

Value-at-Risk Allocation Problem:

Use the following 3-dimensional multivariate Gaussian framework

$$(X_1, X_2, X_3) =_D N_3(0, \Sigma), \quad [\Sigma]_{ij} = \begin{cases} 1 & i = j \\ \rho & i \neq j \end{cases}, \quad \rho \in (-1/2, 1)$$

Under this framework each random variable can be thought of as a sub-portfolio p&l variable, e.g. Equities, Credit, and Rates, and the total portfolio random variable (representing total portfolio p&l) is given by

$$X := X_1 + X_2 + X_3 =_D N(0, \underline{1}' \Sigma \underline{1}), \quad \text{where } \underline{1} = (1, 1, 1)' \in \mathbb{R}^3$$

Choose a fixed value of ρ for the calculations below and note that if $\rho < 0$ some diversification exists in the portfolio, otherwise the p&l of all sub-portfolios will tend to move together, i.e. no diversification.

1. Calculate the Value-at-Risk(0.99) of X . Hint: This is equivalent to -1 times the first quantile.
2. Apply the Euler allocation technique to derive or estimate the allocations for X_1, X_2 and X_3 .
3. Using your knowledge of the Shapley allocation technique, simulate p&l vectors under the joint distribution, e.g. using `numpy.random.multivariate_normal` and then calculate the Shapley allocations for X_1, X_2 and X_3 using a sample quantile estimator, e.g. `scipy.stats.quantile`
4. Calculate the Pro-rata for each sub-portfolio.

Bonus: How do the allocations for X_1, X_2 and X_3 under each method vary as a function of ρ ? Perhaps generate a nice plot to illustrate these concepts.