Level 3 – AS91587
Standard 3.15
3 credits – Internal

Systems of simultaneous equations

Lesson One: Solving 2D systems of equations

Systems of equations
If two or more equations are to be considered at once then it is a system of equations. Often, we are interested in the solutions to the system of equations.

For example, consider the following system of two equations:

$$x - y = 5$$
$$x + 2y = 8$$

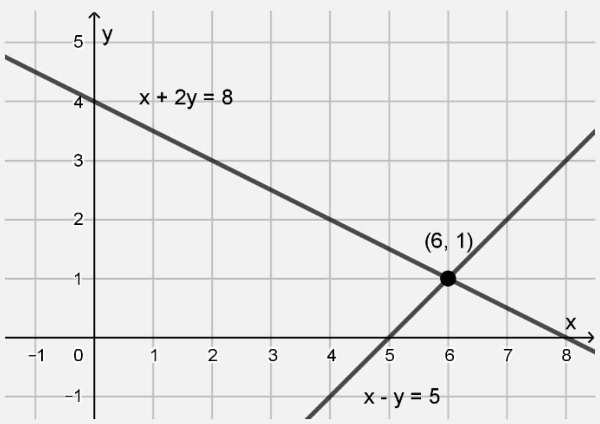
There are infinitely many solutions to the first equation including $(7, 2)$, $(2, -3)$, $(-2, -7)$, etc. There are also infinitely many solutions to the second equation including $(8, 0)$, $(4, 2)$, $(-2, 5)$, etc. After some trial and error you will find that the only solution that solves *both* equations is $(6, 1)$.

We call this the solution to the system of equations.

Use trial and error to calculate the solution to the following system of equations:

$$x + y = 0$$
$$3x - y = 12$$

Graphing solutions to systems of equations
The equations in a system as well as their solutions have a nice geometrical representation. Consider the following geometrical representation of the earlier example:



Each equation can be graphed as a linear graph. Each coordinate that each linear graph passes through is a solution to that equation. The point of intersection is a solution to both equations; that is, the solution to the system of equations.

The elimination method of solving systems of equations
One method to solve systems of equations is to eliminate one of the variable. Consider the following system of equations. For clarity's stake, they are labelled equation (1) and equation (2).

$$2x + 5y = 12 \qquad (1)$$
$$-2x + 3y = 4 \qquad (2)$$

You can add equation (1) to equation (2) by adding each term one-by-one, i.e.:

$$2x + 5y = 12 \qquad (1)$$
$$\underline{+(-2x + 3y = 4)} \qquad (2)$$
$$8y = 16$$

This gives you a new equation where x has been eliminated:

$$8y = 16$$
$$y = 2$$

Now that you have the y value, you need the x value. By substituting y into either equation (1) or (2) you get:

$$2x + 5 \times (2) = 12 \qquad (1)$$
$$2x = 2$$
$$x = 1$$

So the solution for this system of equations is $(1, 2)$. Check this is true using your graphic calculator.

Solve this system of equations by subtracting one equation from the other. Check your answer using your graphic calculator.

$$3x + 7y = 30 \qquad (1)$$
$$3x - 2y = 3 \qquad (2)$$

You may need to scale one or both of the equations to be able to eliminate a variable. For example, consider the following system of equations:

$$5x + 4y = 36 \qquad (1)$$
$$10x + 3y = 2 \qquad (2)$$

Equation (1) needs to be multiplied by 2 in order for x to be eliminated. Subtracting equation (2) from the new equation (1) gives:

$$\begin{aligned} 10x + 8y &= 72 & (1) \times 2 \\ \underline{-(10x + 3y = 2)} & & -(2) \\ 5y &= 70 \\ y &= 14 \end{aligned}$$

Substituting this y value into either equation (1) or equation (2) gives:

$$\begin{aligned} 5x + 4 \times (14) &= 36 & (1) \\ 5x + 56 &= 36 \\ 5x &= -20 \\ x &= -4 \end{aligned}$$

Therefore, the solution to the system of equations is $(-4, 14)$. Check this is true using your graphic calculator.

Solve this system of equations. Check your answer using your graphic calculator.

$$4x - 6y = 10$$
$$6x + 6y = 20$$

Lesson one questions

Solve the following systems of equations. Check your answers using your graphic calculator.

Q1.

$$2x + y = 5 \qquad (1)$$
$$x - 3y = -8 \qquad (2)$$

Q2.

$$6x + y = 10 \qquad (1)$$
$$x + 4y = -6 \qquad (2)$$

Q3.

$$-x - 3y = -2 \qquad (1)$$
$$8x - 6y = -44 \qquad (2)$$

Q4.

$$4x + y = -30 \quad (1)$$
$$-6x + 5y = 6 \quad (2)$$

Q5.

$$9x + y = -82 \quad (1)$$
$$-3x + y = 14 \quad (2)$$

Q6.

$$-2x + 5y = -47 \quad (1)$$
$$-3x + 7y = -66 \quad (2)$$

Q7.

$$-3x - 3y = -33 \quad (1)$$
$$9x + 4y = 69 \quad (2)$$

Q8.

$$7x - 4y = -77 \quad (1)$$
$$x - 6y = -49 \quad (2)$$

Q9.

$$2x - 9y = 35 \quad (1)$$
$$-3x + 2y = 5 \quad (2)$$

Lesson one advanced questions

Solve the following systems of equations. Check your answers using your graphic calculator. Consider dividing each equation by its greatest common factor to simplify it before solving.

Q10.

$y = 5x + 1$ (1)

$x - 2y = 34$ (2)

Q11.

$16x - 72y = 160$ (1)

$-50x + 30y = 280$ (2)

Q12.

$2x + 5 = -y$ (1)

$-x - y = -4$ (2)

Q13.

$32x - 48y = -240$ (1)

$-15x - 18y = -9$ (2)

Q14.

$200x - 75y = 550$ (1)

$54x - 45y = 198$ (2)

Q15.

$-60x - 30y = 330$ (1)

$56x + 35y = -273$ (2)

Lesson Two: Forming systems of equations

Systems of equations can occur in applications such as word problems. In these cases, you'll need to write the equations that describe the situation. For example, consider the following word problem:

Tickets for a concert are either child or adult. There were 300 tickets sold in total. A child ticket costs $15 and an adult ticket costs $25. The total revenue from ticket sales was $7000. How many child and adult tickets got sold?

To form a system of equations, the following steps are a good guide:

1. Identify what you're being asked to calculate. In this case, let

$$x = child\ tickets$$
$$y = adult\ tickets$$

2. Form one of the equations by focusing on one of the totals given in the information. In this case, focus on the total number of tickets sold:

$$x + y = 300 \qquad (1)$$

3. Use the other total given in the information to form the second equation. In this case, focus on the total revenue:

$$15x + 25y = 7000 \qquad (2)$$

This system can be solved and the solution is $(50, 250)$. It is important that you give the solution in context, i.e. "they sold 50 child tickets and 250 adult tickets".

Use the above steps to form and solve a system of equations to answer the following question: In basketball, hoops are worth either two or three points (excluding free throws). A team gets 54 hoops in total. Their total score at the end of the game was 132. How many two-pointers and three-pointers did they get? Make sure to give your solution in context.

Lesson two questions

Form and solve systems of equations to answer the following questions. Make sure to give your solutions in context.

Q1.The admission fee for a carnival is $6 for children and $10 for adults. A total of 800 people attended and $6 040 was made in ticket sales. How many children and adults attended?

Q2.Two people have $100 between them. One has $15 more than the other. How much does each person have?

Q3. Two people have a combined age of 50. Four years ago, one was twice as old as the other. What are their current ages?

Q4. A boy and his dad are 26 years apart. His dad is three times older than him. How old are they?

Q5. Two netball teams played a game in which 76 goals were scored. The winning team scored 17 goals more than the other team. How many goals did each team score?

Q6. Two groups of people go to the swimming pool. The first group consists of 5 adolescents and 1 adult and pays \$25.50. The second group consists of 2 adolescents and 4 adults and pays \$30. What is the price for adolescents and adults?

Q7. You need 5L of a 25% acid solution but you only have a 10% acid solution and a 50% acid solution. How many litres of each solution should you use?

Q8. A customer purchases two trees and five seedling for \$55. Another customer purchases three trees and ten seedlings for \$90. How much does each cost?

Q9. It takes an aeroplane 1.25 hours to travel 500 km (Wellington to Auckland) with a headwind. The return flight has a tailwind and takes 1 hour. The wind speed stayed the same throughout. What is the plane's speed (in still air) and the wind speed? Remember that distance = speed × time.

Lesson two advanced questions

Form and solve systems of equations to answer the following questions. Make sure to give your solutions in context.

Q10. A boat takes 1 hour and 30 minutes to travel 50 km upriver and 45 minutes to make the returns journey. What is the boat's speed (in still water) and the speed of the river's flow?

Q11. You need 6 L of 30% acid solution but you only have a 15% acid solution and a 35% acid solution. How many litres of each solution should you use?

Q12. The sum of the digits in a two-digit number is 9. When the digits are reversed, the two-digit number gets bigger by 27. What is the original two-digit number?

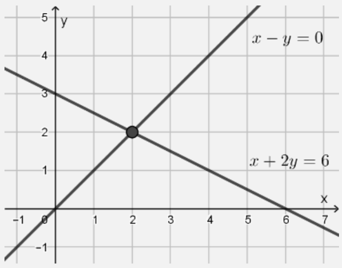
Q13. The sum of the digits in a two-digit number is 8. When the digits are reversed, the two-digit number gets smaller by 54. What is the original two-digit number?

Q14. An athlete wants to consume 7 600 units of vitamin A and 10 700 units of vitamin B. Each gram of supplement 1 contains 60 units of vitamin A and 35 units of vitamin B. Each gram of supplement 2 contains 20 units of vitamin A and 90 units of vitamin B. How many grams of each supplement should they consume?

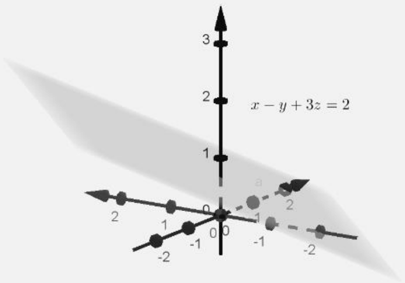
Lesson Three: 3D systems of equations

Comparing 2D and 3D systems of equations

An equation of two variables describes a line in 2D space, i.e. $x + 2y = 6$. It takes two lines to specify a single point at which they intersect.



An equation of three variables describes a plane in 3D space, i.e.



Two planes can meet along a line:

Three planes can meet at a single point:

Therefore, when dealing with three unknowns, three equations are needed to find a solution.

Solving 3x3 systems of equations

3x3 systems of equations have three equations with three unknowns. Solving 3x3 systems uses the same methods as solving 2x2 systems but there are more steps. For example, to solve the following system, start by eliminating one variable:

$$\begin{array}{rl} x + 2y - 4z = -10 & (1) \\ x - y + 2z = 11 & (2) \\ -x + 2z = 1 & (3) \end{array}$$

Adding equations (1) and (3) to eliminate x gives:

$$2y - 2z = -9 \qquad (4)$$

Adding equations (2) and (3) to also eliminate x gives:

$$-y + 4z = 12 \qquad (5)$$

Equations (4) and (5) now make a 2x2 system of equations. Solving this gives:

$$\begin{array}{rl} 2y - 2z = -9 & (4) \\ \underline{+(-2y + 8z = 24)} & +(5) \times 2 \\ 6z = 15 & \\ z = 2.5 & \end{array}$$

Substituting z into either (4) or (5) gives:

$$\begin{array}{rl} 2y - 2 \times (2.5) = -9 & (4) \\ y = -2 & \end{array}$$

Substituting y and z into either (1), (2), or (3) gives:

$$\begin{array}{rl} x + 2 \times (-2) - 4 \times (2.5) = -10 & (1) \\ x = 4 & \end{array}$$

Therefore, the solution to the 3x3 system of equations is the point $(4, -2, 2.5)$. You can check this is true using your graphic calculator.

Solve the following system of equations. Check your answer on your graphic calculator.

$$\begin{array}{rl} 2x - 10y - 5z = -9 & (1) \\ x - y - 2z = -1 & (2) \\ -x + 7y + 3z = 6 & (3) \end{array}$$

Lesson three questions

Solve the following systems of equations. Check your answers using your graphic calculator.

Q1.

$$-4x - 3y + 2z = -2 \quad (1)$$
$$x + y + 6z = -13 \quad (2)$$
$$-x - 3y + 4z = -3 \quad (3)$$

Q2.

$$2x + 9y - 2z = -47 \quad (1)$$
$$-5x - 3y + 6z = 17 \quad (2)$$
$$-2x + 5y + z = -20 \quad (3)$$

Q3.

$$-5x + y - 4z = -27 \quad (1)$$
$$x - 4y + 5z = 18 \quad (2)$$
$$-3x - y + 4z = 3 \quad (3)$$

Form and solve systems of equations to answer the following questions. Make sure to give your solutions in context.

Q4. Dean works in a music shop. Over the last three weeks there has been a sale, and music CDs in the Jumbo bin have been sold for $10, $15 or $20. One of Dean's customers bought a total of six CDs. The total cost for buying the six CDs was $85. The combined number of $15 and $20 CDs that the customer bought was twice as many as the number of $10 CDs that he bought. Find the number of $10, $15 and $20 CDs that the customer bought.

Q5. Dean is paid $283 one week. He paid board (which included all living expenses), invested some and spent the rest. His board cost $35 more than the amounts he invested and spent combined. He invested $20 more than he spent. Find how much money Dean paid for board, how much he invested and how much he spent.

Q6. Marni's friend Jane makes and sells three different types of hand lotion. Over the last three days, she has kept a tally of how many containers of each type she has made. This is summarised in the table below. On the first day, she spent a total of 145 minutes making hand lotions, the second day, 130 minutes, and on the third day she spent two hours making hand lotions. Find the type of hand lotion for which the time taken to produce each container is the smallest.

Type of Hand Lotion	Number of Containers		
	Day 1	Day 2	Day 3
Dewberry	8	2	5
Vitamin E enriched	3	2	0
Lavender	6	10	8

Q7. A high school has a total of 123 senior students involved in its school production. The number of year 13s was twice as many as the combined number of year 11s and 12s. There were five more year 11s than year 12s. Find the number of each year level in the production.

Lesson three advanced questions

Form and solve systems of equations to answer the following questions. Make sure to give your solutions in context.

Q8. Tickets for a high school production were sold for \$5, \$8, and \$15 depending on where the ticket holder will be seated. The total number of tickets sold was 390. The number of \$8 tickets sold was 30 more than twice the number of \$15 tickets sold. The total amount of money received from the ticket sales was \$2 520. Find the number of \$5, \$8, and \$15 tickets sold.

Q9. A landscape gardener is building a new garden. The gardener has ordered 20 m^3 of stones in three different sizes, at a total cost of \$950. The stones are supplied in three sizes: large, medium and small. The price of each size is given in this table:

	Small	Medium	Large
Price (per m^3)	\$40	\$50	\$55

They ordered twice as many medium stones as small stones. Find the quantity of each type that was ordered.

Q10. Find the equation of the parabola that passes through the points $(1, 17)$, $(3, 9)$, and $(4, -1)$.

Q11. Find the equation of the parabola that passes through the points $(-2, 3)$, $(1, -7.5)$, and $(2, -1)$.

Lesson Four: Inconsistent systems and non-unique solutions

Inconsistent systems in 2D

Not all systems of equations can be solved. Consider the following 2x2 system:

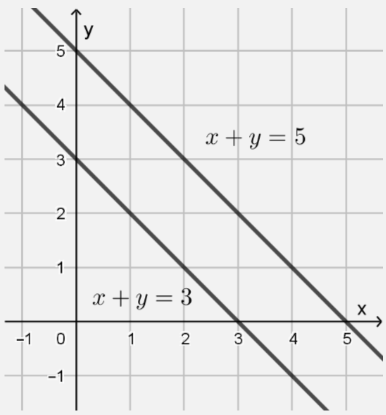
$$\begin{aligned} x + y &= 3 \qquad &(1)\\ x + y &= 5 \qquad &(2) \end{aligned}$$

There are no values for x and y that will satisfy both equations at once. Any such system is called inconsistent.

Inconsistent systems will always return a contradiction such as $0 = k$ where k is any non-zero number. For example, when trying to solve the above system:

$$\begin{aligned} x + y &= 3 \qquad &(1)\\ \underline{-(x + y = 5)} & \qquad &-(2)\\ 0 &= -2 & \end{aligned}$$

Graphing both equations gives another perspective :



Inconsistent systems in 2D are parallel lines so there is no point of intersection. You can identify equations of parallel lines as they have the same coefficients but a different constant.

Show that the following system is inconsistent:

$$\begin{aligned} x - 2y &= 4 \qquad &(1)\\ -2x + 4y &= 10 \qquad &(2) \end{aligned}$$

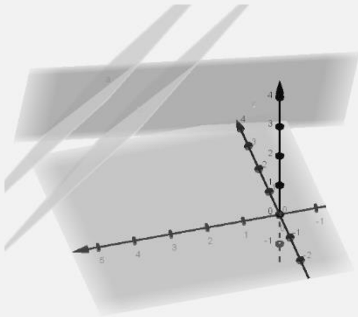
Inconsistent systems in 3D

3D systems that are inconsistent will also return a contradiction ($0 = k$) but there are several variations.

Parallel planes

Consider the following system of equations:

$$\begin{aligned} x + y + z &= 10 \qquad &(1)\\ 2x + 2y + 2z &= 16 \qquad &(2)\\ 3x - y + 4z &= 4 \qquad &(3) \end{aligned}$$

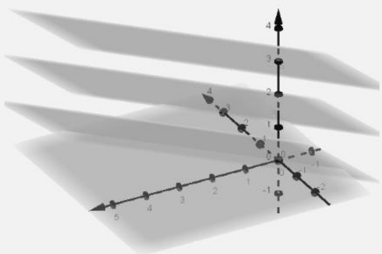


Here, two planes are parallel so there is no point of intersection of all three planes; no solution. You can see that the two planes that are parallel are (1) and (2) because they can be rewritten in a form where they have the same coefficients but different constants:

$$\begin{aligned} x + y + z &= 10 \qquad &(1)\\ -(x + y + z &= 8) \qquad &-(2) \div 2\\ 0 &= 2 & \end{aligned}$$

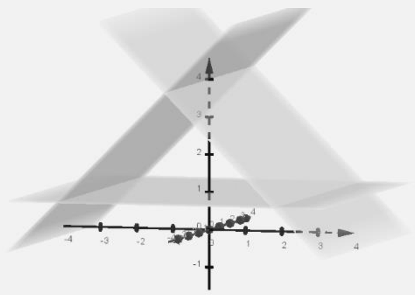
Similarly, you can have all three planes parallel:

$$\begin{aligned} x - y + 2z &= 3 \qquad &(1)\\ 3x - 3y + 6z &= 18 \qquad &(2)\\ -2x + 3y - 4z &= 0 \qquad &(3) \end{aligned}$$



Here, all three planes can be rewritten in a form with the same coefficients but different constants so any combination of two equations will result in a contradiction.

Triangular prism
Even if none of the planes are parallel to each other, the system can still be inconsistent if they form a triangular prism like so:



In this case, the line of intersection between two planes is parallel to the third plane.

Show that the following system is inconsistent and give a geometrical interpretation (i.e. state whether there are two or more parallel planes or the planes form a triangular prism).

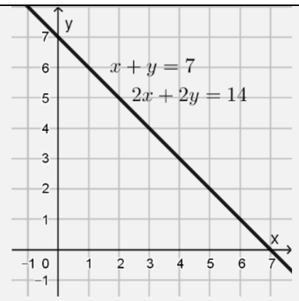
$$\begin{aligned} x + y + z &= 4 && (1) \\ 2x + 3y + 2z &= 8 && (2) \\ 3x + 4y + 3z &= 16 && (3) \end{aligned}$$

Non-unique solutions
If a system has more than one solution, it is said to have non-unique solutions. Systems with non-unique solutions will always return a tautology, i.e. $0 = 0$.

In 2D, this can happen when both equations are the same:

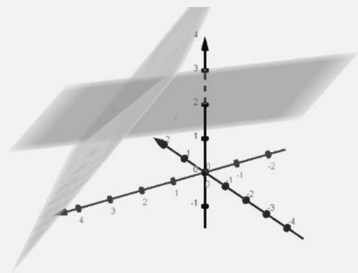
$$\begin{aligned} x + y &= 7 && (1) \\ 2x + 2y &= 14 && (2) \end{aligned}$$

Clearly, there are many values that solve this system. In fact, there are infinitely many.



In 3D, a system has non-unique solutions if two planes are identical or all three planes intersect along the same line. Consider the following system of equations:

$$\begin{aligned} x + y + z &= 10 && (1) \\ -2x - 2y - 2z &= -20 && (2) \\ 3x - y + 4z &= 2 && (3) \end{aligned}$$

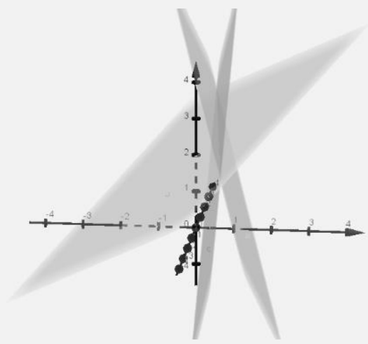


Equations (1) and (2) are the same equation so there are infinitely many solutions, i.e.

$$(1) \times 2 = (2)$$

Even if all three planes are distinct (not the same), there can be non-unique solutions if they all intersect along the same line:

$$\begin{aligned} x + 4y + z &= 5 && (1) \\ x - 2y + 2z &= 4 && (2) \\ x + 22y - 2z &= 8 && (3) \end{aligned}$$



This system can be shown to have non-unique solutions:

$$\begin{aligned} x + 4y + z &= 5 && (1) \\ \underline{-(x - 2y + 2z = 4)} && -(2) \\ 6y - z &= 1 && (4) \end{aligned}$$

$$\begin{array}{rl} x + 4y + z = 5 & (1) \\ \underline{-(x + 22y - 2z = 8)} & -(3) \\ -18y + 3z = -3 & (5) \\ \\ 6y - z = 1 & (4) \\ \underline{+(6y + z = -1)} & +(5) \div 3 \\ 0 = 0 & \end{array}$$

Show that the following system has non-unique solutions and give a geometrical interpretation.

$$\begin{array}{rl} 2x + 3y + z = 5 & (1) \\ x - 2y + 2z = 4 & (2) \\ 6x + 9y + 3z = 15 & (3) \end{array}$$

Describing the form of non-unique solutions

Even though there are infinitely many solutions, not every possible set of values is a solution. In fact, the solutions all lie on the same line in 3D space. It's possible to describe this line. For example, take the system from earlier:

$$\begin{array}{rl} x + 4y + z = 5 & (1) \\ x - 2y + 2z = 4 & (2) \\ x + 22y - 2z = 8 & (3) \\ \\ 6y - z = 1 & (4) \end{array}$$

Replace one of the variables with a parameter such as k or t. You could imagine that this parameter is a slider that you adjust and observe the changes in the other two variables. For example, let $z = t$. Therefore:

$$6y - t = 1 \qquad (4)$$

$$y = \frac{t}{6} + \frac{1}{6}$$

$$x + 4\left(\frac{t}{6} + \frac{1}{6}\right) + t = 5 \qquad (1)$$

$$x = -\frac{5t}{3} + \frac{13}{3}$$

Now you have all three variables defined in terms of this new parameter t.

$$x = -\frac{5t}{3} + \frac{13}{3}$$

$$y = \frac{t}{6} + \frac{1}{6}$$

$$z = t$$

where t is any number. This is the form that the solutions must take. For example, $t = 0$ gives the solution:

$$x = \frac{13}{3}$$

$$y = \frac{1}{6}$$

$$z = 0$$

Confirm that this satisfies all three equations.

Similarly, $t = 1$ gives the solution:

$$x = \frac{7}{3}$$

$$y = \frac{1}{3}$$

$$z = 1$$

and so on.

If there is a context given, it is likely that all three variables need to be positive. In these cases, find the upper and lower limits for t such that all three variables are positive.

Clearly, if $t < 0$, z will be negative so the lower limit for t is 0.

But as t gets large, x gets less positive and closer to zero. To calculate when it reaches zero:

$$x = -\frac{5t}{3} + \frac{13}{3}$$

$$0 = -\frac{5t}{3} + \frac{13}{3}$$

$$\frac{5t}{3} = \frac{13}{3}$$

$$t = \frac{13}{5}$$

Therefore, the form of the non-unique solutions is:

$$x = -\frac{5t}{3} + \frac{13}{3}$$

$$y = \frac{t}{6} + \frac{1}{6}$$

$$z = t$$

where $0 \le t \le \frac{13}{5}$

Model solution for 3x3 systems with non-unique solutions

The system of equations to be solved is:

$$\begin{array}{lrcrcrcr} (1) & 2x & + & 4y & + & 6z & = & 1000 \\ (2) & 3x & + & 7y & + & 10z & = & 1600 \\ (3) & 5x & + & 9y & + & 14z & = & 2400 \end{array}$$

where $x, y, z \geq 0$

Equation (1) has a greatest common factor of 2. Dividing by this gives:

$$\begin{array}{lrcrcrcr} (1) & x & + & 2y & + & 3z & = & 500 \\ (2) & 3x & + & 7y & + & 10z & = & 1600 \\ (3) & 5x & + & 9y & + & 14z & = & 2400 \end{array}$$

Attempt to find a unique solution

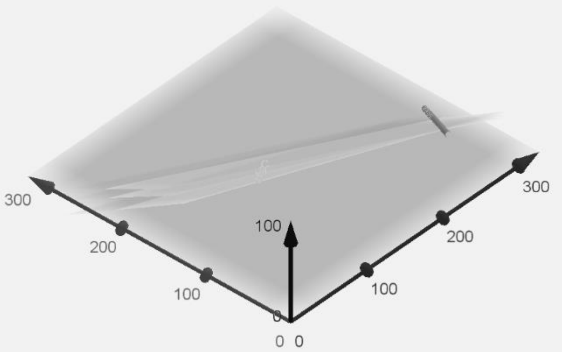
$$\begin{array}{lrcrcrcr} 3 \times (1) & 3x & + & 6y & + & 9z & = & 1500 \\ -(2) & -3x & - & 7y & - & 10z & = & -1600 \\ \hline & & & -y & - & z & = & -100 \\ (4) & & & y & + & z & = & 100 \end{array}$$

$$\begin{array}{lrcrcrcr} 5 \times (1) & 5x & + & 10y & + & 15z & = & 2500 \\ -(3) & -5x & - & 9y & - & 14z & = & -2400 \\ \hline (5) & & & y & + & z & = & 100 \end{array}$$

Equations (4) and (5) are identical meaning that the solutions are non-unique.

Geometrical interpretation

The non-unique solutions result from all three planes intersecting along the same line, i.e.:



Describing the form of the infinite solutions

Let $z = t$. Using (4), we obtain:

$$y + z = 100$$
$$y + t = 100$$
$$y = 100 - t$$

Using (1), we obtain:

$$x + 2y + 3z = 500$$
$$x + 2(100 - t) + 3t = 500$$
$$x + 200 - 2t + 3t = 500$$
$$x = 300 - t$$

Therefore, the solutions take the form:

$$(300 - t, 100 - t, t)$$

Contextual restrictions

The context requires all three variables to be non-negative, i.e. $x, y, z \geq 0$

If $t = 0$, then the corresponding solution is $(300,\ 100,\ 0)$. This is the smallest value t can take as any smaller value will result in a negative z value.

If t gets too big, y will become negative. This happens when:

$$y = 0$$
$$100 - t = 0$$
$$t = 100$$

So the maximum value t can take is 100. The solution corresponding to this value of t is $(200,\ 0,\ 100)$.

Conclusion

The solutions to this system of equations are all combinations of $x, y,$ and z that are of the form

$$(300 - t, 100 - t, t) \quad 0 \leq t \leq 100$$

for some parameter t.. This results in x, y, and z being restricted to the following ranges:

- $200 \leq x \leq 300$
- $0 \leq y \leq 100$
- $0 \leq z \leq 100$

Lesson four questions

Show whether the following systems are inconsistent or have non-unique solutions and give a geometrical interpretation.

Q1.

$$\begin{aligned} x + y + z &= 7 \quad &(1)\\ 3x + 3y + 3z &= 10 \quad &(2)\\ 2x + 2y + 2z &= 12 \quad &(3) \end{aligned}$$

Q2.

$$\begin{aligned} 11x - y + 3z &= 4 \quad &(1)\\ 2x + 3y - z &= -3 \quad &(2)\\ x - 16y + 8z &= 19 \quad &(3) \end{aligned}$$

Q3.

$2x - 10y + 6z = 7$ (1)
$5x - 25y + 15z = -9$ (2)
$3x + 7y - 5z = 4$ (3)

Q4.

$2x + 6y - 4z = 10$ (1)
$2x - 5y - 3z = -4$ (2)
$3x - 2y - 5z = 1$ (3)

Lesson four advanced questions

Show whether the following systems are inconsistent or have non-unique solutions and give a geometrical interpretation.
If there are non-unique solutions, describe their form.

Q5.

$$x + y + z = 2 \quad (1)$$
$$3x + 3y + 3z = 6 \quad (2)$$
$$2x + 4y - 5z = 6 \quad (3)$$

Q6.

$$x + y + z = 3 \quad (1)$$
$$2x - 3y + 3z = 7 \quad (2)$$
$$3x - 2y + 4z = 6 \quad (3)$$

Q7. Find the values of a and b that make the following system have non-unique solutions.

$$2x + 4y + 5z = 17 \quad (1)$$
$$4x + ay + 3z = b \quad (2)$$
$$8x + 7y + 13z = 40 \quad (3)$$

Q8. Find the values of A and B that make the following system inconsistent.

$$4x + 2y - 10z = 17 \quad (1)$$
$$-6x - 3y + 15z = 32 \quad (2)$$
$$5y - 7 = Ax + Bz \quad (3)$$

Practice internal 1: Elaine's equations

Elaine sees some equations in a book and they remind her of systems of equations she has used in her mathematics class. The first two equations are:

$$2x + 2z = 3y + 1$$
$$y = 4z + 8$$

Elaine needs a third equation and decides to use three different methods to create it. Use each of these methods to create the third equation:

- use her pin number, 3287, as the coefficients of x, y, and z, and the constant term in that order when the equation is in the form $ax + by + cz = d$
- multiply all coefficients and the constant in the first equation by 3
- use the first equation but change the constant from 1 to 6.

For each system of equations: solve them and give a geometric interpretation of the solution. Where there are non-unique solutions, give the general form of those solutions.

Practice internal 2: Roger's Rabbits

A breeder of pedigree rabbits, Roger, wants to control the vitamin intake of his prize show rabbits. The rabbits are fed a mixture of foods from three companies: Healthy, Budget, and Organic.

Roger wants his rabbits' daily vitamin intake to be:

- 1 000 micrograms (μg) of vitamin A
- 1 600 milligrams (mg) of vitamin C
- 2 400 milligrams (mg) of vitamin E.

Each gram of Healthy contains 2 μg of vitamin A, 3 mg of vitamin C, and 5 mg of vitamin E.
Each gram of Budget contains 4 μg of vitamin A, 7 mg of vitamin C, and 9 mg of vitamin E.
Each gram of Organic contains 5 μg of vitamin A, 10 mg of vitamin C, and 14 mg of vitamin E.

How many grams of food from each company should Roger feed his rabbits in order to meet their exact daily vitamin requirement?

If Organic increases the amount of vitamin A in its food to 6 μg, investigate the number of grams of food from each company Roger should feed his rabbits.

What quantity of vitamin A would encourage Roger to buy more Organic food?

Answers

Lesson one answers

From the notes

Use trial and error to calculate the solution to the following system of equations:

$$x + y = 0$$
$$3x - y = 12$$

The solution is $(3, -3)$

Solve this system of equations by subtracting one equation from the other. Check your answer using your graphic calculator.

$$3x + 7y = 30 \qquad (1)$$
$$3x - 2y = 3 \qquad (2)$$

Subtracting equation (2) from equation (1) gives:

$$3x + 7y = 30 \qquad (1)$$
$$\underline{-(3x - 2y = 3)} \qquad -(2)$$
$$9y = 27$$
$$y = 3$$

Substituting the y value back into equation (1) or (2) gives:

$$3x + 7 \times (3) = 30 \qquad (1)$$
$$3x = 9$$
$$x = 3$$

So the solution is $(3, 3)$

Solve this system of equations. Check your answer using your graphic calculator.

$$4x - 6y = 10 \qquad (1)$$
$$6x + 6y = 20 \qquad (2)$$

Multiply (1) by 3 and (2) by 2 then subtract:

$$12x - 18y = 30 \qquad (1) \times 3$$
$$\underline{-(12x + 12y = 40)} \qquad -(2) \times 2$$
$$-30y = -10$$
$$y = \frac{1}{3}$$

Substitute into either (1) or (2):

$$4x - 6 \times \left(\frac{1}{3}\right) = 10 \qquad (1)$$
$$4x = 12$$
$$x = 3$$

So the solution is $\left(3, \frac{1}{3}\right)$

Lesson one questions

Solve the following systems of equations. Check your answers using your graphic calculator.

Q1.

$$2x + y = 5 \qquad (1)$$
$$x - 3y = -8 \qquad (2)$$

$$2x + y = 5 \qquad (1)$$
$$\underline{-(2x - 6y = -16)} \qquad -(2) \times 2$$
$$7y = 21$$
$$y = 3$$

$$2x + (3) = 5 \qquad (1)$$
$$x = 1$$

The solution is $(1, 3)$

Q2.

$$6x + y = 10 \qquad (1)$$
$$x + 4y = -6 \qquad (2)$$

$$6x + y = 10 \qquad (1)$$
$$\underline{-(6x + 24y = -36)} \qquad -(2) \times 6$$
$$-23y = 46$$
$$y = -2$$

$$6x + (-2) = 10 \qquad (1)$$
$$x = 2$$

The solution is $(2, -2)$

Q3.

$$-x - 3y = -2 \qquad (1)$$
$$8x - 6y = -44 \qquad (2)$$

$$-8x - 24y = -16 \qquad (1) \times 8$$
$$\underline{+(8x - 6y = -44)} \qquad +(2)$$
$$-30y = -60$$
$$y = 2$$

$$-x - 3 \times (2) = -2 \qquad (1)$$
$$x = -4$$

The solution is $(-4, 2)$

Q4.

$$4x + y = -30 \quad (1)$$
$$-6x + 5y = 6 \quad (2)$$

$$\begin{array}{rl} 12x + 3y = -90 & (1) \times 3 \\ \underline{+(-12x + 10y = 12)} & +(2) \times 2 \\ 13y = -78 & \\ y = -6 & \end{array}$$

$$4x + (-6) = -30 \quad (1)$$
$$x = -6$$

The solution is $(-6, -6)$

Q5.

$$9x + y = -82 \quad (1)$$
$$-3x + y = 14 \quad (2)$$

$$\begin{array}{rl} 9x + y = -82 & (1) \\ \underline{+(-9x + 3y = 42)} & +(2) \times 3 \\ 4y = -40 & \\ y = -10 & \end{array}$$

$$9x + (-10) = -82 \quad (1)$$
$$x = 8$$

The solution is $(8, -10)$

Q6.

$$-2x + 5y = -47 \quad (1)$$
$$-3x + 7y = -66 \quad (2)$$

$$\begin{array}{rl} -6x + 15y = -141 & (1) \times 3 \\ \underline{-(-6x + 14y = -132)} & -(2) \times 2 \\ y = -9 & \end{array}$$

$$-2x + 5 \times (-9) = -47 \quad (1)$$
$$x = 1$$

The solution is $(1, -9)$

Q7.

$$-3x - 3y = -33 \quad (1)$$
$$9x + 4y = 69 \quad (2)$$

$$\begin{array}{rl} -9x - 9y = -99 & (1) \times 3 \\ \underline{+(9x + 4y = 69)} & +(2) \\ -5y = -30 & \\ y = 6 & \end{array}$$

$$-3x - 3 \times (6) = -33 \quad (1)$$
$$x = 5$$

The solution is $(5, 6)$

Q8.

$$7x - 4y = -77 \quad (1)$$
$$x - 6y = -49 \quad (2)$$

$$\begin{array}{rl} 7x - 4y = -77 & (1) \\ \underline{-(7x - 42y = -343)} & -(2) \times 7 \\ 38y = 266 & \\ y = 7 & \end{array}$$

$$7x - 4 \times (7) = -77 \quad (1)$$
$$x = -7$$

The solution is $(-7, 7)$

Q9.

$$2x - 9y = 35 \quad (1)$$
$$-3x + 2y = 5 \quad (2)$$

$$\begin{array}{rl} 6x - 27y = 105 & (1) \times 3 \\ \underline{+(-6x + 4y = 10)} & +(2) \times 2 \\ -23y = 115 & \\ y = -5 & \end{array}$$

$$2x - 9 \times (-5) = 35 \quad (1)$$
$$x = -5$$

The solution is $(-5, -5)$

Lesson one advanced questions

Solve the following systems of equations. Check your answers using your graphic calculator. Consider dividing each equation by its greatest common factor to simplify it before solving.

Q10.

$$y = 5x + 1 \qquad (1)$$
$$x - 2y = 34 \qquad (2)$$

$$-5x + y = 1 \qquad (1)$$
$$\underline{+(5x - 10y = 170)} \qquad +(2) \times 5$$
$$-9y = 171$$
$$y = -19$$

$$(-19) = 5x + 1 \qquad (1)$$
$$x = -4$$

The solution is $(-4, -19)$

Q11.

$$16x - 72y = 160 \qquad (1)$$
$$-50x + 30y = 280 \qquad (2)$$

$$2x - 9y = 20 \qquad (1) \div 8$$
$$-5x + 3y = 28 \qquad (2) \div 10$$

$$10x - 45y = 100 \qquad (1) \times 5$$
$$\underline{+(-10x + 6y = 56)} \qquad +(1) \times 2$$
$$-39y = 156$$
$$y = -4$$

$$2x - 9 \times (-4) = 20 \qquad (1)$$
$$x = -8$$

The solution is $(-8, -4)$

Q12.

$$2x + 5 = -y \qquad (1)$$
$$-x - y = -4 \qquad (2)$$

$$2x + y = -5 \qquad (1)$$
$$\underline{+(-2x - 2y = -8)} \qquad +(2) \times 2$$
$$-y = -13$$
$$y = 13$$

$$2x + 5 = -(13) \qquad (1)$$
$$x = -9$$

The solution is $(-9, 13)$

Q13.

$$32x - 48y = -240 \qquad (1)$$
$$-15x - 18y = -9 \qquad (2)$$

$$2x - 3y = -15 \qquad (1) \div 16$$
$$-5x - 6y = -3 \qquad (2) \div 3$$

$$10x - 15y = -75 \qquad (1) \times 5$$
$$\underline{+(-10x - 12y = -6)} \qquad +(2) \times 2$$
$$3y = -81$$
$$y = -27$$

$$2x - 3 \times (-27) = -15 \qquad (1)$$
$$x = -48$$

The solution is $(-48, -27)$

Q14.

$$200x - 75y = 550 \qquad (1)$$
$$54x - 45y = 198 \qquad (2)$$

$$8x - 3y = 22 \qquad (1) \div 25$$
$$6x - 5y = 22 \qquad (2) \div 9$$

$$24x - 9y = 66 \qquad (1) \times 3$$
$$\underline{-(24x - 20y = 88)} \qquad -(2) \times 4$$
$$11y = -22$$
$$y = -2$$

$$8x - 3 \times (-2) = 22 \qquad (1)$$
$$x = 2$$

The solution is $(2, -2)$

Q15.

$$-60x - 30y = 330 \qquad (1)$$
$$56x + 35y = -273 \qquad (2)$$

$$-2x - y = 11 \qquad (1) \div 30$$
$$8x + 5y = -39 \qquad (2) \div 7$$

$$-8x - 4y = 44 \qquad (1) \times 4$$
$$\underline{+(8x + 5y = -39)} \qquad +(2)$$
$$y = 5$$

$$-2x - (5) = 11 \qquad (1)$$
$$x = -2$$

The solution is $(-2, 5)$

Lesson two answers

From the notes

Use the above steps to form and solve a system of equations to answer the following question: In basketball, hoops are worth either two or three points (excluding free throws). A team gets 54 hoops in total. Their total score at the end of the game was 132. How many two-pointers and three-pointers did they get?

Let:

$$x = \textit{number of two-pointers}$$
$$y = \textit{number of three-pointers}$$

Based on the total number of hoops, we know that:

$$x + y = 54 \qquad (1)$$

Based on the total score, we know that:

$$2x + 3y = 132 \qquad (2)$$

Solving the system of equations:

$$\begin{array}{rl} 2x + 2y = 108 & (1) \times 2 \\ \underline{-(2x + 3y = 132)} & -(2) \\ -y = -24 & \\ y = 24 & \end{array}$$

$$\begin{array}{rl} x + (24) = 54 & (1) \\ x = 30 & \end{array}$$

They got 30 two-pointers and 24 three-pointers.

Lesson two questions

Form and solve systems of equations to answer the following questions. Make sure to give your solutions in context.

Q1. The admission fee for a carnival is \$6 for children and \$10 for adults. A total of 800 people attended and \$6 040 was made in ticket sales. How many children and adults attended?

$$x = child$$
$$y = adult$$

$$\begin{array}{rl} x + y = 800 & (1) \\ 6x + 10y = 6\,040 & (2) \end{array}$$

$$\begin{array}{rl} 6x + 6y = 4\,800 & (1) \times 6 \\ \underline{-(6x + 10y = 6\,040)} & -(2) \\ -4y = -1\,240 & \\ y = 310 & \end{array}$$

$$\begin{array}{rl} x + (310) = 800 & (1) \\ x = 490 & \end{array}$$

490 children and 310 adults attended.

Q2.Two people have \$100 between them. One has \$15 more than the other. How much does each person have?

Let:

$$x = \textit{person 1's money}$$
$$y = \textit{person 2's money}$$

$$\begin{array}{rl} x + y = 100 & (1) \\ x - y = 15 & (2) \end{array}$$

$$\begin{array}{rl} x + y = 100 & (1) \\ \underline{+(x - y = 15)} & +(2) \\ 2x = 115 & \\ x = 57.5 & \end{array}$$

$$\begin{array}{rl} (57.5) + y = 100 & (1) \\ y = 42.5 & \end{array}$$

One has \$57.50 and the other has \$42.50

Q3. Two people have a combined age of 50. Four years ago, one was twice as old as the other. What are their current ages?

Let:

$$x = \textit{person 1's current age}$$
$$y = \textit{person 2's current age}$$

$$\begin{array}{rl} x + y = 50 & (1) \\ 2(x - 4) = y - 4 & (2) \end{array}$$

$$\begin{array}{rl} 2x + 2y = 100 & (1) \times 2 \\ \underline{-(2x - y = 4)} & -(2) \\ 3y = 96 & \\ y = 32 & \end{array}$$

$$\begin{array}{rl} x + (22) = 50 & (1) \\ x = 22 & \end{array}$$

One person is 22 and the other is 32 years old.

Q4. A boy and his dad are 26 years apart. His dad is three times older than him. How old are they?

Let $x =$ *the dad's age and* $y =$ *the boy's age*

$$\begin{array}{rl} x - y = 26 & (1) \\ x = 3y & (2) \end{array}$$

$$\begin{array}{rl} x - y = 26 & (1) \\ \underline{-(x - 3y = 0)} & -(2) \\ 2y = 26 & \\ y = 13 & \end{array}$$

$$\begin{array}{rl} x - (13) = 26 & (1) \\ x = 39 & \end{array}$$

The dad is 39 and the boy is 13 years old.

Q5. Two netball teams played a game in which 76 goals were scored. The winning team scored 18 goals more than the other team. How many goals did each team score?

Let x = Team 1's goals and y = Team 2's goals

$$\begin{array}{rl} x + y = 76 & (1) \\ \underline{+(x - y = 18)} & +(2) \\ 2x = 94 & \\ x = 47 & \end{array}$$

$$\begin{array}{r} (47) + y = 76 \\ y = 29 \end{array}$$

One team scored 47 and the other scored 29 goals.

Q6. Two groups of people go to the swimming pool. The first group consists of 5 adolescents and 1 adult and pays $25.50. The second group consists of 2 adolescents and 4 adults and pays $30. What is the price for adolescents and adults?

Let x = adolescent price and y = adult price

$$\begin{array}{rl} 5x + y = 25.5 & (1) \\ 2x + 4y = 30 & (2) \end{array}$$

$$\begin{array}{rl} 10x + 2y = 51 & (1) \times 2 \\ \underline{-(10x + 20y = 150)} & -(2) \times 5 \\ -18y = -99 & \\ y = 5.5 & \end{array}$$

$$\begin{array}{rl} 5x + (5.5) = 25.5 & (1) \\ x = 4 & \end{array}$$

An adolescent pays $4 and an adult pays $5.50

Q7. You need 8 L of a 25% acid solution but you only have a 10% acid solution and a 50% acid solution. How many litres of each solution should you use?

Let x = litres of 10% solution and y = litres of 50% solution

$$\begin{array}{rl} x + y = 8 & (1) \\ 0.1x + 0.5y = 2 & (2) \end{array}$$

$$\begin{array}{rl} x + y = 8 & (1) \\ \underline{-(x + 5y = 20)} & -(2) \times 10 \\ -4y = -12 & \\ y = 3 & \end{array}$$

$$\begin{array}{rl} x + (3) = 8 & (1) \\ x = 5 & \end{array}$$

You need 5 litres of the 10% solution and 3 litres of the 50% solution.

Q8. A customer purchases two trees and five seedling for $55. Another customer purchases three trees and ten seedlings for $90. How much does each cost?

Let x = cost of trees and y = cost of seedlings

$$\begin{array}{rl} 2x + 5y = 55 & (1) \\ 3x + 10y = 90 & (2) \end{array}$$

$$\begin{array}{rl} 6x + 15y = 165 & (1) \times 3 \\ \underline{-(6x + 20y = 180)} & -(2) \times 2 \\ -5y = -15 & \\ y = 3 & \end{array}$$

$$\begin{array}{rl} 2x + 5 \times (3) = 55 & (1) \\ x = 20 & \end{array}$$

Trees cost $20 and seedlings cost $3.

Q9. It takes an aeroplane 1.25 hours to travel 500 km (Wellington to Auckland) with a headwind. The return flight has a tailwind and takes 1 hour. The wind speed stayed the same throughout. What is the plane's speed (in still air) and the wind speed? Remember that distance = speed × time.

Let x = the plane's speed in still air and y = wind speed

The total speed of the plane with a headwind is $x - y$. It travels at this speed for 1.25 hours. Using the distance formula gives:

$$\begin{array}{rl} distance = speed \times time & \\ 500 = (x - y) \times 1.25 & \\ 1.25x - 1.25y = 500 & (1) \end{array}$$

Similarly, the return journey with a tailwind gives:

$$\begin{array}{rl} distance = speed \times time & \\ 500 = (x + y) \times 1 & \\ x + y = 500 & (2) \end{array}$$

Solving this system of equations gives:

$$\begin{array}{rl} 1.25x - 1.25y = 500 & (1) \\ \underline{+(1.25x + 1.25y = 625)} & +(2) \times 1.25 \\ 2.5x = 1125 & \\ x = 450 & \end{array}$$

$$\begin{array}{rl} (450) + y = 500 & (2) \\ y = 50 & \end{array}$$

The plane speed in still air is 450 km/h and the wind speed is 50 km/h

Lesson two advanced questions

Form and solve systems of equations to answer the following questions. Make sure to give your solutions in context.

Q10. A boat takes 1 hour and 30 minutes to travel 50 km upriver and 45 minutes to make the returns journey. What is the boat's speed (in still water) and the speed of the river's flow?

Let $x =$ the boat's speed (in still water) and $y =$ the speed of the river

$$(x-y)\times 1.5 = 50 \qquad (1)$$
$$(x+y)\times 0.75 = 50 \qquad (2)$$

$$\begin{aligned} 1.5x - 1.5y &= 50 && (1)\\ \underline{+(1.5x+1.5y=100)} & && +(2)\times 2\\ 3x &= 150 \\ x &= 50 \end{aligned}$$

$$\begin{aligned} 0.75\times(50) + 0.75y &= 50 && (2)\\ y &= \frac{50}{3}\\ &= 16.7\ (1\text{ d.p.}) \end{aligned}$$

The boat's speed (in still water) is 50 km/h and the river's speed is 16.7 km/h

Q11. You need 6 L of 30% acid solution but you only have a 15% acid solution and a 35% acid solution. How many litres of each solution should you use?

Let $x =$ litres of 15% acid solution and $y =$ litres of 35% acid solution

$$x + y = 6 \qquad (1)$$
$$0.15x + 0.35y = 1.8 \qquad (2)$$

$$\begin{aligned} 15x + 15y &= 90 && (1)\times 15\\ \underline{-(15x+35y=180)} & && -(2)\times 100\\ -20y &= -90 \\ y &= 4.5 \end{aligned}$$

$$\begin{aligned} x + (4.5) &= 6 && (1)\\ x &= 1.5 \end{aligned}$$

You need 1.5 litres of the 15% solution and 4.5 litres of the 35% solution.

Q12. The sum of the digits in a two-digit number is 9. When the digits are reversed, the two-digit number gets bigger by 27. What is the original two-digit number?

Let $x =$ the ten's digit and $y =$ the unit's digit

$$x + y = 9 \qquad (1)$$
$$10x + y = 10y + x - 27 \qquad (2)$$

$$\begin{aligned} 9x + 9y &= 81 && (1)\times 9\\ \underline{+(9x-9y=-27)} & && +(2)\\ 18x &= 54 \\ x &= 3 \end{aligned}$$

$$\begin{aligned} (3) + y &= 9 && (1)\\ y &= 6 \end{aligned}$$

The original number is 36.

Q13. The sum of the digits in a two-digit number is 8. When the digits are reversed, the two-digit number gets smaller by 54. What is the original two-digit number?

Let $x =$ the ten's digit and $y =$ the unit's digit

$$x + y = 8 \qquad (1)$$
$$10x + y = x + 10y + 54 \qquad (2)$$

$$\begin{aligned} 9x + 9y &= 72 && (1)\times 9\\ \underline{+(9x-9y=54)} & && +(2)\\ 18x &= 126 \\ x &= 7 \end{aligned}$$

$$\begin{aligned} (7) + y &= 8 && (1)\\ y &= 1 \end{aligned}$$

The original number is 71.

Q14. An athlete wants to consume 7 600 units of vitamin A and 10 700 units of vitamin B. Each gram of supplement 1 contains 60 units of vitamin A and 35 units of vitamin B. Each gram of supplement 2 contains 20 units of vitamin A and 90 units of vitamin B. How many grams of each supplement should they consume?

Let $x =$ the amount of supplement 1 to take and $y =$ the amount of supplement 2

Focusing on vitamin A gives $60x + 20y = 7\,600$ *(1)*
Focusing on vitamin B gives $35x + 90y = 10\,700$ *(2)*

$$\begin{aligned} 3x + y &= 380 && (1)\div 20\\ 7x + 18y &= 2\,140 && (2)\div 5 \end{aligned}$$

$$\begin{aligned} 21x + 7y &= 2\,660 && (1)\times 7\\ \underline{-(21x+54y=6\,420)} & && -(2)\times 3\\ -47y &= -3\,760 \\ y &= 80 \end{aligned}$$

$$\begin{aligned} 3x + (80) &= 380 && (1)\\ x &= 100 \end{aligned}$$

They should have 100 g of supplement 1 and 80 g of supplement 2.

Lesson three answers

From the notes

Solve the following system of equations. Check your answer on your graphic calculator.

$$\begin{aligned} 2x - 10y - 5z &= -34 && (1) \\ x - y - 2z &= -10 && (2) \\ -x + 7y + 3z &= 22 && (3) \end{aligned}$$

$$\begin{aligned} 2x - 10y - 5z &= -34 && (1) \\ 2x - 2y - 4z &= -20 && (2) \times 2 \\ -2x + 14y + 6z &= 44 && (3) \times 2 \end{aligned}$$

$$\begin{aligned} 2x - 10y - 5z &= -34 && (1) \\ \underline{-(2x - 2y - 4z} &\underline{= -20)} && -(2) \times 2 \\ -8y - z &= -14 && (4) \end{aligned}$$

$$\begin{aligned} x - y - 2z &= -10 && (2) \\ \underline{+(-x + 7y + 3z} &\underline{= 22)} && +(3) \\ 6y + z &= 12 && (5) \end{aligned}$$

$$\begin{aligned} -8y - z &= -14 && (4) \\ \underline{+(6y + z} &\underline{= 12)} && +(5) \\ -2y &= -2 \\ y &= 1 \end{aligned}$$

$$\begin{aligned} -8(1) - z &= -14 && (4) \\ z &= 6 \end{aligned}$$

$$\begin{aligned} 2x - 10(1) - 5(6) &= -34 && (1) \\ x &= 3 \end{aligned}$$

The solution is $(3, 1, 6)$

Lesson three questions

Solve the following systems of equations using either elimination or matrix methods. Check your answers using your graphic calculator.

Q1.

$$\begin{aligned} -4x - 3y + 2z &= -2 && (1) \\ x + y + 6z &= -13 && (2) \\ -x - 3y + 4z &= -3 && (3) \end{aligned}$$

$$\begin{aligned} -4x - 3y + 2z &= -2 && (1) \\ \underline{+(4x + 4y + 24z} &\underline{= -52)} && +(2) \times 4 \\ y + 26z &= -54 && (4) \end{aligned}$$

$$\begin{aligned} -4x - 3y + 2z &= -2 && (1) \\ \underline{-(-4x - 12y + 16z} &\underline{= -12)} && -(3) \times 4 \\ 9y - 14z &= 10 && (5) \end{aligned}$$

$$\begin{aligned} 9y + 234z &= -486 && (4) \times 9 \\ \underline{-(9y - 14z} &\underline{= 10)} && -(5) \\ 248z &= -496 \\ z &= -2 \end{aligned}$$

$$\begin{aligned} y + 26(-2) &= -54 && (4) \\ y &= -2 \end{aligned}$$

$$\begin{aligned} -4x - 3(-2) + 2(-2) &= -2 && (1) \\ x &= 1 \end{aligned}$$

The solution is $(1, -2, -2)$

Q2.

$$\begin{aligned} 2x + 9y - 2z &= -47 && (1) \\ -5x - 3y + 6z &= 17 && (2) \\ -2x + 5y + z &= -20 && (3) \end{aligned}$$

$$\begin{aligned} 10x + 45y - 10z &= -235 && (1) \times 5 \\ \underline{+(-10x - 6y + 12z} &\underline{= 34)} && +(2) \times 2 \\ 39y + 2z &= -201 && (4) \end{aligned}$$

$$\begin{aligned} 2x + 9y - 2z &= -47 && (1) \\ \underline{+(-2x + 5y + z} &\underline{= -20)} && +(3) \\ 14y - z &= -67 && (5) \end{aligned}$$

$$\begin{aligned} 39y + 2z &= -201 && (4) \\ \underline{+(28y - 2z} &\underline{= -134)} && +(5) \times 2 \\ 67y &= -335 \\ y &= -5 \end{aligned}$$

$$\begin{aligned} 39(-5) + 2z &= -201 && (4) \\ z &= -3 \end{aligned}$$

$$\begin{aligned} 2x + 9(-5) - 2(-3) &= -47 && (1) \\ x &= -4 \end{aligned}$$

The solution is $(-4, -5, -3)$

Q3.

$$\begin{aligned} -5x + y - 4z &= -27 && (1) \\ x - 4y + 5z &= 18 && (2) \\ -3x - y + 4z &= 3 && (3) \end{aligned}$$

$$\begin{aligned} -5x + y - 4z &= -27 && (1) \\ \underline{+(5x - 20y + 25z} &\underline{= 90)} && +(2) \times 5 \\ -19y + 21z &= 63 && (4) \end{aligned}$$

$$\begin{aligned} 3x - 12y + 15z &= 54 && (2) \times 3 \\ \underline{+(-3x - y + 4z} &\underline{= 3)} && +(3) \\ 11y + 19z &= 57 && (5) \end{aligned}$$

$$\begin{aligned} -209y + 231z &= 693 && (4) \times 11 \\ \underline{+(209y + 361z} &\underline{= 1\,083)} && +(5) \times 19 \\ 592z &= 1\,776 \\ z &= 3 \end{aligned}$$

$$\begin{aligned} -19y + 21(3) &= 63 && (4) \\ y &= 0 \end{aligned}$$

$$\begin{aligned} -5x + (0) - 4(3) &= -27 && (1) \\ x &= 3 \end{aligned}$$

The solution is $(3, 0 ,3)$

Form and solve systems of equations to answer the following questions. Make sure to give your solutions in context.

Q4. Dean works in a music shop. Over the last three weeks there has been a sale, and music CDs in the Jumbo bin have been sold for \$10, \$15 or \$20. One of Dean's customers bought a total of six CDs. The total cost for buying the six CDs was \$85. The combined number of \$15 and \$20 CDs that the customer bought was twice as many as the number of \$10 CDs that he bought. Find the number of \$10, \$15 and \$20 CDs that the customer bought.

Let: x = \$10 CDs, y = \$15 CDs, and z = \$20 CDs

$$\begin{aligned} x + y + z &= 6 && (1) \\ 10x + 15y + 20z &= 85 && (2) \\ y + z &= 2x && (3) \end{aligned}$$

$$\begin{aligned} 10x + 10y + 10z &= 60 && (1) \times 10 \\ \underline{-(10x + 15y + 20z = 85)} & && -(2) \\ -5y - 10z &= -25 && (4) \end{aligned}$$

$$\begin{aligned} 2x + 2y + 2z &= 12 && (1) \times 2 \\ \underline{+(-2x + y + z = 0)} & && +(3) \\ 3y + 3z &= 12 && (5) \end{aligned}$$

$$\begin{aligned} -15y - 30z &= -75 && (4) \times 3 \\ \underline{+(15y + 15z = 60)} & && +(5) \times 5 \\ -15z &= -15 \\ z &= 1 \end{aligned}$$

$$\begin{aligned} -5y - 10(1) &= -25 && (4) \\ y &= 3 \end{aligned}$$

$$\begin{aligned} x + (3) + (1) &= 6 && (1) \\ x &= 2 \end{aligned}$$

The customer bought 2 \$10 CDs, 3 \$15 CDs, and 1 \$20 CD.

Q5. Dean is paid \$283 one week. He paid board (which included all living expenses), invested some and spent the rest. His board cost \$35 more than the amounts he invested and spent combined. He invested \$20 more than he spent. Find how much money Dean paid for board, how much he invested and how much he spent.

Let x = board cost, y = amount invested, and z = amount spent

$$\begin{aligned} x + y + z &= 283 && (1) \\ x &= y + z + 35 && (2) \\ y &= z + 20 && (3) \end{aligned}$$

$$\begin{aligned} x + y + z &= 283 && (1) \\ \underline{-(x - y - z = 35)} & && -(2) \\ 2y + 2z &= 248 && (4) \end{aligned}$$

$$\begin{aligned} 2y - 2z &= 40 && (3) \times 2 \\ \underline{+(2y + 2z = 248)} & && +(4) \\ 4y &= 288 \\ y &= 72 \end{aligned}$$

$$\begin{aligned} (72) - z &= 20 && (3) \\ z &= 52 \end{aligned}$$

$$\begin{aligned} x + (72) + (52) &= 283 \\ x &= 158 \end{aligned}$$

Dean paid \$158 for board, invested \$72, and spent \$52

Q6. Marni's friend Jane makes and sells three different types of hand lotion. Over the last three days, she has kept a tally of how many containers of each type she has made. This is summarised in the table below. On the first day, she spent a total of 145 minutes making hand lotions, the second day, 130 minutes, and on the third day she spent two hours making hand lotions. Find the type of hand lotion for which the time taken to produce each container is the smallest.

Type of Hand Lotion	Number of Containers		
	Day 1	Day 2	Day 3
Dewberry	8	2	5
Vitamin E enriched	3	2	0
Lavender	6	10	8

Let x = time to make dewberry, y = time to make vitamin E enriched, and z = time to make lavender

$$\begin{aligned} 8x + 3y + 6z &= 145 && (1) \\ 2x + 2y + 10z &= 130 && (2) \\ 5x + 8z &= 120 && (3) \end{aligned}$$

$$\begin{aligned} 16x + 6y + 12z &= 290 && (1) \times 2 \\ \underline{-(6x + 6y + 30z = 390)} & && -(2) \times 3 \\ 10x - 18z &= -100 && (4) \end{aligned}$$

$$\begin{aligned} 10x + 16z &= 240 && (3) \times 2 \\ \underline{-(10x - 18z = -100)} & && -(4) \\ 34z &= 340 \\ z &= 10 \end{aligned}$$

$$\begin{aligned} 5x + 8(10) &= 120 && (3) \\ x &= 8 \end{aligned}$$

$$\begin{aligned} 8(8) + 3y + 6(10) &= 145 && (1) \\ y &= 7 \end{aligned}$$

The vitamin E enriched lotion takes the least time to make; 7 minutes

Q7. A high school has a total of 123 senior students involved in its school production. The number of year 13s was twice as many as the combined number of year 11s and 12s. There were five more year 11s than year 12s. Find the number of each year level in the production.

Let $x =$ years 11s, $y =$ year 12s, and $z =$ year 13s.

$$\begin{aligned} x + y + z &= 123 && (1)\\ z &= 2(x+y) && (2)\\ x &= y + 5 && (3) \end{aligned}$$

$$\begin{aligned} x + y + z &= 123 && (1)\\ -2x - 2y + z &= 0 && (2)\\ x - y &= 5 && (3) \end{aligned}$$

$$\begin{aligned} x + y + z &= 123 && (1)\\ \underline{-(-2x - 2y + z = 0)} & && -(2)\\ 3x + 3y &= 123 && (4) \end{aligned}$$

$$\begin{aligned} 3x - 3y &= 15 && (3)\times 3\\ \underline{+(3x + 3y = 123)} & && +(4)\\ 6x &= 138 &&\\ x &= 23 && \end{aligned}$$

$$\begin{aligned} (23) - y &= 5 && (3)\\ y &= 18 && \end{aligned}$$

$$\begin{aligned} (23) + (18) + z &= 123 && (1)\\ z &= 82 && \end{aligned}$$

There were 23 year 11s, 18 year 12s, and 82 year 13s.

Lesson three advanced questions

Form and solve systems of equations to answer the following questions. Make sure to give your solutions in context.

Q8. Tickets for a high school production were sold for \$5, \$8, and \$15 depending on where the ticket holder will be seated. The total number of tickets sold was 390. The number of \$8 tickets sold was 30 more than twice the number of \$15 tickets sold. The total amount of money received from the ticket sales was \$2 520. Find the number of \$5, \$8, and \$15 tickets sold.

Let $x =$ \$5 tickets, $y =$ \$8 tickets and $z =$ \$15 tickets.

$$\begin{aligned} x + y + z &= 390 && (1)\\ y &= 2z + 30 && (2)\\ 5x + 8y + 15z &= 2\,520 && (3) \end{aligned}$$

$$\begin{aligned} x + y + z &= 390 && (1)\\ y - 2z &= 30 && (2)\\ 5x + 8y + 15z &= 2\,520 && (3) \end{aligned}$$

$$\begin{aligned} 5x + 5y + 5z &= 1\,950 && (1)\times 5\\ \underline{-(5x + 8y + 15z = 2\,520)} & && -(3)\\ -3y - 10z &= -570 && (4) \end{aligned}$$

$$\begin{aligned} 3y - 6z &= 90 && (2)\times 3\\ \underline{+(-3y - 10z = -570)} & && +(4)\\ -16z &= -480 &&\\ z &= 30 && \end{aligned}$$

$$\begin{aligned} y - 2(30) &= 20 && (2)\\ y &= 80 && \end{aligned}$$

$$\begin{aligned} x + (80) + (30) &= 390 && (1)\\ x &= 280 && \end{aligned}$$

They sold 280 \$5 tickets, 80 \$8 tickets, and 30 \$15 tickets.

Q9. A landscape gardener is building a new garden. The gardener has ordered 20 m³ of stones in three different sizes, at a total cost of \$950. The stones are supplied in three sizes: large, medium and small. The price of each size is given in this table:

	Small	Medium	Large
Price (per m³)	\$40	\$50	\$55

They ordered twice as many medium stones as small stones. Find the quantity of each type that was ordered.

Let $x =$ small, $y =$ medium, and $z =$ large.

$$\begin{aligned} x + y + z &= 20 && (1)\\ 40x + 50y + 55z &= 950 && (2)\\ 2x &= y && (3) \end{aligned}$$

$$\begin{aligned} 55x + 55y + 55z &= 1\,100 && (1)\times 55\\ \underline{-(40x + 50y + 55z = 950)} & && -(2)\\ 15x + 5y &= 150 && (4) \end{aligned}$$

$$\begin{aligned} 10x - 5y &= 0 && (3)\times 5\\ \underline{+(15x + 5y = 150)} & && +(4)\\ 25x &= 150 &&\\ x &= 6 && \end{aligned}$$

$$\begin{aligned} 2(6) - y &= 0 && (3)\\ y &= 12 && \end{aligned}$$

$$\begin{aligned} (6) + (12) + z &= 20 && (1)\\ z &= 2 && \end{aligned}$$

They ordered 6m³ of small stones, 12m³ of medium stones and 2m³ of large stones.

Q10. Find the equation of the parabola that passes through the points $(1, 17)$, $(3, 9)$, and $(4, -1)$.

The equation of a parabola is $y = ax^2 + bx + c$

$$\begin{aligned} 17 &= a(1)^2 + b(1) + c && (1)\\ 9 &= a(3)^2 + b(3) + c && (2)\\ -1 &= a(4)^2 + b(4) + c && (3) \end{aligned}$$

$$\begin{aligned} a + b + c &= 17 && (1)\\ 9a + 3b + c &= 9 && (2)\\ 16a + 4b + c &= -1 && (3) \end{aligned}$$

$$\begin{aligned} 9a + 3b + c &= 9 && (2) \\ \underline{-(a + b + c = 17)} & && -(1) \\ 8a + 2b &= -8 && (4) \end{aligned}$$

$$\begin{aligned} 16a + 4b + c &= -1 && (3) \\ \underline{-(a + b + c = 17)} & && -(1) \\ 15a + 3b &= -18 && (5) \end{aligned}$$

$$\begin{aligned} 24a + 6b &= -24 && (4) \times 3 \\ \underline{-(30a + 6b = -36)} & && -(5) \times 2 \\ -6a &= 12 \\ a &= -2 \end{aligned}$$

$$\begin{aligned} 8(-2) + 2b &= -8 && (4) \\ b &= 4 \end{aligned}$$

$$\begin{aligned} (-2) + (4) + c &= 17 && (1) \\ c &= 15 \end{aligned}$$

The equation is $y = -2x^2 + 4x + 15$

Q11. Find the equation of the parabola that passes through the points $(-2, 3)$, $(1, -7.5)$, and $(2, -1)$.

$$\begin{aligned} 3 &= a(-2)^2 + b(-2) + c && (1) \\ -7.5 &= a(1)^2 + b(1) + c && (2) \\ -1 &= a(2)^2 + b(2) + c && (3) \end{aligned}$$

$$\begin{aligned} 4a - 2b + c &= 3 && (1) \\ a + b + c &= -7.5 && (2) \\ 4a + 2b + c &= -1 && (3) \end{aligned}$$

$$\begin{aligned} 4a - 2b + c &= 3 && (1) \\ \underline{-(4a + 2b + c = -1)} & && -(3) \\ -4b &= 4 \\ b &= -1 \end{aligned}$$

$$\begin{aligned} 4a - 2b + c &= 3 && (1) \\ \underline{-(a + b + c = -7.5)} & && -(2) \\ 3a - 3b &= 10.5 \\ 3a - 3(-1) &= 10.5 \\ a &= 2.5 \end{aligned}$$

$$\begin{aligned} (2.5) + (-1) + c &= -7.5 && (2) \\ c &= -9 \end{aligned}$$

The equation is $y = 2.5x^2 - x - 9$

Lesson four answers

From the notes

Show that the following system is inconsistent:

$$\begin{aligned} x - 2y &= 4 && (1) \\ -2x + 4y &= 10 && (2) \end{aligned}$$

$$\begin{aligned} -2x + 4y &= -8 && (1) \times -2 \\ \underline{-(-2x + 4y = 10)} & && -(2) \\ 0 &= -18 \end{aligned}$$

Therefore the system is inconsistent.

Show that the following system is inconsistent and give a geometrical interpretation (i.e. state whether there are two or more parallel planes or the planes form a triangular prism).

$$\begin{aligned} x + y + z &= 4 && (1) \\ 2x + 3y + 2z &= 8 && (2) \\ 3x + 4y + 3z &= 16 && (3) \end{aligned}$$

$$\begin{aligned} 2x + 2y + 2z &= 8 && (1) \times 2 \\ \underline{-(2x + 3y + 2z = 8)} & && -(2) \\ -y &= 0 \\ y &= 0 && (4) \end{aligned}$$

$$\begin{aligned} 3x + 3y + 3z &= 12 && (1) \times 3 \\ \underline{-(3x + 4y + 3z = 16)} & && -(3) \\ -y &= -4 \\ y &= 4 && (5) \end{aligned}$$

The system in inconsistent because (4) and (5) are contradictory. None of the equations have the same (or multiples of the same) coefficients so none of the planes are parallel. Therefore the planes must form a triangular prism.

Show that the following system has non-unique solutions and give a geometrical interpretation.

$$\begin{aligned} 2x + 3y + z &= 5 && (1) \\ x - 2y + 2z &= 4 && (2) \\ 6x + 9y + 3z &= 15 && (3) \end{aligned}$$

$$\begin{aligned} 6x + 9y + 3z &= 15 && (3) \\ \underline{-(6x + 9y + 3z = 15)} & && -(1) \times 3 \\ 0 &= 0 \end{aligned}$$

Therefore the system has non-unique (or infinite) solutions. (1) and (3) are multiples of each other so they represent the same plane.

Lesson four questions

Show whether the following systems are inconsistent or have non-unique solutions and give a geometrical interpretation.

Q1.

$$\begin{aligned} x + y + z &= 7 && (1) \\ 3x + 3y + 3z &= 10 && (2) \\ 2x + 2y + 2z &= 12 && (3) \end{aligned}$$

$$\begin{aligned} 3x + 3y + 3z &= 21 && (1) \times 3 \\ \underline{-(3x + 3y + 3z = 10)} & && -(2) \\ 0 &= 11 \end{aligned}$$

Therefore the system is inconsistent. All three planes are parallel (because the coefficients of all three planes are multiples of each other).

Q2.

$$\begin{aligned} 11x - y + 3z &= 4 && (1) \\ 2x + 3y - z &= -3 && (2) \\ x - 16y + 8z &= 19 && (3) \end{aligned}$$

$$\begin{aligned} 2x + 3y - z &= -3 && (2) \\ \underline{-(2x - 32y + 16z = 38)} & && -(3) \times 2 \\ 35y - 17z &= -41 && (4) \end{aligned}$$

$$\begin{aligned} 11x - y + 3z &= 4 && (1) \\ \underline{-(11x - 176y + 88z = 209)} & && -(3) \times 11 \\ 175y - 85z &= -205 && (5) \end{aligned}$$

$(4) = (5) \div 5$ *so there are infinitely many solutions. None of the planes are identical so they must all intersect along the same line.*

Q3.

$$\begin{aligned} 2x - 10y + 6z &= 7 && (1) \\ 5x - 25y + 15z &= -9 && (2) \\ 3x + 7y - 5z &= 4 && (3) \end{aligned}$$

$$\begin{aligned} 10x - 50y + 30z &= 35 && (1) \times 5 \\ \underline{-(10x - 50y + 30z = -18)} & && -(2) \times 2 \\ 0 &= 53 && \end{aligned}$$

Therefore the system is inconsistent. Planes (1) and (2) are parallel.

Q4.

$$\begin{aligned} 2x + 6y - 4z &= 10 && (1) \\ 2x - 5y - 3z &= -4 && (2) \\ 3x - 2y - 5z &= 1 && (3) \end{aligned}$$

$$\begin{aligned} 2x + 6y - 4z &= 10 && (1) \\ \underline{-(2x - 5y - 3z = -4)} & && -(2) \\ 11y - z &= 14 && (4) \end{aligned}$$

$$\begin{aligned} 6x + 18y - 12z &= 30 && (1) \times 3 \\ \underline{-(6x - 4y - 10z = 2)} & && -(3) \times 2 \\ 22y - 2z &= 28 && (5) \end{aligned}$$

$(4) \times 2 = (5)$ *so there are infinitely many solutions. None of the planes are identical so they must all intersect along the same line.*

Lesson four advanced questions

Show whether the following systems are inconsistent or have non-unique solutions and give a geometrical interpretation. **If there are non-unique solutions, describe their form.**

Q5.

$$\begin{aligned} x + y + z &= 2 && (1) \\ 3x + 3y + 3z &= 6 && (2) \\ 2x + 4y - 5z &= 6 && (3) \end{aligned}$$

$$\begin{aligned} 3x + 3y + 3z &= 6 && (1) \times 3 \\ \underline{-(3x + 3y + 3z = 6)} & && -(2) \\ 0 &= 0 && \end{aligned}$$

Therefore the system has infinitely many solutions because planes (1) and (2) are the same. Let $z = t$*, then:*

$$\begin{aligned} 2x + 2y + 2z &= 4 && (1) \times 2 \\ \underline{-(2x + 4y - 5z = 6)} & && -(3) \\ -2y + 7z &= -2 && (4) \\ -2y + 7t &= -2 && \\ 2y &= 7t + 2 && \\ y &= 3.5t + 1 && \end{aligned}$$

and:

$$\begin{aligned} x + (3.5t + 1) + (t) &= 2 && (1) \\ x &= -4.5t + 1 && \end{aligned}$$

The solutions are of the form:

$$\begin{aligned} x &= -4.5t + 1 \\ y &= 3.5t + 1 \\ z &= t \end{aligned}$$

where t *is any number.*

Q6.

$$\begin{aligned} x + y + z &= 3 && (1) \\ 2x - 3y + 3z &= 7 && (2) \\ 3x - 2y + 4z &= 6 && (3) \end{aligned}$$

$$\begin{aligned} 2x + 2y + 2z &= 6 && (1) \times 2 \\ \underline{-(2x - 3y + 3z = 7)} & && -(2) \\ 5y - z &= -1 && (4) \end{aligned}$$

$$\begin{aligned} 3x + 3y + 3z &= 9 && (1) \times 3 \\ \underline{-(3x - 2y + 4z = 6)} & && -(3) \\ 5y - z &= 3 && (5) \end{aligned}$$

Equations (4) and (5) are inconsistent so there are no solutions. None of the planes are parallel so they must form a triangular prism.

Q7. Find the values of a and b that make the following system have non-unique solutions.

$$\begin{aligned} 2x + 4y + 5z &= 17 && (1) \\ 4x + ay + 3z &= b && (2) \\ 8x + 7y + 13z &= 40 && (3) \end{aligned}$$

$$\begin{aligned} 4x + 8y + 10z &= 34 && (1) \times 2 \\ \underline{-(4x + ay + 3z = b)} & && -(2) \\ (8 - a)y + 7z &= 34 - b && (4) \end{aligned}$$

$$\begin{aligned} 8x + 7y + 13z &= 40 && (3) \\ \underline{-(8x + 2ay + 6z = 2b)} & && -(2) \times 2 \\ (7 - 2a)y + 7z &= 40 - 2b && (5) \end{aligned}$$

For non-unique solutions, we need $(4) \equiv (5)$*, i.e.:*

$$\begin{aligned} 8 - a &= 7 - 2a & \qquad 34 - b &= 40 - 2b \\ a &= -1 & b &= 6 \end{aligned}$$

Q8. Find the values of A and B that make the following system inconsistent.

$$\begin{aligned} 4x + 2y - 10z &= -17 && (1) \\ -6x - 4y + 10z &= 32 && (2) \\ 5y - 7 &= Ax + Bz && (3) \end{aligned}$$

Rewriting equation (3):

$$-Ax + 5y - Bz = 7 \qquad (3)$$

Solving:

$$\begin{aligned} 8x + 4y - 20z &= -34 && (1) \times 2 \\ \underline{+(-6x - 4y + 10z = 32)} & && +(2) \\ 2x - 10z &= -2 && (4) \end{aligned}$$

$$\begin{aligned} 20x + 10y - 50z &= -85 && (1) \times 5 \\ \underline{-(-2Ax + 10y - 2Bz = 14)} & && -(3) \times 2 \\ (20 + 2A)x + (2B - 50)z &= -99 && (5) \end{aligned}$$

For an inconsistent system, the coefficients of (4) and (5) need to be equal, i.e.:

$$20 + 2A = 2 \qquad\qquad -10 = 2B - 50$$

$$A = -9 \qquad\qquad B = 20$$

Practice internal 1 answers

First method

$$\begin{aligned} 2x - 3y + 2z &= 1 && (1) \\ y - 4z &= 8 && (2) \\ 3x + 2y + 8z &= 7 && (3) \end{aligned}$$

$$\begin{aligned} 6x - 9y + 6z &= 3 && (1) \times 3 \\ \underline{-(6x + 4y + 16z = 14)} & && -(3) \times 2 \\ -13y - 10z &= -11 && (4) \end{aligned}$$

$$\begin{aligned} 5y - 20z &= 40 && (2) \times 5 \\ \underline{-(-26y - 20z = -22)} & && +(4) \times 2 \\ 31y &= 62 \\ y &= 2 \end{aligned}$$

$$\begin{aligned} 2 - 4z &= 8 && (2) \\ z &= -1.5 \end{aligned}$$

$$\begin{aligned} 2x - 3(2) + 2(-1.5) &= 1 && (1) \\ x &= 5 \end{aligned}$$

The solution is $(5, 2, -1.5)$*. This is the point at which all three planes intersect.*

Second method

$$\begin{aligned} 2x - 3y + 2z &= 1 && (1) \\ y - 4z &= 8 && (2) \\ 6x - 9y + 6z &= 3 && (3) \end{aligned}$$

Because $(1) = (3) \div 3$*, they represent the same plane so there will be infinitely many solutions, e.g.:*

$$\begin{aligned} 6x - 9y + 6z &= 3 && (1) \times 3 \\ \underline{-(6x - 9y + 6z = 3} & && -(3) \\ 0 &= 0 \end{aligned}$$

Let $z = t$*:*

$$\begin{aligned} y - 4t &= 8 && (2) \\ y &= 4t + 8 \end{aligned}$$

$$\begin{aligned} 2x - 3(4t + 8) + 2(t) &= 1 && (1) \\ x &= 5t + 12.5 \end{aligned}$$

The solutions take the form of $(5t + 12.5, 4t + 8, t)$ *where* t *is any number.*

Third method

$$\begin{aligned} 2x - 3y + 2z &= 1 && (1) \\ y - 4z &= 8 && (2) \\ 2x - 3y + 2z &= 6 && (3) \end{aligned}$$

Because equations (1) and (3) are identical except for their constant, they will be parallel planes so the system will be inconsistent, e.g.:

$$\begin{aligned} 2x - 3y + 2z &= 1 && (1) \\ \underline{-(2x - 3y + 2z = 6)} & && -(3) \\ 0 &= -5 \end{aligned}$$

Achieved	Merit	Excellence
Has solved at least one system of equations to find either: • The point of intersection • That the system is inconsistent • That the system has non-unique solutions Has given a geometric interpretation of at least one system of equations.	Has solved all three systems of equations and given geometric interpretations for all three.	As for merit, plus: Has given the general form of the solutions for the system with non-unique solutions.

Practice internal 2 answers

Let x = amount of Healthy, y = amount of Budget, and z = amount of Organic

$$\begin{aligned} 2x + 4y + 5z &= 1\,000 && (1)\\ 3x + 7y + 10z &= 1\,600 && (2)\\ 5x + 9y + 14z &= 2\,400 && (3) \end{aligned}$$

$$\begin{aligned} 6x + 12y + 15z &= 3\,000 && (1)\times 3\\ \underline{-(6x + 14y + 20z = 3\,200)} & && -(2)\times 2\\ -2y - 5z &= -200 &&\\ 2y + 5z &= 200 && (4) \end{aligned}$$

$$\begin{aligned} 10x + 20y + 25z &= 5\,000 && (1)\times 5\\ \underline{-(10x + 18y + 28z = 4\,800)} & && -(3)\times 2\\ 2y - 3z &= 200 && (5) \end{aligned}$$

$$\begin{aligned} 2y + 5z &= 200 && (4)\\ \underline{-(2y - 3z = 200)} & && -(5)\\ 8z &= 0 &&\\ z &= 0 && \end{aligned}$$

$$\begin{aligned} 2y + 5(0) &= 200 && (4)\\ y &= 100 && \end{aligned}$$

$$\begin{aligned} 2x + 4(100) + 5(0) &= 1\,000 && (1)\\ x &= 300 && \end{aligned}$$

Roger should feed his rabbits 300g of Healthy food, 100g of Budget food and no Organic food.

If Organic increases the amount of vitamin A in its food to 6 μg:

$$\begin{aligned} 2x + 4y + 6z &= 1\,000 && (1)\\ 3x + 7y + 10z &= 1\,600 && (2)\\ 5x + 9y + 14z &= 2\,400 && (3) \end{aligned}$$

$$\begin{aligned} 6x + 12y + 18z &= 3\,000 && (1)\times 3\\ \underline{-(6x + 14y + 20z = 3\,200)} & && -(2)\times 2\\ -2y - 2z &= -200 &&\\ y + z &= 100 && (4) \end{aligned}$$

$$\begin{aligned} 10x + 20y + 30z &= 5\,000 && (1)\times 5\\ \underline{-(10x + 18y + 28z = 4\,800)} & && -(3)\times 2\\ 2y + 2z &= 200 &&\\ y + z &= 100 && (5) \end{aligned}$$

$(4) = (5)$ *so there are infinitely many solutions. None of the planes are parallel so they must all intersect along the same line. Let* $z = t$*:*

$$\begin{aligned} y + (t) &= 100 && (4)\\ y &= 100 - t && \end{aligned}$$

$$\begin{aligned} 2x + 4(100 - t) + 6(t) &= 1\,000 && (1)\\ x &= 300 - t && \end{aligned}$$

Neither $x, y, z < 0$ *so* t *has to be between 0 and 100 as when* $t = 0$*, the solution is* $(300, 100, 0)$ *which is the same solution as before. And when* $t = 100$*, the solution is* $(200, 0, 100)$.

No amount of vitamin A in the Organic food will force Roger to buy more. In fact, for most amounts of vitamin A, Roger will buy no Organic food. But if the amount is exactly 6 μg then there is a range of solutions that means he could buy up to 100 g of Organic food.

Achieved	Merit	Excellence
Formed and solved a system of equations to state how much of each food Roger should feed his rabbits.	Shown that the new system of equations has infinitely many solutions and given one possible solution.	Given the general form of the general solutions (given contextual constraints, i.e. all amounts need to be positive) and has discussed the quantity of vitamin A that would encourage Roger to buy more Organic food.

3.15 Simultaneous equations log

My goal for 3.15 simultaneous equations is: ____

Lessons	Questions completed and marked	Advanced questions completed and marked
Lesson One: 2D systems of equations		
Lesson Two: Forming systems of equations		
Lesson Three: 3D systems of equations		
Lesson Four: Inconsistent systems and non-unique solutions		

Practice internals	Completed and marked	Result
Practice internal 1		
Practice internal 2		