LECTURE #9

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EXAMPLE: OPINION FORMATION

Social Networks

- Opinion formation in social networks.
- A group of interacting individuals in a social network that may <u>support</u> or <u>dismiss</u> an established theory.
- Individuals endowed with two separate opinions: a <u>publicly</u> pronounced and a <u>privately</u> held opinion for/against the established theory.

Social Networks

- Consider a fixed homogeneous group of $2L$ individuals who react in the same manner to a given situation.
- An individual holds simultaneously a public and a private opinion that each takes values ½ (for the established theory) and -½ (against the established theory).
- X_1 is the net public opinion (sum of the publicly held opinions of all individuals).
- X_2 is the net private opinion (sum of the privately held opinions of all individuals).

Social Networks

- We are dealing with 2 species interacting with 4 reactions:

$$X_1 + X_2 \to 2X_1 + X_2$$
$$X_1 + X_2 \to X_2$$
$$X_1 + X_2 \to X_1 + 2X_2$$
$$X_1 + X_2 \to X_1$$

Social Networks

- Reactions

$$X_1 + X_2 \rightarrow 2X_1 + X_2$$
$$X_1 + X_2 \rightarrow X_2$$

model the influence of net private opinion X_2 on the net public opinion X_1 that results in a single individual changing their public opinion in support of (first reaction) or against (second reaction) the established theory.

- In this case, the net private opinion remains unchanged, whereas, the net public opinion is increased by one in the first reaction [due to a value change from -½ (against) to ½ (for)] and decreased by one in reaction 2 [due to a value change from ½ (for) to -½ (against)].

Social Networks

- Reactions

$$X_1 + X_2 \rightarrow X_1 + 2X_2$$
$$X_1 + X_2 \rightarrow X_1$$

model the influence of net public opinion X_1 on the net private opinion X_2 that results in a single individual changing their private opinion in support of (first reaction) or against (second reaction) the established theory.

Social Networks

- The following propensity functions have been suggested:

$$X_1 + X_2 \to 2X_1 + X_2$$
$$X_1 + X_2 \to X_2$$
$$X_1 + X_2 \to X_1 + 2X_2$$
$$X_1 + X_2 \to X_1$$

$$\pi_1(\mathbf{x}) = \kappa_1(L - x_1)\exp(a_1 x_1 + a_2 x_2)$$
$$\pi_2(\mathbf{x}) = \kappa_1(L + x_1)\exp(-a_1 x_1 - a_2 x_2)$$
$$\pi_3(\mathbf{x}) = \kappa_2(L - x_2)\exp(a_3 x_1)$$
$$\pi_4(\mathbf{x}) = \kappa_2(L + x_2)\exp(-a_3 x_1)$$

Social Networks

$$X_1 + X_2 \to 2X_1 + X_2 \qquad \pi_1(\mathbf{x}) = \kappa_1 (L - x_1) \exp(a_1 x_1 + a_2 x_2)$$
$$X_1 + X_2 \to X_2 \qquad \pi_2(\mathbf{x}) = \kappa_1 (L + x_1) \exp(-a_1 x_1 - a_2 x_2)$$
$$X_1 + X_2 \to X_1 + 2X_2 \qquad \pi_3(\mathbf{x}) = \kappa_2 (L - x_2) \exp(a_3 x_1)$$
$$X_1 + X_2 \to X_1 \qquad \pi_4(\mathbf{x}) = \kappa_2 (L + x_2) \exp(-a_3 x_1)$$

- x_1, x_2 represent the net values of all publicly and privately held opinions.
- $-L \le x_1, x_2 \le L$, where $-L$ and L represent total disapproval and total approval of the established theory.

Social Networks

$$X_1 + X_2 \to 2X_1 + X_2$$
$$X_1 + X_2 \to X_2$$
$$X_1 + X_2 \to X_1 + 2X_2$$
$$X_1 + X_2 \to X_1$$

$$\pi_1(\mathbf{x}) = \kappa_1(L - x_1)\exp(a_1 x_1 + a_2 x_2)$$
$$\pi_2(\mathbf{x}) = \kappa_1(L + x_1)\exp(-a_1 x_1 - a_2 x_2)$$
$$\pi_3(\mathbf{x}) = \kappa_2(L - x_2)\exp(a_3 x_1)$$
$$\pi_4(\mathbf{x}) = \kappa_2(L + x_2)\exp(-a_3 x_1)$$

- $a_1 \geq 0$ controls pressure inflicted on public opinion.
- $a_2 \geq 0$ controls the influence of privately held beliefs on publicly stated opinions.
- a_3 controls the reinforcement (for $a_3 > 0$) or the weakening (for $a_3 < 0$) effect that the net public opinion has on the net private opinion in support of the established theory.

Example: Opinion Formation

- The simplicity of this model permits us to solve the underlying ME using a numerical approach.

- The complexity introduced by the nonlinear nature of the propensity functions allows us to illustrate some intricate behavior.

Example: Opinion Formation

- We consider two parameterizations of the model corresponding to a liberal case and a non-liberal case.

$L = 40$ (80 individuals)

$\kappa_1 = 1/2 \text{ day}^{-1}\text{individual}^{-1}$

$\kappa_2 = 1 \text{ day}^{-1}\text{individual}^{-1}$

$\alpha_1 = 0 \; (3/80) \text{ individual}^{-1}$ (pressure inflicted on public opinion)

$\alpha_2 = 1/80 \; (1/40) \text{ individual}^{-1}$ (influence of privately held beliefs on publicly stated opinions)

$\alpha_3 = 1/80 \; (-1/320) \text{ individual}^{-1}$ [the net public opinion has a reinforcing (weakening) effect on the net private opinion]

$X_1(0) = X_2(0) = 0$ (completely neutral initial net publicly and privately held opinions)

Numerical Solution (KSA Method)

- We can numerically solve the ME by employing the KSA method.

- This method is more appropriate than the IE method, since the DAs of the underlying reactions can grow rapidly, whereas the populations $X_1(t)$ (net public opinion) and $X_2(t)$ (net private opinion) are <u>bounded</u>, taking values between -40 and 40.

Numerical Solution (KSA Method)

LIBERAL CASE

see video-8-1.mov

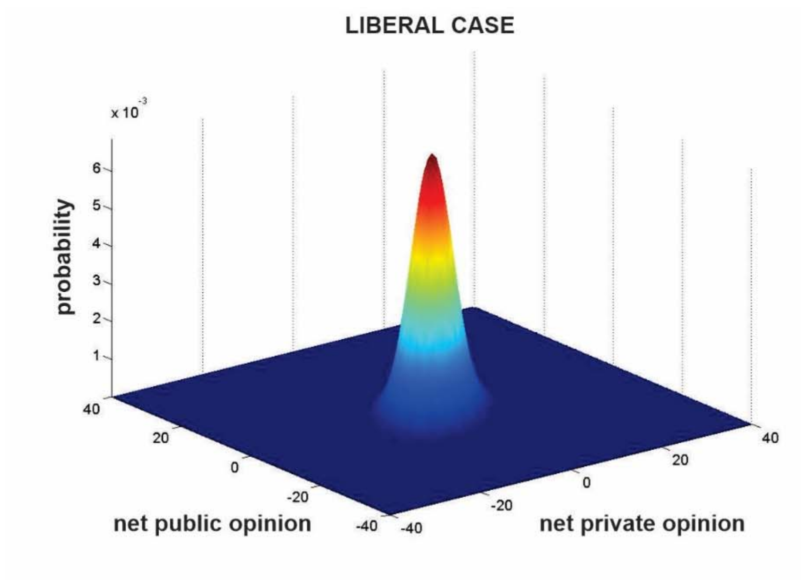
Numerical Solution (KSA Method)

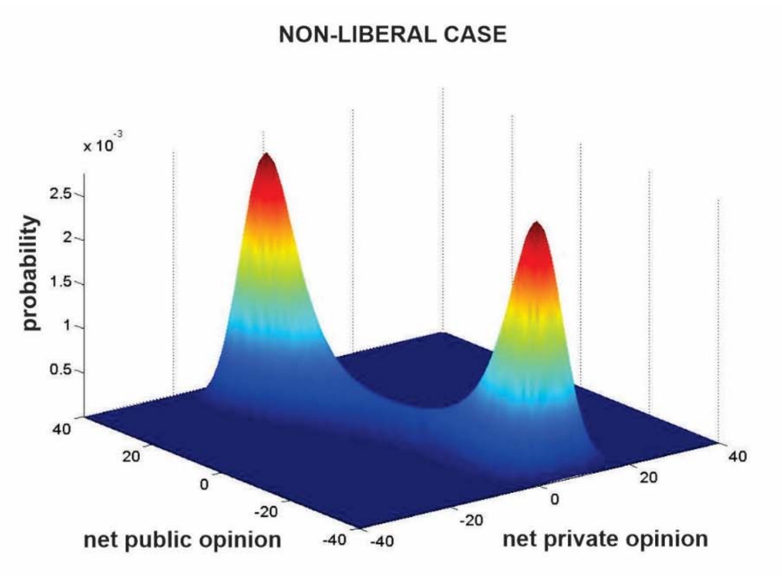
NON-LIBERAL CASE

see video-8-2.mov

Numerical Solution (KSA Method)

The joint probability distributions of the net public and private opinions in the <u>liberal</u> and <u>non-liberal</u> cases at steady-state computed by the KSA method.

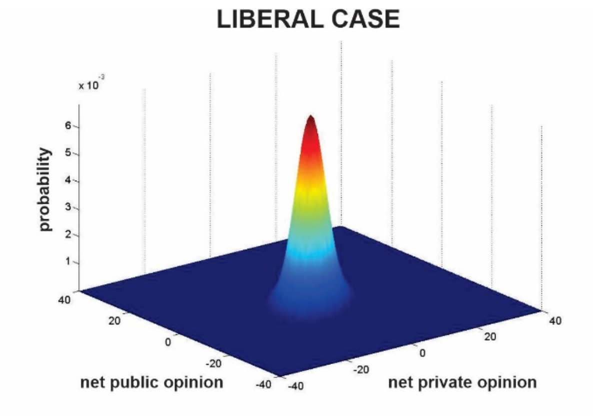


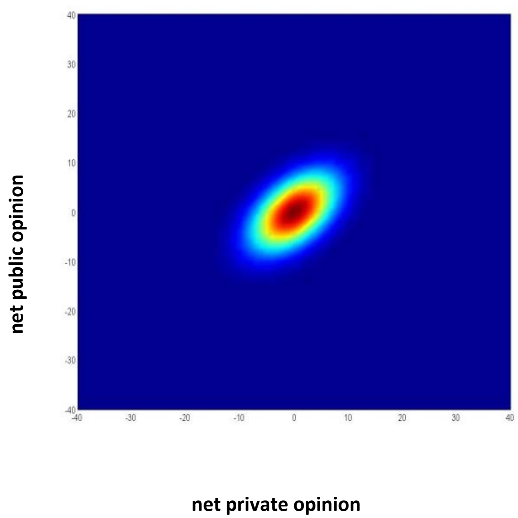


UNIMODAL stationary distribution with the mode located at the point of <u>zero</u> net public and private opinions

BIMODAL stationary distribution with the modes corresponding to two different states: one in which a large number of individuals publicly support the established theory, while a small number of individuals are privately against this theory, and one in which a large number of individuals are publicly opposing the established theory, while a small number of individuals privately support it.

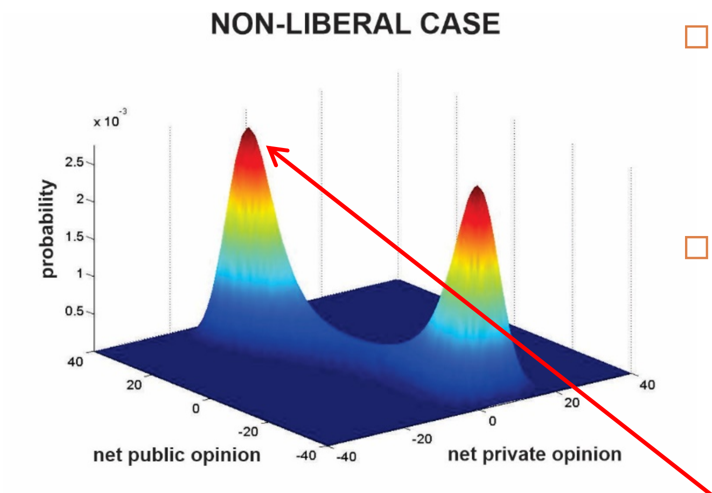
Numerical Solution (KSA Method)





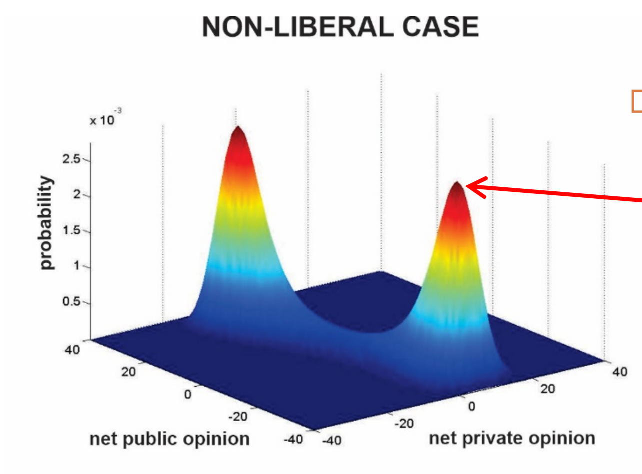
- In the <u>liberal</u> case, the stationary joint probability distribution of public and private opinions is <u>unimodal</u> and almost identical to a <u>sampled normal</u> distribution.
- This distribution characterizes the fact that there is no need for an individual to hide their private opinion (thus the correlation between the public and private opinions).
- Consequently, the net opinions nearly balance around the origin, which represents an equal number of privately held and publicly pronounced opinions for or against the established theory.

Numerical Solution (KSA Method)



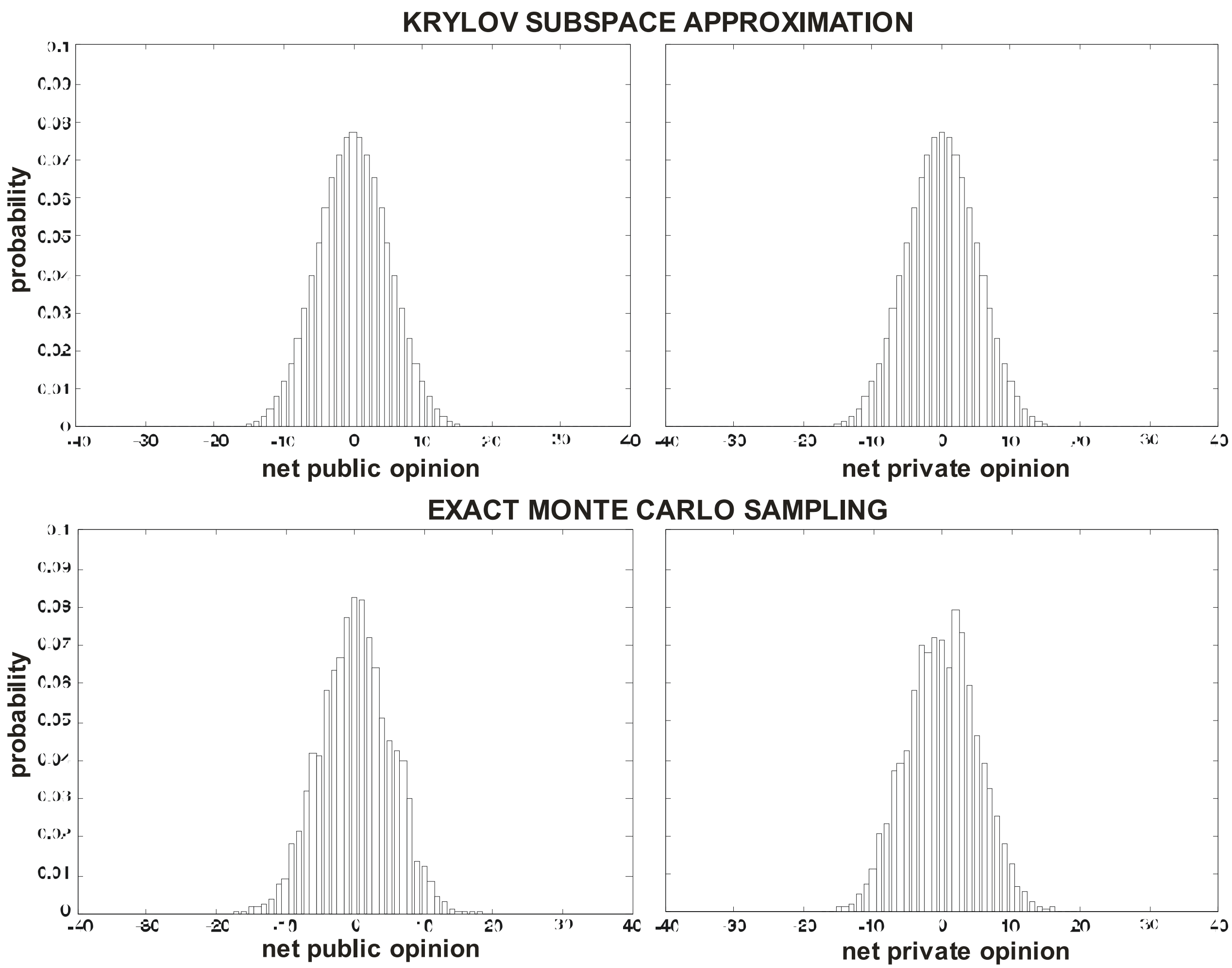
- In the <u>non-liberal</u> case, the stationary joint probability distribution of public and private opinions is <u>bimodal</u>.
- Here, weakly dissident individuals tend to privately disapprove the established idea, but pressure on public opinion ensures that most individuals publicly approve this idea.
- Consequently, dissidence is not strong enough to destabilize the established theory and the network will operate around the peak located at point $(30, -6)$ with strong net public opinion in support of the established theory and a rather weak private opinion against this theory.

Numerical Solution (KSA Method)



- In the non-liberal case, the population of individuals may move to another peak located at point $(-30,6)$ with strong net public opinion against the established theory and a rather weak private opinion in support of this theory.

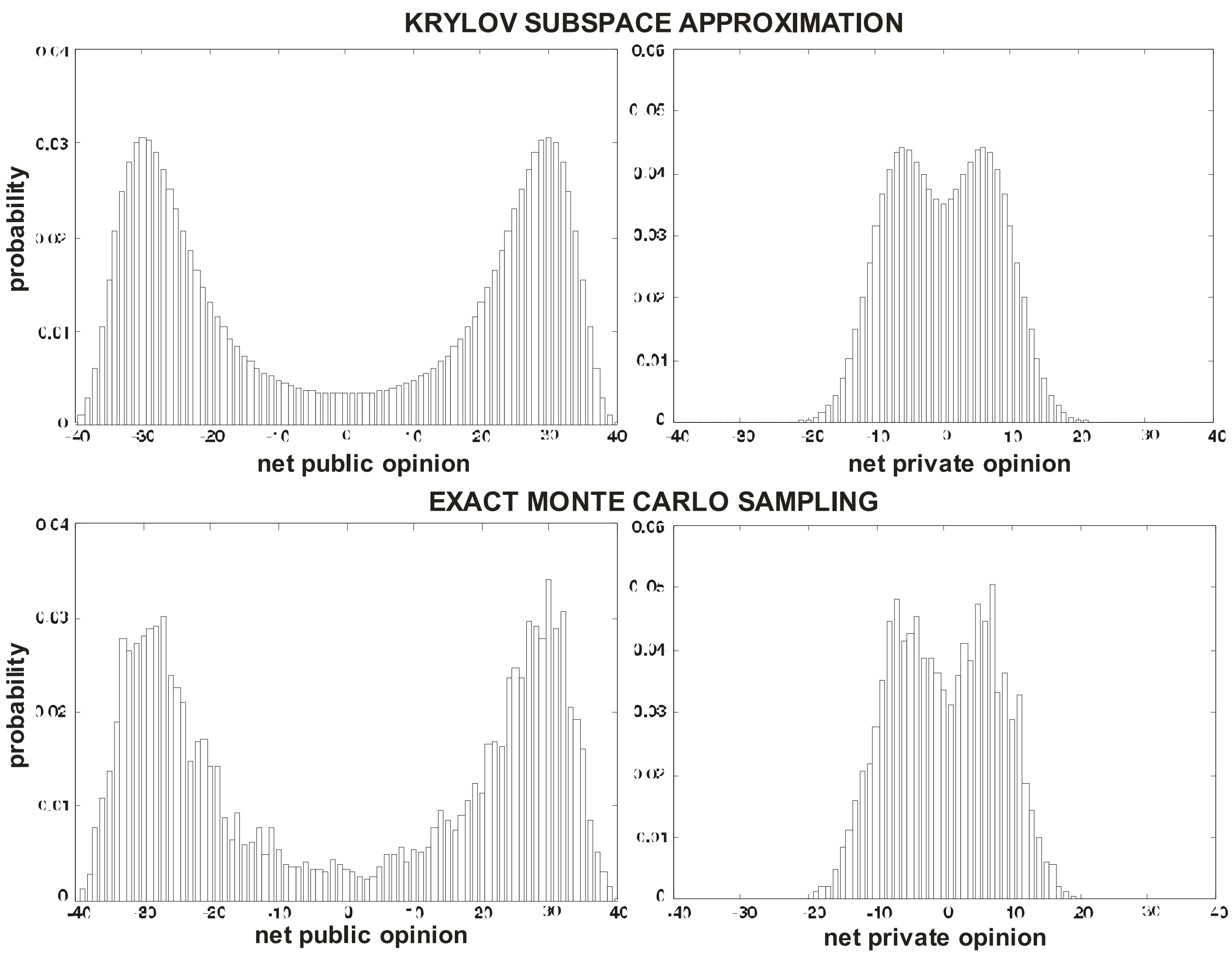
Monte Carlo Sampling



The stationary marginal probability distributions of the net public and private opinions in the <u>liberal</u> case.

4000 trajectories
230 sec CPU time (exact)
30 sec CPU time (GL)
75 sec CPU time (PL)

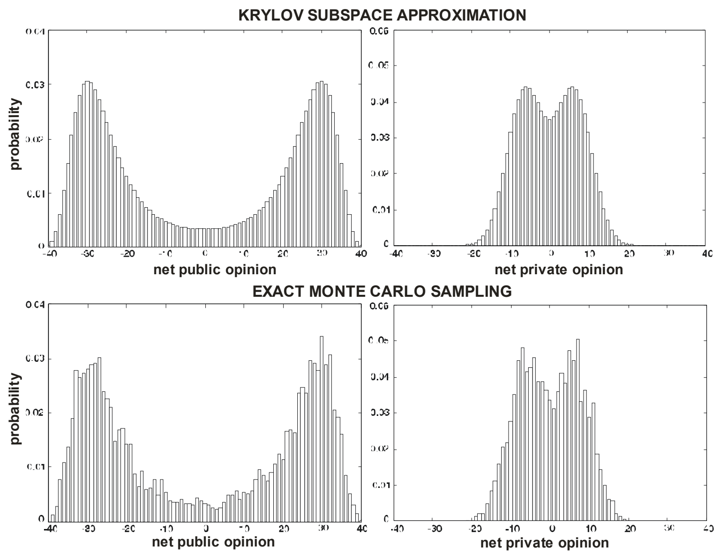
Monte Carlo Sampling



The stationary marginal probability distributions of the net public and private opinions in the <u>non-liberal</u> case.

4000 trajectories
230 sec CPU time (exact)
30 sec CPU time (GL)
75 sec CPU time (PL)

Monte Carlo Sampling – Remarks

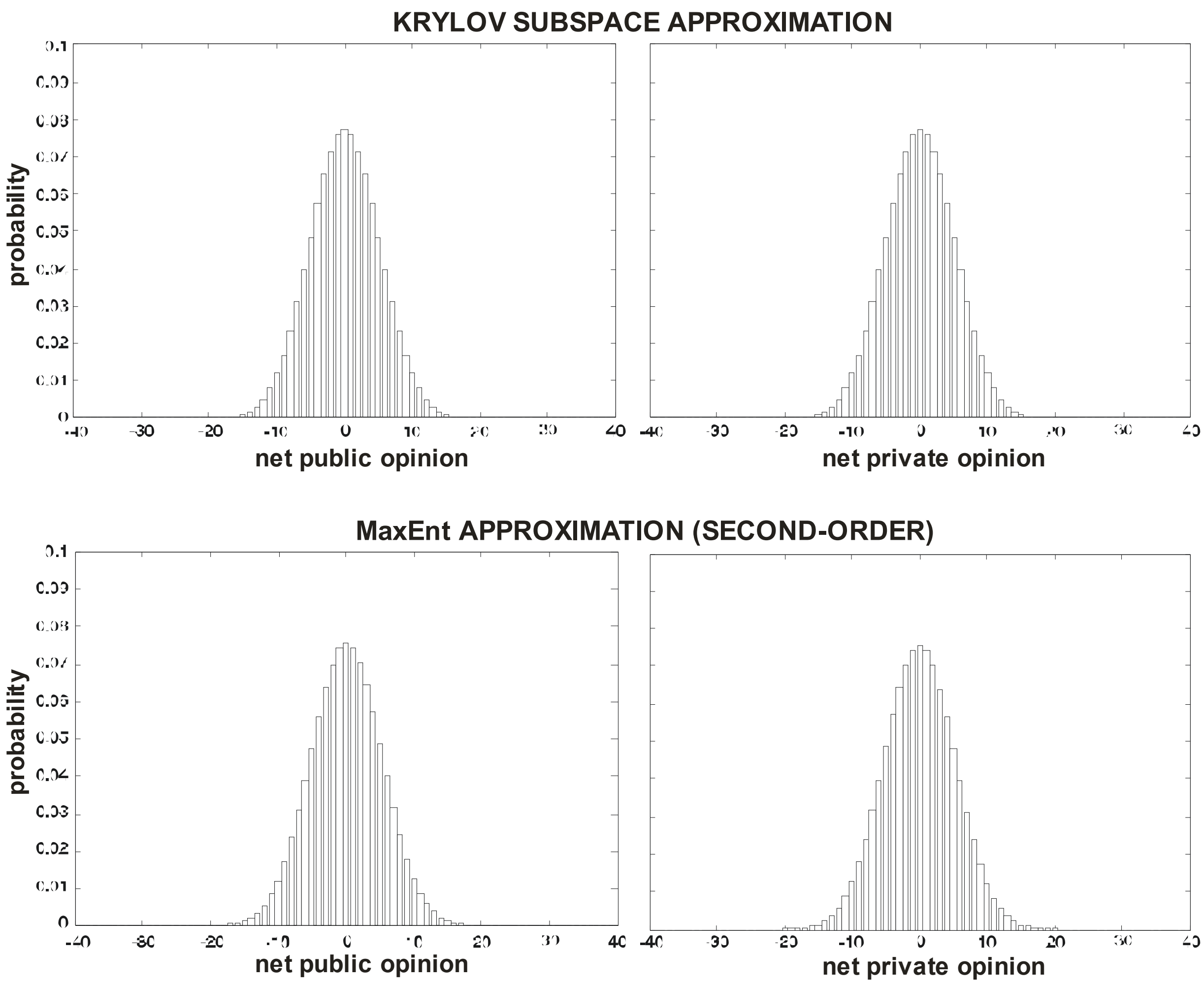


- Monte Carlo sampling is a simple and robust procedure for solving the ME which, in principle, allows us to compute almost any statistical summary of interest with arbitrary precision.
- This simple and elegant procedure comes with a large computational cost, since the number of trajectories required to obtain a sufficiently accurate estimate is usually very large.
- Even for estimating the univariate marginal probability distributions of the net public and private opinions, 4000 trajectories do not seem to be sufficient, since the symmetric and bimodal nature of these distributions can be obscured by estimation errors.
- It is often very difficult to know a priori how many samples must be used to sufficiently estimate the qualitative and quantitative properties of a statistical summary of interest or to verify a posteriori when convergence has occurred.
- The Monte Carlo method scales far better than the KSA method, since trajectories can be sampled from the ME with a relative ease, even in the case of large Markovian reaction networks for which numerical methods, such as the KSA or IE methods, cannot be used.
- For large networks, the number of trajectories that can be sampled from the ME in a reasonable time will certainly not be adequate for accurately computing complex population statistics, such as probability distributions, but they may be sufficient for estimating certain moments (e.g., the means and covariances) of the DA and population processes with a desired precision.

Monte Carlo/MaxEnt Approximation

- We can use Monte Carlo sampling to approximate only moments of the net public and private opinions, and not the marginal distributions themselves, and then employ MaxEnt to approximate these probability distributions.
- The main advantage of this approach is that it can approximate marginal probability distributions better than when employing only Monte Carlo sampling using the same number of sample trajectories.

Monte Carlo/MaxEnt Approximation



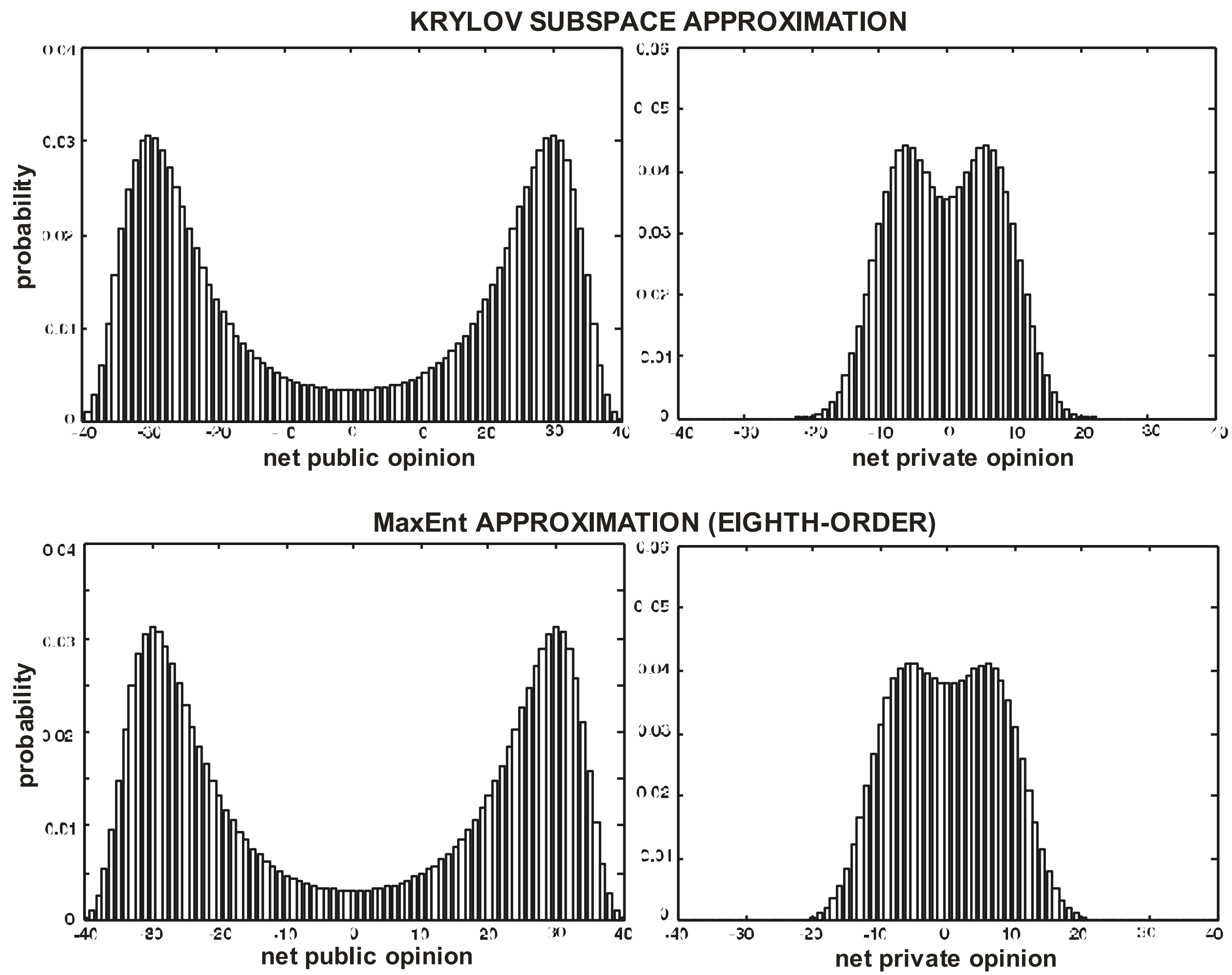
The stationary marginal probability distributions of the net public and private opinions in the <u>liberal</u> case.

4000 trajectories (PL)
Estimation of means and variances

75 sec CPU time

compare these results with the ones on slide 19!

Monte Carlo/MaxEnt Approximation



The stationary marginal probability distributions of the net public and private opinions in the <u>non-liberal</u> case.

4000 trajectories (PL)
Estimation of first <u>eight</u> moments

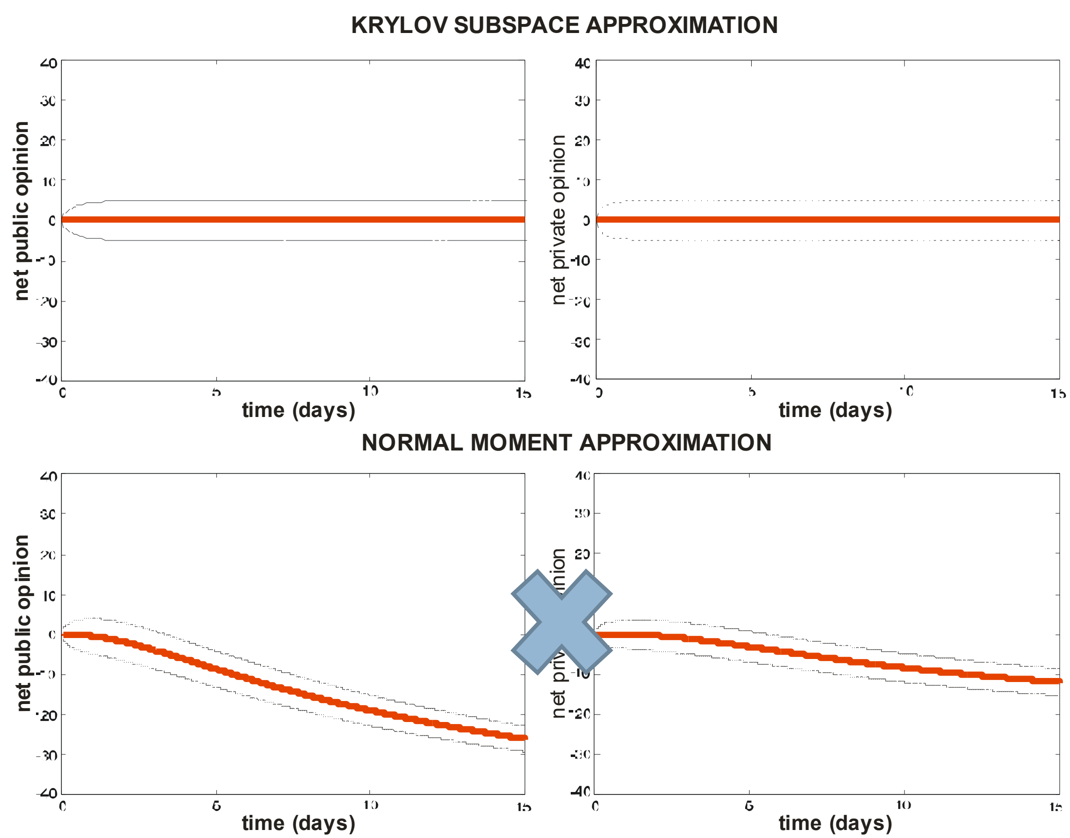
75 sec CPU time

compare these results with the ones on slide 20!

MA/MaxEnt Approximation

- Instead of Monte Carlo, you may use the MA method with an appropriate closure scheme in order to estimate moments required by MaxEnt.
- However, you must exercise a lot of <u>caution</u> when you do so.
- Use of the MA method is much easier in the liberal case than in the non-liberal case.
- The non-liberal case requires <u>at least eighth-order moments</u> to sufficiently characterize its bimodal stationary distribution.
- Due to the exponential nature of the propensity functions, their effect on the moment equations can persist through infinitely many derivatives, which can make the task of finding an appropriate moment closure scheme very difficult.
- Since the stationary joint probability distributions of the net public and private opinions in the liberal case approximate well a sampled Gaussian distribution, we may be able to approximate the means and covariances by using the <u>normal</u> MA scheme.

MA/MaxEnt Approximation

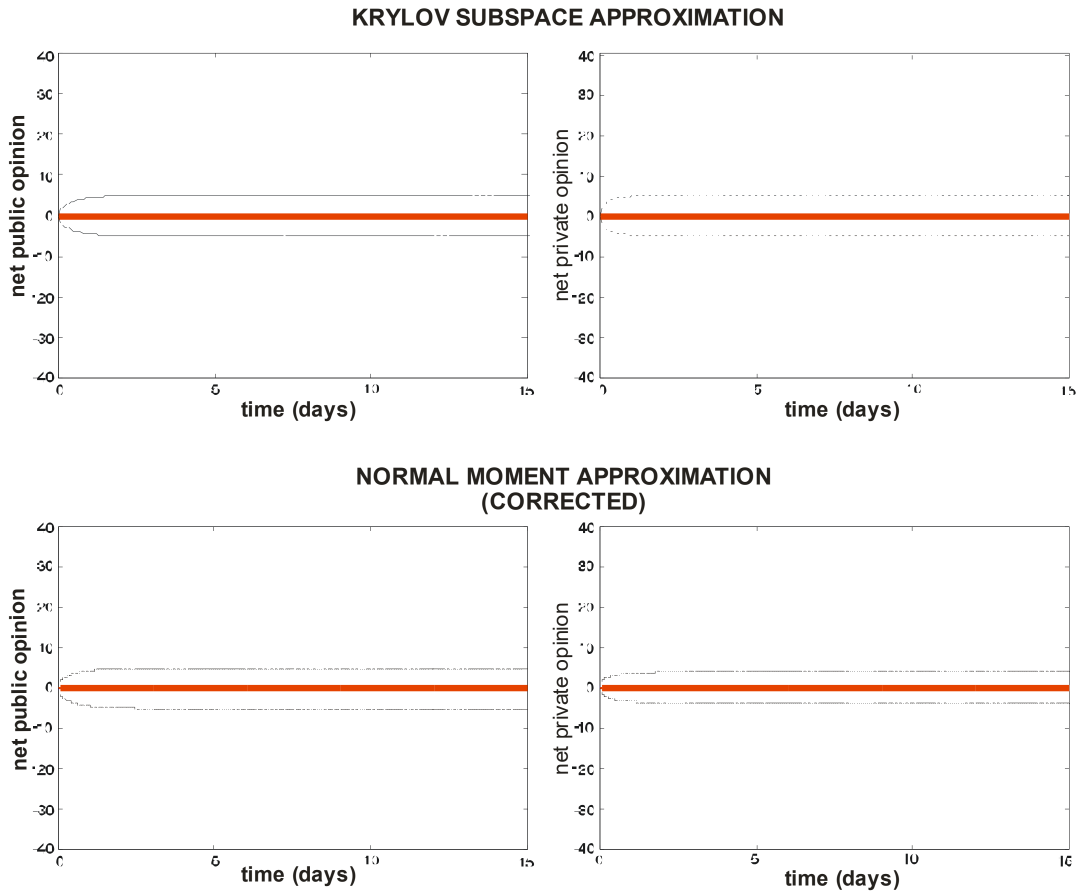


The means (solid red lines) and the ± 1 standard deviations (dashed black lines) of the net public and private opinions in the <u>liberal</u> case.

0.6 sec CPU time

The main culprit here is the fact that the normal MA scheme is derived for <u>quadratic</u> propensity functions whose higher-order derivatives vanish, which along with the Gaussian assumption results in a decoupling of the means and covariances from higher-order central moments.

MA/MaxEnt Approximation



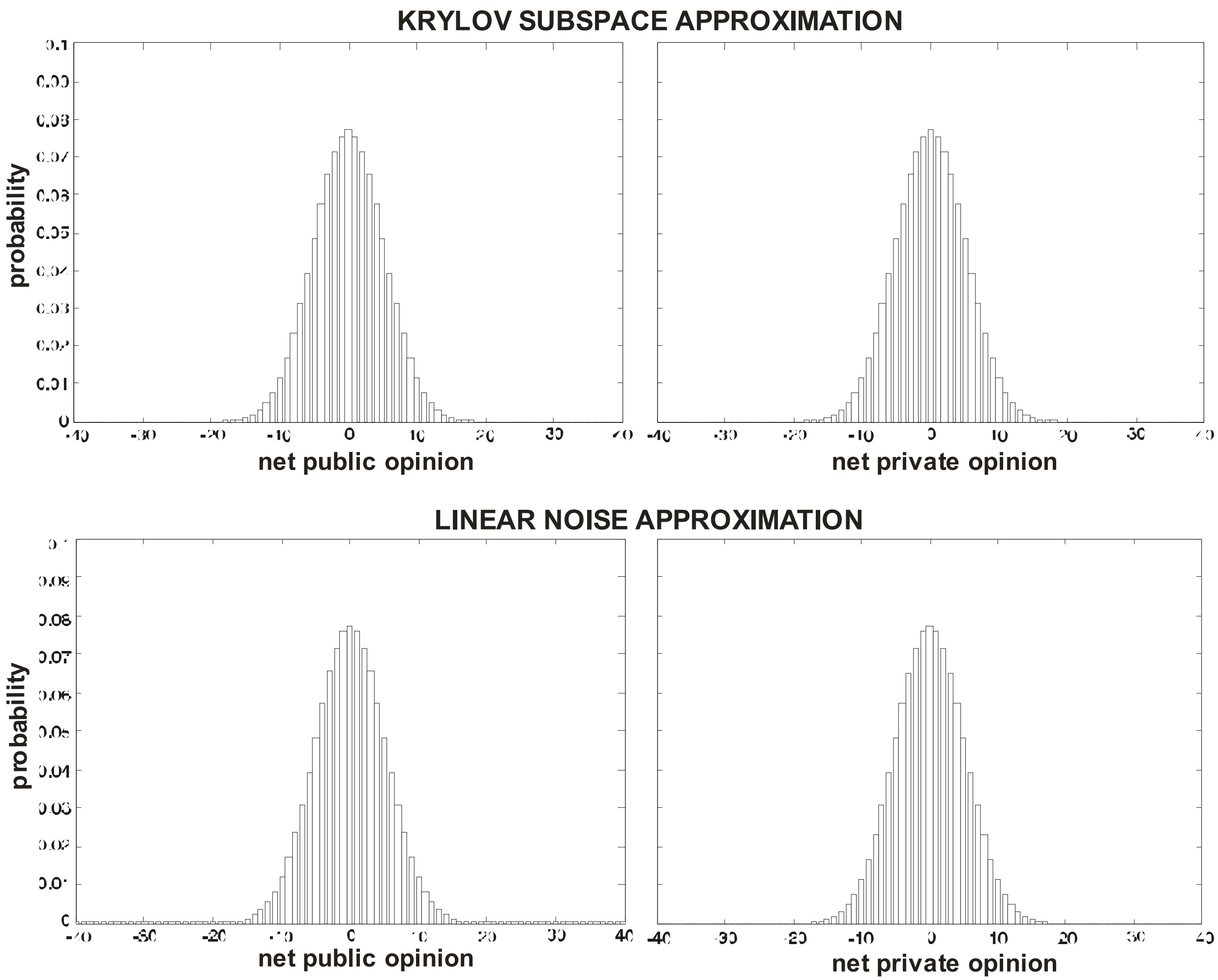
In the liberal case, the Gaussian assumption is approximately valid, but errors accumulate, since the underlying equations neglect to account for non-vanishing higher-order derivatives of the propensity functions.

We can however improve the closure scheme using Jensen's inequality, due to the convexity of the propensity functions.

The depicted results clearly demonstrate the effectiveness of this correction.

$$\frac{d\boldsymbol{\mu}_Z(t)}{dt} = \alpha_m(\boldsymbol{\mu}_Z(t)) + T_m(\boldsymbol{\mu}_Z(t)) \quad \rightarrow \quad \frac{d\boldsymbol{\mu}_Z(t)}{dt} = \alpha_m(\boldsymbol{\mu}_Z(t)) + \max\{0, T_m(\boldsymbol{\mu}_Z(t))\}$$

Linear Noise Approximation



The stationary marginal probability distributions of the net public and private opinions in the <u>liberal</u> case.

$\Omega = L$ ("size" of the net public or private opinions)

0.35 sec CPU time

Linear Noise Approximation – Remarks

- Application of the LNA method for solving the ME associated with the non-liberal case is not possible due to the <u>bimodal</u> nature of the stationary joint probability distribution.
- If there were more individuals in the social network (i.e., for larger values of Ω , then the LNA method could produce a more accurate result.
- On the other hand, a significantly smaller number of individuals may dramatically reduce the accuracy of the method, since the statistical properties of the network may appreciably deviate from normality.
- Despite its clear computational advantage, use of the LNA method is hampered by the absence of a strategy to effectively determine for which values of Ω the resulting normal approximation is accurate.