Level 1 – AS91037

Standard 1.12

4 Credits – External

Demonstrate understanding of chance and data

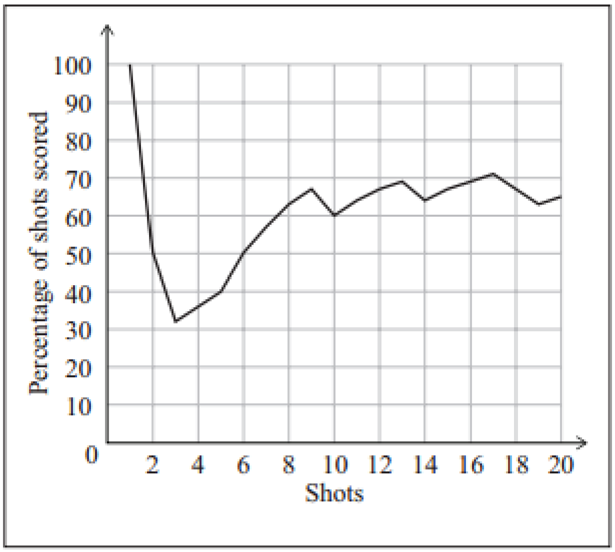
Contents

Sample question: Basketball

Part One: Levi has been wondering about how good he is at . Levi thinks that his of scoring is 50% since he can either miss or score on the shot. Do you agree?

Part Two: Levi's father says that, when he was the same age, his chance of was 60%. If Levi's father had taken shots when he was the same age, how many shots would he have scored on?

Part Three: Levi begins an . He takes some shots and records whether he scores or he misses. A graph of his results is below. Based on this experiment, what is the probability that Levi will score on his next shot [shot 21]? How can you be that this probability is correct?



Homework tracking

As you do homework for this standard, keep track of it all right here.

Week	What do you need to do?	What did you do?	What day did you do it?
Term 2, Week 7	*e.g. 2-3 more questions on page 5*	*e.g. questions 4-6*	*e.g. Monday (11/4)*
Term 2, Week 8			
Term 2, Week 9			
Term 2, Week 10			

Lesson Zero: Our inability to intuitively understand probability

<u>The birthday problem</u>

"How many people do you need for it to be   that there's a birthday in common between two people?" We're not interested in the   , only the   .[1]

Answer this with experimental probability. Go through all of the students in this class then move on to students in other classes until you get a birthday in common.

How many people did it take? Write it in the table below.

You can do a similar   by generating random numbers. Generate random numbers from 1 to 365 [one for each day of the year] until you get a double-up. How many did it take? Do it multiple times with some friends until you have 5 results.

Simulation #	People
(class)	
1	
2	
3	
4	
5	

<u>23 and Football Birthdays - Numberphile</u>

https://youtu.be/a2ey9a70yY0

You can visualise this theoretically:

You can ask a similar question: how many people do you need for it to be likely that someone shares a birthday with you?

[1] For those that pay attention to details, we're not counting twins and we're not considering leap years. If someone is born on Feb 29, we'll exlude them from our experiment.

The Monty Hall problem

You're in a   . There are three doors and behind the doors are two goats and a car. You pick whichever door you want. The host,   , opens one of the doors and reveals what's behind it. He can never open the door you've picked and he would never reveal the car. He then offers you the opportunity to switch to the last remaining door.

For example, you choose door 2. Monty reveals a goat behind door 1 and offers you the opportunity to switch to door 3.

What should you do? Or does it not matter because it's a 50-50 chance?

Again, let's use   probability. Simulate this with a friend; one of you being the contestant and the other being Monty Hall. Do it 10 times switching and 10 times staying. The person being Monty Hall needs to remember the rules for which door to reveal.

Switch

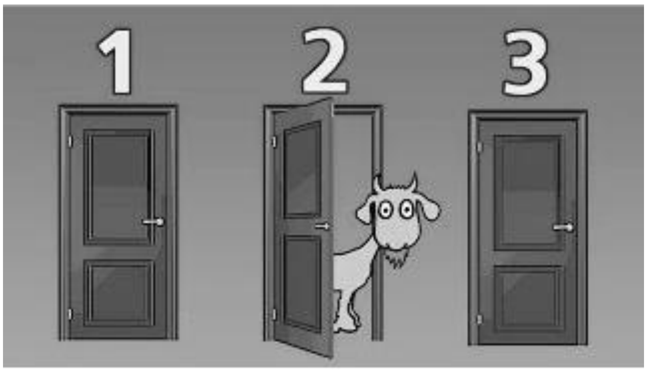
Simulation #	Result
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Stay

Simulation #	Result
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

You can also use   probability to explore this situation. This is explained in the following video.

Monty Hall Problem – Numberphile



https://youtu.be/4Lb-6rxZxx0

Why did the ancients locate their monuments to form exact triangles? How did they do that to such precision? Why do songs have lyrics that work backwards as well as forwards? How does Wingdings predict events?

Matt Parker explains.

LMS Popular Lecture Series 2010, Clutching at Random Straws, Matt Parker

https://youtu.be/sf5OrthVRPA [63 min]

Lesson One: Theoretical and experimental probability

Theoretical probability [noun]: What you expect to happen based on theory.
Experimental probability [noun]: What you expect to happen based on experiments.

Theoretical and experimental probability both help us understand what you can   to happen. Unfortunately, neither theoretical nor experimental probability can tell us what will   happen. Sorry about that. Probability is the world of guesses.

Mathematicians decided that probabilities will be on a scale from 0 [   ] to 1 [   ] or 0% to 100%. Obviously you almost never deal with absolute impossibilities or certainties, for example there's a small but non-zero chance you'll be hit on the head by a meteorite in the next few minutes. Therefore you usually deal with probabilities somewhere between the   .

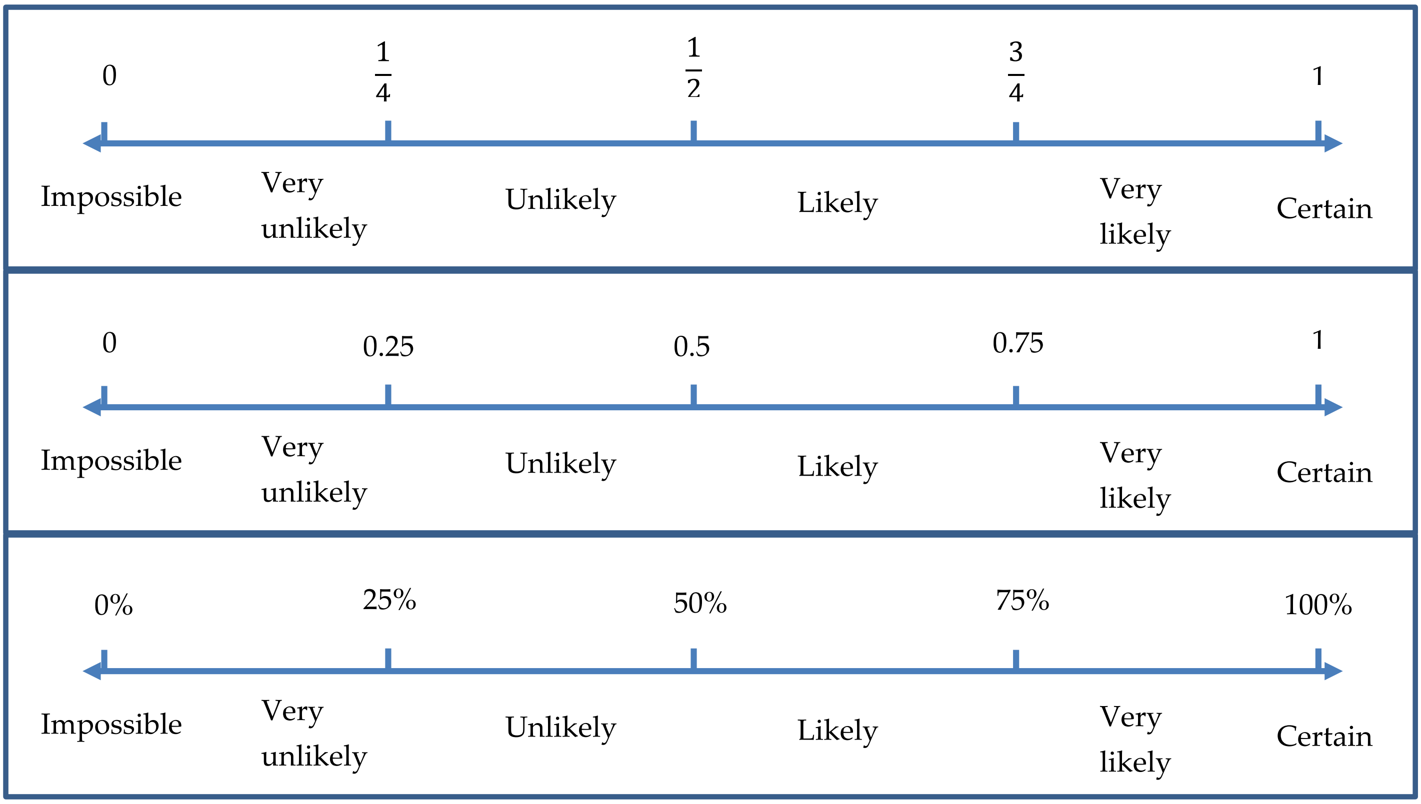
We denote the probability of an event happening as P(event). So the probability of event A occurring is P(A), the probability of flipping a   is P(heads), and the probability of being hit on the head by a meteorite in the next few minutes is P(hit on the head by a meteorite in the next few minutes). Although you could just shorten it to P(meteorite).

<u>Fractions, decimals, and percentages</u>

You can choose whether you write probabilities as fractions, decimals, or percentages. Each option has its own advantage:

- Fractions: Easy to   and to use in calculations
- Decimals: Easy to   different probabilities
- Percentages: Easy for the general public to

See these number lines:



<u>Sample space</u>
In theoretical probability, you can write down every possibility in what's called the **sample space.**

Sample space [noun]: *All of the possible outcomes*

For example the sample space of throwing two different coloured dice [red and white] has 36 possibilities:

		Red					
		1	2	3	4	5	6
White	1	1,1	1,2	1,3	1,4	1,5	1,6
	2	2,1	2,2	2,3	2,4	2,5	2,6
	3	3,1	3,2	3,3	3,4	3,5	3,6
	4	4,1	4,2	4,3	4,4	4,5	4,6
	5	5,1	5,2	5,3	5,4	5,5	5,6
	6	6,1	6,2	6,3	6,4	6,5	6,6

If you can list all   in the sample space, you can use theoretical probability. If you can't list them, you can't use theoretical probability.

Theoretical and experimental probability questions

Q1. What's the difference between theoretical and experimental probability? Reword the notes above.

Q2. Which one best predicts what will actually happen?

Note: Write probabilities like a mathematician would, e.g. $(black) = \frac{1}{2}$, $P(total = 7) = \frac{1}{6}$

Q3. Two different coloured dice are thrown, white and red. What's the probability that:
a) The total is 7
b) The total is more than 7
c) The total is at least 7
d) Getting a double
e) The number on the red die is more than the number on the white die
f) Only prime numbers are shown on the dice

Q4. A pack of cards is shuffled. What's the probability that the card drawn is a:
a) Black card
b) Yellow card
c) Card with a number on it
d) Card without a number on it
e) Jack
f) Jack of hearts
g) Number card between 4 and 7 inclusive

Q5. Two identical dice are thrown.
a) Draw the sample space for this situation

What's the probability that:
b) The total is 7
c) The total is more than 7
d) The total is at least 7
e) Getting a double
f) The number on one die is more than the number on the other die
g) Only prime numbers are shown on the dice

Lesson Two: Conditional probability and expected value

Conditional probability

Conditional probability involves calculating probabilities given a . For example, imagine drawing two cards from a deck. The first card was the 7 of spades. What's the probability the second card is red? There are only 51 cards left and 26 red cards so the probability is $\frac{26}{51}$, slightly more than .

You can say that $P(2nd\ card = red\ given\ that\ 1st\ card = black) = \frac{26}{51}$

Unsurprisingly, mathematicians have decided to the 'given that' by using a symbol.

$P(2nd\ card = red \mid 1st\ card = black) = \dfrac{26}{51}$

You could even go one step further if you would like.

$P(red \mid black) = \dfrac{26}{51}$

Another example is rolling two dice with the condition that the total is 8 or less. The new sample space is:

Red die

		1	2	3	4	5	6
White die	1	1,1	1,2	1,3	1,4	1,5	1,6
	2	2,1	2,2	2,3	2,4	2,5	2,6
	3	3,1	3,2	3,3	3,4	3,5	
	4	4,1	4,2	4,3	4,4		
	5	5,1	5,2	5,3			
	6	6,1	6,2				

Conditional probability questions

Q1. Why did you use theoretical probability in the lesson one questions rather than experimental probability?

Q2. How is conditional probability different to normal probability?

Note: Write probabilities like a mathematician would, e.g. $P(red \mid black) = \frac{26}{51}$

Q3. Some cards are drawn from a deck. Calculate the probability that the card drawn is a:
a) Queen given that two 10s have been drawn
b) Number card given that 8 picture cards have been drawn
c) Number card given that 8 number cards have been drawn
d) Ace given that 2 aces have been drawn
e) King given that all the red cards have been drawn
f) King given that a red card has been drawn

Q4. Fill in the totals in the table below.

	Boy	Girl	Total
Year 11	10	4	
Year 12	5	6	
Year 13	5	2	
Total			

Calculate the probability that a randomly chosen student is:
a) A boy
b) A girl
c) A year 11
d) A year 11 girl
e) A boy given that they're year 11
f) A girl given that they're year 12
g) A year 12 given that they're a girl
h) A boy given that they're year 13
i) A year 13 given that they're a boy

Q5. Two identical dice are thrown. Calculate the following theoretical probabilities:
a) The total is at least 9
b) The total is a prime number

Lesson Three: Combining probabilities and probability trees

There are two ways to combine probabilities:

1. OR [e.g. P(A) or P(B), P(rain) or P(sunny)]
2. AND [e.g. P(A) and P(B), P(rain) and P(sunny)]

OR

Technically, you've already used the 'or' combination. In the lesson one questions when you were calculating the probability of rolling a total of 7 on you were actually calculating the probability of rolling 1 and 6 OR 2 and 5 OR 3 and 4 OR 4 and 3 OR 5 and 2 OR 6 and 1.

That's a really confusing way to write it. Let's write it better:

$P(total = 7) = P(1{,}6)$ OR $P(2{,}5)$ OR OR $P(4{,}3)$ OR $P(5{,}2)$ OR

When combining probabilities with 'or' you simply add the probabilities together.

$$P(A\ or\ B) =$$

For example:

$$\begin{aligned} P(total = 7) &= P(1{,}6) + P(2{,}5) + P(3{,}4) + P(4{,}3) + P(5{,}2) + P(6{,}1) \\ &= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} \\ &= \\ &= \frac{1}{6} \end{aligned}$$

AND

You can also combine events using 'and'. For example, what's the probability that you roll a 6 and then a 1 by rolling a die ? There are 36 outcomes [remember the sample space?] and only one way to roll a 6 then a 1 so the probability is $\frac{1}{36}$.

In general, when combining probabilities with 'and' you simply the probabilities together.

$$P(A\ and\ B) = P(A) \times P(B)$$

For example:

$$\begin{aligned} P(6\ and\ 1) &= P(6) \times P(1) \\ &= \frac{1}{6} \times \frac{1}{6} \\ &= \frac{1}{36} \end{aligned}$$

Probability trees

You can visualise combining events with probability trees. Probability trees are another way of listing the sample space of a situation.

For example, construct a probability tree for rolling a die twice:

First roll	Second roll

When you combine probabilities in the same event, you're using the 'or' combination, e.g. $P(5\ or\ 6) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$. When you combine probabilities in different events you're using the 'and' combination, e.g. $P(first = \ 6\ and\ second = \ odd) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$

Combining events questions

Q1. A family has three children.

a) What's the probability that all three are girls?
b) What's the probability of at least one girl?
c) What's the probability of at least two girls?

Q2.

a) What's the probability of flipping 10 heads in a row?
b) What's the probability of flipping alternating heads and tails 10 times? [in other words, heads, tails, heads tails, heads, tails, … etc. for 10 flips]

Q3. Draw four cards from a deck with replacement.[2]

a) What's the probability of getting all spades?
b) What's the probability of getting no spades?
c) What's the probability of getting one spade?

Q4. Draw four cards from a deck without replacement

a) What's the probability of getting all spades?
b) What's the probability of getting no spades?
c) What's the probability of getting one spade?
d) What's the probability of getting at least one spade?

[2] With replacement means putting the card back after you've drawn it.

Lesson Four: Expected value and expected return

Expected value

Expected value is the you expect something to occur. For example if you have a $\frac{3}{5}$ [60%] chance of scoring in basketball and you have 10 shots, you'd expect to get 6 of them in.

$$Expected\ value = P(event) \times number\ of\ attempts$$

Expected return

This is really another example of . Imagine drawing two cards from a deck and checking their colour. Now imagine you're doing this for money. If you get two reds, you get \$2. If you get one red and one black, you get \$1. And if you get two blacks, you lose \$1.

How much money do you expect to make or lose ? In other words, would you expect to make or lose money in the long run?

Multiply the of each outcome by the for that outcome, e.g.

2 reds: $\frac{1}{4} \times \$2 = \$0.50$

1 red 1 black: $\frac{1}{2} \times \$1 = \$0.50$

2 blacks: $\frac{1}{4} \times -\$1 = -\$0.25$

Each outcome is 'worth' an amount *on average.* In other words, the 2 reds outcome is worth \$0.50 on average. The 1 red 1 black outcome is also worth \$0.50 on average. And 2 blacks is worth -\$0.25 on average. Adding them together, you get $\$0.50 + \$0.50 - \$0.25 = \$0.75$.

Therefore you expect to make 75c each time you play. In other words, you expect to make money in the long run.

This is how casinos and lotteries make their money. In a simplified situation, there is only one jackpot worth \$1,000 with a 0.05% chance of success. The game costs \$2 to play. The value of each outcome is:

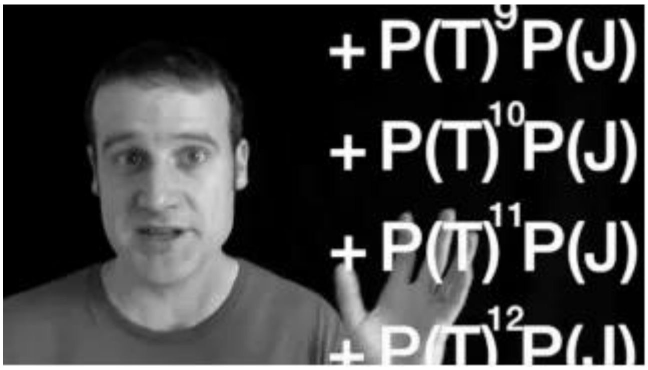
- Jackpot: $0.05\% \times \$1{,}000 = \frac{5}{10{,}000} \times \$1{,}000 = \$0.50$
- Failure: $99.95\% \times -\$2 = \frac{9{,}995}{10{,}000} \times -\$2 = -\$1.999$

Adding these together, you get $\$0.50 - \$1.999 = -\$1.499 = -\$1.50$ (2dp). Therefore, you expect to lose \$1.50 on average each time you play. You may and win the jackpot but on average, you've just thrown away \$1.50.

In reality, there are many smaller prizes with a higher chance of success to lure people in but the same principle applies.

For more about this, including infinite geometric series, see the video below.

How to get infinitely many lottery tickets



https://youtu.be/pOx_daJT_8c

Expected value and expected return questions

Q1. You toss two coins at the same time. If you repeat this 50 times, how many times do you expect to get double tails? How many times do you expect to get one head and one tail?

Q2. The chance of catching a fast frisbee is 70%. In a training session, you receive about 5 fast frisbees each minute and the training lasts for 40 minutes. How many do you expect to drop?

Q3. What is the probability of the following events?
a) Flipping 5 tails in a row with a coin?
b) Flipping 5 in a row [either heads or tails]?
c) Flipping 10 heads in a row?

Q4. For every part of question 1, consider if I gave you $100 every time you did what was asked. How much do you expect to earn on average?

Q5. Let's make the challenge in question 2 fair. How much would you need to pay me each time you lost to drop the expected value to $0 for each attempt?

Lesson Five: Experimental probability

Experimental probability is just as good as probability for predicting probability and giving us an idea of what will happen in the future. However, you'd probably prefer to use theoretical probability because it's much easier to . I mean, would you really roll a die 100 times to confirm the probability of each outcome is $\frac{1}{6}$? Wouldn't you rather just calculate it?

On the other hand, you'd want to use experimental probability when it is impractical or impossible to calculate the probability theoretically. For example, what's the chance that a meteorite will hit you on the head in the next few minutes? A theoretical approach would surely involve and all of the asteroids currently in our solar system [if you were doing it properly, you'd track all of the asteroids that will be near Earth over your lifetime]. Alternatively, an experimental approach would involve counting how many people have been hit by a meteorite over a certain period of time and dividing by the total number that could've been hit, in other words, the hit rate.

Experimental probability is also known as the "long-run relative frequency" and it's to know what that means. Let's break it down word-by-word, starting at the end of the phrase [because, reasons].

Frequency [noun]: *how often something happens, e.g. being hit by a meteorite or getting a basketball shot in.*

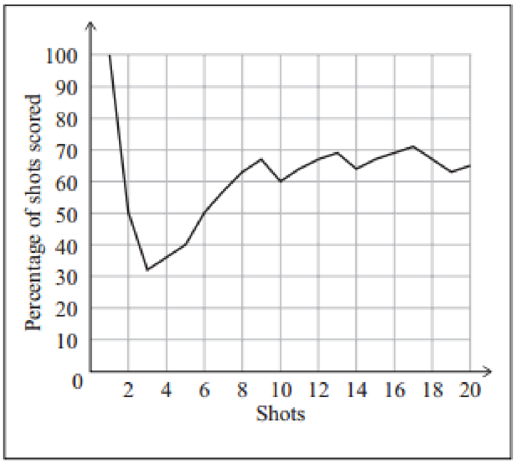
Relative [adjective]: *Compared to the total number of times attempted.*

Long-run [noun]: *Over a . If you only counted the first few basketball shots you wouldn't get any useful data. Only after 'enough' attempts will you get a good picture.*

In other words, the long-run relative frequency involves counting how many times something happens and comparing it to [dividing it by] the total number of attempts and this is done many, many times.

Take another look at part three of the sample question [reprinted here].

Part Three: Levi begins an experiment. He takes some shots and records whether he scores or he misses. A graph of his results is below. Based on this experiment, what is the probability that Levi will score on his next shot [shot 21]? How confident can you be that this probability is correct?



Experimental probability questions

Q1. What's the difference between theoretical and experimental probability? [Do I need to remind you to show off what you know?]

Q2. Which one best predicts what will actually happen?

Q3. Fill out the totals in the table below and answer the following questions: [need I encourage you to show off your mathematical notation?]

	Left-handed	Right-handed	Ambidextrous	Total
Year 4	4	233	26	
Year 5	10	1,166	173	
Year 6	90	1,327	16	
Year 7	144	2,379	21	
Year 8	210	2,937	30	
Year 9	466	6,570	48	
Year 10	445	5,532	58	
Year 11	376	3,111	244	
Year 12	16	1,181	172	
Year 13	7	764	108	
Total				

Source: A random sample from censusatschool.org.nz

a) What is the probability that a random student picked from this sample is left-handed?
b) What about right-handed?
c) What about ambidextrous?
d) What about ambidextrous given that they're year 11?
e) What about year 11 given that they're ambidextrous?
f) Which year group has the highest relative frequency of left-handed students?
g) Which year group has the highest relative frequency of right-handed students?
h) Which year group has the highest relative frequency of ambidextrous students?