Classical computation on a quantum computer
(Nielsen and Chuang 3.2.5)

A quantum computer can efficiently simulate a classical computer, by compiling everything down to NAND gates

quantum computer $\longrightarrow$ reversible classical computer $\longrightarrow$ general classical computer

This seems to imply that we can run a classical subroutine as part of a quantum computation

But it's not quite as easy as this

Consider only reversible circuits (since unitary circuits are of this form)

Consider a circuit that maps an input z to an output f(z), and also ouputs z to ensure reversibility

In general it will also output 'garbage' g(z)

$$(z, 0) \longrightarrow (z, f(z), g(z))$$

If the input was not included in the output, the garbage must be required for unitarity. But this is not the case here

So g(z) is just some undeleted remnants of the computation

For a classical computation, this garbage can be ignored or deleted

For a quantum computation, we have superpositions to worry about

$$U|z\rangle = |f(z)\rangle \quad \Rightarrow \quad U\sum_z c_z |z\rangle = \sum_z c_z |f(z)\rangle$$

$$U' |z\rangle \otimes |0\rangle^{\otimes l} = |f(z)\rangle \otimes |g(z)\rangle \Rightarrow U' \sum_z c_z |z\rangle \otimes |0\rangle^{l} = \sum_z c_z |f(z)\rangle \otimes |g(z)\rangle$$

The operations U and U' are fundamentally different. U' entangles the computation to garbage in an ancilla, which could mess up required interference effects

So, can a function f(z) that can be computed efficiently with a classical computer also be computed efficiently with a reversible classical computer that produces no garbage?

Yes! By means of 'uncomputation'

$(z, 0, 0, 0)$ We use 4 registers, one with input, rest set to 0

$\rightarrow (z, f(z), g(z), 0)$ Then do the computation, including garbage output

Then copy f(x) to the fourth register $\rightarrow (z, f(z), g(z), f(z))$

$\rightarrow (z, 0, 0, f(z))$

Finally invert the computation (but not the copy) to reset the second and third registers

Same complexity, but no garbage

Note: we assumed above that the additional registers were initialized in the zero state

$$(z, 0, 0, 0) \rightarrow (z, 0, 0, f(z))$$

This need not be true in general, they can be initialized in any state

$$(z, a, b, c) \rightarrow (z, 0, 0, f(z))$$

This allows us to have nontrivial a, b and c if we want (pre-stored constants, mathematical convenience)

Coming back to the quantum realm (and ignoring any systems that start and finish in the same state) an efficient classical computation

$$z \rightarrow (f(z), g(z)) \quad \text{[garbage present in general]}$$

Implies an efficient quantum computation

$$|z, 0\rangle \rightarrow |z, f(z)\rangle \quad \text{[no garbage]}$$