Portfolio Risk Allocation

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In addition to measuring risk at the portfolio level, it is often necessary to understand what the underlying drivers of the observed risk are. For example, in a large complex portfolio it may not be so obvious what the drivers of the portfolio volatility are, or even less obvious what the primary drivers of tail-risk metrics may be. However, once the 'sinks' and 'sources' have been identifed, the risk manager will have obtained superior insight into the portfolio risk dynamics. Such an analysis takes us into the realm of Risk Allocation methods.

<u>Portfolio Structure</u>:

- The real-valued random variable X denotes total portfolio P&L
- The portfolio can be partitioned into mutually exclusive and exhaustive components, e.g. $X = X_1 + X_2 + \cdots + X_m$, where X_i denotes the P&L contribution from the i^{th} sub-portfolio

For example, a portfolio of single-name equities, e.g. common stock, can be partitioned via industries, regions, or credit grades.

<u>Allocation Problem</u>: Given a risk metric $\rho(\cdot)$, e.g. volatility, Value-at-Risk, or Expected Shortfall, we want to decompose or break the total portfolio risk down in an additive manner which has a financially meaningful explanation.

$$\rho(X) = \phi_1(X_1, X_2, \cdots, X_m) + \phi_2(X_1, X_2, \cdots, X_m) + \cdots + \phi_m(X_1, X_2, \cdots, X_m)$$

where $\phi_i \equiv f_i(X_1, X_2, \cdots, X_m)$ represents the risk allocation to the i^{th} sub-portfolio and the details of f_i depend on a given methodology.

While there are multple risk allocation techniques available in the literature, there are three of particular interest we will explore: i) Euler Allocation, ii) Pro-rata, and iii) Shapley Allocation.

However, for any particlar risk metric the applicability of a given allocation method is dependent on its theorectical properties. We will now explore this idea further.

Many familiar risk metrics exhibit desirable properties. However, in order to make these notions rigorous [1] defined the class of Coherent Risk metrics, wherein the following properties are satisfied:

Coherent Risk Metric

1. Monotonicity – $X \leq Y$ a.s. $\Rightarrow \rho(X) \geq \rho(Y)$
2. Sub-additivity – $\rho(X+Y) \leq \rho(X) + \rho(Y)$
3. Positive Homogenity – $\forall a > 0, \;\; \rho(aX) = a\rho(X)$
4. Translation Invariance – $\forall a > 0, \;\; \rho(a+X) = \rho(X) - a$

1 Euler Allocation

Definition 1. *A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be homogenous of order k if $\forall x \in \mathbb{R}^n$ and $a > 0$:*

$$f(ax) = a^k f(x) \tag{1}$$

Theorem 1. *Euler's Theorem for homogeneoous function:* $f \in \mathcal{C}^{(1)}(\mathbb{R}^n)$ *is homogenous of degree* k *if and only if:*

$$kf(x) = x \cdot \nabla f(x) \tag{2}$$

This is a very useful geometric property which essentially says that f is proportional to the directional derivative of its argument everywhere. In light of the Coherent Risk metric property of positive homogeneity there is hope that given a risk metric $\rho(\cdot)$ admits a derivative, in a functional sense, then Euler's Homogeneity Theorem applies and the metric admits an Euler Allocation.

In order to assist the derivations below, let's suppose we can adjust our portfolio by weighting each component via $\omega \in \mathbb{R}^m$. Then the total portfolio P&L is given by:

$$X(\omega) := \sum_{i=1}^{m} \omega_i X_i$$

Then taking a fixed risk metric ρ we can define the following real-valued function

$$f_\rho(\omega_1, \omega_2, \cdots, \omega_m) := \rho(X(\omega))$$

Example 1. ***Portfolio volatility:***

Define $f : \mathbb{R}^m \to \mathbb{R}$ *as* $f_\sigma(\omega_1, \omega_2, \cdots, \omega_m) = \sigma(X(\omega))$*:*

$$\sigma(X(\omega)) = Var\left(\sum_{i=1}^{m} \omega_i X_i\right)^{1/2} = \left(\sum_{i=1}^{m} \omega_i^2 \sigma_i^2 + \sum_{i \neq j} \omega_i \omega_j \sigma_{i,j}\right)^{1/2} = \left(\omega' \Sigma \omega\right)^{1/2},$$

where $\Sigma \in M_{n,n}(\mathbb{R})$ *denotes the covariance matrix for* $(X_1, X_2, \cdots, X_n)$*.*

Given $a > 0$

$$\sigma(aX(\omega)) = \left(a^2 \omega' \Sigma \omega\right)^{1/2} = a\sigma(X(\omega))$$

Hence, the function $f_\sigma(\omega)$ *is positive 1-homogenous. Howeveer, it is also clear that* f_σ *is also differentiable. Thus, Euler's Homogeneity Theorem applies and the gradient takes the following form:*

$$\nabla f_\sigma(\omega) = \frac{\Sigma \omega}{(\omega' \Sigma \omega)^{1/2}}$$

and we see immediately that $\omega' \nabla f_\sigma(\omega) = \left(\omega' \Sigma \omega\right)^{1/2} = \sigma(X(\omega))$*.*

The Euler Allocation for each component of the portfolio takes the form:

$$\phi(\sigma)_i^{Euler} = \frac{\omega_i e_i' \Sigma \omega}{(\omega' \Sigma \omega)^{1/2}}, \qquad i = 1, 2, \cdots, n$$

where $e_i \in \mathbb{R}^n$ *denotes the* i^{th} *standard basis vector, e.g.* $e_i = \underbrace{(0, \cdots, 0, 1, 0, \cdots, 0)}_{\text{0 except for ith position}}$

Example 2. ***Portfolio Value-at-Risk:***

Fixing the confidence level $\alpha \in (0, 1)$ *for the Value-at-Risk, e.g. typically 0.95 or 0.99 in practice, define* $f_{VaR}(\omega) := VaR_\alpha(X\omega))$*.*

It is a well known fact that VaR *is positive 1-homogenous. However, it is not so obvious that* f_{VaR} *is differentiable and thus the Euler Homogeneity Theorem applies. Nonetheless, under general regularity conditions on the distribution of* $(X_1, X_2, \cdots, X_n)$ *it can be shown that:*

$$\frac{\partial f_{VaR}(\omega)}{\partial \omega_i} = \mathbb{E}\left[X_i \middle| X(\omega) = VaR_\alpha(X(\omega))\right]$$

From here it is clear that $\omega^{'} \nabla f_{VaR}(\omega) = VaR_\alpha(\omega)$ and hence the Euler Allocation for each component of the portfolio takes the form:

$$\phi(VaR)_i^{Euler} = \omega_i \mathbb{E}\left[X_i \middle| X(\omega) = VaR_\alpha(X(\omega))\right], \qquad i = 1, 2, \cdots, n$$

The Expected Shortfall, sometimes referred to a Conditional VaR or CVaR, also admits an Euler Allocation.

2 Pro-rata

Another commonly used allocation methodology is the Pro-rata. This approach is very general in that essentially all risk metrics, coherent or not, will admit a Pro-rata allocation.

$$\phi(\rho)_i^{Pro-rata} := \frac{\rho(\omega_i X_i)}{\rho(\omega_1 X_1) + \cdots + \rho(\omega_n X_n)} \cdot \rho(X(\omega)), \qquad i = 1, 2, \cdots, n$$

However, computational the simplicity of this approach comes at the cost of i) interpretabilty, ii) inability to detect diversification, and iii) it can give rise to a free-rider effect.

3 Shapley

References

[1] Philippe Artzner, Freddy Delbaen, Jean-Marc Eber, and David Heath. Coherent measures of risk. *Mathematical Finance*, 9(3):203–228, 1999.