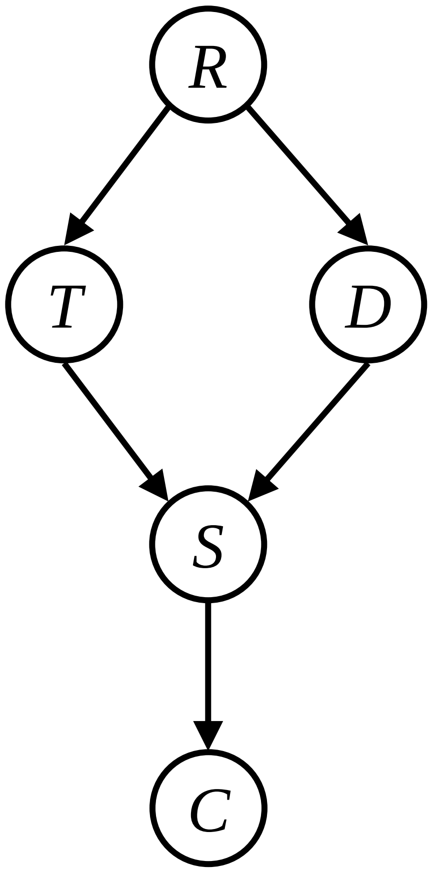


Inference Complexity

- Inference complexity will solely depend on the size of the joint distribution, or number of query and hidden variables
- But we do not have to wait to sum over all variables at the end!

- Better idea: Perform summation over each variable independently
- Factors not dependent on X can be *taken out* of a summation over X

- Ex: $uwy + uwz + uxy + uxz + vwy + vwz + vxy + vxz$ has 16 multiplies and 7 adds
- $(u + v)(wy + wz + xy + xz)$ has 5 multiplies and 4 adds
- $(u + v)(w + x)(y + z)$ has 2 multiplies and 3 adds

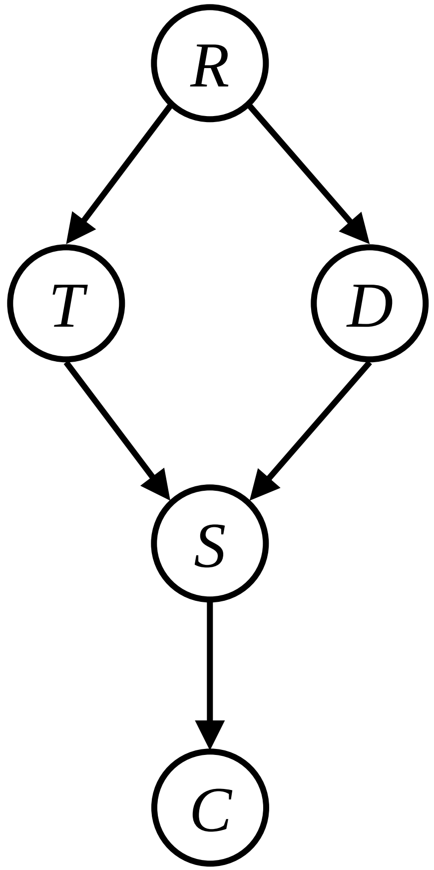


Variable Elimination

- Idea: Move summations as far *inwards* as possible
- Marginalization is done starting inside and moving outward

$$P(S|r) \propto P(S,r) = \sum_{t,d} P(r)P(t|r)P(d|r)P(S|t,d)$$
$$= P(r)\sum_{t} P(t|r)\sum_{d} P(d|r)P(S|t,d)$$

$$P(S|c) \propto P(S,c) = \sum_{r,t,d} P(r)P(t|r)P(d|r)P(S|t,d)P(c|S)$$
$$= P(c|S)\sum_{r} P(r)\sum_{t} P(t|r)\sum_{d} P(d|r)P(S|t,d)$$



Example: Variable Elimination

$$P(R|+c,+d) \propto P(R)P(+d|R)\sum_t P(t|R)\sum_s P(s|t,+d)P(+c|s) = f_1(R)f_2(R)\sum_t f_3(R,T)\sum_s f_4(T,S)f_5(S)$$

T	S	$f_6(T,S)$
+t	+s	(0.1)(0.8)
+t	-s	(0.9)(0.3)
-t	+s	(0.2)(0.8)
-t	-s	(0.8)(0.3)

$$\sum_s$$

T	$f_7(T)$
+t	0.35
-t	0.4

$$\times f_3(R,T)$$

R	T	$f_8(R,T)$
+r	+t	(0.7)(0.35)
+r	-t	(0.3)(0.4)
-r	+t	(0.6)(0.35)
-r	-t	(0.4)(0.4)

$$\sum_t$$

R	$f_9(R)$
+r	0.365
-r	0.37

$$\times f_1(R) \times f_2(R)$$

R	$f_{10}(R)$
+r	(0.5)(0.7)(0.365)
-r	(0.5)(0.6)(0.37)

Max table size is 2^2 rows instead of 2^3

Total operations:
12 multiplies, 4 adds

Variable Ordering

- Elimination ordering does not affect correctness of inference, but *does* greatly affect computational efficiency!

$$P(S, c) = \sum_{r,t,d} f_1(R) f_2(T, R) f_3(D, R) f_4(S, T, D) f_5(S)$$

- R then T then D: $f_5(S) \sum_d \sum_t f_4(S, T, D) \sum_r \underbrace{f_1(R) f_2(T, R) f_3(D, R)}_{\text{8 rows}}$
- 22 multiplies, 10 adds

- T then D then R: $f_5(S) \sum_r f_1(R) \sum_d f_3(D, R) \sum_t \underbrace{f_2(T, R)\, f_4(S, T, D)}_{\text{16 rows}}$
- 30 multiplies, 14 adds

Improving Complexity

- Elimination complexity depends on size of the largest constructed CPT
- NP-hard in the worst case, as this can reduce to a satisfiability problem

- *Greedy* variable ordering can be a good heuristic: Select the next variable that minimizes the size of the constructed CPT
- Still no guarantee of optimal variable ordering

- If Bayes net is a **polytree** (replace all directed edges with undirected edges), elimination can be *linear* if we eliminate *leaves first, then root*

Summary

- Inference in Bayesian networks: Computing distributions over query variables given evidence variables (and marginalizing hidden variables)

- Inference by enumeration: Compute full joint distribution of all relevant variables using chain rule, then marginalize hidden variables

- Variable elimination: Alternate between building up and summing out CPTs to reduce computational complexity
- Overall still a NP-hard problem