THE UNIVERSITY *of* EDINBURGH
School of Physics
and Astronomy

Mechanics, Geometry and the Momentum Map

MPhys Project Report

David Sola Gil

Submitted for the 40pt MPhys Project course PHYS11016
December 2, 2021

Abstract

Mechanics is at the heart of our physical understanding of the universe. Nonetheless, given a mechanics problem, chances are it will be unsolvable. What are the conditions for solvability? If solvable, how can we find a solution? In the framework of Hamiltonian dynamics, conserved quantities are an essential piece of the solvability puzzle. To study how these arise, we develop the geometrical theory underlying Hamilton's canonical equations: Symplectic Geometry. Within this approach, the second question will be answered by a mathematical construction: the Momentum Map. Symmetries yield conserved quantities, and independently conserved quantities are the key to solve the problem.

Supervisor: Professor Harry W. Braden

Personal statement

As you will observe in the report, there is a large theoretical background upholding the ideas for integrability and Symplectic Geometry. Since both Hamiltonian dynamics and Differentiable Manifolds are courses that I had not taken at the beginning of this academic year, I spent most of the time during the first semester familiarising myself with the background material, whilst developing both a mathematical and physical intuition on it. During the first 8 weeks, I would read chapters from mainly two books: [1] and [2], carry out some computations, and discuss the general ideas in the weekly meetings. These discussions turned out to be very enlightening and pushed me away from subtle misconceptions. The field of differential geometry was almost entirely novel to me. Performing calculations while taking the Differentiable Manifolds course certainly allowed me to acquire a deeper understanding of the crucial concepts.

However, as I approached the definition of momentum map, the descriptions in [1] of all the constructions related to it were too abstract for me to grasp. To this day, I still believe that differential geometry is a topic that takes time to develop intuition for. Thus, following a recommendation from my supervisor, I focused on reading [3] instead, a more elementary book that provides multiple examples and has an extensive discussion of flows and Lie Derivatives, required to understand momentum maps. Parallelly, I checked the more intuitive topic of Hamiltonian dynamics reading Roger Horsley's notes, and did some research about the Laplace-Runge-Lenz vector. Its computation was a crucial step towards understanding the conditions on conserved quantities required to solve the problem.

By the last weeks of November, I had managed to understand the basic notions behind Differentiable Manifolds and Symplectic Geometry. During this month, I also learnt the basics of Mathematica coding language with the aim of developing the notebook in the appendix D. Methods to compute vector fields, Poisson brackets and to check for the independence of conserved quantities were coded in a notebook to ease the calculations in the project.

Thanks to the previous work, I was then able to go back to the momentum map definition and to start to comprehend its usefulness and generality. Meetings during these weeks were crucial to start linking the first two parts of the project: mechanics and geometry. The next step was to compute the momentum map of several actions, which is what I dedicated most of the Christmas break to. The multiple identifications when working with Lie groups and the notion of a cotangent lift were not intuitive to me. I spent a couple of weeks doing some literature review, comparing papers where similar computations were carried out, and extracting the intuition on how to link symmetries to conserved quantities.

In the last week of January, after several talks with my supervisor, ideas began to fall into place. It was also a great exercise to try to explain the concepts to my closest relatives and friends during the Christmas break, as that forced me to distil the ideas to their essence. Once I had managed to compute the momentum maps for the most basic actions, we moved onto developing the last topic: symplectic reduction.

During the last twenty days of February, I read multiple papers on reduction and applied it to the problem which had undergone all the tests: the Kepler orbit problem. We also briefly discussed some other problems such as Euler's two fixed centres problem, to avoid being misled by the great amount of symmetry Kepler orbit problem has.

The final three weeks of the project were spent tying up all concepts and main ideas I had acquired. My goal was to produce a comprehensive document, with the Kepler orbit problem as the guiding thread, and that explores and marries the three main topics which give the name to this project. The result is this report.

Acknowledgments

To begin with, I would like to thank my MPhys project supervisor Professor Harry W. Braden for the amount of time he has devoted not only to develop an MPhys project, but also to make sure that I am learning as much as I can in the process. He was the first Personal Tutor I had, and his passion and high regard for mathematical physics have had a huge influence on me. He has not only taught me, but nudged me into wanting to learn by myself, by always challenging my engraved conceptions and mathematical thoughts.

I would also like to thank my friend and colleague Mike Chatzittofi, for the illuminating discussions on some of the topics of the project. I take great pride in saying that I have shared my undergraduate years at the university with him and the rest of the Cypriots.

Finally, I would like to thank both of my parents and my brother. During the self-isolation times we are going through due to COVID outbreak we have managed to support and look after each other. Without them listening to my, as described by them: "weird and obscure waffles", it would have been completely impossible for me to arrive at most of the crucial insights.

Contents

1 Introduction

Mechanics is one of the richest branches of human knowledge and it is intrinsic to our existence. Even though it was investigated unconsciously, the most basic notions of mechanics granted our ancestors a physical advantage over the rest of the animal kingdom, thanks to which our predecessors perpetuated and developed. The expansion and conquest of the world as species, allowed us to stop worrying about our survival and start wondering instead. Back in the Ancient Greece, came about the concern for the bright dots populating the night skies and this inquisitiveness led to the discovery of the first formal laws of dynamics [4].

Ever since, we have been thinking about the changes in nature, trying to predict how everything that surrounds us evolves with time and what motivates such transformations. Kepler, Galileo and Huygens are some of the people that greatly contributed to the development of mechanics, yet the first mayor revolution of physics is rightfully attributed to Isaac Newton, with his fundamental laws of dynamics. Together with Leibniz, they developed the mathematical language of change: Calculus. Despite the posterior revolutions of Lagrange and Hamilton, to this day the starting point of physics is rather similar: we establish a set of differential equations and via integration methods, we try to find their solution to understand the evolution of the system.

Given an arbitrary system, chances are that we cannot find an analytical solution to the problem. We can write down the equations that establish the conditions, nonetheless most of the times we cannot integrate. This impossibility of finding a solution has put some brightest minds to work, to try to determine which systems we can solve and what are the conditions for a system to be integrable. As an example, both Euler and Langrange worked on the orbit of a planet around two fixed Suns, known as the Euler problem in the astronomical literature, and they both determined that the system was could be solved via integration. However, despite having each other in high regard, they disputed their colleague's approach [5]. To this day, we still lack a universal definition of "integrability", as the determining whether a problem can be solved via integration or not is rather non-trivial [6].

Parallel to the puzzle on integrability and the evolution of dynamics, the notion of conserved quantities was also developed. The first theory of conservation was developed in the fourteenth century under the name of "theory of impetus" [4], by the French philosopher Buridan. After the first stone had been laid, several physical quantities were found to be conserved, with the clear protagonist being the energy. Via experimentation, we could verify that momenta and energy were conserved, but it took us five centuries to precisely determine the origin of conserved quantities and what these share in common. It was in 1918, when Émily Noether published the articles that provoked another major revolution of Physics. In these, she established that *conserved quantities* come from what she called a symmetry of the system. Energy comes from the symmetry in time of the system, linear momentum from a translational symmetry and angular momentum from a rotational symmetry [7].

These two seemingly unrelated topics: symmetries and integrability, have a deep connection when considering the Hamiltonian dynamics framework. In this report, our aim is to make this relation explicit.

We will begin with all the mathematical background required for the project. After explaining some terminology and conventions, we will introduce the topic of differential geometry, required to study dynamical system from a coordinate-free perspective, and the notion of a Lie group, which formalises the notion of symmetry that Noether developed.

Once we have covered the mathematical background, we will explore and link the two main topics discussed in the introduction: solvability of mechanical problems and symmetry, having the Kepler orbit problem as our guiding thread. Firstly, we will determine conditions to be able to integrate a system in the Hamiltonian approach, comprehend what the conditions are and when these are satisfied. Then, we will explore the mathematical constructions required to study the structure underlying Hamiltonian dynamics: Symplectic Geometry. Within this framework, we will introduce a powerful mathematical tool called the *momentum map*, which will bond symmetries to the solvability of any problem in mechanics. Finally, we will compute some examples of momentum maps, and briefly explain how this construction provides us with a method to reduce the dimensions of the dynamical system of interest, known as *symplectic reduction*.

2 Background

The main aim of this section is to set up notation and to establish some unseen key concepts in a Physics undergraduate degree. We will begin by reviewing some general mathematical definitions and conventions employed throughout the report. Then, we will introduce the differential geometry definitions and tools employed in this project, whilst providing an intuition and an interpretation on these. Finally, we will establish the notion of the action of a Lie group and review some examples which will later arise as symmetries of specific systems.

2.1 Mathematical concepts and conventions

2.1.1 Basic definitions

Definition 2.1. *Consider a function $f : U \to \mathbb{R}^n$, where $U \subseteq \mathbb{R}^n$ is open subset and $n \in \mathbb{Z}^+$. If f is infinitely differentiable we call f a* ***smooth function****, and write $f \in C^\infty(U)$. Finally, if $U = \mathbb{R}^n$, we write $f \in C^\infty$ [8].*

Definition 2.2. *Let V, W be two vector spaces, and W^* be the dual vector space of W. Given a bilinear map $F : V \times W \to \mathbb{R}$, for any $v \in V$ we can construct the map $F^\flat : V \to W$ defined by $F^\flat(v) := F(v, -)$. Then, the map F is called* ***weakly non-degenerate*** *if $F^\flat$ is injective. This is equivalent to having $F(v, w) = 0$ for all $w \in W \implies v = 0$ [1].*

2.1.2 The Einstein Summation convention

In this report, instead of working with equations in terms of vectors, we will consider equations of their components, thus having mathematical expressions containing quantities with up and down indices. In our **Einstein summation convention**, we drop the Σ symbol for sums and any term with a pair of repeated indices is implicitly summed over [8]. Note that in this report, we do not distinguish between up and down indices when applying this convention. As some examples, consider the expressions

$$\sum_{i=1}^{n} q^i p_i \equiv q^i p_i, \quad \sum_{i,j,k} \epsilon_{ijk} q^j p_k \equiv \epsilon_{ijk} q^j p_k, \quad \text{and} \quad \sum_{i=1}^{n} p_i p_i \equiv p_i p_i.$$

Unless stated otherwise, whenever indices appear in an expression, we are using this convention.

2.1.3 Integration by quadratures

In this section we describe the procedure of **integration by quadratures**, required to understand a key theorem in the report. Let $t \in \mathbb{R}$ and $x, f \in C^\infty$. Consider the differential equation

$$\frac{d^2x}{dt^2} = \frac{\partial f(x(t))}{\partial x}.$$

Multiplying both sides by dx/dt gives

$$\frac{dx}{dt}\frac{d^2x}{dt^2} = \frac{\partial f(x(t))}{\partial x}\frac{dx}{dt} \iff \frac{1}{2}\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2\right] = \frac{d}{dt}(f(x(t))) \iff \left(\frac{dx}{dt}\right)^2 = 2f(x(t)) + \mathcal{C},$$

where $\mathcal{C}$ is a constant coming from integrating with respect to time. Taking the positive or negative root on both sides yields the separable O.D.E.

$$\frac{dx}{dt} = \pm\sqrt{2f(x(t)) + \mathcal{C}},$$

where $\mathcal{C}$ is a real constant. Although we keep both signs here for illustrative purposes, when solving a physical problem we have to be careful about which root we choose, as such choice may lead to inconsistencies. Now consider the real intervals $[x_0, x_f]$ and $[t_0, t_f]$ in which $\sqrt{2f(x) + \mathcal{C}}$ is non-vanishing. Integrating the equation above, we obtain

$$\int_{x_0}^{x_f} \frac{dx}{\pm\sqrt{2f(x) + \mathcal{C}}} = \int_{t_0}^{t_f} dt. \tag{1}$$

This procedure of taking the differential equation we started with and turning it into an equality of integrals is what we call **integration by quadratures**.

2.2 Differential geometry: basic notions and constructions

2.2.1 Definition of manifold and associated spaces

As we have mastered calculus in $\mathbb{R}^n$, the key idea of differentiable geometry is to extend the notion of "differentiability" to more exotic spaces. We do this via the concept of a **manifold**.

For the purposes of this report, an n-dimensional **differentiable manifold** M is a set that in patches, looks like $\mathbb{R}^n$. This means that in an open set containing each of the point of interest $\sigma \in M$, i.e. **locally**, we have the smooth coordinate functions $(x^1, x^2, \dots, x^n)$ which map $\sigma \in M$ to a point in $\mathbb{R}^n$. We know how to integrate and differentiate in $\mathbb{R}^n$, so we break up the exotic spaces into pieces that we can work with. As examples of common manifolds, apart from the trivial case of $\mathbb{R}^n$, we can consider the n-sphere S^n, any open subset of $\mathbb{R}^n$ or of a manifold, or even the spaces where we carry out Special and General Relativity [3].

Once we have chosen a set of coordinates $(x^1, x^2, \dots, x^n)$ for the manifold, we construct the **tangent space** at each $\sigma \in M$, denoted by $T_\sigma M$ at σ. This is an n-dimensional real vector space, where if $v \in T_\sigma M$

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_\sigma, \left.\frac{\partial}{\partial x^2}\right|_\sigma, \dots, \left.\frac{\partial}{\partial x^n}\right|_\sigma \right\} \text{ is a basis for } T_\sigma M \implies v = v^i \left.\frac{\partial}{\partial x^i}\right|_\sigma .$$

We call v a **tangent vector**, with the $v^i \in \mathbb{R}$ being its (real) coefficients. Since the basis vectors are partial derivatives, this space holds all the **directional derivatives**. We can apply v to a function $f \in C^\infty$, giving

$$v(f) = v^i \left.\frac{\partial f}{\partial x^i}\right|_\sigma = v^i \left.\frac{\partial f}{\partial x^i}\right|_\sigma = (v^1, v^2, \dots, v^n) \cdot (\nabla f)_\sigma,$$

which matches the directional derivative of f along $(v^1, v^2, \cdots, v^n)$ and evaluated at σ. Apart from directional derivatives, these can also be regarded as **tangent vectors** to smooth curves on our manifold. If $\epsilon > 0$ and $(-\epsilon, \epsilon) \subset \mathbb{R}$, a **smooth curve** in M passing through σ is a smooth function $c : (-\epsilon, \epsilon) \to M$ with $c(0) = \sigma$. The **velocity** at σ of the curve c is the tangent vector $c'(0)$ defined by

$$c'(0)(f) := \left.\frac{d}{dt}(f \circ c)(t)\right|_{t=0} .$$

In fact, every $X_\sigma \in T_\sigma M$ can be expressed as $c'(0)$ for some curve c through σ [3]. Note that every point in the manifold will yield a different tangent space, as the derivatives are evaluated at distinct points. We can consider, however, the disjoint union of all tangent spaces $TM := \sqcup_{\sigma \in M} T_\sigma M$, which we call the **tangent bundle**.

This $2n$-dimensional manifold has induced coordinates $(x^1, x^2, \dots, x^n, v^1, \dots, v^n)$ where the x^i's are the coordinates of σ and v^i's the components of the tangent vector. In addition to the induced coordinates, it comes with a canonical projection map $\pi : TM \to M$ that for each $v \in T_\sigma M$ gives

$$\pi(v) = \pi\left(v^i \frac{\partial}{\partial x^i}\bigg|_\sigma\right) = \sigma \implies \pi \star (x^1, x^2, \ldots, x^n, v^1, \ldots, v^n) = (x^1, \ldots, x^n),$$

where I used the symbol "$\star$" to reflect that there are omitted coordinate maps.

In the tangent bundle lies the most important construction for the study of dynamics. A **vector field** is a smooth map $X : M \to TM$ that assigns to each point on the manifold a unique vector on the corresponding tangent space. The space of vector fields on a manifold is denoted by $\mathfrak{X}(M)$ and instead of having real v^i components as in tangent vectors, we have X^i smooth functions. Thus, denoting by $x = (x^1, x^2, \ldots, x^n)$ the coordinates of σ, we may explicitly write

$$X(\sigma) =: X^i(x) \frac{\partial}{\partial x^i}\bigg|_\sigma .$$

More often than not the point σ at which we are evaluating the vector field bears no significant meaning. In these cases, we "abuse notation" and do not explicitly state the point of evaluation of the derivatives. Besides, σ can be written as a lower-index, i.e. $X_\sigma \equiv X(\sigma)$. In connection to the projection map, note that $\pi \circ X = \mathrm{id}_M$, as if we consider X at any $\sigma \in M$, we have that $\pi(X_\sigma) = \sigma$. Finally, as remarkable properties, we know that adding up two vector fields will yield another vector field and that a vector field multiplied by a smooth function is a vector field as well. In mathematical terms, we say that $\mathfrak{X}(M)$ is a C^∞**-module** [3].

Dual to each $T_\sigma M$, we have the **cotangent space** $T^*_\sigma M$, containing all the linear maps from $T_\sigma M$ to $\mathbb{R}$. For finite dimensional vector spaces, we know that their dual vector spaces must have the same dimension. Then $T_\sigma M$ is also n-dimensional for each σ and we denote its basis elements by

$$\{(dx^1)_\sigma, (dx^2)_\sigma, \ldots, (dx^n)_\sigma\} \text{ with } (dx^i)_\sigma\left(\frac{\partial}{\partial x^j}\bigg|_\sigma\right) = \delta^i_j, \text{ a } \textbf{dual basis}.$$

Denoting by p_j the real coefficients, the general expression of $\alpha \in T^*_\sigma M$ is given by

$$\alpha = p_j (dx^j)_\sigma \implies \langle \alpha, v \rangle := \alpha(v) = p_i v^j dx^i\left(\frac{\partial}{\partial x^j}\right) = p_i v^i,$$

where the bracket $\langle -, - \rangle$ represents the **canonical dual pairing**. In this case, we are pairing $T^*_\sigma M$ with $T_\sigma M$. As we did before with tangent spaces, we can also assemble all the cotangent spaces by considering $T^*M := \sqcup_{\sigma \in M} T^*_\sigma M$, giving what we call the **cotangent bundle**.

This is a $2n$-dimensional manifold with induced coordinates $(x^1, x^2, \ldots, x^n, p_1, \ldots, p_n)$. In addition, it has a corresponding canonical projection map $\tilde{\pi} : T^*M \to M$ that for any $\alpha \in T^*_\sigma M$ yields

$$\tilde{\pi}(\alpha) = \tilde{\pi}(p_j (dx^j)_\sigma) = \sigma \implies \tilde{\pi} \star (x^1, x^2, \ldots, x^n, p_1, \ldots, p_n) = (x^1, \ldots, x^n).$$

In dual correspondence to the tangent bundle, the cotangent bundle has a built-in crucial kind of smooth map called **one-forms**. A one-form is a map $\alpha : M \to T^*M$ which assigns to each point on the manifold an element from the cotangent space. The space of one-forms on the manifold is denoted by $\Omega^1(M)$, and instead of having p_i coefficients, we again have α_i real smooth functions. This definition yields another canonical pairing between vector fields and one-forms

$$\alpha = \alpha_i dx^i \implies \langle \alpha, X \rangle := \alpha(X) = p_i X^j dx^i \left(\frac{\partial}{\partial x^j} \right) = p_i X^i.$$

Here we abused notation, as the points of evaluation of both dx^i's and ∂_{q^i}'s need to match. To refer to the one-form evaluated at a certain point σ, we write α_σ. Note that, again $\tilde{\pi} \circ \alpha = \mathrm{id}_M$ and that $\Omega^1(M)$ is also a C^∞-module.

2.2.2 Maps relating manifolds

Definition 2.3. *A continuous map $F : M^m \to N^n$ between manifolds is **smooth map** if after choosing local coordinates in open sets containing $\sigma \in M$ and $F(\sigma) \in N$, the coordinate expression for F is a C^∞ function from $\mathbb{R}^m$ to $\mathbb{R}^n$. In such case, we write $F \in C^\infty(M)$. Moreover, if the smooth map is a bijection, we call it a **diffeomorphism** [8].*

Smooth maps induce maps that connect the respective tangent and cotangent spaces.

Definition 2.4. *The **push-forward** at $\sigma \in M$ of a smooth map $F : M \to N$ is the linear map $T_\sigma F : T_\sigma M \to T_{F(\sigma)} N$ defined by*

$$[T_\sigma F(X_\sigma)](f) = X_\sigma(f \circ F), \text{ for all } f \in C^\infty(N) \text{ and } X_\sigma \in T_\sigma M. \tag{2}$$

In coordinates (x^i) around σ and (y^j) around $F(\sigma)$, the push-forward is given by

$$(T_\sigma F) \left(\left. \frac{\partial}{\partial x^i} \right|_\sigma \right) = \left. \frac{\partial F^a}{\partial x^i} \right|_\sigma \left. \frac{\partial}{\partial y^a} \right|_{F(\sigma)}, \tag{3}$$

where F^a is the a-th component of the F map. Finally, the push-forwards assemble naturally on TM to give the map $TF : TM \to TM$ defined by $TF(X_\sigma) := T_{\pi(X_\sigma)} F(X_\sigma)$ [8].

Regarding X_σ as the tangent vector to a smooth curve $c : (-\epsilon, \epsilon) \to M$ with $c(0) = \sigma$, then $T_\sigma F(X_\sigma)$ is the velocity at $F(\sigma)$ of the curve given by the composition $F \circ c : (-\epsilon, \epsilon) \to N$. Hence, the push-forward sends the tangent vectors of $c(t)$ to those of $F \circ c(t)$ (Fig.1a), and if F is a diffeomorphism, it establishes a $1-1$ relation.

Just as derivatives, push-forwards obey a chain rule.

Proposition 2.5. ***Chain rule for manifolds**. Let M, N and L be differentiable manifolds and consider $F : M \to N$, $G : N \to L$ smooth maps. Then, for all $\sigma \in M$ [8],*

$$T_\sigma(G \circ F) = T_{F(\sigma)} G \circ T_\sigma F. \tag{4}$$

While push-forwards link tangent spaces via $F : M \to N$, another kind of map relating the constructions from N to those on M is induced.

Definition 2.6. *Let $f \in C^\infty(N)$ and $\alpha \in \Omega^1(N)$, so that $\alpha_\sigma \in T^*_\sigma N$. Given $F : M \to N$ smooth, we define*

1. *The **pull-back of a function** $F^*f := f \circ F$, so that F^*f lives in $C^\infty(M)$.*

2. *The **pull-back** $F^*\alpha : M \to T^*M$ as the dual map to the push-forward*

$$F^*\alpha_\sigma := (T_\sigma F)^*\alpha_{F(\sigma)} = \alpha_{F(\sigma)} \circ T_\sigma F.$$

Note that $(F^*)_\sigma : T^*_{F(\sigma)}N \to T^*_\sigma M$. Choosing local coordinates (x^i) around $\sigma \in M$ and (y^a) around $F(\sigma) \in N$, we have that

$$F^*(dy^a)_{F(\sigma)} = \left.\frac{\partial F^a}{\partial x^i}\right|_\sigma (dx^i)_\sigma. \tag{5}$$

*Finally, we have the special case where $M = N$ and F is a diffeomorphism. In this case, we write $T^*F := F^*$ and call this induced map the **cotangent lift** of F. The map $T^*F : T^*M \to T^*M$ is a diffeomorphism and has the property that $\tilde{\pi} = F \circ \tilde{\pi} \circ T^*F$ [8, 9].*

Finally, the push-forwards provide us with a way to construct submanifolds.

Theorem 2.7. ***Regular Value Theorem**. Let $F : M^{m+n} \to N^m$ be a smooth map between manifolds. We say that $c \in N$ is a **regular value** of F if for all $\sigma \in F^{-1}(c)$, the push-forward $T_\sigma F : T_\sigma M \to T_c N$ is surjective. If c is a regular value of F, then $F^{-1}(c)$ is an n-dimensional submanifold of M and for all $\sigma \in F^{-1}(c)$, we have [8]*

$$T_\sigma F^{-1}(c) = \ker(T_\sigma F).$$

As a summary, it follows a commutative diagram with the spaces and maps we have constructed so far. For the N manifold, $\rho : TN \to N$ and $\tilde{\rho} : T^*N \to N$ are the corresponding canonical projections, $\beta : N \to T^*N$ is a one-form and $Y : N \to TN$ a vector space. Notice that T^*F goes in the direction opposite to both F and TF.

$$\begin{tikzcd}
T^*M \arrow[r, bend left, "\tilde{\pi}"] & M \arrow[l, bend left, "\alpha"] \arrow[r, bend right, "X"'] \arrow[d, "F"] & TM \arrow[l, bend right, "\pi"'] \arrow[d, "TF"] \\
T^*N \arrow[u, "T^*F"] \arrow[r, bend left, "\tilde{\rho}"] & N \arrow[l, bend left, "\beta"] \arrow[r, bend right, "Y"'] & TN \arrow[l, bend right, "\rho"']
\end{tikzcd}$$

2.2.3 K-forms and their operations

We can now proceed to operate on one-forms with a map known as **wedge product**, to construct a new mathematical object.

Definition 2.8. *A* ***k-multilinear alternating form*** *or* ***k-form*** *on a real n-dimensional vector space V, is a k-multilinear map $\phi : \underbrace{V \times \cdots \times V}_{\text{(k-times)}} \to \mathbb{R}$ which vanishes whenever two of its arguments coincide [8].*

Now let $V^* = T^*M$ and denote by $\Omega^k(M)$ the $C^\infty(M)$-module of k-forms. By convention, we say that $\Omega^0(M) = C^\infty(M)$, so that elements of $\Omega^0(M)$ are smooth functions.

Definition 2.9. *If $\alpha_1, \ldots, \alpha_k \in \Omega^1(M)$ and $X_1, \ldots, X_n \in \mathfrak{X}(M)$, then the* ***wedge product*** *of k one-forms is defined by*

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(X_1, \ldots, X_k) = \det \left[\begin{pmatrix} \alpha_1(X_1) & \ldots & \alpha_1(X_k) \\ \vdots & \ddots & \vdots \\ \alpha_k(X_1) & \ldots & \alpha_k(X_k) \end{pmatrix} \right].$$

Regarding T^*M as the dual vector space to TM, we can denote by $\Omega^k(M)$ the $C^\infty(M)$-module of smooth multilinear alternating maps $\alpha : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{(k-times)} \to C^\infty(M)$, mapping

$$\alpha(X_1, \ldots, X_k)(\sigma) \mapsto \alpha_\sigma((X_1)_\sigma, \ldots, (X_k)_\sigma), \quad \textit{for all } \sigma \in M \ [8].$$

The previous notion of pull-back also extends to k-forms by pulling back on each of the vector fields.

Definition 2.10. *Let $\sigma \in M$, $F : M \to N$ be a smooth map and $\alpha \in \Omega^k(N)$. Then its pull-back $F^*\alpha \in \Omega^k(M)$ by F is defined by [8]*

$$(F^*\alpha)(X_1, \ldots, X_k)(\sigma) = \alpha_{F(\sigma)}((T_\sigma F)(X_1)_\sigma, \ldots, (T_\sigma F)(X_k)_\sigma).$$

Applying this definition to the wedge product of one-forms yields

$$F^*(\alpha_1 \wedge \cdots \wedge \alpha_k) = F^*\alpha_1 \wedge \cdots \wedge F^*\alpha_k.$$

Apart from taking the wedge product, we can modify the **degree** k of the k-forms: either by decreasing it by one, mapping a k-form to a $(k-1)$-form; or increasing it by one, by mapping the k-form to a $(k+1)$-form.

Definition 2.11. *Let $X \in \mathfrak{X}(M)$. The* ***interior product*** *with X is the $C^\infty(M)$-linear map $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ defined by [8]*

$$\iota_X \alpha(X_2, \ldots, X_k) = \alpha(X, X_2 \ldots, X_k). \tag{6}$$

Proposition 2.12. *Let $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. Then we have that [8]*

1. $\iota_X \circ \iota_X = 0$.
2. $\iota_{fX} = f\iota_X$ *for all* $f \in C^\infty(M)$.

Definition 2.13. ***The exterior derivative*** $d : \Omega^k(M) \to \Omega^{k+1}(M)$ *for* $k \geq 1$ *is given*

$$d\alpha := \sum_{(i_1,i_2,\ldots,i_k)} d\alpha_{(i_1,i_2,\ldots,i_k)} \wedge (dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}), \tag{7}$$

with $1 \leq i_1 < \cdots < i_k \leq n$ *labelling all the smooth functions. If* $f \in \Omega^0(M)$ *with* $k = 0$*, after choosing coordinates* (x^i) *for the manifold, we have [8]*

$$df = \frac{\partial f}{\partial x^i} dx^i. \tag{8}$$

The exterior derivative obeys three crucial properties.

Theorem 2.14. *Let* $\alpha \in \Omega^k(N)$*,* $F : M \to N$ *a smooth map between manifolds consider the exterior derivative* $d : \Omega^k(N) \to \Omega^{k+1}(N)$*. Then, we have [8]*

1. $d(d\alpha) = 0$ *or equivalently,* $d^2 = 0$.
2. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$. ***Leibniz rule***
3. $dF^*\alpha = F^* d\alpha$.

2.2.4 Integral curves and Lie derivatives.

Despite having a differentiable structure, manifolds do not have an inherent notion of distance. Moreover, the basic operation of addition may result in points that do not even belong in the manifold. Take the example of the circle S^1. Adding two of the (x, y) pole points: $(-1, 0) + (1, 0) = (0, 0) \notin S^1$.

Nonetheless, vector fields defined on our manifold induce a path on the manifold itself, along which we can "move on the manifold". If you have a unique tangent arrow at each point of a surface, you can take infinitesimal steps following the direction of the arrows at each point. Thus, the way to obtain this path on the manifold, i.e. a curve, is via integration.

Definition 2.15. *A smooth map* $\gamma : (\alpha, \beta) \to M$ *is an* ***integral curve*** *of a vector field* $X \in \mathfrak{X}(M)$ *if for all* $t \in (\alpha, \beta)$*, we have*

$$T_t\gamma\left(\frac{d}{dt}\right) = X_{\gamma(t)}. \tag{9}$$

The velocity vector of the curve agrees with the vector field along the curve. After choosing coordinates (x^i) *and denoting a general curve* $\gamma(t) = (x^1(t), \ldots, x^n(t))$*, we have [8]*

$$T\gamma\left(\frac{d}{dt}\right) = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} = X^i(x(t))\frac{\partial}{\partial x^i} \iff \frac{dx^i}{dt} = X^i(x(t)).$$

The last equation is a first order differential equation. Then, given the initial point of the integral curve, its solution is unique in an open set containing such point [8]. Uniqueness tells us that the integral curves of a vector field will not intersect near the initial point: *for each vector field there is a unique induced way of (locally) moving along the manifold.*

We can choose a specific $\sigma \in M$ to be the initial point of a curve.

Definition 2.16. *Let X be a vector field defined on an open subset $U \subset M$ with coordinates. Define $\gamma : (-\epsilon, \epsilon) \times W \to U$ such that for each $\sigma \in W$, $\gamma(t, \sigma)$ is the integral curve of X starting at σ. Use the notation $\gamma_t(\sigma) := \gamma(t, \sigma)$. By uniqueness, if $s, t \in (-\epsilon, \epsilon)$ then*

$$\gamma_t(\gamma_s(\sigma)) = \gamma_{s+t}(\sigma).$$

*The map $\gamma : (-\epsilon, \epsilon) \times W \to U$ is the **local flow** generated by X. If a local flow is defined on $\mathbb{R} \times M$ is called a **global flow**, and the vector field is called **complete** [8].*

In this project, we will be dealing with complete vector fields, so we only need to focus on global flows. Given any $t \in \mathbb{R}$, if γ is global it is also defined at $-t \in \mathbb{R}$ and we have

$$\gamma_t \circ \gamma_{-t} = id_M.$$

But γ is smooth, so this must be a diffeomorphism for each $\sigma \in M$. We call this family of maps a **one-parameter group of diffeomorphisms**.

Definition 2.17. *A **one-parameter group of diffeomorphisms** on a manifold M is a smooth map $\gamma : \mathbb{R} \times M \to M$ that defines a diffeomorphism $\gamma_t := \gamma(t, -) : M \to M$ for all $t \in \mathbb{R}$. In particular, $\gamma_0 = id_M$ and for all $s_1, s_2 \in \mathbb{R}$, the composition of two of these maps obeys $\gamma_{s_1} \circ \gamma_{s_2} = \gamma_{s_1+s_2}$ [8].*

This induced diffeomorphism allows us to shift any construction on M along the integral curves of the vector field. Thus, we can define a new kind of derivative.

Definition 2.18. *Let $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$ with local flow γ_t. The **Lie Derivative** along X of f is defined by*

$$\pounds_X(f) := \frac{d}{dt}\Big((\gamma_t)^* f\Big) = \frac{d}{dt}\Big(f \circ \gamma_t\Big) = X(f),$$

where last equality follows from (9) applied to a general f [3].

Now denote by "$\gamma_t \cdot a$" the change of the construction "a" on the manifold along the flow γ_t. For instance, $\gamma_t \cdot f = f \circ \gamma_t$. Once we have defined the Lie Derivative for a function, imposing **equivariance**, i.e. that every construction transforms equivalently under "γ_t", we can deduce the Lie Derivative for vector fields, one-forms and k-forms (see appendix C).

In particular, for $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, we have that [8]

$$(\pounds_X Y) f = X(Y(f)) - Y(X(f)) =: [X, Y](f), \tag{10}$$

where $[X, Y] \in \mathfrak{X}(M)$ is known as the **Lie bracket** of X and Y (Fig.1b). For $\alpha \in \Omega^1(M)$, we have

$$(\pounds_X \alpha)(Y) = X(\alpha(Y)) - \alpha([X, Y]).$$

The Lie derivative obeys the following general properties.

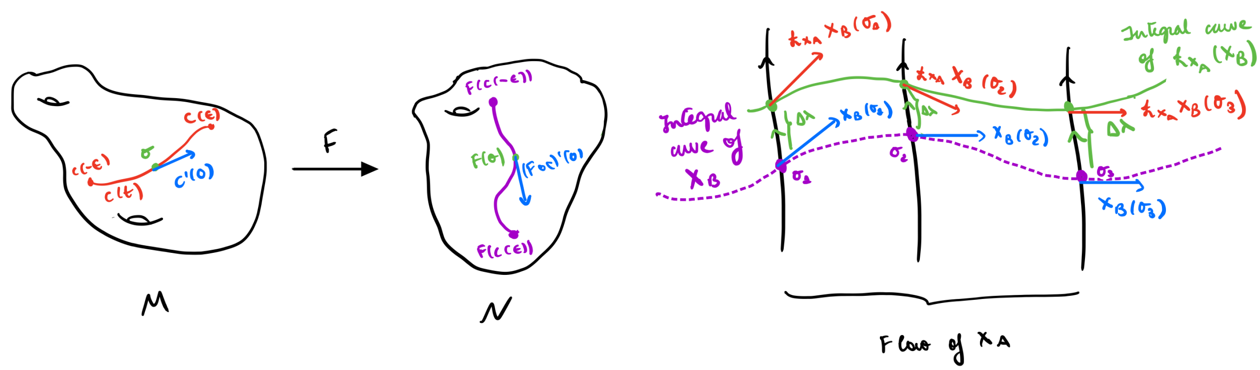
Theorem 2.19. *Let $X, Y \in \mathfrak{X}(M)$, $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$. Then*

1. $\pounds_X(\alpha \wedge \beta) = \pounds_X \alpha \wedge \beta + \alpha \wedge \pounds_X \beta$.
2. $\pounds_X(d\alpha) = d(\pounds_X \alpha)$.
3. *The Lie derivative of any kind of construction on M does not change the type of construction. E.g. $\pounds_X Y \in \mathfrak{X}(M)$ and $\pounds_X \alpha \in \Omega^k(M)$ [8].*

Finally, we state without proof an crucial toolkit of equations which will greatly simplify differential algebra in the upcoming parts of the report.

Theorem 2.20. ***Cartan Formulae.*** *For all $X, Y \in \mathfrak{X}(M)$ and $\alpha \in \Omega^k(M)$, we have that, for any construction in M (using α as an example) [3]*

$$\pounds_X = d \circ \iota_X + \iota_X \circ d \implies \pounds_X \alpha = \iota_X d\alpha + d\iota_X \alpha. \tag{I}$$
$$[\pounds_X, \iota_Y] = \iota_{[X,Y]} \implies \pounds_X \iota_Y \alpha = \iota_Y \pounds_X \alpha + \iota_{[X,Y]} \alpha. \tag{II}$$
$$[\pounds_X, \pounds_Y] = \pounds_{[X,Y]} \implies \pounds_X \pounds_Y \alpha = \pounds_Y \pounds_X \alpha + \pounds_{[X,Y]} \alpha. \tag{III}$$



(a) Representation of the push-forward of a map F.

(b) Pictorial representation of the Lie derivative

Figure 1. Plots to provide geometrical intuition on operations. Fig.(1a) represents how the tangent vector at $\sigma \in M$ to the curve $c(t)$ is mapped to the tangent vector at $\sigma \in N$ to the curve $F \circ c(t)$ via the push-forward of F. Fig.(1b) graphically represents $\pounds_{X_A}(X_B)$. Notice how the integral curve of X_B is shifted along the flow of X_A by a distance $\Delta\lambda$ and the tangent vectors change.

2.3 Continuous symmetries: Lie groups and Lie algebras

2.3.1 The Lie group and its Lie algebra

Definition 2.21. *A* ***Lie Group*** *is a manifold G where we identify a special point $e \in G$ called the* ***identity****. In a Lie Group, the product $\mu : G \times G \to G$, sending $(g, h) \mapsto gh$, is a C^∞ map. In particular, $\mu(e, g) = \mu(g, e) = g$, for all $g \in G$. We also define the* ***inverse*** *map $I : G \to G$ as $I(g) = g^{-1}$, which is also smooth [8].*

Physicists think of these as continuous groups, whose elements are described in terms of one or multiple parameters. We have already encountered several Lie groups during our degree, and even though these have multiple representations, we will focus on their matrix representation. In this representation, the product is defined by matrix multiplication and the identity element is the corresponding $n \times n$ **identity matrix** $\mathbb{1}_n$. Some examples are

- The **group of translations** in $\mathbb{R}^n$, denoted by $G = T^n$. For translations by a vector $\underline{x} \in \mathbb{R}^n$, elements of this group are of the form

$$T^n(\underline{x}) = \begin{pmatrix} \mathbb{1}_n & \underline{x} \\ \underline{0} & 1 \end{pmatrix},$$

and we have that

$$T^n(\underline{x})T^n(\underline{y}) = \begin{pmatrix} \mathbb{1}_n & \underline{x} \\ \underline{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbb{1}_n & \underline{y} \\ \underline{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_n & \underline{x} + \underline{y} \\ \underline{0} & 1 \end{pmatrix} = T^n(\underline{x} + \underline{y}),$$

as we intuitively expect. From this computation, we immediately deduce that

$$[T^n(\underline{x})]^{-1} = T^n(-\underline{x}) = \begin{pmatrix} \mathbb{1}_n & -\underline{x} \\ \underline{0} & 1 \end{pmatrix}.$$

- The group of n-dimensional **orthogonal transformations** $SO(n)$.

$$SO(n) = \{A \in Mat(n, \mathbb{R}) |\ A\underline{x} \cdot A\underline{y} = \underline{x} \cdot \underline{y} \text{ for all } \underline{x}, \underline{y} \in \mathbb{R}^n \text{ and } \det(A) = 1\}.$$

For the case $n = 2$, its elements are the 2-dimensional anticlockwise rotation matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \text{ for all } \theta \in \mathbb{R}.$$

For the case $n = 3$, its elements are of the form of a 3-dimensional anticlockwise rotation, which are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ for all } \theta \in \mathbb{R}. \tag{11}$$

We normally apply these transformations to a particular space. For instance, we rotate or translate vectors in $\mathbb{R}^3$. We call this application of a Lie group on other manifold **action**.

Definition 2.22. *Let M be a manifold and let G be a Lie group.* ***A (left) action*** *of G on M is a smooth mapping $\Phi : G \times M \to M$ such that [1]*

1. *$\Phi_e(x) = x$, for all $x \in M$, i.e. $\Phi_e = Id_M$.*
2. *$\Phi_g \circ \Phi_h(x) = \Phi_{gh}(x)$.*

For example, the action of $SO(3)$ in $\mathbb{R}^3$ is given by

$$\Phi_A(\underline{x}) = A\underline{x}, \text{ for all } A \in SO(3) \text{ and } \underline{x} \in \mathbb{R}^3.$$

Another example of action is given by the flow of a complete vector field. Denote this flow by $F_\lambda : M \to M$. The properties of a one-parameter group of diffeomorphisms (definition 2.17) coincide with the conditions to define an action. If we think of $\mathbb{R}$ as the 1-dimensional group of translations, for each $\lambda \in \mathbb{R}$ the flow F_λ defines an action of $\mathbb{R}$ on M.

Since we can combine elements on the Lie group by multiplying them, we also have two actions of a Lie group on itself.

Definition 2.23. *Let $g \in G$. The* ***left translation*** *L_g is defined as $L_g(h) = gh$, for all $h \in G$. Similarly, the* ***right translation*** *R_g is defined by $R_g(h) = hg$ for all $h \in G$. These actions commute, and are diffeomorphisms with diffeomorphic push-forwards [1]. In addition, L_g gives an action on G for all $g \in G$, whereas $R_{g^{-1}}$ gives an action on G for all $g \in G$ [1].*

We now have a look at a specific kind of vector field which is closely related to Lie groups.

Definition 2.24. *A vector field $\tilde{X} \in \mathfrak{X}(G)$ is called* ***left-invariant*** *if for all $g \in G$, we have that*

$$T_h L_g[\tilde{X}(h)] = \tilde{X}(gh), \text{ for all } h \in G.$$

We denote the space of left-invariant vector fields by $\mathfrak{X}_L(G)$ [1].

Proposition 2.25. *These left-invariant vector fields are* ***closed under the Lie bracket****: If $\tilde{X}, \tilde{Y}$ are left invariant, so is $[\tilde{X}, \tilde{Y}]$ [8].*

Since Lie Groups are manifolds, as we argued in section 2.2.1 these must have a tangent space at each point, and we can consider T_eG in particular. For each $\xi \in T_eG$, define

$$\tilde{X}_\xi(g) = T_e L_g(\xi).$$

Then, we have that

$$\tilde{X}_\xi(gh) = T_e L_{gh}(\xi) = T_e(L_g \circ L_h)(\xi) = T_h L_g(T_e L_h(\xi)) = T_h L_g(\tilde{X}_\xi(h)),$$

where in the third equality I used the chain rule (4). This implies that $\tilde{X}_\xi$ is left invariant, so we have constructed a map from T_eG to $\mathfrak{X}_L(M)$. Besides, evaluation at the identity yields

$$\tilde{X}_\xi(e) = T_eL_e(\xi) = Id_{T_eG}(\xi) = \xi.$$

So we have an inverse map from $\mathfrak{X}_L(G)$ to T_eG as well. Since both maps are linear, *the set of left invariant vector fields is isomorphic to the tangent space at the identity, when treating these as vector spaces* [3]. From the definition of Lie bracket in $\mathfrak{X}_L(G)$ given by (10), we may define for all $\xi, \eta \in T_eG$

$$[\xi, \eta] := [\tilde{X}_\xi, \tilde{X}_\eta](e).$$

This is what we call the **Lie algebra** of a Lie Group.

Definition 2.26. *The* ***Lie algebra*** $\mathfrak{g}$ *of G is the vector space T_eG equipped with the Lie bracket*

$$[\xi, \eta] := [\tilde{X}_\xi, \tilde{X}_\eta](e)$$

where $\xi, \eta \in T_eG$ and $\tilde{X}_\xi, \tilde{X}_\eta$ are as defined above [1].

We now compute the Lie algebras of the Lie groups we gave as previous examples. The Lie algebra of a group is denoted by the name of the group in lower-case characters.

• The **Lie algebra of the group of translations** in $\mathbb{R}$, denoted by $\mathfrak{g} = t^1$. The n-dimensional case follows from applying this procedure to each of the entries of the translation vector. Here $\lambda \in \mathbb{R}$ and we can take a curve in T through the identity $\mathbb{1}_2$ given by

$$T(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \text{ where } \lambda \in \mathbb{R}.$$

This is just scalar translation by λ. Now, differentiating with respect to λ we obtain

$$T'(0) = \frac{d}{d\lambda} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \Bigg|_{\lambda=0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which is what we call the **generator** of the Lie algebra. This means that if we think of the Lie algebra as a real vector space, then this is the only element in the basis, as t^1 is one dimensional [1]. Thus, elements here are of the form

$$\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}, \text{ for some } \xi \in \mathbb{R}.$$

• The **Lie algebra of the 3-dimensional orthogonal group** $so(3)$. In this case, the rotation matrices from (11) can be thought of curves starting at the identity and going along each of the three possible "independent" directions of the Lie group. Thus the three generators of the Lie algebra are [10]

$$\frac{d}{d\theta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \Bigg|_{\theta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the other two terms also follow by differentiating the corresponding rotation matrices.

2.3.2 The exponential map

Differentiation at the identity allows us to obtain the Lie algebra from the Lie group. In general, we cannot recover the entire Lie group from the algebra. Nonetheless, there exists a map that allows us to send Lie algebra elements to the part of the Lie group that is connected to the identity.

Definition 2.27. *Let $G \subseteq GL(n, \mathbb{R})$. The **exponential map** $\exp : \mathfrak{g} \to G$ for the matrix Lie algebra is given by*

$$\exp(\xi) := \sum_{k=0}^{\infty} \frac{\xi^k}{k!},$$

for each $\xi \in \mathfrak{g}$. Besides, we can obtain a one-parameter group of diffeomorphisms by considering

$$\exp(\lambda\xi) := \sum_{k=0}^{\infty} \frac{\lambda^k \xi^k}{k!},$$

since $exp(0) = \mathbb{1}_n$ and if we multiply two exponentials, we add up their exponents. This map provides a 1-1 relation between elements in our Lie algebra and the connected part of the Lie group [10].

As an illustrative example, we can recover the entire Lie group of 1-dimension translations via this map. Observe that

$$\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This means that the $k-$th power of the matrix vanishes for $k \geq 2$. Then, taking the exponential map yields

$$\exp\left[\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}\right] = \sum_{k=1}^{\infty} \frac{1}{k!} \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}^k = \mathbb{1}_2 + \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \in T^1.$$

So the element in t^1 associated to $\xi \in \mathbb{R}$ is mapped to the element corresponding to ξ in the Lie group. Bearing in mind the Taylor series for cos and sin, we can also exponentiate the generators from (11) times θ and recover the corresponding rotation matrices [10].

2.3.3 Actions of the Lie group and the Lie algebra

Apart from acting on itself via left and right translations, given a Lie group G we can construct an action on its own Lie algebra $\mathfrak{g}$.

Definition 2.28. *The **adjoint action** of G on $\mathfrak{g}$ is the map $Ad : G \times \mathfrak{g} \to \mathfrak{g}$, defined by*

$$Ad_g(\xi) := T_e(L_g \circ R_{g^{-1}})(\xi).$$

*We also define the **coadjoint action** by $CoAd_g := Ad^*_{g^{-1}}$, so that for $\alpha \in \mathfrak{g}^*$ and $\xi \in G$, we have [1]*

$$\begin{aligned}
&< CoAd_g(\alpha) CoAd_h(\alpha), \xi >=< Ad^*_{g^{-1}} Ad^*_{h^{-1}}(\alpha), \xi >=< Ad^*_{h^{-1}}(\alpha), Ad_{g^{-1}}\xi > \\
&=< \alpha, Ad_{h^{-1}} Ad_{g^{-1}}\xi >=< \alpha, Ad_{h^{-1}g^{-1}}\xi >=< \alpha, Ad_{(gh)^{-1}}\xi >=< Ad^*_{(gh)^{-1}}(\alpha), \xi > \\
&=< CoAd_{gh}(\alpha), \xi > .
\end{aligned}$$

Finally, since for each $\lambda \in \mathbb{R}$ and $\xi \in \mathfrak{g}$ we have that $\exp(\lambda\xi) \in G$, we can apply any given action with $g = \exp(\lambda\xi)$. This grants us a way to observe how our Lie algebra induces an action on a manifold.

Definition 2.29. *Let Φ_g be define a (left) action for all $g \in G$. For $\xi \in \mathfrak{g}$, the map $\Phi(\exp(\lambda\xi), \sigma) : \mathbb{R} \times M \to M$ is an $\mathbb{R}$-action on M. In other words, $\Phi_{\exp(\lambda\xi)} : M \to M$ is a flow on M and defines an action of $\mathfrak{g}$ on M. The corresponding vector field on M is given by*

$$\xi_M(\sigma) := \frac{d}{dt}\left(\Phi_{\exp(\lambda\xi)}(\sigma)\right)\Bigg|_{\lambda=0},$$

*and is called the **infinitesimal generator** of the action corresponding to ξ [1].*

3 Mechanics and symmetries

3.1 The Hamiltonian framework

In dynamics, we think of the 3-dimensional space as $\mathbb{R}^3$. Within this space, consider a general system of N particles, which are described by their position coordinates $\underline{r_k}$ with $k \in \{1, 2, ..., N\}$. The change of these positions with time can be constrained in various ways, but we will focus on the **holonomic constraints**. These constraints are the ones that can be expressed as an algebraic equation. If ψ is a holonomic constraint, then we have

$$\psi(\underline{r_1}, ..., \underline{r_N}, t) = 0. \tag{12}$$

The physical quantities we need to specify to determine the state of the system are called the **degrees of freedom**. For a system of N particles in 3-dimensional space, we need to specify the position of each of these particles in space, giving us a total of $3N$ degrees of freedom. Note, however, that each holonomic constraint provides us with an equation. Therefore, we can solve for one of the physical quantities in terms of the rest, allowing us to reduce the amount of degrees of freedom by one. The upshot is that *each holonomic constraint can be used to reduce the amount of degrees of freedom in our system by one* [11].

If we have $3N - f$ holonomic constraints, then we can reduce the amount of degrees of freedom of our system from $3N$ to f. Thus the system will be determined by f independent coordinates, called the **generalised coordinates** and denoted by $q^1, ..., q^f$. Once we have set up the system with these coordinates, we call the set of all possible physical configurations the **configuration space**, and represent it by Q. This is the starting point of modern dynamics.

In the Lagrangian framework we compute the **generalised velocities**, defined by

$$\dot{q}^i = \frac{dq^i}{dt}, \tag{13}$$

where q^i denotes the i-th generalised coordinate. Assuming that $\dot{q}^i$'s do not depend on q^i's, we work on the **state space**, given by the pairs of positions and velocities $(q^i, \dot{q}^i)$. In this space, we compute the Lagrangian $\mathcal{L} = T - V$, where T is the kinetic energy of the system and V is the potential energy. From D'Alembert's principle or Hamilton's principle, we obtain Langrange's equations [11]:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \qquad \text{for all } i \in 1, ..., f. \tag{14}$$

These allow us to solve for the motion in terms of the generalised coordinates, and then substitute back if we wish to obtain the equations of motions in the coordinates we were initially working with.

In the Hamiltonian approach, we shift our interest from the generalised velocities $\dot{q}^i$ to the **generalised momenta** p_i, defined by

$$p_i := \frac{\partial L}{\partial \dot{q}^i}. \tag{15}$$

These are taken to be independent from the generalised velocities $\dot{q}^i$. With this definition, we construct the **Hamiltonian** function H from the Lagrangian, by means of the **Legendre transform** [1]

$$H(q^i, p_i, t) = \dot{q}^i p_i - L(q^i, \dot{q}^i, t). \tag{16}$$

In this project, we will be mainly focusing in time-independent Lagrangians and Hamiltonians. Hamiltonian dynamics takes place in the **phase space**, given by pairs of points (q^i, p_i) (the distinction between different spaces will become important in the future). Within this space, we can obtain the time evolution of each of the coordinates by solving **Hamilton's canonical equations** [1]

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{17}$$

A crucial construction, which supports the underlying structure of Hamiltonian dynamics, is the **Poisson bracket**. For any two smooth functions F and G, we define their Poisson bracket as

$$\{F, G\} := \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q^i}\frac{\partial F}{\partial p_i}. \tag{18}$$

We say that F, G are in **involution** provided that $\{F, G\} = 0$. Given this construction and using (17) we can immediately write

$$\dot{q}^i = \{q^i, H\}, \text{ and } \dot{p}_i = \{p_i, H\}. \tag{19}$$

Finally, consider a function $X(\underline{q}, \underline{p})$, where $\underline{q}$ is shorthand for all q^i's and $\underline{p}$ for all p_i's, i.e. a function defined in phase space that is time-independent. Using the chain rule and (17), we obtain

$$\frac{dX}{dt} = \frac{\partial X}{\partial q^i}\frac{dq^i}{dt} + \frac{\partial X}{\partial p_i}\frac{dp_i}{dt} = \frac{\partial X}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial X}{\partial p_i}\frac{\partial H}{\partial q^i} = \{X, H\}. \tag{20}$$

Thus, it suffices to check the bracket of X with H to establish whether X is a conserved quantity. Even though the construction of the Poisson bracket seems rather arbitrary, it will turn out to be essential in the study of symmetries and we will consider a more formal definition of the Poisson bracket in section 3.4.3. Finally, do bear in mind that we are restricting ourselves to time-independent quantities and a time-independent H. More terms arise in the time-dependent case [11].

3.2 Independent conserved quantities in the Hamiltonian framework

The phase space where 3-dimensional dynamics occurs is a $6N$-dimensional space, where N is the number of particles. For the case $N = 1$, we represent its time evolution in phase space by a 6-dimensional curve encoding its position and momenta: $(\underline{q}(t), \underline{p}(t)) = (q^1(t), q^2(t), q^3(t), p_1(t), p_2(t), p_3(t))$. To solve the problem, i.e. to determine the particle's motion in phase space, our final goal is to find expressions for each of these time-dependent functions.

If every quantity that depends on positions and momentum components is arbitrarily changing, there is no way for us to predict what is going to happen. Fortunately, we know that several conserved quantities exist for most physical problems. Amongst all the conserved quantities, those that do not hold redundant information can be used as a holonomic constraints, to reduce by one the degrees of freedom of the system. We call such conserved quantities **independent**.

Definition 3.1. *Consider the 3-dimensional phase space for one particle, described by the coordinates $\{q^1, q^2, q^3, p_1, p_2, p_3\}$. Let k be an non-zero integer, and consider the set of smooth functions $\{f_1, f_2, ..., f_k\}$ in phase space. Define the Jacobian matrix J by*

$$J := \begin{pmatrix} \frac{\partial f_1}{\partial q^1} & \frac{\partial f_1}{\partial q^2} & \frac{\partial f_1}{\partial q^3} & \frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial p_2} & \frac{\partial f_1}{\partial p_3} \\ \frac{\partial f_2}{\partial q^1} & \frac{\partial f_2}{\partial q^2} & \frac{\partial f_2}{\partial q^3} & \frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial p_2} & \frac{\partial f_2}{\partial p_3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_k}{\partial q^1} & \frac{\partial f_k}{\partial q^2} & \frac{\partial f_k}{\partial q^3} & \frac{\partial f_k}{\partial p_1} & \frac{\partial f_k}{\partial p_2} & \frac{\partial f_k}{\partial p_3} \end{pmatrix}$$

*Treating T^*M as a vector space, by equation (8) the matrix J contains the real coefficients of the df_i "vectors". In that sense, J does not have full rank (in this case, its rank is less than 5) if and only if the exterior derivatives of smooth functions $\{f_1, f_2, ..., f_k\}$ are **linearly dependent** [12]. This is what characterises the independence of smooth functions. We can functions that obey this condition **functionally independent**.*

So, given a set of quantities (functions) in phase space, it suffices to check the rank of the Jacobian of these. To ease computations in the project, I devised a Mathematica notebook that performs the necessary computations, the code is in the appendix D.

As we have deduced, conserved quantities are fundamental to solve the problem, yet in this case, having six independently conserved quantities will fix everything. If we want to observe dynamics, we can have at most at most five independently conserved quantities. We now have a look at a well-known physics problem which will serve as a check of this deduction.

3.3 The Kepler orbit problem in the Hamiltonian framework

3.3.1 Setup of the problem

Let $\{q^1, q^2, q^3\}$ denote the cartesian coordinates and $\{p_1, p_2, p_3\}$ the corresponding momenta in each perpendicular direction (Fig.2a). Notice that the Hamiltonian for both the Kepler orbit problem or the hydrogen atom is given by

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) - \frac{\kappa}{\sqrt{(q^1)^2 + (q^2)^2 + (q^3)^2}} = \frac{1}{2m}(p_j p_j) - \frac{\kappa}{\sqrt{q^i q^i}},$$

so all the upcoming developments are applicable to both problems. Focusing on the Kepler orbit problem, we let $\kappa = GMm$, with m being the mass of the planet and M being the mass of the fixed Sun, and $M >> m$. By definition, the components of the angular momentum in cartesian coordinates, are given by

$$L_1 = q^2 p_3 - q^3 p_2, \quad L_2 = q^3 p_1 - q^1 p_3, \quad \text{and} \quad L_3 = q^1 p_2 - q^2 p_1.$$

Equivalently, in index summation convention: $L_i = \epsilon_{ijk} q^j p_k$. Then, with the identity for the product of two Levi-Civita symbols [1], we find that

$$\begin{aligned}
\{L_i, L_j\} &= \frac{\partial L_i}{\partial q^m}\frac{\partial L_j}{\partial p_m} - \frac{\partial L_i}{\partial p_m}\frac{\partial L_j}{\partial q^m} \\
&= \epsilon_{iab}\delta_{am}p_b\epsilon_{jrs}q^r\delta_{sm} - \epsilon_{iab}q^a\delta_{bm}\epsilon_{jrs}p_s\delta_{mr} = \epsilon_{bis}\epsilon_{sjr}q^r p_b - \epsilon_{iar}\epsilon_{rsj}q^a p_s \\
&= (\delta_{bj}\delta_{ir} - \delta_{br}\delta_{ij})q^r p_b - (\delta_{is}\delta_{aj} - \delta_{as}\delta_{ij})q^a p_s = q^i p_j - q^j p_i = \epsilon_{ijk}L_k.
\end{aligned}$$

Using Hamilton's equations (17), we obtain the E.O.M.s

$$\dot{p}_i = \frac{-\kappa q^i}{((q^1)^2 + (q^2)^2 + (q^3)^2)^{\frac{3}{2}}}, \quad \text{and } \dot{q}^i = \frac{p_i}{m} \implies p_i = m\dot{q}^i. \tag{21}$$

Evaluating the Poisson bracket with the Hamiltonian, we can check that all L_i's are conserved:

$$\dot{L}_i = \{L_i, H\} = \frac{\partial L_i}{\partial q^m}\frac{\partial H}{\partial p_m} - \frac{\partial L_i}{\partial p_m}\frac{\partial H}{\partial q^m} = \epsilon_{ijk}p_j p_k - \epsilon_{ijk}\frac{q^j q^k}{(q^l q^l)^{\frac{3}{2}}} = 0,$$

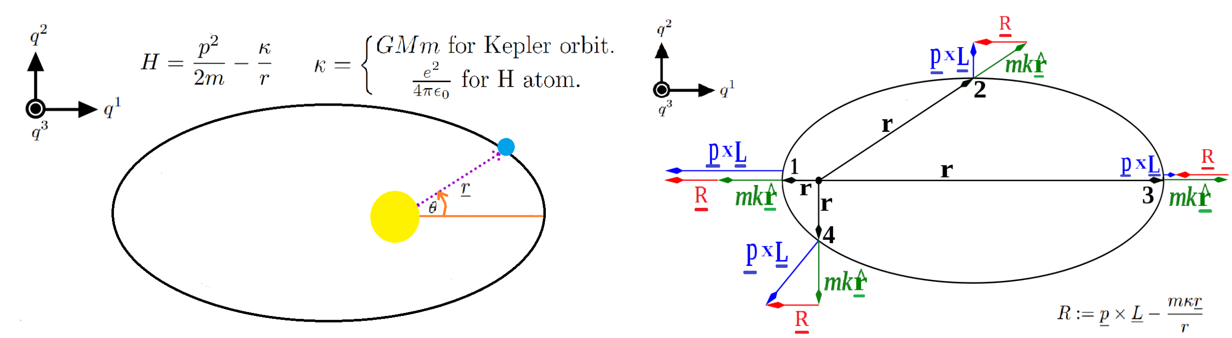
due to symmetry of $p_j p_k$ and anti-symmetry of ϵ_{ijk} in j and k. Moreover, $L^2 = L_1^2 + L_2^2 + L_3^2$ is a conserved quantity, as we have

$$\begin{aligned}\dot{L^2} &= \{L^2, H\} = \{L_1^2 + L_2^2 + L_3^2, H\} = \{L_1^2, H\} + \{L_2^2, H\} + \{L_3^2, H\} \\ &= L_1\{L_1, H\} + \{L_1, H\}L_1 + L_2\{L_2, H\} + \{L_2, H\}L_2 + L_3\{L_3, H\} + \{L_3, H\}L_3 = 0,\end{aligned}$$

using the previous result. So all the L_i's are conserved, L^2 and H are conserved. This gives us in total 4 independently conserved quantities, as L^2 is defined in terms of the L_i's. These are the quantities we would normally use to solve the problem. Nonetheless, there is an additional conserved quantity: the **Laplace-Runge-Lenz vector** $\underline{R}$ (LRL vector), defined by

$$\underline{R} := \underline{p} \times \underline{L} - \frac{m\kappa\underline{r}}{r}, \tag{22}$$

This vector points along the axis of symmetry of the orbit and it is conserved (proof in appendix A).



(a) Kepler orbit/H-atom problem.

(b) Laplace vector for a closed orbit.

Figure 2. Plots related to the Kepler orbit problem. Fig.2a represents the setup of the problem, with the q^3-axis coming out of the page. Fig.2b shows how the LRL vector points along the axis of symmetry of an elliptical orbit.

3.3.2 Independently conserved quantities for the Kepler Orbit problem

Following the definition of linear independence, we can readily check that the set of functions $\{L_1, L_2, L_3, H\}$ that we often contemplate are functionally independent. Computing the required Jacobian in Mathematica, we obtained

$$\begin{pmatrix} 0 & p_3 & -p_2 & 0 & -q^3 & q^2 \\ -p_3 & 0 & p_1 & q^3 & 0 & -q^1 \\ p_2 & -p_1 & 0 & -q^2 & q^1 & 0 \\ \frac{kq^1}{((q^1)^2+(q^2)^2+(q^3)^2)^{3/2}} & \frac{kq^2}{((q^1)^2+(q^2)^2+(q^3)^2)^{3/2}} & \frac{kq^3}{((q^1)^2+(q^2)^2+(q^3)^2)^{3/2}} & \frac{p_1}{m} & \frac{p_2}{m} & \frac{p_3}{m} \end{pmatrix}$$

It suffices to find a 3×3 matrix with a non-vanishing determinant to prove that the matrix has full rank. Considering the following determinant

$$\begin{vmatrix} q^3 & 0 & -q^1 \\ -q^2 & q^1 & 0 \\ \frac{p_1}{m} & \frac{p_2}{m} & \frac{p_3}{m} \end{vmatrix} = \frac{1}{m} \begin{vmatrix} q^3 & 0 & -q^1 \\ -q^2 & q^1 & 0 \\ p_1 & p_2 & p_3 \end{vmatrix} = \frac{1}{m}[q^3q^1p_3 - q^1q^2p_2 + p_1(q^1)^2],$$

which does not vanish in every point of phase space. Hence, the Jacobian matrix has full rank and by definition 3.1, the conserved quantities $\{L_1, L_2, L_3, H\}$ are linearly independent. We still have the 3 components of the LRL vector to consider. Evaluating the Jacobian in Mathematica, we arrived at the conclusion that only one of these components can be added to the previous set of linearly independent conserved quantities, and we are entirely free to choose which one.

Instead of going through the cumbersome computation of many determinants, we can provide two explicit equations relating the components of the LRL vector to the rest of the quantities from the list. In total, we end up with 5 independently conserved quantities.

We need to find to equations relating these to reduce the number from 7 to 5 supposedly independent conserved quantities. The first relation arises from considering

$$\underline{R} \cdot \underline{L} = \left(\underline{p} \times \underline{L} - \frac{m\kappa\underline{r}}{r}\right) \cdot \underline{L} = (\underline{p} \times \underline{L}) \cdot \underline{L} - \frac{m\kappa\underline{r}}{r} \cdot (\underline{r} \times \underline{p}) = 0. \tag{23}$$

Thus $\underline{R}$ is perpendicular to $\underline{L}$, as we properly drew on the diagram (Fig.2b). For the other constraint we need to work a little harder. We first make a couple of explicit computations. By definition

$$L^2 = (\underline{r} \times \underline{p}) \cdot (\underline{r} \times \underline{p}) = \underline{r} \cdot [\underline{p} \times (\underline{r} \times \underline{p})] = \underline{r} \cdot [\underline{r}(\underline{p} \cdot \underline{p}) - (\underline{r} \cdot \underline{p})\underline{p}] = r^2p^2 - (\underline{p} \cdot \underline{r})^2. \tag{24}$$

Also, from the expression of the conserved Hamiltonian, which in this case matches the total energy, we have that

$$H = E = \frac{p^2}{2m} - \frac{\kappa}{r} \implies 2mE = p^2 - \frac{2m\kappa}{r}. \tag{25}$$

With these equations in mind we can compute the magnitude of the LRL vector squared.

By (10) we have

$$\begin{aligned}
R^2 = \underline{R}\cdot\underline{R} &= \left(\underline{p}\times\underline{L} - \frac{m\kappa\underline{r}}{r}\right)\cdot\left(\underline{p}\times\underline{L} - \frac{m\kappa\underline{r}}{r}\right)\\
&= (\underline{p}\times\underline{L})\cdot(\underline{p}\times\underline{L}) - \frac{2m\kappa}{r}(\underline{p}\times\underline{L})\cdot\underline{r} + \frac{m^2\kappa^2}{r^2}(\underline{r}\cdot\underline{r})\\
&= (\underline{p}\times(\underline{r}\times\underline{p}))\cdot(\underline{p}\times(\underline{r}\times\underline{p})) - \frac{2m\kappa}{r}(\underline{p}\times(\underline{r}\times\underline{p}))\cdot\underline{r} + m^2\kappa^2\\
&= [\underline{r}(\underline{p}\cdot\underline{p}) - (\underline{r}\cdot\underline{p})\underline{p}]\cdot(\underline{p}\times(\underline{r}\times\underline{p})) - \frac{2m\kappa}{r}(\underline{p}\times(\underline{r}\times\underline{p}))\cdot\underline{r} + m^2\kappa^2.
\end{aligned}$$

Note that in the first squared bracket, $\underline{p}$ is perpendicular to $\underline{p}\times(\underline{r}\times\underline{p})$, and hence only the first term will survive. This fact, combined with the results from (24) and (25), allows us to obtain

$$\begin{aligned}
R^2 &= (\underline{r}p^2)\cdot(\underline{p}\times(\underline{r}\times\underline{p})) - \frac{2m\kappa}{r}(\underline{p}\times(\underline{r}\times\underline{p}))\cdot\underline{r} + m^2\kappa^2 = p^2L^2 - \frac{2m\kappa}{r}L^2 + m^2\kappa^2\\
&= \left(p^2 - \frac{2m\kappa}{r}\right)L^2 + m^2\kappa^2 = 2mEL^2 + m^2\kappa^2.
\end{aligned}$$

Therefore, we have proven the following relations:

$$\underline{R}\cdot\underline{L} = 0 \quad \text{and} \quad R^2 = 2mEL^2 + m^2\kappa^2.$$

From these, we can conclude that we have at most five independently conserved quantities: $\{E, L_1, L_2, L_3\}$ and only one of the components of $\underline{R}$; since the other two can be obtained using the two expressions above. Note that R^2 cannot be an independently conserved quantity either since the expression above involves the product of E and $L^2 = {L_1}^2 + {L_2}^2 + {L_3}^2$. Bear in mind that more choices for a set of five independently conserved quantities are possible. For instance, Mathematica yields that the sets $\{E, L_1, L_2, R_1, R_2\}$ and $\{E, R_1, R_2, R_3, L_1\}$ are also linearly independent. The crucial realisation is that the maximal amount of independently conserved quantities is always the same: at most 5 independently conserved quantities, as we had argued in section 3.2.

3.3.3 Conditions on integrability: the Liouville-Arnold theorem

Even though we have establish an upper-bound on the number of independently conserved quantities for a system, experience tells us that we do not require to find all of these to obtain a solution. In the case of the Kepler orbit problem (or the hydrogen atom), it suffices to consider the set $\{H, L_3, L^2\}$ [11, 13].

The amount and properties of the conserved quantities we need are formally established in the following theorem.

Theorem 3.2. ***Liouville-Arnold theorem (simplified)***. *Consider a dynamical system with f degrees of freedom, i.e. with a $2f$-dimensional phase space. If f independently conserved quantities in pairwise involution are known, the canonical equations with Hamiltonian function H can be integrated by quadratures. We call such system* ***integrable*** *[14].*

Hence, we just need to have 3 quantities in involution for the Kepler orbit problem to be able to find a solution. This is why we have considered $\{H, L_3, L^2\}$ throughout our degree, because these tell us that the system is integrable. The solution for an orbit is usually given in the form

$$r(\theta) = \frac{l}{1 + \epsilon\cos(\theta)},$$

where $l = \frac{L^2}{mk}$ and $\epsilon = \frac{l}{r_0} - 1$, with r_0 being the radial initial position. Depending on the value of ϵ, we can determine the conic section that traced out by our orbit and hence predict the movement of the planet around the Sun [15]. Moreover, a solution may be written down in terms of "action-angle coordinates" [11]. But why do we not find a global solution $r(t)$ in the literature?

Our Hamiltonian canonical equations do establish time-dependence for each coordinate, and the system is integrable. However, *the condition of integrability by quadratures only ensures that we can find a solution in an open interval of* t, as I will prove in the following section.

3.3.4 The impossibility of a one-function time-dependent solution

In polar coordinates, the radial equation for the Kepler orbit reads [11, 15]

$$\frac{\dot{p}_r}{m} = \ddot{r} = \frac{L^2}{m^2r^3} + \frac{-k}{mr^2}.$$

We now proceed to integrate by quadratures. Multiplying by $\dot{r}$ yields

$$\dot{r}\ddot{r} = \frac{L^2}{m^2r^3}\dot{r} - \frac{k}{mr^2}\dot{r} \quad\iff\quad \frac{1}{2}\frac{d}{dt}(\dot{r}^2) = \frac{d}{dt}\left(\frac{-L^2}{2m^2r^2} + \frac{k}{mr}\right),$$

and integrating once with respect to time, we obtain

$$\frac{1}{2}\dot{r}^2 = \frac{-L^2}{2m^2r^2} + \frac{k}{mr} + \mathcal{C}, \tag{26}$$

where $\mathcal{C}$ is an integration constant. Since we know that the answer will be a conic section from previous computations, at some point $\dot{r}(0) = 0$. For simplicity, we can choose as our initial conditions $r(0) = r_0 > 0$ and $\dot{r}(0) = 0$, giving

$$0 = \frac{-L^2}{2m^2r_0^2} + \frac{\kappa}{mr_0} + \mathcal{C} \quad\iff\quad \mathcal{C} = \frac{L^2}{2m^2r_0^2} - \frac{\kappa}{mr_0}.$$

Recall that the total energy for the orbital motion is given by [15]

$$\frac{E}{m} = \frac{\dot{r}^2}{2} + \frac{L^2}{2m^2r^2} - \frac{\kappa}{mr}, \text{ and using } \dot{r}(0) = 0 \implies \frac{E}{m} = \frac{L^2}{2m^2r_0^2} - \frac{\kappa}{mr_0} = \mathcal{C},$$

since energy is conserved. At any of the turning points of the orbit, the kinetic energy vanishes. Then either $r_0 = r_{\min}$ or $r_0 = r_{\max}$ in the case of closed orbits, and $r_0 = r_{\min}$ for hyperbolic and parabolic orbits. Without loss of generality, we assume that here r_0 is the minimum radial distance $r_{\min}$. Thus, we can solve for $\dot{r}$ in (26), giving

$$\dot{r}^2 = \frac{-L^2}{m^2r^2} + \frac{2k}{mr} + \frac{2E}{m} = 2\left(\frac{-L^2}{2m^2r^2} + \frac{k}{mr}\right) + \frac{2E}{m},$$

and following the procedure from section 2.1.3, we have that

$$\int_{r_a}^{r} \frac{dr'}{\pm\sqrt{\frac{-L^2}{m^2r'^2} + \frac{2k}{mr'} + 2\frac{E}{m}}} = \frac{m}{\sqrt{2}}\int_{r_a}^{r} \frac{r'dr'}{\pm\sqrt{Er'^2 + kr' - \frac{L^2}{2m}}} = \int_{t_a}^{t} dt',$$

where $t_a \neq 0$ and $r_a = r(t_a) > r_0$ and does not cancel out the denominator. Let $F(t,r)$ be the function

$$F(t,r) := \frac{m}{\sqrt{2}}\int_{r_a}^{r} \frac{r'dr'}{\pm\sqrt{Er'^2 + kr' - \frac{L^2}{2m}}} - \int_{t_a}^{t} dt'.$$

By the Fundamental Theorem of Calculus [16], on the interval that does not include any of the turning points (or roots of the denominator), say $[r_a, r_b]$ with $r_b < r_{\max}$, the partial derivative with respect to r is given by

$$\frac{\partial F}{\partial r} = \frac{r}{\pm\sqrt{Er^2 + kr - \frac{L^2}{2m}}},$$

which is clearly smooth and non-vanishing in $[r_a, r_b]$. By the Implicit Function Theorem [16], after integration, we should be able to invert $t(r)$ into $r(t)$, but only in an open interval of t where r is away from the turning points, i.e. locally.

Even though theorem 3.2 tells us that we can write down an integral, *integrability does not imply that we can find a one-function explicit solution for our problem.* The best we can hope for is that, after extending the local descriptions of $r(t)$, these may coincide at the turning points at which we can glue the local descriptions together and obtain a general $r(t)$. But we will always have to extend and glue at least two functions.

The issue is provoked by the change of sign of $\dot{r}$. When we take the root, we impose a sign and the definition of $\dot{r}$ excludes the point $\dot{r} = 0$. In the process of continuously going from positive sign solutions to negative sign solutions, we are going through $\dot{r} = 0$, but we are forbidden to do that since such points cancel out the denominator in our integrand.

As a brief summary, the study of the Kepler orbit problem has served as a sanity check. So far, we have concluded that a dynamical system with f degrees of freedom can have at most $2f - 1$ independently conserved quantities, and that we only require f of these in involution for the system to be integrable. Nonetheless, integrability only guarantees a local solution of the problem. Thus, having enough independently conserved quantities in involution ensures that we can locally solve the dynamical system of interest.

We now need to divert away from the Kepler orbit problem and develop the mathematical tools required to understand how a great amount of conserved quantities arise.

3.4 Development of the symplectic geometry approach

3.4.1 The phase space as a cotangent bundle

Our goal at this point is to uncover the underlying structure of Hamiltonian dynamics, whilst acquiring a better intuition of what the Poisson bracket of two functions represents. To achieve so we must combine the acquired knowledge of differential geometry and Hamiltonian dynamics.

The first connection comes from the spaces where Hamiltonian dynamics is developed. Our $n-$*dimensional* configuration space Q for one particle, with points denoted by $(q^1, q^2, \ldots, q^n)$, perfectly matches our definition of n-dimensional manifold from section 2.2.1. The state space with points $(q^1, q^2, \ldots, q^n, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^n)$ represents its respective tangent bundle TQ, with $\dot{q}^i$'s being the velocities of curves through the configuration space. Similarly, due to the duality of momenta to velocities, the phase space where dynamics occurs turns out to be the cotangent bundle T^*Q, with points given by $(q^1, q^2, \ldots, q^n, p_1, p_2, \ldots, p_n)$ [1].

3.4.2 Construction of the symplectic form

Within the phase space T^*Q, the cotangent bundle of Q, we automatically obtain a local definition of a vital one-form: the **Liouville one-form**.

Definition 3.3. *Let M be a manifold and T^*M its cotangent bundle. The Liouville one-form θ is defined as follows. If $\alpha \in T_p^*M$, then $\theta_\alpha \in T_\alpha^*(T^*M)$ sends $v \in T_\alpha(T^*M)$ to the real number*

$$\theta_\alpha(v) := \alpha(T_\alpha \tilde{\pi}(v)), \tag{27}$$

where $T_\alpha \tilde{\pi} : T_\alpha(T^*M) \to T_pM$ is the push-forward of the projection map. This one-form on the cotangent bundle, has the following properties [9]

- *If $\alpha \in \Omega^1(M)$, then regarding α as a smooth map $\alpha : M \to T^*M$, we have that $\alpha^*\theta = \alpha$.*
- *Relative to coordinates (q^i, p_i) we have that $\theta = p_i dq^i$.*
- *If $F : M \to M$ is a diffeomorphism and $T^*F : T^*M \to T^*M$ the induced lift, then we have that $(T^*F)^*\theta = \theta$.*

With this a tautological one-form in our phase space as a starting point, by defining $\omega = -d\theta$, we obtain the local expression we were looking for.

Theorem 3.4. *The two-form $\omega := -d\theta = \sum_{i=1}^n dq^i \wedge dp_i$ provides a weakly non-degenerate, bilinear and anti-symmetric form defined in phase space T^*Q. Moreover, locally it is a closed 2-form.*

Proof. We will show that the theorem above holds. By properties of differential one-forms ω is bilinear. Consider two arbitrary tangent vectors fields $v, w \in T(T^*Q)$. Then, using the coordinates we have chosen for the cotangent bundle, these can be expressed as

$$v = v^j \frac{\partial}{\partial q^j} + v_j \frac{\partial}{\partial p_j} \quad \text{and} \quad w = w^k \frac{\partial}{\partial q^k} + w_k \frac{\partial}{\partial p_k}.$$

If we have $\omega(v, w) = 0$ for all $w \in T(T^*Q)$, by the definition of the wedge product, this means that

$$\omega(v, w) = 0 \implies v^i w_i - v_i w^i = 0,$$

but since w is arbitrary, so are w^i's and w_i, hence the only way for this to hold is to have v_i's and v^i's all zero. Thus we also have weak non-degeneracy. Finally, by the theorem 2.14 we have that locally

$$d\omega = -d^2\theta = 0.$$

□

When this two-form is globally closed, it characterises a special variety of manifold.

Definition 3.5. *A* ***symplectic manifold*** *is a pair (M, ω) where M is a manifold and ω is a closed weakly non-degenerate two-form on M. The two-form ω is called the* ***symplectic form*** *[1].*

Notice that we impose that ω is closed, not locally, but in the entire manifold M. If Q is a vector space, then so will be T^*Q and thus all our constructions can be applied globally, in particular the coordinate expression for the symplectic form ω. That is why for vector spaces, the condition of the symplectic form vanishing globally is automatic, we do not really need to impose it. So if we are working on $Q = \mathbb{R}^3$, this imposition is irrelevant for us.

However, not all the manifolds where Physics takes place are vector spaces. Imagine that the motion is constrained to move in a specific region of space, as a bead moving in a ring or inside a cylindrical pipe. Then, we need to impose that ω is closed in the entire manifold M, as we can only assert that it vanishes locally, after choosing the canonical coordinates. As some illustrative examples, consider [1]

- The cylinder $S^1 \times \mathbb{R}$ with coordinates (θ, p), a symplectic manifold with $\omega = d\theta \wedge dp$.
- The torus $\mathcal{T}^2$ with periodic coordinates (θ, ϕ), a symplectic manifold with $\omega = d\theta \wedge d\phi$.
- The two-sphere S^2 of radius r, a symplectic manifold with $\omega = r^2 \sin(\theta) d\theta \wedge d\phi$ on the sphere as the symplectic form.

The computations of this project will be carried out in $Q = \mathbb{R}^n$. This is a vector space for all $n \in \mathbb{Z}$, and thus ω is closed. Yet it is important to highlight this difference for future research.

3.4.3 Hamiltonian dynamics in the symplectic approach

Using our symplectic form and given any $f \in C^\infty(T^*Q)$, we can construct a unique vector field $X_f \in \mathfrak{X}(T^*Q)$[1]. This is done by considering how $\iota_{X_f}\omega$ would act on an arbitrary $Y \in \mathfrak{X}(T^*Q)$. Impose that

$$\omega(X_f, Y) = df(Y) = Y(f), \tag{28}$$

where the last equality comes from the exterior derivative definition Using the coordinates (q^i, p_i) for T^*Q, the vector fields can be locally expressed as

$$X_f = (X_f)^i \frac{\partial}{\partial q^i} + (X_f)_i \frac{\partial}{\partial p_i} \quad \text{and} \quad Y = Y^i \frac{\partial}{\partial q^i} + Y_i \frac{\partial}{\partial p_i},$$

where X^i, X_i, Y^i and Y_i are smooth functions of the coordinates. Evaluating each side of (28) separately, we have

$$\omega(X_f, Y) = dq^i(X_f)dp_i(Y) - dp_i(X_f)dq^i(Y) = (X_f)^i Y_i - (X_f)_i Y^i, \text{ and}$$
$$df(Y) = \frac{\partial f}{\partial q^i} Y^i + \frac{\partial f}{\partial p_i} Y_i.$$

Comparing both expressions, we can conclude that, given the symplectic form ω and a smooth function f, the general expression for a vector field is

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}.$$

We give to these special vector fields arising from smooth functions a special name.

Definition 3.6. *A vector field $Y \in \mathfrak{X}(T^*Q)$ is called **hamiltonian vector field** if*

$$Y = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}, \tag{29}$$

for some $f \in C^\infty(T^*Q)$. If we know the function explicitly, we denote Y by X_f [1].

Locally, there is a $1-1$ correspondence between symplectic forms and our previous notion of Poisson bracket.

Theorem 3.7. *Define $\{-,-\} : C^\infty(T^*Q) \times C^\infty(T^*Q) \to C^\infty(T^*Q)$, by [9]*

$$\{f, g\} := -X_f(g) = \omega(X_f, X_g). \tag{30}$$

*This is the **Poisson bracket** defined in terms of the symplectic form. Once we have chosen (q^i, p_i) coordinates, this definition matches our previous notion of Poisson brackets (18).*

[1] The subscript "f" indicates the function where this type of vector field is obtained from, not the evaluation point (normally "σ"). Besides, we do not write "$\sim$" on these to distinguish them from the left-invariant vectors of section 2.3.1.

Proof. We check that the defined bracket obeys the local coordinate expression, as this is not done in [1]. Expanding the R.H.S. of (30) yields

$$-X_f(g) = -\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}\right)(g) = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} = \{f, g\},$$

by comparison with (18). For the second equality, using the definition of exterior derivative and (28) we have

$$-X_f(g) = -dg(X_f) = -\omega(X_g, X_f) = \omega(X_f, X_g),$$

as we wanted to show. $\square$

Thus, all the Poisson brackets of smooth functions in phase space can be rewritten in terms of our symplectic form evaluated at the corresponding hamiltonian vector fields. In particular, by evaluating $\omega(X_f, X_H) = \{f, H\} = \dot{f}$ we can determine whether an arbitrary smooth function $f(\underline{q}, \underline{p})$ is conserved.

In addition to that, the hamiltonian vector field encodes the information of Hamilton's canonical equations. As explained in section 2.2.4, vector fields provide us with a construction to move along the manifold: their integral curves. Consider the vector field generated by the Hamiltonian H and denote its integral curve by $c(t) = (q^i(t), p_i(t))$. By definition of the integral curve, we have that

$$\dot{c}(t) = (\dot{q}^i(t), \dot{p}_i(t)) = X_H = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i}\right).$$

We have just deduced that *the flow of the vector field associated to the Hamiltonian tells us how the dynamical system evolves with time*, how the particle moves in phase space as time advances.

There is also another kind of vector fields we are interested in, the ones that preserve our symplectic form.

Definition 3.8. *A vector field $Y \in \mathfrak{X}(T^*Q)$ is **symplectic** if $\pounds_Y \omega = 0$ [9].*

All the vectors fields that are constructed from a function in phase preserve the symplectic form. This includes the vector field X_H.

Proposition 3.9. *Hamiltonian vector fields are symplectic. If $Y \in \mathfrak{X}(T^*Q)$ is symplectic, then for any $f \in C^\infty(M)$, we have that the Lie bracket $[Y, X_f] = -X_{Y(f)}$ [9].*

Proof. Given X_f, we have that, by Cartan's Formula (I)

$$\pounds_{X_f}\omega = \iota_{X_f} d\omega + d\iota_{X_f}\omega = 0 + d(df) = 0.$$

Thus X_f is symplectic. For the second part, start with the R.H.S. and use Cartan's Formulae (II) to obtain

$$\iota_{[Y,X_f]}\omega = [\pounds_Y, \iota_{X_f}]\omega = \pounds_Y \iota_{X_f}\omega - \iota_{X_f}\pounds_Y\omega = \pounds_Y \iota_{X_f}\omega,$$

as Y is symplectic. By vector field definition (28) and the fact that the exterior derivative commutes with the Lie derivative

$$\iota_{[Y,X_f]}\omega = -\pounds_Y df = d(-\pounds_Y(f)) = \iota_{-X_{Y(f)}}\omega.$$

Finally, weakly non-degeneracy of ω tells us that

$$[Y, X_f] = -X_{Y(f)},$$

as we wanted to show. □

By setting $Y = X_g$, we get an equation relating the Lie bracket, the hamiltonian vector fields and the Poisson bracket.

Corollary 3.10. *If f, g are smooth functions in T^*Q, the Lie bracket of two hamiltonian vector fields is also hamiltonian and it is given by*

$$[X_f, X_g] = -X_{\{f,g\}}. \tag{31}$$

Setting $\{f, g\} = 0$ and using weak non-degeneracy of ω, we have that

$$[X_f, X_g] = 0 \iff \{f, g\} = 0.$$

Recall that in section 2.2.4 we established the geometrical meaning of the Lie derivative. Combining the geometrical meaning with the corollary above, we arrive at the following insight: *if the Poisson bracket of two quantities vanishes, we can move along the integral curve generated by one of their respective fields whilst holding the other quantity fixed.* In particular, if $[X_H, X_f] = 0$, the quantity f will be fixed along flow of X_H, which determines the time evolution of the dynamical system.

Finally, we prove some properties of the symplectic form and the Poisson bracket

Proposition 3.11. *Let f, g and $m \in C^\infty(T^*Q)$ and $\lambda \in \mathbb{R}$. Denote by X_f, X_g and X_m the corresponding Hamiltonian vector fields. The following properties hold:*

- ***Bilinearity**. $\omega(\lambda X_f, X_g) = \lambda\omega(X_f, X_g) \iff \{\lambda f, g\} = \lambda\{f, g\}$. Note that for the case of ω, we may also have $\lambda \in C^\infty(T^*Q)$.*
- ***Anti-symmetry**. $\omega(X_f, X_g) = -\omega(X_g, X_f) \iff \{f, g\} = -\{g, f\}$.*
- ***Jacobi's identity**. $\omega(X_f, [X_g, X_m]) + \omega(X_g, [X_m, X_f]) + \omega(X_m, [X_f, X_g]) = 0$*

$$\iff \{f, \{g, m\}\} + \{g, \{m, f\}\} + \{m, \{f, g\}\} = 0.$$

- ***Leibniz's identity***. $\omega(X_{fg}, X_m) = f\omega(X_g, X_m) + g\omega(X_f, X_m)$

$$\iff \{fg, m\} = f\{g, m\} + g\{f, m\}.$$

Proof. We will prove the identities for the symplectic form and then use (30) to obtain the expressions in terms of Poisson brackets.

The first identity follows from bilinearity and anti-symmetry of wedge product, combined with the fact that one-forms are a $C^\infty(T^*Q)$-module.

For Jacobi's identity, recall that $d\omega$ is a 3-form, but as ω is closed, we have that

$$d\omega(X_f, X_g, X_m) = 0 \implies \iota_{X_m}\iota_{X_g}\iota_{X_f}d\omega = 0,$$

where I expressed the L.H.S. using the interior product. Now, by Cartan Formulae (I) we have

$$\iota_{X_m}\iota_{X_g}\iota_{X_f}d\omega = \iota_{X_m}\iota_{X_g}(\pounds_{X_f}\omega - d\iota_{X_f}\omega) = \iota_{X_m}\iota_{X_g}\pounds_{X_f}\omega - \iota_{X_m}\iota_{X_g}d\iota_{X_f}\omega.$$

And using (II) in the first term and (I) in the second term yields

$$\begin{aligned}\iota_{X_m}\iota_{X_g}\iota_{X_f}d\omega &= \iota_{X_m}(\pounds_{X_f}\iota_{X_g}\omega - \iota_{[X_f,X_g]}\omega) - \iota_{X_m}(\pounds_{X_g}\iota_{X_f}\omega - d\iota_{X_g}\iota_{X_f}\omega) \\ &= \iota_{X_m}\pounds_{X_f}\iota_{X_g}\omega - \iota_{X_m}\iota_{[X_f,X_g]}\omega - \iota_{X_m}\pounds_{X_g}\iota_{X_f}\omega + \iota_{X_m}d\iota_{X_g}\iota_{X_f}\omega. \end{aligned} \tag{32}$$

By means of the Cartan Formulae, we want to express (32) in a form where, in each term, all the interior products act on ω first. Using Cartan Formula (II) in the first and third term of (32) gives

$$\begin{aligned}\iota_{X_m}\iota_{X_g}\iota_{X_f}d\omega = \pounds_{X_f}\iota_{X_m}\iota_{X_g}\omega + \iota_{[X_f,X_m]}\iota_{X_g}\omega - \iota_{X_m}\iota_{[X_f,X_g]}\omega - \\ - \pounds_{X_g}\iota_{X_m}\iota_{X_f}\omega - \iota_{[X_g,X_m]}\iota_{X_f}\omega + \iota_{X_m}d\iota_{X_g}\iota_{X_f}\omega.\end{aligned} \tag{33}$$

Focusing on the last term, we have

$$\iota_{X_m}d\iota_{X_g}\iota_{X_f}\omega = d\iota_{X_m}\iota_{X_g}\iota_{X_f}\omega - \pounds_{X_m}\iota_{X_g}\iota_{X_f}\omega = -\pounds_{X_m}\iota_{X_g}\iota_{X_f}\omega, \tag{34}$$

since ω is a two-form. Thus equation (33) becomes

$$\begin{aligned}\iota_{X_m}\iota_{X_g}\iota_{X_f}d\omega = \pounds_{X_f}\iota_{X_m}\iota_{X_g}\omega - \pounds_{X_g}\iota_{X_m}\iota_{X_f}\omega - \pounds_{X_m}\iota_{X_g}\iota_{X_f}\omega - \\ + \iota_{[X_f,X_m]}\iota_{X_g}\omega - \iota_{X_m}\iota_{[X_f,X_g]}\omega - \iota_{[X_g,X_m]}\iota_{X_f}\omega.\end{aligned} \tag{35}$$

The terms involving the Lie derivatives can be rewritten as

$$\pounds_{X_f}\iota_{X_m}\iota_{X_g}\omega = X_f(\{g, m\}) = d(\{g, m\})(X_f) = \omega(-X_{\{g,m\}}, X_f) = \omega(X_f, [X_g, X_m]),$$

where I used the definition of vector field and (30). With this in mind, the first three terms of (35) become

$$\omega(X_f, [X_g, X_m]) + \omega(X_g, [X_m, X_f]) + \omega(X_m, [X_f, X_g]),$$

and evaluating the interior products of the three last terms, we get

$$\begin{aligned}&-\omega(X_g,[X_f,X_m])-\omega([X_f,X_g],X_m)+\omega(X_f,[X_g,X_m])\\ &=\omega(X_g,[X_m,X_f])+\omega(X_m,[X_f,X_g])+\omega(X_f,[X_g,X_m]),\end{aligned}$$

matching the expression we obtained for the terms involving Lie derivatives. Thus, we can conclude that

$$\begin{aligned}&d\omega(X_f,X_g,X_m)=0\\ \iff\quad &2*[\omega(X_g,[X_m,X_f])+\omega(X_m,[X_f,X_g])+\omega(X_f,[X_g,X_m])]=0\\ \iff\quad &\omega(X_g,[X_m,X_f])+\omega(X_m,[X_f,X_g])+\omega(X_f,[X_g,X_m])=0,\end{aligned}$$

so Jacobi's identity holds. To obtain the Poisson bracket identity, use (31) and then (30) to write each term above as

$$\omega(X_g,[X_m,X_f])=\omega(X_g,-X_{\{m,f\}})=-\{g,\{m,f\}\}.$$

For Leibniz's identity, we consider the L.H.S. of the equation. After applying (30), using the product rule for partial derivatives

$$\begin{aligned}\omega(X_{fg},X_m)&=-\omega(X_m,X_{fg})=X_m(fg)=fX_m(g)+gX_m(f)\\ &=-f\omega(X_m,X_g)-g\omega(X_m,X_f)=f\omega(X_g,X_m)+g\omega(X_f,X_m),\end{aligned}$$

as we wanted to show. The Poisson bracket expression follows immediately from (30). □

We will now go back to the Kepler Orbit problem to reproduce some results in this new framework. This will allow us to establish a connection between the symmetry of the problem and conserved quantities.

3.4.4 The Kepler Orbit problem in the symplectic approach

In section 3.3 we managed to compute the following relations

$$\{L_1,L_2\}=L_3,\quad \{L_2,L_3\}=L_1,\quad \{L_3,L_1\}=L_2\quad\text{and}\quad \{L_i,H\}=0,\ \text{for all } i\in\{1,2,3\}.$$

If we evaluate the Poisson bracket of any pair of the L_i's, we will obtain plus or minus the remaining component from the set $\{L_1,L_2,L_3\}$, depending on which entry we choose to put each L_i. E.g. $\{L_1,L_2\}=L_3$, but $\{L_2,L_1\}=-L_3$. The results from previous section can be used to interpret what this relation between angular momentum components means. In terms of Lie brackets, by (31) we have that

$$[X_{L_1},X_{L_2}]=-X_{L_3},\quad [X_{L_2},X_{L_3}]=-X_{L_1},\quad\text{and}\quad [X_{L_3},X_{L_1}]=-X_{L_2}.$$

We will focus on explaining one equation, the rest follow by symmetry. By the first equation, we have that $[X_{L_1},X_{L_2}]=-X_{L_3} \implies \pounds_{X_{L_1}}(X_{L_2})=-X_{L_3}$.

Let $\mu(t)$ denote one of the integral curves of X_{L_2}. Shift by a (small) distance all the points of $\mu(t)$ along the integral curves of X_{L_1}, and then compute the new tangent vectors. Then, by the equation above, we will obtain the tangent vectors that construct X_{L_3}, but pointing in the opposite direction because of the minus sign. This means that the movement along two of the vector fields from the set $\{X_{L_1}, X_{L_2}, X_{L_3}\}$, can be employed to determine the flow of the remaining vector field. *The flows of these vector fields do not span three independent directions in phase space, but two.* This idea is formalized in **Frobenius' Theorem**.

Theorem 3.12. ***Frobenius' Theorem***. *If a set of m smooth vector fields defined in a region U of an m-dimensional manifold M have the Lie Brackets with one another, all of which are linear combinations of the m vector fields, then the integral curves of the fields mesh to form a family of submanifolds.*

Each submanifold has dimension equal to the dimension of the vector space these fields define at any point, which is at most m.

*Each point of the set U is on one and only one submanifold, provided that the dimension of the vector space defined by the fields is the same everywhere in U. The family of submanifolds fills U and we call it a **foliation** of U. Each of these submanifolds is a **leaf** of the foliation, where we can perform integration [3].*

Since $\{X_{L_1}, X_{L_2}, X_{L_3}\}$ are globally defined, their integral curves foliate our entire phase space, thus determining various submanifolds. Using Mathematica, these vector fields are explicitly

$$\begin{aligned}
X_{L_1} &= -q^3\frac{\partial}{\partial q^2} + q^2\frac{\partial}{\partial q^3} - p_3\frac{\partial}{\partial p_2} + p_2\frac{\partial}{\partial p_3},\\
X_{L_2} &= q^3\frac{\partial}{\partial q^1} + -q^1\frac{\partial}{\partial q^3} + p_3\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial p_3},\\
X_{L_3} &= -q^2\frac{\partial}{\partial q^1} + q^1\frac{\partial}{\partial q^2} - p_2\frac{\partial}{\partial p_2} + p_1\frac{\partial}{\partial p_2}.
\end{aligned}$$

Consider the function $R : Q \to \mathbb{R}; (q^1, q^2, q^3) \mapsto (q^1)^2 + (q^2)^2 + (q^3) - l^2$. Then, the level set $R^{-1}(0)$ defines a 2-sphere of radius l in configuration space. Its exterior derivative yields

$$dR = 2q^1dq^1 + 2q^2dq^2 + 2q^3dq^3.$$

This is clearly surjective into $\mathbb{R}$ for every (q^1, q^2, q^3), as $(q^1, q^2, q^3) = 0 \notin F^{-1}(0)$. Hence 0 is a regular value of R. By the Regular Value Theorem (theorem 2.7), we know that the kernel of dR represents the tangent space of $R^{-1}(0)$, the tangent space of a 2-sphere in this case. And using the expressions we have just computed for the vector fields of the components of angular momentum, we have that

$$dR(X_{L_1}) = dR(X_{L_2}) = dR(X_{L_3}) = 0.$$

This means that, when restricted to their ∂q^i components[2], *the three vector fields corresponding to the three angular momenta are tangent to 2-spheres, which foliate the configuration space.*

Moreover, this is uncovering another property of the system we are considering. Since $\{L_i, H\} = 0 \implies [X_H, X_{L_i}] = \pounds_{X_H}(X_{L_i}) = 0$, we know that each of the angular momenta are fixed as the system evolves with time. However, using the anti-symmetry of the Lie bracket, this also means that $[X_{L_i}, X_H] = \pounds_{X_{L_i}}(X_H) = 0$, so the flows that determine how the system evolves with time in phase space are invariant under being dragged by the flows of X_{L_i}. But we have just proven that the flows of X_{L_i} foliate 2-spheres. Thus the dynamics of the system do not change after being dragged along spheres, or equivalently, *the system is invariant under a rotation.*

Thus the symplectic form has allowed us, not only to give an interpretation of non-commuting Poisson brackets, but also to relate a symmetry of the system to a conserved quantity. We define a mathematical tool to make this connection explicit in the following section.

4 The Momentum Map

In this section we formally develop the link between symmetries and conserved quantities: **the momentum map**. Before stating the definition of the momentum map, we explain how actions in phase space can be obtained from actions in configuration space.

4.1 The cotangent lift of an action

As we argued on section 2.3.1, actions represent the continuous transformations of the phase space where the dynamics occur. If this action does not modify H, we call such action a **symmetry**. We tend to think of actions in the configuration space Q, rather than in the cotangent space T^*Q where the Hamiltonian H is defined. For instance, when we think of a rotation, we generally define how to rotate the spatial components, but not the components of momentum. Nonetheless, the structure in the cotangent bundle inherited from the manifold itself, allows us promote any diffeomorphism in our configuration space to the phase space, as we saw in section 2.2.2. The Liouville one-form is preserved by these lifts.

Theorem 4.1. *Let $F : M \to M$ be a diffeomorphism and θ be the Liouville one-form. The one-form θ is invariant under the pull-back of the T^*F diffeomorphism [9]*

$$(T^*F)^*\theta = \theta.$$

[2] The map $dP = (p_1)^2+(p_2)^2+(p_3)^2-l^2$ yields a similar result, the ∂p_i components foliate the momentum space in 2-spheres. As we will see in section 4.3.3, a rotation in Q induces a rotation of each of the 2-spheres, so the symmetry argument works in the entire phase space actually.

Thus, lifting actions from our configuration space Q with this approach automatically preserves the structure of dynamics.

Corollary 4.2. *Consider an action Φ on the configuration space Q. The lift of the action $T^*\Phi$ preserves the symplectic form and thus the Poisson bracket. We call actions that preserve the symplectic form* ***canonical****.*

Proof. Recall that if Φ is an action, by definition 2.22 we immediately get that it is smooth and it has an inverse, so we have a diffeomorphism. By theorem 2.6, we know that $T^*\Phi$ is a diffeomorphism and that $(T^*\Phi)^*\theta = \theta$. This fact, combined with our local definition of the symplectic form $\omega := -d\theta$, yields

$$(T^*\Phi)^*\omega = (T^*\Phi)^*(-d\theta) = -d[(T^*\Phi)^*\theta] = -d\theta = \omega.$$

Hence, the pull-back by the cotangent lift of any action leaves ω invariant. $\square$

When computing the lift of a canonical action, we need to be wary about the direction of the map. Let G be a Lie group acting on Q by Φ_g for each $g \in G$. In section 2.2.2, we established that the cotangent lift $T^*\Phi_g$ goes in the opposite direction to Φ_g, so that points of the manifold where we would end up would not match.

$$\begin{tikzcd}
T^*Q \arrow[d, "\tilde{\pi}"'] \arrow[r, shift right, "T^*\Phi_{g^{-1}}"'] & T^*Q \arrow[l, shift right, "T^*\Phi_g"'] \arrow[d, "\tilde{\pi}"] \\
Q \arrow[r, shift left, "\Phi_g"] & Q \arrow[l, shift left, "\Phi_{g^{-1}}"]
\end{tikzcd}$$

As we can observe in the commutative diagram above, when we are thinking about the lift of the Φ_g action, we actually want to compute $(T^*\Phi_g)^{-1} = T^*\Phi_{g^{-1}}$ to map the same points in Q. Hence, *the (left) lift of the action is given by the cotangent lift of the inverse action.* Explicitly, for some $(\underline{q}, \underline{p}) \in T^*Q$, the lift of the action Φ_g acting on coordinates is given by

$$T^*\Phi_{g^{-1}} \star (\underline{q}, \underline{p}) = (\Phi_g(\underline{q}), \underline{p'}),$$

where "$\star$" denotes that there is an omitted coordinate map and $\underline{p'}$ are the momenta components after applying the lift of the action. With this construction in mind, we can dive into the definition of a momentum map.

4.2 The definition of a momentum map

The following definitions are stated in terms of **Poisson manifolds**, which are a generalisation of symplectic manifolds. A Poisson manifold P is a manifold endowed with a Poisson bracket which obeys the properties from definition 3.11. In this report we will restrict ourselves to symplectic manifolds, which are Poisson manifolds due to our developments from 3.4.3. The proper definition for Poisson manifolds is in appendix B.

Definition 4.3. *Let the Lie algebra $\mathfrak{g}$ act canonically on a Poisson manifold P. Suppose that there exists a linear map $J : \mathfrak{g} \to C^\infty(P)$ such that*

$$X_{J(\xi)} = \xi_P, \quad \textit{for all } \xi \in \mathfrak{g}. \tag{36}$$

*The dual map $\underline{J} : P \to \mathfrak{g}^*$ defined by $\langle \underline{J}(z), \xi \rangle = J(\xi)(z)$ is called a **momentum map** of the action.*

These allow us to think about Noether's theorem in terms of actions of Lie groups on our Poisson manifold.

Theorem 4.4. ***Hamiltonian version of Noether's theorem****. If the Lie algebra $\mathfrak{g}$ acts canonically on the Poisson manifold P, admits a momentum mapping $\underline{J} : P \to \mathfrak{g}^*$ and the Hamiltonian is $\mathfrak{g}$-invariant, i.e. $\xi_P(H) = 0$ for all $\xi \in \mathfrak{g}$, then $\underline{J}$ is a constant of the motion for H. This, in geometrical terms means that if φ_t is the flow of X_H, then [1]*

$$\underline{J} \circ \varphi_t = \underline{J}.$$

If the Lie algebra action comes from a canonical left Lie group action Φ, then the invariance hypothesis on H is implied by [1]

$$H \circ \Phi_g = H, \quad \textit{for all } g \in G. \tag{37}$$

Each Lie Group G, whose Lie algebra does not modify the Hamiltonian, provides us with a symmetry. From each of these symmetries we can construct the momentum map, and obtain one or multiple conserved quantities. Thus, by means of a geometrical interpretation of the structure underlying Hamiltonian dynamics, we have established a general method to find constants from symmetries in the system.

We now explicitly compute the momentum map for various examples, whose results are stated as examples in [1, 2], but not worked out explicitly in the literature.

4.3 Computations of momentum maps

4.3.1 Momentum map for a general Hamiltonian

On a Poisson manifold P, consider the flow $F_t : P \to P$ of the complete Hamiltonian vector field X_H. Think the vector flow as an action of $\mathbb{R}$ on P, as explained at the end of section

2.3.1. There is no need for a lift in this case as the Hamiltonian flow us tells us how to move in phase space.

The action of $\mathbb{R}$ via addition is represented by the 1-dimensional group of translations T^1 and its respective Lie algebra t^1. We will denote elements of $\mathfrak{g} \cong \mathbb{R}$ with a tilde, so that $\tilde{1} \in \mathfrak{g}$, which corresponds to $0 \in \mathbb{R}$, represents the identity. Besides, instead of writing the entire 2×2 matrix to refer the element of the Lie algebra, I will write the scalar it is associated to, so that the exponential map behaves as the identity map in this notation (also explained in section 2.3.1). We compute the infinitesimal action of $\tilde{1}$ on a point $\sigma \in P$ to obtain

$$\tilde{1}_P(\sigma) = \frac{d}{dt} F_{\exp(t\tilde{1})}(\sigma)\bigg|_{t=0} = \frac{d}{dt} F_{t\tilde{1}}(\sigma)\bigg|_{t=0} = \frac{d}{dt} F_{t+0}(\sigma)\bigg|_{t=0} = X_H(\sigma). \tag{38}$$

by the definition of flow. By equation (36), $\tilde{1}_P(\sigma) = X_{J(\tilde{1})} = X_H$, hence we conclude that $J(\tilde{1}) = H$. By linearity of J, we can deduce that

$$J(\tilde{k}) = \tilde{k}H,$$

for any $\tilde{k} \in \mathfrak{g}$. In the expressions below, I will keep $\underline{J}(\sigma)$ underlined to distinguish it from $J(\tilde{k})$, but it is a scalar function. Imposing the duality condition

$$\underline{J}(\sigma)\tilde{k} = J(\tilde{k})(\sigma) = \tilde{k}H \quad \Longleftrightarrow \quad k + \underline{J}(\sigma) = k + H(\sigma) \quad \Longleftrightarrow \quad \underline{J} = H.$$

Hence our momentum map is the Hamiltonian. Recall that the flow of X_H encodes the time evolution of the dynamical system. We have just proven that, if the system is invariant under time evolution, the Hamiltonian (its total energy for most cases) is conserved. This is another statement of one of the most fundamental laws of physics: conservation of energy.

4.3.2 Momentum map for linear translations

Consider the configuration space $Q = \mathbb{R}^3$ and the action of the translation Lie group T^3 on Q, given by

$$\Phi_{\underline{x}}(\underline{q}) = \underline{x} + \underline{q},$$

for all $\underline{x}, \underline{q} \in \mathbb{R}^3$. Notice that I am denoting elements in the Lie group by their corresponding vector. Since we want to study dynamics on the phase space $P = T^*\mathbb{R}^3$, we need the lift of the translation action. The push-forward for $\underline{v} \in T_qQ$ is given by

$$T\Phi_{\underline{x}}(\underline{v}) = \frac{d}{dt}\Big(\Phi_{\underline{x}}(\underline{q} + t\underline{v})\Big)\bigg|_{t=0} = \frac{d}{dt}\Big(\underline{x} + \underline{q} + t\underline{v}\Big)\bigg|_{t=0} = \underline{v} \implies T\Phi_{\underline{x}} \star (\underline{q}, \underline{v}) = (\underline{q} + \underline{x}, \underline{v}).$$

To obtain the lift, we want to compute $T^*\Phi_{-\underline{x}}$. Bearing in mind that the push-forward leaves $\underline{v}$ unchanged, we can readily deduce that

$$T^*\Phi_{-\underline{x}} \star (\underline{q}, \underline{p}) = (\underline{q} + \underline{x}, \underline{p}).$$

Remember that the lift points in the same direction as $\Phi_{\underline{x}}$, so $\underline{q} \mapsto \underline{q} + \underline{x}$ as well, despite the fact that we are considering the lift for $-\underline{x}$. We can now focus our attention on $\mathfrak{g} = t^3$, denoting each of its elements by the corresponding associated vector $\underline{\xi} \in \mathbb{R}^3$. In section 2.3.1, we saw that exp maps the element in t^3 associated to $\underline{\xi}$ to the element in T^3 associated to the same vector. Under this convention $\exp(\lambda\underline{\xi}) = -\lambda\underline{\xi}$ and we compute

$$\underline{\xi}_P((\underline{q},\underline{p})) = (T^*\Phi_{\exp(-\lambda\underline{\xi})} \star (\underline{q},\underline{p}))\Big|_{\lambda=0} = \frac{d}{d\lambda}(T^*\Phi_{-\lambda\underline{\xi}} \star (\underline{q},\underline{p}))\Big|_{\lambda=0} = \frac{d}{d\lambda}(\underline{q} + \lambda\underline{\xi}, p)\Big|_{\lambda=0} = (\underline{\xi}, 0).$$

Using (36) and the definition of vector field, we must have that

$$\frac{\partial J(\underline{\xi})}{\partial p_i} = \xi^i, \qquad -\frac{\partial J(\underline{\xi})}{\partial q^i} = 0.$$

Imposing linearity in J and setting the integration constant to zero (we are free to choose such constant [1]), we obtain

$$J(\underline{\xi}) = \underline{\xi} \cdot \underline{p},$$

where the dot denotes the usual inner product in $\mathbb{R}^3$. Identifying the dual of $\mathbb{R}^3$ by means of this inner product, we can conclude that our momentum map is $\underline{J} = \underline{p}$, the linear momentum. This is conservation of linear momentum: *if the Hamiltonian is invariant under spatial translations, the linear momentum is conserved.*

4.3.3 Momentum map for three dimensional rotations

Again, our configuration space is $Q = \mathbb{R}^3$, and the action of the rotation group $SO(3)$ on $\mathbb{R}^3$ is given by

$$\Phi_A(\underline{q}) = A\underline{q},$$

for any $A \in SO(3)$ and $\underline{q} \in \mathbb{R}^3$. Following the same procedure as before, we find the corresponding cotangent lift. For an arbitrary $\underline{v} \in T_q Q$, we compute

$$T\Phi_A(\underline{v}) = \frac{d}{d\lambda}\Big(\Phi_A(\underline{q} + \lambda\underline{v})\Big)\Big|_{\lambda=0} = \frac{d}{d\lambda}\Big(A\underline{q} + \lambda A\underline{v}\Big)\Big|_{\lambda=0} = A\underline{v} \implies T\Phi_A \star (\underline{q},\underline{v}) = (A\underline{q}, A\underline{v}).$$

Going back to the theorem 2.6, for any $\alpha \in T^*_{A^{-1}q}Q$ and $\underline{v} \in T_q Q$, we must have

$$[T^*\Phi_{A^{-1}}(\alpha)](\underline{v}) = \alpha(T\Phi_{A^{-1}}(\underline{v})) = \alpha(A^{-1}\underline{v}),$$

but since $\alpha \in T^*_{A^{-1}q}Q \approxeq (\mathbb{R}^3)^*$, using the inner product to identify duals with their respective vectors and looking at the momentum vector $\underline{p}$ instead of α, we have

$$[T^*\Phi_{A^{-1}}(\underline{p})](\underline{v}) = \underline{p} \cdot A^{-1}\underline{v} = (A^{-1})^T\underline{p} \cdot \underline{v}.$$

Hence, using that $A^T = A^{-1}$ because $A \in SO(3)$, we can conclude

$$T^*\Phi_{A^{-1}}(\underline{p}) = (A^{-1})^T\underline{p} = A\underline{p} \implies T^*\Phi_{A^{-1}} \star (\underline{q}, \underline{p}) = (A\underline{q}, A\underline{p}).$$

Computing the infinitesimal action for $\hat{\xi} \in so(3)$ yields

$$\hat{\xi}_P((\underline{q}, \underline{p})) = \frac{d}{d\lambda}(T^*\Phi_{\exp(-\lambda(\hat{\xi}))} \star (\underline{q}, \underline{p}))\Big|_{\lambda=0} = \frac{d}{d\lambda}(\exp(\lambda\hat{\xi})\underline{q}, \exp(\lambda\hat{\xi})\underline{p})\Big|_{\lambda=0} = (\hat{\xi}\underline{q}, \hat{\xi}\underline{p}). \quad (39)$$

The elements of $so(3)$ can be represented as 3×3 anti-symmetric real matrices, but we can identify its elements with vectors in $\mathbb{R}^3$ and describe their action on other vectors by taking the cross-product. This is done via the **hat map** $\wedge : \mathbb{R}^3 \to so(3)$, which establishes an isomorphism to $\mathbb{R}^3$ defined by [1]

$$\underline{v} = (v^1, v^2, v^3) \mapsto \hat{v} = \begin{pmatrix} 0 & -v^3 & v^2 \\ v^3 & 0 & -v^1 \\ -v^2 & v^1 & 0 \end{pmatrix}. \quad (40)$$

Under this isomorphism, we are identifying $\hat{v}\underline{w} = \underline{v} \times \underline{w}$ for all $\underline{w} \in \mathbb{R}^3$. By (36), (39) and the hamiltonian vector field definition, we obtain the following differential equations

$$\frac{\partial J(\hat{\xi})}{\partial p_i} = \hat{\xi}^i_j q^j, \qquad -\frac{\partial J(\hat{\xi})}{\partial q^i} = \hat{\xi}^i_j p^j,$$

where we wrote the matrix product $(\hat{\xi}\underline{q})^i = \hat{\xi}^i_j q^j$. We want to solve these having J linear in ξ. A solution, noting that $-\hat{\xi} = \hat{\xi}^T$, is given by

$$J(\hat{\xi})(\underline{q}, \underline{p}) = (\hat{\xi}\underline{q}) \cdot \underline{p} = (\underline{\xi} \times \underline{q}) \cdot \underline{p} = (\underline{q} \times \underline{p}) \cdot \underline{\xi}.$$

Thus, using the same inner product identification, we can conclude that $\underline{J}(\underline{q}, \underline{p}) = \underline{q} \times \underline{p}$, the angular momentum. *If the Hamiltonian is invariant under rotations, the angular momentum is conserved.*

We have thus proven that in problems with spherical symmetry, such as the Kepler Orbit problem, the angular momentum is conserved. The momentum map explicitly represents the link from spherical symmetry to conservation of angular momenta we described in section 3.4.4.

Now, we step away from well-known results and compute more general momentum maps.

4.3.4 Momentum map for se(2) group acting on the phase space plane

To begin with, we define the special euclidean group $SE(2)$ and state some of its properties. The group $SE(2)$ consists of matrices of the form

$$(R_\theta, \underline{a}) := \begin{pmatrix} R_\theta & \underline{a} \\ 0 & 1 \end{pmatrix}, \text{ where } \underline{a} \in \mathbb{R}^2 \text{ and } R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The identity is given by the 3×3 identity matrix and the inverse by

$$(R_\theta, \underline{a})^{-1} := \begin{pmatrix} R_\theta & \underline{a} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} R_{-\theta} & -R_{-\theta}\,\underline{a} \\ 0 & 1 \end{pmatrix}.$$

Taking the derivative with respect to θ at the identity of R_θ, we can deduce that $se(2)$ consists of 2×2 matrices given by [1]

$$\begin{pmatrix} -\omega\mathbb{J} & -\underline{v} \\ 0 & 0 \end{pmatrix}, \text{ where } \omega \in R,\ \underline{v} = (v^1, v^2)^T \in \mathbb{R}^2 \text{ and } \mathbb{J} = -\mathbb{J}^T = -\mathbb{J}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For matrices A, B in matrix Lie groups, the canonical pairing $\langle A, B\rangle := \mathrm{Tr}(A^T B)$ [3], so the elements of $se^*(2)$ are given by

$$\begin{pmatrix} \frac{\mu\mathbb{J}}{2} & 0 \\ \underline{\alpha} & 0 \end{pmatrix}, \text{ where } \mu \in \mathbb{R} \text{ and } \alpha \in (\mathbb{R}^3)^* \cong \mathbb{R}^3.$$

We can identify algebras with $\mathbb{R}^3$ using the maps

$$\begin{pmatrix} -\omega\mathbb{J} & \underline{v} \\ 0 & 0 \end{pmatrix} \mapsto (\omega, \underline{v}) \in \mathbb{R}^3 \text{ and } \begin{pmatrix} \frac{\mu\mathbb{J}}{2} & 0 \\ \underline{\alpha} & 0 \end{pmatrix} \mapsto (\mu, \underline{\alpha}) \in \mathbb{R}^3,$$

so that the pairing after this identification in $\mathbb{R}^3$ is given by

$$\langle(\mu, \underline{\alpha}), (\omega, \underline{v})\rangle = \mu\omega + \underline{\alpha} \cdot \underline{v}. \tag{41}$$

A final comment before starting the computations on the exponential map acting on this group. Evaluating the powers of the matrix in Mathematica yields

$$\exp((\omega, \underline{v})) = \begin{pmatrix} \cos(\omega) & -\sin(\omega) & \frac{v^1\sin(\omega)+v^2(\cos(\omega)-1)}{\omega} \\ \sin(\omega) & \cos(\omega) & \frac{v^1(1-\cos(\omega))+v^2\sin(\omega)}{\omega} \\ 0 & 0 & 1 \end{pmatrix} = (R_\omega, \underline{\tilde{v}}), \tag{42}$$

where

$$\underline{\tilde{v}}_\omega = \left(\frac{v^1\sin(\omega) + v^2(\cos(\omega) - 1)}{\omega}, \frac{v^1(1 - \cos(\omega)) + v^2\sin(\omega)}{\omega}\right).$$

We now compute the momentum map corresponding to the action of $SE(2)$ on the plane configuration space described by points $(q, p) = \underline{z} \in \mathbb{R}^2$. Note that there is no need to compute lifts, as $SE(2)$ acts directly on $\mathbb{R}^2$ by

$$\Phi_{(R_\theta,\underline{a})}(\underline{z}) = R_\theta\underline{z} + \underline{a}.$$

Consider $\xi \in se(2)$ given by

$$\xi = \begin{pmatrix} -\omega\mathbb{J} & \underline{v} \\ 0 & 0 \end{pmatrix} \mapsto (\omega, \underline{v}) \in \mathbb{R}^3,\ \text{with } \underline{v} = (v^1, v^2).$$

[3]This pairing is canonical as we can apply it for any real $(m \times n)$-matrices A and B. Notice that $A^T B$ yields an square matrix, and $Tr(A^T B)$ defines an inner-product for real matrices, as it is checked in [17].

The corresponding infinitesimal action gives

$$\xi_{\mathbb{R}^2}(\underline{z}) = \frac{d}{d\lambda}\left(\Phi_{\exp(\lambda(\omega,\underline{v}))}\right)(\underline{z})\Big|_{\lambda=0} = \frac{d}{d\lambda}\left(\Phi_{(R_{\lambda\omega},\underline{\tilde{v}}_{\lambda\omega})}\right)(\underline{z})\Big|_{\lambda=0} = \frac{d}{d\lambda}\left(R_{\lambda\omega}\underline{z} + \underline{\tilde{v}}_{\lambda\omega}\right)\Big|_{\lambda=0}$$

$$= \frac{d}{d\lambda}\left[\begin{pmatrix} \cos(\lambda\omega) & -\sin(\lambda\omega) \\ \sin(\lambda\omega) & \cos(\lambda\omega) \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} \frac{v^1\sin(\lambda\omega)+v^2(\cos(\lambda\omega)-1)}{\omega} \\ \frac{v^1(1-\cos(\lambda\omega))+v^2\sin(\lambda\omega)}{\omega} \end{pmatrix}\right]\Big|_{\lambda=0}.$$

Taking the derivative and evaluating at $\lambda = 0$ yields

$$= \left[\omega\begin{pmatrix} -\sin(\lambda\omega) & -\cos(\lambda\omega) \\ \cos(\lambda\omega) & -\sin(\lambda\omega) \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} v^1\cos(\lambda\omega) - v^2\sin(\lambda\omega) \\ v^1\sin(\lambda\omega)) + v^2\cos(\lambda\omega) \end{pmatrix}\right]\Big|_{\lambda=0}$$

$$= \omega\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} -\omega p + v^1 \\ \omega q + v^2 \end{pmatrix}.$$

Using equation (36), we obtain the two differential equations

$$\frac{\partial J(\underline{\xi})}{\partial p} = -\omega p + v^1 \qquad -\frac{\partial J(\underline{\xi})}{\partial q} = \omega q + v^2.$$

Integration whilst keeping the solution linear in $\underline{\xi}$, gives us

$$J(\underline{\xi})(q,p) = -\frac{\omega}{2}(p^2+q^2) + pv^1 - qv^2 = (\omega, \underline{v}) \cdot \left(\frac{-1}{2}(p^2+q^2), p, -q\right).$$

By means of (41), this can be rewritten as

$$\langle(\omega,\underline{v}), \left(\frac{-1}{2}(p^2+q^2), p, -q\right)\rangle \implies \underline{J}(q,p) = \left(\frac{-1}{2}(p^2+q^2), p, -q\right),$$

by the definition of momentum map. This matches the result for the action of $se(2)$ on $\mathbb{R}^2$ from the exercise 11.4-3 in [1]. Note that this vector provides us with three conserved quantities.

4.3.5 Momentum map for 3-dimensional euclidean group

Here we regard the action $SE(3) \cong SO(3) \ltimes \mathbb{R}^3$ acting on $Q = \mathbb{R}^3$. From its definition, every $g \in SE(3)$ can be represented as $(A, \underline{x})$, with $A \in SO(3)$ and $\underline{x} \in \mathbb{R}^3$ and with multiplication and inverses given by [1]

$$(A,\underline{x})(B,\underline{y}) = (AB, A\underline{y} + \underline{x}). \quad \text{ and } (A,\underline{x})^{-1} = (A^{-1}, -A^{-1}\underline{x}). \tag{43}$$

Then the group action is given by

$$\Phi_{(A,\underline{x})}(\underline{q}) = A\underline{q} + \underline{x},$$

a rotation followed by a translation. Contrary to the previous case, we do need to compute a lift, as the action is defined on Q. We immediately know where the point $\underline{q}$ will be sent and from $SO(3)$ computations, translations do not modify the momenta yet rotations do rotate the momentum vector. Hence, we deduce that the cotangent lift of the action is

$$T^*\Phi_{(A,\underline{x})^{-1}} \star (\underline{q},\underline{p}) = (A\underline{q}+\underline{x}, A\underline{p}). \tag{44}$$

The elements of the Lie algebra $se(3)$ can be represented by $(\hat{\omega},\underline{a})$ with $\hat{\omega} \in so(3)$ and $\underline{a} \in \mathbb{R}^3$ [2], where the hat map is as in (40). Note that this automatically provides us with an identification to $\mathbb{R}^6$ by means of the isomorphism $(\hat{\omega},\underline{a}) \mapsto (\underline{\omega},\underline{a})$, with

$$\langle(\underline{\omega},\underline{a}),(\underline{\psi},\underline{b})\rangle = \underline{\omega}\cdot\underline{\psi} + \underline{a}\cdot\underline{b}.$$

As a final computation to set up the problem, we consider out the exponent map of an element from $se(3)$. Let $\xi \in se(3)$ be represented by $(\hat{\omega},\underline{a})$. By Rodrigues' Formula, the exponential map of ξ yields

$$\exp(\xi) = (\exp(\hat{\omega}), V\underline{a}), \text{ with } V = \mathbb{1}_3 + \frac{1-\cos(|\underline{\omega}|)}{|\underline{\omega}|^2}\hat{\omega} + \frac{|\underline{\omega}|-\sin(|\underline{\omega}|)}{|\underline{\omega}|^3}\hat{\omega}^2. \tag{45}$$

With this setup in mind, we compute the infinitesimal action. Let $\xi \in se(3)$ be represented by $(\hat{\omega},\underline{a})$. Then

$$\hat{\xi}_P((\underline{q},\underline{p})) = \frac{d}{d\lambda}(T^*\Phi_{\exp(-\lambda(\hat{\omega},\underline{a}))} \star (\underline{q},\underline{p}))\Big|_{\lambda=0} = \frac{d}{d\lambda}(T^*\Phi_{\exp((-\lambda\hat{\omega},-\lambda\underline{a}))} \star (\underline{q},\underline{p}))\Big|_{\lambda=0}. \tag{46}$$

By (45), we have that $\exp((-\lambda\hat{\omega},-\lambda\underline{a})) = (\exp(-\lambda\hat{\omega}), -\lambda V_\lambda \underline{a})$, where V_λ denotes the λ dependence from $-\lambda\hat{\omega}$ explicitly. To apply (44), we need to express $(\exp(-\lambda\hat{\omega}), -\lambda V_\lambda\underline{a})$ as the inverse of an element. We know that if $g \in G$, then $g = (g^{-1})^{-1}$. Hence, taking the inverse twice and using (43) yields

$$\exp((-\lambda\hat{\omega},-\lambda\underline{a})) = [(\exp(-\lambda\hat{\omega}), -\lambda V_\lambda\underline{a})^{-1}]^{-1} = (\exp(\lambda\hat{\omega}), \exp(\lambda\hat{\omega})\lambda V_\lambda\underline{a})^{-1}.$$

The cotagent lift of this inverse element as described in (44), when substituted in (46) yields

$$\frac{d}{d\lambda}(T^*\Phi_{\exp(-\lambda(\hat{\omega},\underline{a}))} \star (\underline{q},\underline{p}))\Big|_{\lambda=0} = \frac{d}{d\lambda}(\exp(\lambda\hat{\omega})\underline{q} + \exp(\lambda\hat{\omega})\lambda V_\lambda\underline{a}, \exp(\lambda\hat{\omega})\underline{p})\Big|_{\lambda=0}. \tag{47}$$

We compute the derivative of each type of term separately. The derivatives of exponent maps multiplying the vectors $\underline{q}$ and $\underline{p}$ yield

$$\frac{d}{d\lambda}(\exp(\lambda\hat{\omega})\underline{q})|_{\lambda=0} = \exp(0)\hat{\omega}\underline{q} = \hat{\omega}\underline{q} \quad \text{and} \quad \frac{d}{d\lambda}(\exp(\lambda\hat{\omega})\underline{p})|_{\lambda=0} = \exp(0)\hat{\omega}\underline{p} = \hat{\omega}\underline{p}.$$

Also, by (45) the matrix λV_λ is

$$\lambda V_\lambda = \left[\lambda\mathbb{1}_3 + \frac{\cos(\lambda|\underline{\omega}|)-1}{|\underline{\omega}|^2}\hat{\omega} + \frac{\lambda|\underline{\omega}|-\sin(\lambda|\underline{\omega}|)}{|\underline{\omega}|^3}\hat{\omega}^2\right].$$

From here we deduce that $\lambda V_\lambda|_{\lambda=0} = 0$. Then, taking the derivative gives

$$\frac{d}{d\lambda}\left[\lambda \mathbb{1}_3 + \frac{\cos(\lambda|\underline{\omega}|)-1}{|\underline{\omega}|^2}\hat{\omega} + \frac{\lambda|\underline{\omega}| - \sin(\lambda|\underline{\omega}|)}{|\underline{\omega}|^3}\hat{\omega}^2\right]\Bigg|_{\lambda=0}$$
$$= \left[\mathbb{1}_3 + \frac{-|\underline{\omega}|\sin(\lambda|\underline{\omega}|)}{|\underline{\omega}|^2}\hat{\omega} + \frac{|\underline{\omega}| - |\underline{\omega}|\cos(\lambda|\underline{\omega}|)}{|\underline{\omega}|^3}\hat{\omega}^2\right]\Bigg|_{\lambda=0} = \mathbb{1}_3 + 0 + \frac{|\underline{\omega}| - |\underline{\omega}|}{|\underline{\omega}|^3} = \mathbb{1}_3.$$

So we can evaluate the term

$$\frac{d}{d\lambda}(\exp(\lambda\hat{\omega})\lambda V_\lambda \underline{a})|_{\lambda=0} = [\hat{\omega}\exp(0)[(\lambda V_\lambda)]|_{\lambda=0}]\underline{a} + \exp(0)\frac{d}{d\lambda}\left[(\lambda V_\lambda)|_{\lambda=0}\right]\underline{a} = \underline{a}.$$

Finally, substituting the derivatives in (47) we conclude that

$$\hat{\xi}_P((\underline{q},\underline{p})) = \frac{d}{d\lambda}(\exp(\lambda\hat{\omega})\underline{q} + \exp(\lambda\hat{\omega})\lambda V_\lambda \underline{a}, \exp(\lambda\hat{\omega})\underline{p})|_{\lambda=0} = (\hat{\omega}\underline{q} + \underline{a}, \hat{\omega}\underline{p}).$$

By means of (36), we obtain the following differential equations

$$\frac{\partial J(\hat{\xi})}{\partial p_i} = \hat{\omega}^i_j q^j + a^i, \qquad -\frac{\partial J(\hat{\xi})}{\partial q^i} = \hat{\omega}^i_j p^j,$$

where we wrote the i-th component of the matrix product $(\hat{\omega}\underline{q})^i = \hat{\omega}^i_j q^j$. Bearing in mind that the solution for J must be linear in $\hat{\xi}$ or in its representation in $\mathbb{R}^6$: $(\underline{\omega},\underline{a})$, we can combine the results from previous computations to obtain the solution

$$J(\hat{\omega})(\underline{q},\underline{p}) = (\hat{\omega}\underline{q})\cdot\underline{p} + \underline{p}\cdot\underline{a} = (\underline{\omega}\times\underline{q})\cdot\underline{p} + \underline{p}\cdot\underline{a} = (\underline{q}\times\underline{p})\cdot\underline{\omega} + \underline{p}\cdot\underline{a},$$

and denoting by $\cdot_{\mathbb{R}^6}$ the inner product in $\mathbb{R}^6$, this can be rewritten as

$$= ((\underline{q}\times\underline{p}),\underline{p})\cdot_{\mathbb{R}^6}(\underline{\omega},\underline{a}) = ((\underline{q}\times\underline{p}),\underline{p})\cdot_{\mathbb{R}^6}\underline{\xi}.$$

By the same inner product identification, we can conclude that $\underline{J}(\underline{q},\underline{p}) = ((\underline{q}\times\underline{p}),\underline{p})$. This is a rather general statement which matches what we found in the previous two examples. *If our system is invariant under both rotations and translations, both the angular momentum and the linear momentum must be independently conserved.*

We have thus shown that the momentum map provides us with general method to find conserved quantities from symmetries in our system. As we discussed in section 3.1, each independently conserved quantity allows us to reduce the degrees of freedom of the system. In the final part of this project, we will employ momentum maps to explicitly simplify the dynamics of the system.

4.4 Symplectic reduction

4.4.1 The ideas behind reduction

Symmetries give us conserved quantities in the form of momentum maps. We now think of how to explicitly reduce the dimensions of the space where the dynamics take place, the phase space. The key idea is the following: *we can restrict ourselves to a fixed value of the momentum map we are considering, as the flow of X_H preserves the momentum map.* So we can work out dynamics for a fixed value of the momentum map instead.

Consider a symplectic manifold (M, ω) with G a Lie Group acting symplectically on M via Φ_g, for all $g \in G$. Denote by $\underline{J} : M \to \mathfrak{g}^*$ the momentum map for this action. Several subtle conditions are required for the upcoming theorem to hold, although we will consider an example where are of these are true. For completeness, the requirements are

- Ad^***-equivariance** of $\underline{J}$, which is automatically granted if G is compact [1].
- The action preserves compact sets (**proper**), and for all $x, y \in M$ we have a $g \in G$ with $\Phi_g(x) = y$ (**transitive**) [18].

Fix the momentum map to some value $\mu \in \mathfrak{g}^*$ and suppose that μ is a regular value of $\underline{J}$. By the Regular Value Theorem (theorem 2.7), this means $\underline{J}^{-1}(\mu)$ is a submanifold of M with dimension $\dim(M) - \dim(\mathfrak{g}^*) = \dim(M) - \dim(G)$. If we manage to express all of our constructions in terms of coordinates of this submanifold, we would have reduced the dynamics of the system to a system with less degrees of freedom.

But we can go one step further. Consider the orbit of the lifted action $G \cdot \sigma$ and the stabiliser G_μ of μ under the coadjoint action

$$G \cdot \sigma = \{\Phi_g(\sigma) | g \in G\}, \text{ and } G_\mu := \{g \in G | Ad^*_{g^{-1}}\mu = \mu\}.$$

Notice that the stabiliser is a subgroup of G, whilst each orbit is a submanifold of the phase space and these submanifolds also foliate the phase space [1]. The subset $G \cdot \sigma$ represents where each $\sigma \in P$ can be sent via G. In particular, $\sigma \in G \cdot \sigma$ as $\Phi_e(\sigma) = \sigma$. Amongst all these points from orbits, we know that the flow of X_H will go through those that yield $\underline{J}(\sigma) = \mu$, so they must live in $\underline{J}^{-1}(\mu)$. In fact, one can show that each $G \cdot \sigma$ intersects $\underline{J}^{-1}(\mu)$ cleanly, giving the set [18]

$$G_\mu \cdot \sigma = (G \cdot \sigma) \cap (\underline{J}^{-1}(\mu)).$$

Since the Hamiltonian is invariant under G, all the points in $\underline{J}^{-1}(\mu)$ related by an element from G_μ will yield the same dynamics as a result, these are equivalent. Hence, we can carry out a second reduction by identifying all these points under an equivalence relation denoted by "$\sim$". Let $\sigma_1, \sigma_2 \in \underline{J}^{-1}(\mu)$. Then

$$\sigma_1 \sim \sigma_2 \iff \Phi_g(\sigma_1) = \sigma_2, \text{ for some } g \in G_\mu.$$

After this identification, we know that dynamics will occur in $\underline{J}^{-1}(\mu)/G_\mu$. The upcoming theorem tells us that this space is a submanifold, where the reduced version of ω can also be obtained.

Theorem 4.5. ***Marsden-Weinstein-Meyer theorem***. *Let (M, ω) be a symplectic manifold, where a Hamiltonian H is defined. Let the Lie group G act on M. If H is G-invariant, and the action defines a momentum map $\underline{J}$ fulfilling all the conditions we have discussed above, we have [18, 19]*

- ***Symplectic reduction***. *Let $\mu \in \mathfrak{g}^*$, and denote by G_μ the stabilizer of the coadjoint action for μ, which acts properly and transitively on $\underline{J}^{-1}(\mu)$. Then $\underline{J}^{-1}(\mu)$ is a G_μ-invariant submanifold of M, and the orbit space ${\underline{J}^{-1}(\mu)}/{G_\mu}$ is a symplectic manifold with a reduced symplectic form defined by*

$$\rho_\mu^* \omega_\mu = \zeta_\mu^* \omega,$$

where $\zeta_\mu : \underline{J}^{-1}(\mu) \to M$ and $\rho_\mu : \underline{J}^{-1}(\mu) \to {\underline{J}^{-1}(\mu)}/{G_\mu}$.

- ***Reduction of dynamics***. *The Hamiltonian is also simplified to a corresponding expression in ${\underline{J}^{-1}(\mu)}/{G_\mu}$. The **reduced Hamiltonian** h_μ is given by*

$$h_\mu \circ \rho_\mu = H \circ \zeta_\mu.$$

We now apply the symplectic reduction to the problem that has been our guiding thread throughout the entire project.

4.4.2 Symplectic reduction of the Kepler orbit problem

Recalling the setup from section 3.3, we have the Hamiltonian

$$H = \frac{\underline{p} \cdot \underline{p}}{2m} - \frac{\kappa}{\sqrt{\underline{q} \cdot \underline{q}}}, \text{ where } \underline{q} = (q^1, q^2, q^3) \text{ and } \underline{p} = (p_1, p_2, p_3).$$

In section 4.3.3, we computed the lifted action of $SO(3)$ on $T^*\mathbb{R}^3$. Since for any $A \in SO(3)$, mapping $(\underline{q}, \underline{p}) \mapsto (A\underline{q}, A\underline{p})$ gives

$$A \cdot H = \frac{A\underline{p} \cdot A\underline{p}}{2m} - \frac{\kappa}{\sqrt{A\underline{q} \cdot A\underline{q}}} = \frac{\underline{p} \cdot \underline{p}}{2m} - \frac{\kappa}{\sqrt{\underline{q} \cdot \underline{q}}} = H,$$

and thus the momentum map $\underline{J}(\underline{q}, \underline{p}) = \underline{L}$ is conserved. For simplicity, fix $\underline{L}$ and align the axes so that $\underline{L} = (0, 0, L_3)$. Then, the pre-image of $\underline{J}$ is given by

$$\underline{J}^{-1}(\underline{L}) = \{(q^1, q^2, q^3, p_1, p_2, p_3) \in \mathbb{R}^6 | \; q^1 p_2 - q^2 p_1 = L_3, q^3 = p_3 = 0\}. \tag{48}$$

Due to the spherical symmetry of the problem, it will be useful to consider the canonical variables in terms of polar coordinates. In such coordinates, the Hamiltonian is given by [11]

$$H = \frac{p_r^2}{2m} + \frac{p_\theta}{2mr^2} - \frac{\kappa}{r}, \text{ where } p_r = m\dot{r} \text{ and } p_\theta = mr^2\dot{\theta}.$$

We now change from cartesian to these coordinates on the plane with $q^3 = 0$. We have

$$q^1 = r\cos(\theta) \implies p_1 = m\dot{q}^1 = m(\dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)),$$
$$q^2 = r\sin(\theta) \implies p_2 = m\dot{q}^2 = m(\dot{r}\sin(\theta) - r\dot{\theta}\sin(\theta)).$$

Under these changes, the condition on (48) becomes

$$L_3 = mr\cos(\theta)[\dot{r}\sin(\theta) + r\dot{\theta}\cos(\theta)] - mr\sin(\theta)[\dot{r}\cos(\theta) + r\dot{\theta}\sin(\theta)] = mr^2\dot{\theta} = p_\theta \implies$$
$$\implies J^{-1}(\underline{L}) = \{(r,\theta,\phi,p_r,p_\theta,p_\phi) \in \mathbb{R}^6 |\ \phi = \frac{\pi}{2}, p_\phi = 0, p_\theta = mr^2\dot{\theta}\} \cong \{(r,\theta,p_r) \in \mathbb{R}^3\}.$$

Since three of the quantities are fixed, we only need to consider the three remaining variables to describe the pre-image. By theorem 4.5, we have $\zeta_{\underline{L}} : \underline{J}^{-1}(\underline{L}) \to T^*\mathbb{R}^3$, which in this case is defined by

$$\begin{aligned}\zeta_{\underline{L}}[(r,\theta,p_r)] &= (r\cos(\theta), r\sin(\theta), \frac{\pi}{2}, m(\dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)), m(\dot{r}\sin(\theta) - r\dot{\theta}\sin(\theta)), 0) \\ &= (r\cos(\theta), r\sin(\theta), \frac{\pi}{2}, p_r\cos(\theta) - \frac{p_\theta}{r}\sin(\theta), p_r\sin(\theta) - \frac{p_\theta}{r}\dot{\theta}\sin(\theta), 0).\end{aligned}$$

Then, using theorem 2.14 and, the pull-back of ω in $T^*\mathbb{R}^3$ is given by

$$\begin{aligned}\zeta_{\underline{L}}^*\omega &= \zeta_{\underline{L}}^*(dq^i \wedge dp_i) = \zeta_{\underline{L}}^*(dq^i) \wedge \zeta_{\underline{L}}^*(dp_i) = d(q^i \circ \zeta_{\underline{L}}) \wedge d(p_i \circ \zeta_{\underline{L}}) \\ &= d(q^1 \circ \zeta_{\underline{L}}) \wedge d(p_1 \circ \zeta_{\underline{L}}) + d(q^2 \circ \zeta_{\underline{L}}) \wedge d(p_2 \circ \zeta_{\underline{L}}),\end{aligned} \tag{49}$$

as $dq^3 = dp_3 = 0$. So we need to express the remaining cartesian coordinates in terms of the embedding and then take the exterior derivatives. By (8) we obtain that

$$\begin{aligned}&d(q^1 \circ \zeta_{\underline{L}}) = \cos(\theta)dr - r\sin(\theta)d\theta, \quad d(q^2 \circ \zeta_{\underline{L}}) = \sin(\theta)dr + r\cos(\theta)d\theta, \\ &d(p_1 \circ \zeta_{\underline{L}}) = \cos(\theta)dp_r - p_r\sin(\theta)d\theta + \frac{p_\theta}{r^2}\sin(\theta)dr - \frac{p_\theta}{r}\cos(\theta)d\theta, \\ &d(p_2 \circ \zeta_{\underline{L}}) = \sin(\theta)dp_r - p_r\cos(\theta)d\theta - \frac{p_\theta}{r^2}\cos(\theta)dr - \frac{p_\theta}{r}\sin(\theta)d\theta.\end{aligned}$$

The terms in (49), after using the anti-symmetry of wedge product and after some cancellations, become

$$\begin{aligned}d(q^1 \circ \zeta_{\underline{L}}) \wedge d(p_1 \circ \zeta_{\underline{L}}) &= \cos^2(\theta)dr \wedge dp_r - \frac{p_\theta}{r}\cos^2(\theta)dr \wedge d\theta + \frac{p_\theta}{r}\sin^2(\theta)dr \wedge d\theta, \\ d(q^2 \circ \zeta_{\underline{L}}) \wedge d(p_2 \circ \zeta_{\underline{L}}) &= \sin^2(\theta)dr \wedge dp_r - \frac{p_\theta}{r}\sin^2(\theta)dr \wedge d\theta + \frac{p_\theta}{r}\cos^2(\theta)dr \wedge d\theta.\end{aligned}$$

Thus, using the identity $\cos^2(\theta) + \sin^2(\theta) = 1$, we can conclude that

$$\zeta_{\underline{L}}^*\omega = dr \wedge dp_r - \frac{p_\theta}{r}dr \wedge d\theta + \frac{p_\theta}{r}dr \wedge d\theta = dr \wedge dp_r. \tag{50}$$

Notice that $\zeta_{\underline{L}}^*\omega$ does not fulfill the non-degeneracy property because even though θ is a coordinate, there is no $d\theta$ term in its expression. If we consider

$$\partial_\theta := \frac{\partial}{\partial\theta} \in T(\underline{J}^{-1}(\underline{L})) \rightsquigarrow \iota_{\partial_\theta}(\zeta_{\underline{L}}^*\omega) = 0.$$

Thus taking the quotient is crucial to obtain a symplectic form. In this example

$$G_{\underline{L}} = \{A \in SO(3) |\ A\underline{L} = \underline{L}\} = SO(2),$$

i.e. rotations in the $x - y$ plane fix $\underline{L} = (0, 0, L_3)$. Thus, taking the quotient here is equivalent to identifying all the possible values of θ for any r and p_r. Hence the quotient map $\rho_{\underline{L}} : \underline{J}^{-1}(\underline{L}) \to \underline{J}^{-1}(\underline{L}) / G_{\underline{L}}$ is given by $\rho_{\underline{L}}(r, \theta, p_r) := (r, p_r)$, and we need to satisfy the condition $\rho_{\underline{L}}^* \omega_{\underline{L}} = \zeta_{\underline{L}}^* \omega$.

Consider $Y, Z \in \mathfrak{X}(J^{-1}(\underline{L}))$. By definition 2.10, we have that

$$\rho_{\underline{L}}^* \omega_{\underline{L}}(Y, Z) = \omega_{\underline{L}}(T\rho_{\underline{L}}(Y), T\rho_{\underline{L}}(Z)).$$

Linearity of push-forwards, tells us that we only need to compute where the basis elements of $T(\underline{J}^{-1}(\underline{L}))$ are sent. Using (3) with the quotient map $\zeta_{\underline{L}}$, we have that for each $\sigma \in \underline{J}^{-1}(\underline{L})$

$$T_\sigma \rho_{\underline{L}} \left(\frac{\partial}{\partial r}\bigg|_\sigma \right) = \frac{\partial}{\partial r}\bigg|_{\rho_{\underline{L}}(\sigma)}, \quad T_\sigma \rho_{\underline{L}} \left(\frac{\partial}{\partial p_r}\bigg|_\sigma \right) = \frac{\partial}{\partial p_r}\bigg|_{\rho_{\underline{L}}(\sigma)}, \quad T_\sigma \rho_{\underline{L}} \left(\frac{\partial}{\partial \theta}\bigg|_\sigma \right) = 0.$$

This means that the partial derivatives of r and p_r simply mapped to the corresponding point, whilst the partial derivative of θ vanishes. Since we want to have $\rho_{\underline{L}}^* \omega_{\underline{L}} = \zeta_{\underline{L}}^* \omega = dr \wedge dp_r$, the computations from above allow us to conclude that $\omega_{\underline{L}} = dr \wedge dp_r$.

Notice that the degeneracy issue has been solved as $\omega_{\underline{L}}$ is defined in $\underline{J}^{-1}(\underline{L}) / G_{\underline{L}}$. Then, we only require r the radial distance and $p_r = m\dot{r}$ the radial speed to determine the type of orbital motion, which is what we expected.

As a last computation, we find the reduced Hamiltonian $h_{\underline{L}}$. We have that

$$\begin{aligned} H \circ \zeta_{\underline{L}}(r, p_r, \theta) &= \frac{1}{2m} [(p_r \cos(\theta) - \frac{p_\theta}{r} \sin(\theta))^2 + (p_r \sin(\theta) - \frac{p_\theta}{r} \cos(\theta))^2] - \frac{\kappa}{r} \\ &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{\kappa}{r}. \end{aligned}$$

Then, we want

$$h_{\underline{L}} \circ \rho_{\underline{L}} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{\kappa}{r}.$$

Since $p_\theta = L_3$, the R.H.S. does not depend on θ and taking the quotient map changes nothing. Therefore

$$h_{\underline{L}}(r, p_r) = \frac{p_r^2}{2m} + \frac{(L_3)^2}{2mr^2} - \frac{\kappa}{r},$$

making explicit that the dynamics only depends on (r, p_r).

5 Conclusion

In this report we have determined the conditions required for integrability in the Hamiltonian approach, whilst assessing the limitations of the integral solutions. Then, we have adopted the Symplectic Geometry perspective, established a geometrical meaning for the Poisson bracket and constructed the momentum map. This tool allowed both relating the conservation of physical quantities to symmetries and carrying out the symplectic reduction of the system.

We began by introducing the language of coordinate-free physics: Differentiable Manifolds. These allow us to study of physics in more exotic spaces while supporting the notion of change with their differentiable structure. Then we established how transformations of the dynamical systems, such as translations and rotations, must follow a specific set of rules and are formalised via the notion of Lie groups.

Once we had acquired the arsenal of tools we would need, we dived into answering the questions that motivated this project.

Determining whether a general mechanics problem is solvable is a non-trivial task. With the aim of providing a concrete answer, we focused on the Hamilonian dynamics framework, where Liouville and Arnold had developed a theory of integrability. In the phase space, Hamilton's canonical equations provide us with a set of differential equations, and the independently conserved quantities allow us to integrate such equations to find the solution. In particular, if f denotes the number of degrees of freedom of the dynamical system, we can have at most $2f-1$ independently conserved quantities to observe dynamics. Nonetheless, by the Liouville-Arnold theorem, it suffices to have f independently conserved quantities in involution, to be able to integrate by quadratures.

We then put to test our conclusions, exploring the Kepler Orbit problem. We wanted to study the 3-dimensional movement of a planet around a fixed Sun, so that $f = 3$ in this case. Even though H and the components of $\underline{L}$ and $\underline{R}$ are all conserved, we found two equations relating their components. Thus, we proved that we could have at most a set of $2f-1 = 2*3-1 = 5$ independently conserved quantities, for instance $\{H, L_1, L_3, L^2, R^1\}$. Since the independently conserved quantities $\{H, L_3, L^2\}$ Poisson commute, this problem fulfills the criteria from Arnold Liouville theorem to be integrable. Nonetheless, integrability by quadratures only ensures solvability in open intervals around turning and critical points. To construct $r(t)$ for all times, we need to consider open time intervals near these special points separately, integrate in each interval, invert $t(r)$ into $r(t)$ and then glue all the solutions together. A one-function solution may require other set of coordinates not involving time, as for instance $r(\theta)$.

After highlighting the need of conserved quantities, we developed the second part of the project: geometry. Our aim was to understand the structure underlying Hamilton's canonical equations, hoping to acquire some geometrical intuition on how dynamics work and the meaning behind the Poisson bracket construction.

The symplectic form ω defines a Hamiltonian vector field X_f for each $f \in C^\infty(T^*Q)$, with X_H encoding Hamilton's canonical equations. Geometrically, the integral curves of X_f define a set of paths along phase space, which we call flow. Then, every f induces a way to move along phase space via this flow. Given any $f, g \in C^\infty(T^*Q)$, we can geometrically interpret their Poisson bracket if we instead regard how each quantity is modified along the induced flow of the other, or equivalently, evaluate their Lie bracket: $\pounds_{X_f}(X_g) = [X_f, X_g]$.

In particular $[X_f, X_g] = 0 \iff \{f, g\} = 0$. If a quantity Poisson commutes with H, then it is fixed along the flow of X_H, which determines the time evolution of the system. This is what the conservation of a quantity geometrically means. On the other hand, if a set of conserved quantities have vanishing Poisson brackets with one another, Frobenius' Theorem ensures that the flows corresponding to their respective vector fields foliate the manifold. For the case of the Kepler orbit, the flows of the set $\{X_{L_1}, X_{L_2}, X_{L_3}\}$ foliate the space in spheres. Thus $[X_H, X_{L_i}] = 0$ not only tells us that L_i's are conserved, but also that H is invariant under the flows of X_{L_i}. Since the flows of L_i's construct spheres, this implies that H is invariant under rotation. The equation $[X_H, X_{L_i}] = 0$ hinted at a link between rotational invariance and conservation of angular momentum.

To formally determine the link between symmetries and conserved quantities we moved onto the third main topic: the momentum map. This construction allows us to establish Noether's Theorem for Hamiltonian dynamics: *each action that does not modify the Hamiltonian H is a symmetry*, and from each symmetry we can compute a momentum map $\underline{J}$ that is preserved along the flow of X_H. The momentum maps are a mechanism to obtain conserved quantities to solve the problem, although not all the conserved quantities arise from a known symmetry, as for example the LRL vector.

After providing a method to translate actions from the configuration space to the phase space, we explicitly computed the momentum map for various symmetries. We found how H, linear and angular momentum conservation come from the fact that these are actually momentum maps. We also illustrated the power and generality of momentum maps as a tool to find conserved quantities, by computing the non-trivial momentum map of $SE(2)$ on the plane and how the symmetry under $SE(3)$ provides both linear and angular momentum conservation.

Apart from providing conserved quantities, momentum maps serve to explicitly reduce the dimensions of the phase space where dynamics occur. This motivated the last part of the project: symplectic reduction. Given certain conditions on the action, instead of regarding the entire phase space, we can reduce it in a two-step process. First, we fix the momentum map $\underline{J}$ to create a submanifold enclosing all the dynamics, and within this submanifold, we identify the points which yield the same dynamics. The resulting space has less dimensions and both ω and H are expressed in terms of the coordinates of this reduced space.

For the guiding thread through this project, the Kepler orbit problem, we obtained that the entire 6-dimensional phase space is reduced to a 2-dimensional one. Thus the orbit of a planet around the Sun can be established in terms of the radial distance r and the radial momentum p_r, i.e. how fast the planet approaches or distances itself from the origin of the gravitational pull.

In light of the above, we have managed to provide answers to the questions that motivated the project, whilst finding some powerful and beautiful insights. There is a rich structure underlying Hamiltonian dynamics, and all of the topics of the project are intertwined in an unexpected way. Given any problem in mechanics we need independently conserved quantities to solve it, and even though we have shown that not all come from a symmetry, a great deal of these do. Momentum maps turn out to be the general manifestation of conserved quantities coming from symmetries, and not only do they provide us with a general method to find conserved quantities, but also they can be used to explicit reduce the dimensionality of the phase space where Physics occurs. Due to their generality, momentum maps are ubiquitous in physics and constitute a rather active area of research. Quantum Mechanics [20], Supergravity [21, 22] and the general theory of physical symmetries [23] are some of the fields where momentum maps play a crucial role.

This project represents the first step towards understanding the contemporary approach to mechanics via momentum maps. For the sake of concision and clarity, there are various interesting topics that we have not explored in depth.

For instance, the nature of the LRL vector is still unclear. Is it related to a non-obvious symmetry? Where does it come from? We did not dwell on the methods to obtain momentum maps either, which may be generalised for specific cases. For instance, a general method to compute momentum maps of semi-direct product groups such as $SE(3)$ does exist[2]. Finally, we could compute momentum maps for more complex symmetries, like gauge symmetries coming from electromagnetism or quantum mechanical symmetries [1]. There is great scope for further research related to both methods and calculations of momentum maps. This is a tool of growing importance and I do believe that it will contribute to essential findings in the near future.

References

[1] Jerrold E. Marsden and Tudor S. Ratiu. *Introduction to Mechanics and Symmetry: a Basic Exposition of Classical Mechanical Systems*. Springer-Verlag, 1994.

[2] Victor Guillemin and Sternberg. Shlomo. *Symplectic Techniques in Physics*. Cambridge University Press, 1984.

[3] Bernard F. Schutz. *Geometrical Methods of Mathematical Physics*. Cambridge University Press, 2013.

[4] Rene Dugas. *A history of mechanics*. London: Routledge and Paul, 1957.

[5] Diarmuid Ó'Mathúna. *Integrable Systems in Celestial Mechanics*. Boston, MA: Birkhäuser Boston. Imprint: Birkhäuser, 2008.

[6] Alexander. et al. Mikhailov. *Integrability*. Springer Berlin Heidelberg : Imprint: Springer, 2008.

[7] Yvette Kosmann-Schwarzbach. *The Noether Theorems Invariance and Conservation Laws in the 20th Century*. Springer, 2011.

[8] José Figueroa-O'Farrill. *Differentiable Manifolds, lecture notes.* University of Edinburgh, 2019. Accessed Nov 2020.

[9] José Figueroa-O'Farrill. *Differentiable Manifolds, workshops 3 and 5 and solutions.* University of Edinburgh, 2019. Accessed Nov 2020.

[10] Brian Hall. *Lie Groups, Lie Algebras, and Representations.* Springer International Publishing AG, 2015. Vol. 222.

[11] Roger Horsley. *Hamiltonian Dynamics, lecture notes.* University of Edinburgh, 2021. Accessed March 2021.

[12] Ronald A. Walker. Linear dependence of quotients of analytic functions of several variables with the least subcollection of generalized wronskians. *The Ohio State University, Department of Mathematics*, 06 2005.

[13] Luigi Del Debbio. *Principles of Quantum Mechanics, lecture notes.* University of Edinburgh, 2018. Accessed March 2021.

[14] V.I. Arnold. *Mathematical Methods of Classical Mechanics.* Springer-Verlag, 1978.

[15] Arjun Berera. *Introductory Dynamics, lecture notes.* University of Edinburgh, 2017. Accessed Dec 2020.

[16] William Wade. *An Introduction to Analysis.* Pearson Education Ltd., 2013.

[17] Beifang Chen. *Math111 L2: Linear Algebra, Lecture Notes.* Hong Kong University of Science and Technology, 2006.

[18] J. Butterfield. On symplectic reduction in classical mechanics. *Oxford University*, 07 2005.

[19] Miguel Teixidó Román. Hamiltonian methods in stability and bifurcations problems for artifical satellite dynamics. *Universidad Politècnica de Catalunya*, 06 2011.

[20] Adam Sawicki et al. Convexity of momentum map, morse index, and quantum entanglement. *Cornell University*, 2013.

[21] L. Andrianopoli et al. *N=2 Supergravity and N = 2 Super Yang-Mills Theory on General Scalar Manifolds: Symplectic Covariance Gaugings and the Momentum Map.* Journal of Geometry and Physics, 1997. vol. 23, no.2, pp. 111-189.

[22] Igor A Bandos and Tomas Ortin. On the dualization of scalars into (d-2)-forms in supergravity. momentum maps, r-symmetry and gauged supergravity. *The Journal of High Energy Physics*, 2018. vol.2016, no.8, pp. 1-54.

[23] Giuseppe Pucacco and Antonella Marchesiello. An energy–momentum map for the time-reversal symmetric 1:1 resonance with symmetry. *Physica. D*, 2014. vol.271, pp. 10-18.

[24] Alex Alemi. Laplace-runge-lenz vector. *California Institute of Technology*, 06 2006.

Appendices

Note — Appendices are provided for completeness only and any content included in them will be disregarded for the purposes of assessment.

A Computation of the Laplace-Runge-Lenz vector

Starting from Newton's second law, since the force only acts radially, we may write [24]

$$\underline{F} = \underline{\dot{p}} = -\frac{\kappa \underline{r}}{r^3},$$

where $\underline{r}$ is the usual vector in polar coordinates, pointing radially outwards and with magnitude $r = \sqrt{(q^1)^2 + (q^2)^2 + (q^3)^2}$. Now, taking the cross product with $\underline{L}$, we have

$$\underline{\dot{p}} \times \underline{L} = -\frac{\kappa \underline{r}}{r^3} \times (\underline{r} \times \underline{p}) = -\frac{\kappa \underline{r}}{r^3} \times (\underline{r} \times m\underline{\dot{r}}),$$

where $\underline{p} = m\underline{\dot{r}}$. Extracting the mass and evaluating the triple vector product yields

$$\underline{\dot{p}} \times \underline{L} = -\frac{m\kappa}{r^3}(\underline{r}(\underline{r} \cdot \underline{\dot{r}}) - r^2 \underline{\dot{r}}). \tag{51}$$

By the product rule, we have the following identity:

$$\underline{r} \cdot \underline{\dot{r}} = \frac{1}{2}\frac{d}{dt}(\underline{r} \cdot \underline{r}) = \frac{1}{2}\frac{d}{dt}(r^2) = r\dot{r}.$$

Using this result in (51) and applying the chain rule followed by product rule again yields

$$\underline{\dot{p}} \times \underline{L} = -\frac{m\kappa}{r^3}(\underline{r}(r\dot{r}) - r^2\underline{\dot{r}}) = -m\kappa\left(\frac{\dot{r}\underline{r}}{r^2} - \frac{\underline{\dot{r}}}{r}\right) = m\kappa\left(\frac{\underline{\dot{r}}}{r} - \frac{\dot{r}\underline{r}}{r^2}\right) = \frac{d}{dt}\left(\frac{m\kappa\underline{r}}{r}\right).$$

Finally noting that, since angular momentum is conserved, $\underline{\dot{L}} = 0$, we can conclude that

$$\frac{d}{dt}\left(\underline{p} \times \underline{L} - \frac{m\kappa\underline{r}}{r}\right) = 0.$$

Hence the vector

$$\underline{R} := \underline{p} \times \underline{L} - \frac{m\kappa\underline{r}}{r}, \tag{52}$$

is conserved. This vector is called the **Laplace-Runge-Lenz** vector (LRL vector) and as we argued in the report, it always points along the symmetry axis traced by the orbit of the planet (Fig.2b).

B The notion of a Poisson manifold

These are the manifolds equipped with the bare minimum structure to describe Hamiltonian dynamics.

Definition B.1. *Let P be manifold endowed with a Poisson Bracket on $C^\infty(P)$ which obeys the properties from definition 3.11. Then P is called a* ***Poisson manifold****. In particular, every symplectic manifold is a Poisson manifold.*

We can also extend the definition of Hamiltonian vector field to the Poisson manifolds, using the local correspondence between Poisson brackets and the symplectic form. On a Poisson manifold, a Hamiltonian vector field X_H obeys [1]

$$df(X_H) = X_H(f) = \{f, H\},$$

for all $f \in^\infty (P)$.

Finally, the notion of canonical action is also extended to the Poisson bracket.

Definition B.2. *Let P be a Poisson manifold, G a Lie group and $\Phi : G \times P \to P$ a smooth left action of G on P by canonical transformations. Denoting, for $z \in P$, $g \cdot z =: \Phi_g(z)$, so that $\Phi_g : P \to P$, then the action being* ***canonical*** *means that*

$$\Phi^*\{F_1, F_2\} = \{\Phi^* F_1, \Phi^* F_2\},$$

for all $F_1, F_2 \in C^\infty(P)$ and any $g \in G$. This means that pulling back by the action preserves the Poisson bracket, and thus, Hamilton's equations. Note that if (P, ω) is a symplectic manifold, then the action is canonical if and only if $\Phi^*\omega = \omega$.

C Computing the Lie derivative for a vector field

Let $X, Y \in \mathfrak{X}(M)$. Equivariance tells us that

$$(\gamma_t \cdot Y)(\gamma_t \cdot f) = \gamma_t \cdot (Y(f)).$$

Differentiating with respect to t, using the product rule and noting that $Y(f)$ is a function

$$\begin{aligned} &\iff (\pounds_X Y)f + Y(\pounds_X f) = \pounds_X(Y(f)) \\ &\iff (\pounds_X Y)f = \pounds_X(Y(f)) - Y(\pounds_X f) = X(Y(f)) - Y(X(f)) = [X, Y](f), \end{aligned} \tag{53}$$

as we wanted to show.

D Mathematica code

Toolkit for Mechanics, Geometry and Momentum Map MPhys project.

In this document several global functions are coded, which will serve for checks and computations for the project. All the computations are carried out choosing the set of independent variables $\{q^1, q^2, q^3, p_1, p_2, p_3\}$ as coordinates for the phase space.

In[ ]:=
```
ClearAll["Global`*"]

(*Definitions of key functions used in the Kepler Orbit problem*)
SetAttributes[{m, k}, Constant];
r = Sqrt[q1^2 + q2^2 + q3^2];
energy = (p1^2 + p2^2 + p3^2) / (2 m) - k / r;
lx = q2 * p3 - q3 * p2;
ly = q3 * p1 - q1 * p3;
lz = q1 * p2 - q2 * p1;
l2 = lx^2 + ly^2 + lz^2;
rx = p2 * lz - ly * p3 - (m * k * q1) / r;
ry = p3 * lx - p1 * lz - (m * k * q2) / r;
rz = p1 * ly - lx * p2 - (m * k * q3) / r;
r2 = rx^2 + ry^2 + rz^2;
```

In[ ]:=
```
QPSindependence[quantitieslist_] :=
 (*Checks for independence of a list of quantities in phase space*)
 Module[{variables, jacobian},
  variables = {q1, q2, q3, p1, p2, p3};
  jacobian = Simplify[D[quantitieslist, {variables}]];
  Print[MatrixForm[jacobian]];
  If[MatrixRank[jacobian] == Min[Dimensions[jacobian]], Print[
    "The functions are independent"], Print["The functions are not independent"],
   Print["There has been an error in the computation"]
  ]
 ]
QPoissonBracket[quantity1_, quantity2_] :=
 (*Computes the Poisson bracket of
  2 given functions in the considered phase space*)
 Module[{position, momentum, result},
  position = {q1, q2, q3};
  momentum = {p1, p2, p3};
  result = D[quantity1, {position}].D[quantity2, {momentum}] -
    D[quantity1, {momentum}].D[quantity2, {position}];
  Simplify[result]
 ]
```

We now set up some functions to deal with vector fields and forms once we have chosen our basis for the phase space. To interpret the vector outputs, note that we are using $\{\partial_{q^1}, \partial_{q^2}, \partial_{q^3}, \partial_{p_1}, \partial_{p_2}, \partial_{p_3}\}$ as a basis for the tangent space and the basis $\{dq^1, dq^2, dq^3, dp_1, dp_2, dp_3\}$ for the cotangent space. For instance:

- A vector field X: X = {(q1)^2, 0, 0, 0, 0, p2} = (q1)^2∂_{q^1}+ p2∂_{p_3};
- A one-form α: α = {1, 0, 0, p3, 0, 0} = dq^1 + p3dp_1;

```
QObtainVF[quantity_] :=
 (*Computes the hamiltonian vector field corresponding to quantity,
 so that it matches our definition with the symplectic form*)
 Module[{position, momentum, poscomps, momcomps},
  position = {q1, q2, q3};
  momentum = {p1, p2, p3};
  poscomps = D[quantity, {momentum}];
  momcomps = (-1) * D[quantity, {position}];
  Simplify[Join[poscomps, momcomps]]
 ]
QApplyVF[vectorfield_, quantity_] :=
 (*Applies the vector field to a quantity*)
 Module[{variables},
  variables = {q1, q2, q3, p1, p2, p3};
  Simplify[vectorfield.D[quantity, {variables}]]
 ]
QLieDerivativeVF[vectorfield1_, vectorfield2_] :=
 Module[{vffinalmomcomps, vffinalposcomps},
  (*Computes the Lie Derivative of a vector field along another vector field*)
  vffinalposcomps = Map[QApplyVF[vectorfield1, #] &, vectorfield2[[1 ;; 3]]] -
    Map[QApplyVF[vectorfield2, #] &, vectorfield1[[1 ;; 3]]];
  vffinalmomcomps = Map[QApplyVF[vectorfield1, #] &, vectorfield2[[4 ;; 6]]] -
    Map[QApplyVF[vectorfield2, #] &, vectorfield1[[4 ;; 6]]];
  Join[vffinalposcomps, vffinalmomcomps]
 ]
```

```
In[ ]:= (*We now run some tests*)
mat = QPSindependence[{lx, ly, lz, energy}]
QPSindependence[{energy, rx, ry, rz, lx}]
QPoissonBracket[lx, ly]
QPoissonBracket[energy, rx]
QPoissonBracket[energy, r2]
xlx = QObtainVF[lx]
xly = QObtainVF[ly]
xlz = QObtainVF[lz]
QPoissonBracket[r2, l2]
QLieDerivativeVF[xlx, xly]
QApplyVF[xlx, ly]
QApplyVF[xlx, ry]
```

$$\begin{pmatrix} 0 & p3 & -p2 & 0 & -q3 & q2 \\ -p3 & 0 & p1 & q3 & 0 & -q1 \\ p2 & -p1 & 0 & -q2 & q1 & 0 \\ \frac{k\,q1}{(q1^2+q2^2+q3^2)^{3/2}} & \frac{k\,q2}{(q1^2+q2^2+q3^2)^{3/2}} & \frac{k\,q3}{(q1^2+q2^2+q3^2)^{3/2}} & \frac{p1}{m} & \frac{p2}{m} & \frac{p3}{m} \end{pmatrix}$$

```
The functions are independent
```

$$\begin{pmatrix} \frac{k\,q1}{(q1^2+q2^2+q3^2)^{3/2}} & \frac{k\,q2}{(q1^2+q2^2+q3^2)^{3/2}} & \frac{k\,q3}{(q1^2+q2^2+q3^2)^{3/2}} & \frac{p1}{m} & \frac{p2}{m} \\ p2^2+p3^2-\frac{k\,m\,(q2^2+q3^2)}{(q1^2+q2^2+q3^2)^{3/2}} & -p1\,p2+\frac{k\,m\,q1\,q2}{(q1^2+q2^2+q3^2)^{3/2}} & -p1\,p3+\frac{k\,m\,q1\,q3}{(q1^2+q2^2+q3^2)^{3/2}} & -p2\,q2-p3\,q3 & 2\,p2\,q1-p1\,q\ldots \\ -p1\,p2+\frac{k\,m\,q1\,q2}{(q1^2+q2^2+q3^2)^{3/2}} & p1^2+p3^2-\frac{k\,m\,(q1^2+q3^2)}{(q1^2+q2^2+q3^2)^{3/2}} & -p2\,p3+\frac{k\,m\,q2\,q3}{(q1^2+q2^2+q3^2)^{3/2}} & -p2\,q1+2\,p1\,q2 & -p1\,q1-p3\,q\ldots \\ -p1\,p3+\frac{k\,m\,q1\,q3}{(q1^2+q2^2+q3^2)^{3/2}} & -p2\,p3+\frac{k\,m\,q2\,q3}{(q1^2+q2^2+q3^2)^{3/2}} & p1^2+p2^2-\frac{k\,m\,(q1^2+q2^2)}{(q1^2+q2^2+q3^2)^{3/2}} & -p3\,q1+2\,p1\,q3 & -p3\,q2+2\,p2\,q \\ 0 & p3 & -p2 & 0 & -q3 \end{pmatrix}$$

```
The functions are independent
```

Out[ ]= $p2\,q1 - p1\,q2$

Out[ ]= 0

Out[ ]= 0

Out[ ]= {0, -q3, q2, 0, -p3, p2}

Out[ ]= {q3, 0, -q1, p3, 0, -p1}

Out[ ]= {-q2, q1, 0, -p2, p1, 0}

Out[ ]= 0

Out[ ]= {q2, -q1, 0, p2, -p1, 0}

Out[ ]= $-p2\,q1 + p1\,q2$

Out[ ]= $p1\,p3\,q1 + p2\,p3\,q2 - p1^2\,q3 - p2^2\,q3 + \frac{k\,m\,q3}{\sqrt{q1^2+q2^2+q3^2}}$