Level 2 – AS91257– 2.2 – 4 Credits – Internal

Apply graphical methods in solving problems

Context

Level 1	Level 2	Level 3 calculus
	2.2 Graphs 4 credits [Int] →	3.3 Trigonometry 4 credits [Int]

Sample question: Bridges

Use a variety of curves [straight lines, parabolic, cubic, square root, trigonometric, or hyperbolic] to model the London Bridge. Try modelling it for the fictional values of height=12m and length=60m. Then try modelling it with the real values: height=65 m and length=244 m.[1]

Using the same methods, model the Auckland Bridge. The length is 1,020m.[2] The height is unknown but the maximum height of the underside is 43m.[3]

[1] https://en.wikipedia.org/wiki/Tower_Bridge
[2] https://en.wikipedia.org/wiki/Auckland_Harbour_Bridge
[3] http://www.3dnewzealand.com/auckland/city/harbour-bridge/index.html

Passport

Use this to track your progress through this workbook.

Polynomial curves	First time	Revision
Factorised form of parabolas		
Completed square form of parabolas		
Factorised cubics		
Factorised quartics and higher		

Non-polynomial curves

Hyperbolas		
Exponential functions		
Logarithmic functions		
Trigonometric functions		
Trigonometric modelling		

Graphs

Graphical modelling		

Lesson One: Introduction to parabolas

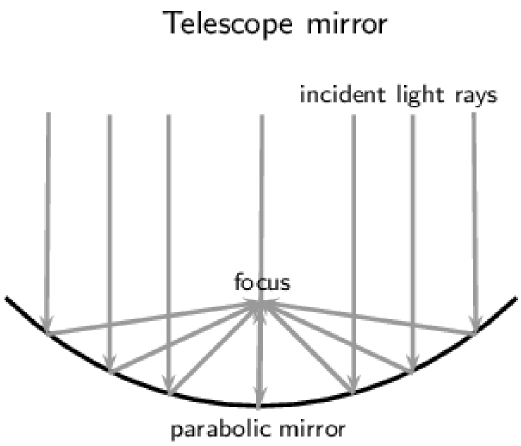
Parabola [noun]: A type of curve. It is one of the conic sections and looks like an arch.

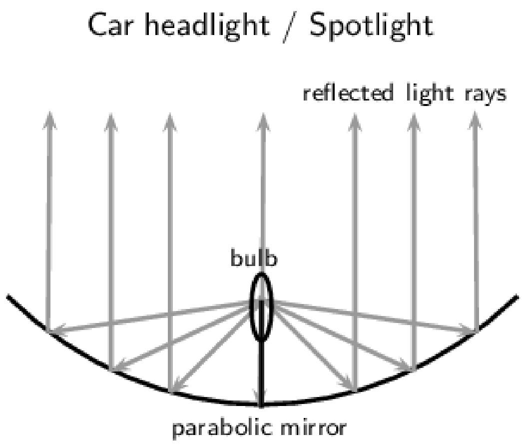
Parabolic [adjective]: Parabola-shaped, e.g. parabolic mirror, parabolic arch.

Parabolas are so insanely useful it's hard to know where to start. One of their uses is in projectiles, e.g. footballs, cannonballs, and Angry Birds. Launch any projectile with any mass on any planet and the projectile will follow a parabolic path. Guaranteed.

Projectiles
This basketball is following a parabolic path because it must.[4] Excitingly, that means we can model its path with a parabola to predict if it will go in the hoop. Check out the Basketball- GeoGebra resources on the 2.2 graphs page.

Headlights
Parabolas also have the property that they focus all straight lines [that hit the parabola head on] into a single focus point. This is useful for burning things with the sun but also works in reverse, e.g. car headlights.

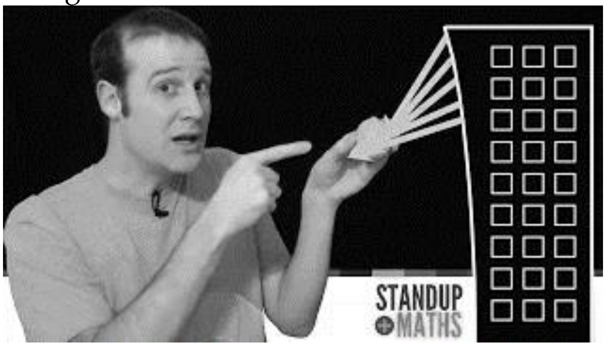




[4] Because it has constant horizontal motion but constant vertical *acceleration*.

Setting things on fire

Paraboloids and The Building which Set Things on Fire



https://youtu.be/owVwjr6pTqc

Parabolic Solar Cooker Test

https://youtu.be/n8gz88mnS0o

Setting things on fire in a big way

Search: "Odeillo solar furnace" (https://en.wikipedia.org/wiki/Odeillo_solar_furnace)

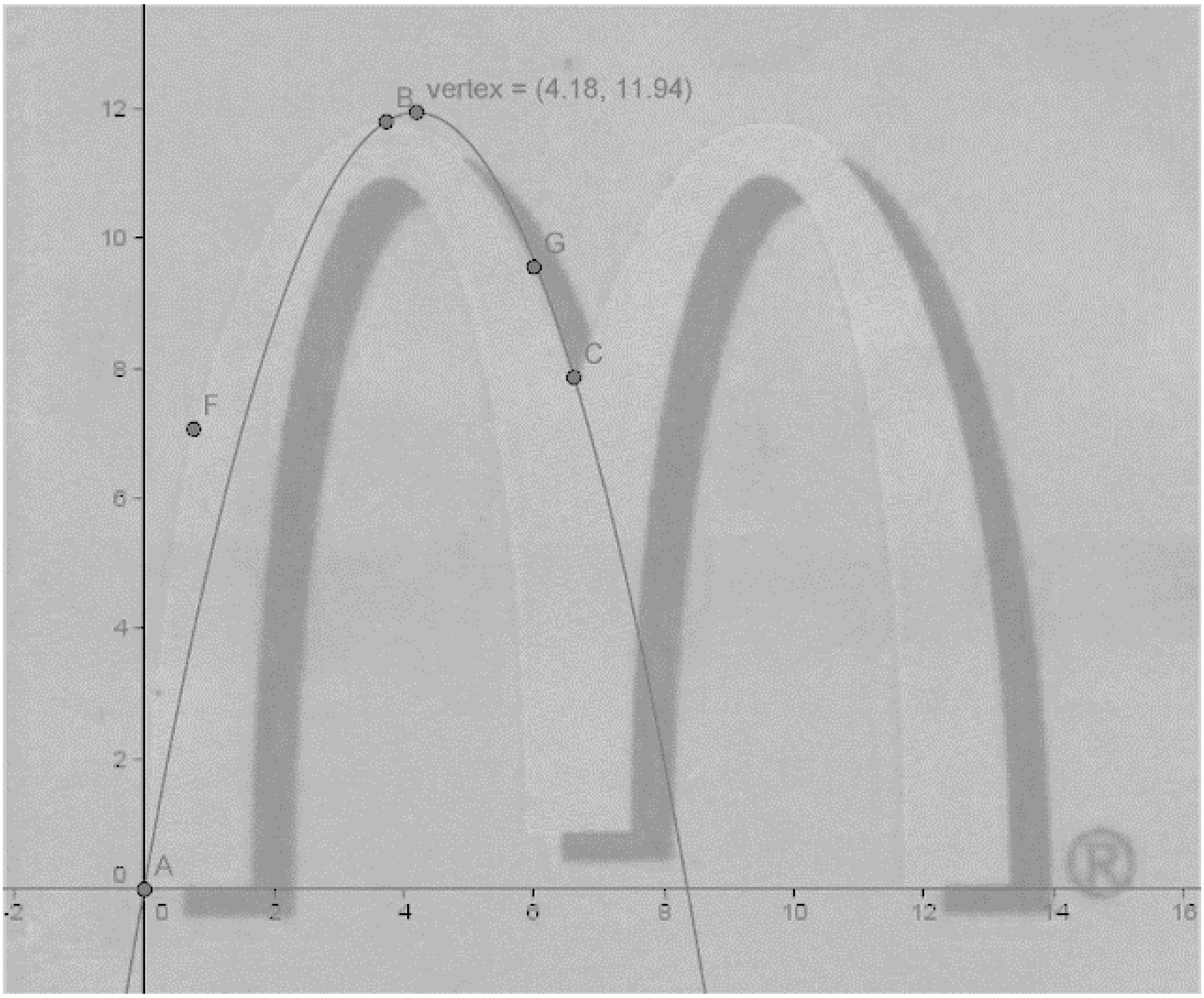
Whisper dishes

Parabolic reflectors can focus light and radiation but they can also focus sound. Outside the Otago Museum in Dunedin there are a pair of "whisper dishes". Speaking into the mouthpiece of one, you can be heard 20-30 metres away[5] at the mouthpiece of the other dish, even if you whisper.

[5] Ish.

Non-parabolas

Not all arches are parabolas. For example, the McDonalds arches aren't parabolas. No parabolic curve will ever model these arches. For more information, search "The Golden Arches Exposed" (http://omega-unlimited.blogspot.co.nz/2011/10/golden-arches-exposed.html).



Parabolas can be expressed in several forms. You will explore three:

- Quadratic form[6]
- Factorised form
- Completed square form

[6] I'm pretty sure I just made this name up.

Lesson Two: Different forms of parabolas

Quadratic form[7]

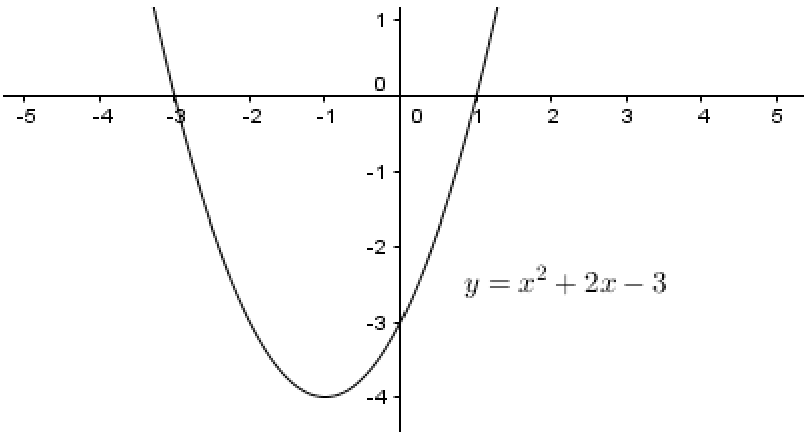
$$y = ax^2 + bx + c$$

where:
a =nothing useful
b =nothing useful
c =nothing useful

Quadratic [noun]: Square-like.

This is the form in which you usually first encounter parabolas. Advantages of this form are:

- You're used to seeing it this way



Factorised form

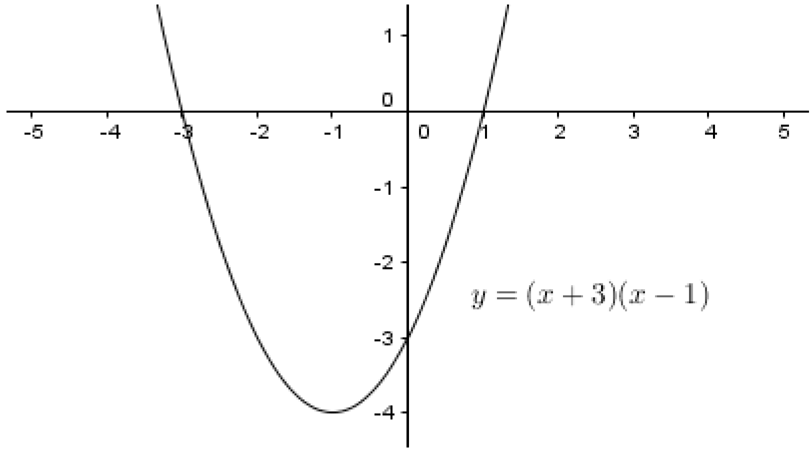
$$y = a(x - b)(x - c)$$

where:
a =scale factor
b =one root
c =the other root

This form is the factorised version of the quadratic form. It involves finding the factors of the quadratic, $(x - a)$ and $(x - b)$. Advantages of this form are:

- It's laughably easy to find the roots of the parabola.
- Once you have the roots, you can calculate the turning point.

[7] This is actually called expanded form but I prefer the name I made up for it: quadratic form.



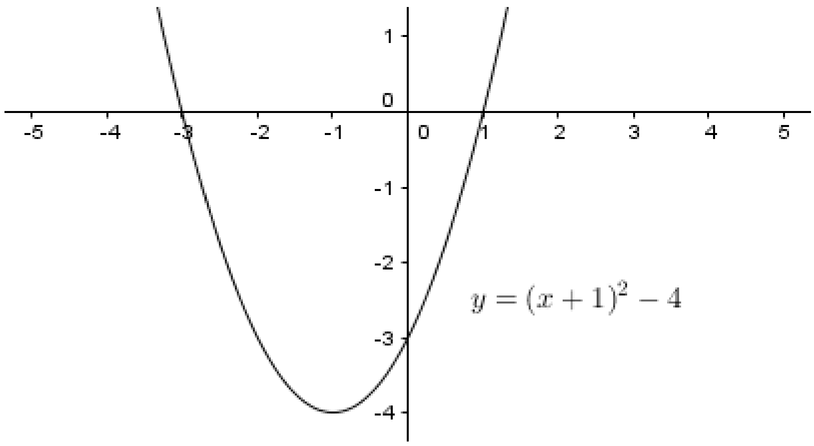
Completed square form

$$y = a(x-b)^2 + c$$

where:
a =scale factor
b =horizontal shift
c =vertical shift

This is the form that Dan Meyer's basketball app on GeoGebra uses except he uses different variable names: $y = a(x-h)^2 + k$. Advantages of this form are:

- It's laughably easy to find the turning point.
- Finding the roots is then just a matter of playing with the scale factor.



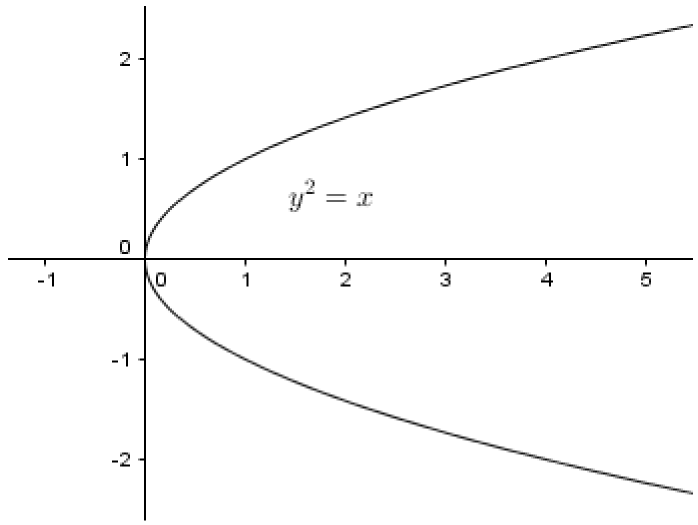
Some general parabola information

- Turning point: The point at which the parabola changes direction. Depending on the orientation, this is either a maximum or a minimum. It's always halfway between the two roots.
- Roots: The points at which the parabola crosses the x-axis, i.e. when $y = 0$. There can be 2 roots, 1 repeated root, or no roots.[8] Also called the x-intercepts.
- y-intercept: The point at which the parabola crosses the y-axis, i.e. when $x = 0$. There is always exactly one y-intercept.[9]

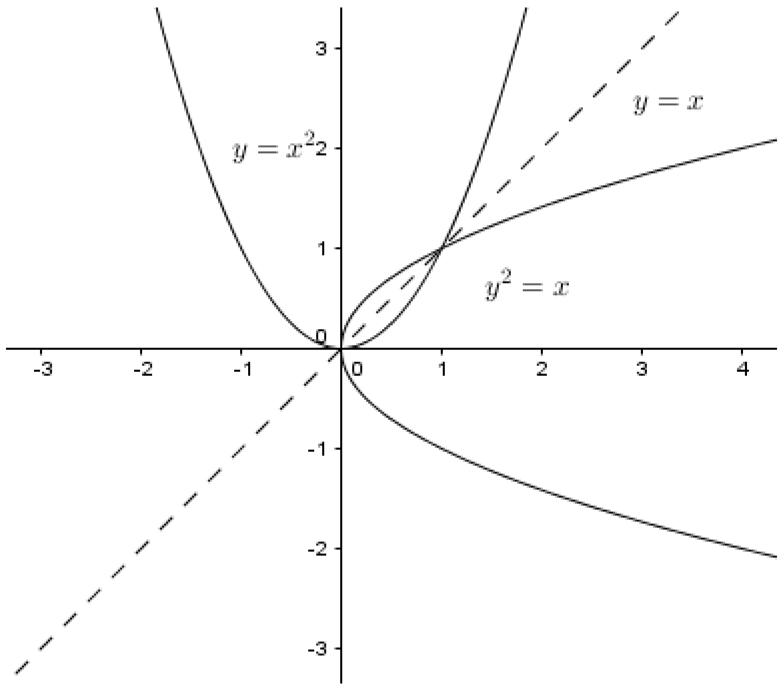
[8] At least, as far as real roots are concerned. There are always two roots of parabolas, though they may be the same [repeated] or complex [involving imaginary numbers].
[9] Because of how functions work; there can ever only be one y value for each x value.

- Drawing parabolas: To draw a parabola, find the three interesting points: the turning point and the two [presumably there's two] roots, then find as many other points as you need to get a sense of the scale. Use all of these points to sketch the graph.

Inverse parabolas
The inverse of a graph is the mirror image in the mirror line $y = x$. For example the inverse parabola looks like this:



The equation of the inverse parabola can be found by swapping x and y in the original equation. This is the same as reflecting it in the mirror line $y = x$.



Or you could think of the inverse operation of squaring which is square root:

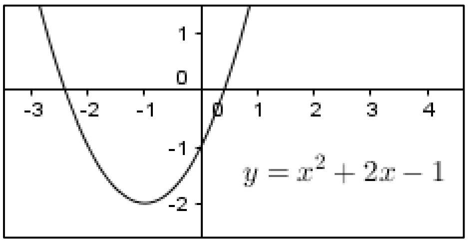
$$y = \sqrt{x}$$

Except this isn't quite right. Why not?

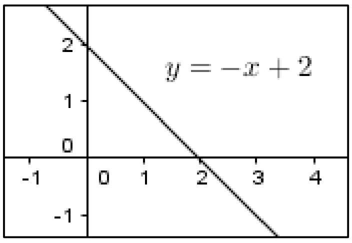
Lesson Three: Higher order polynomials

Polynomial [noun]: An equation of several terms of different powers of the same variable.

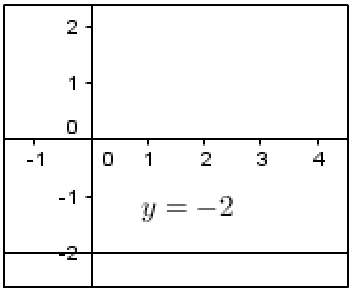
A quadratic is a polynomial of order 2, i.e. the highest power of x is 2: $y = ax^2 + bx + c$



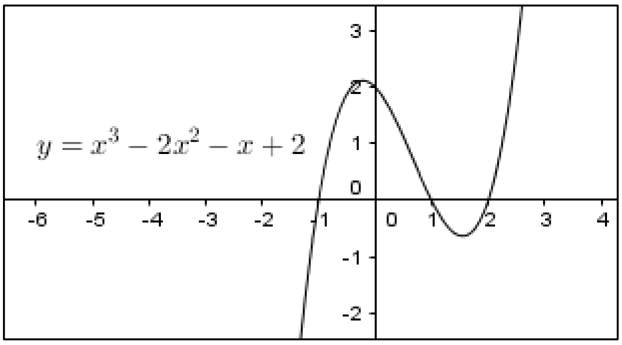
A straight line is a polynomial of order 1:
$y = ax + c$



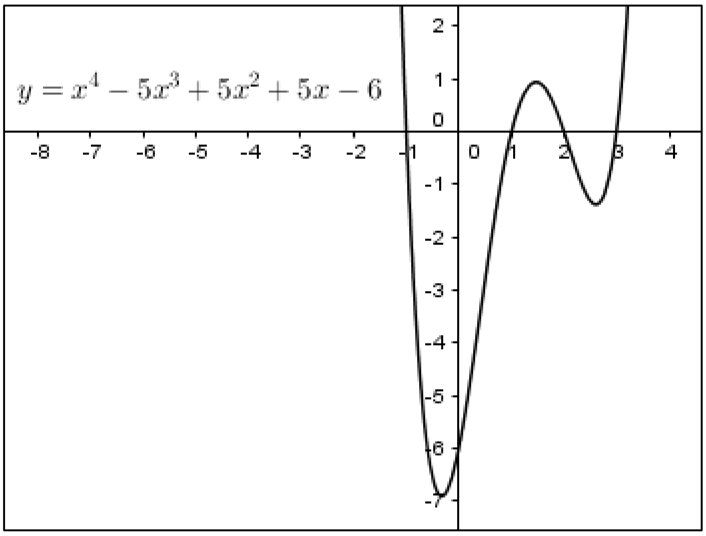
A constant is a polynomial of order 0:
$y = c$



Similarly, a cubic is a polynomial of order 3:
$y = ax^3 + bx^2 + cx + d$



And a quartic is a polynomial of order 4:
$y = ax^4 + bx^3 + cx^2 + dx + e$



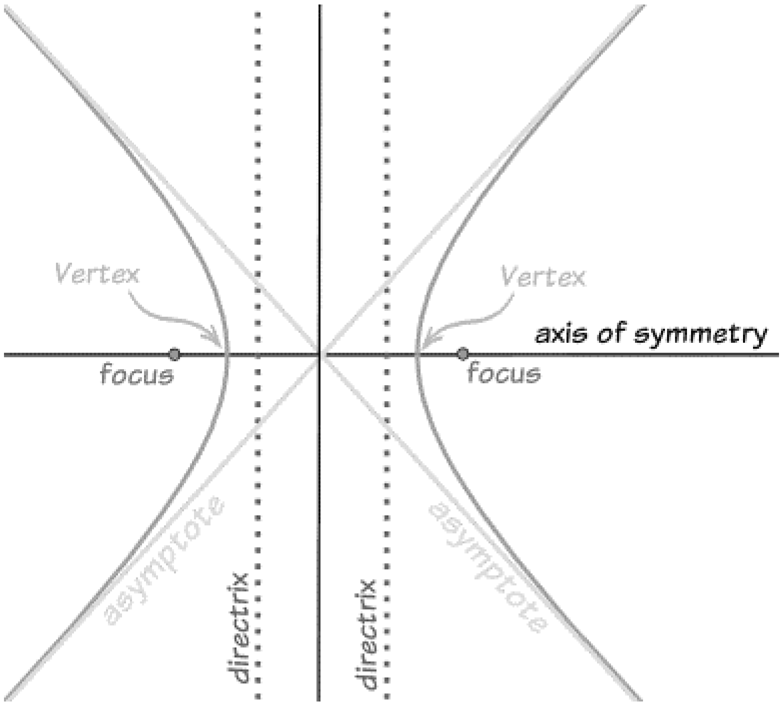
In this standard you will always encounter higher order polynomials in factorised form which immediately shows you where the roots will be. Unfortunately, unlike parabolas, the turning points won't be halfway between the roots. The best way to get a sense of orientation and scale of the polynomial is to find several other points on the curve, or to draw it using a graphics calculator or GeoGebra.

Lesson Four: Hyperbolas

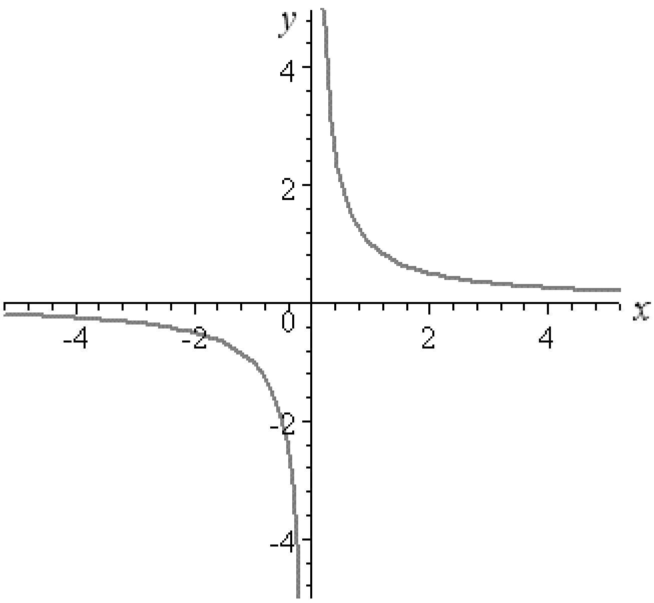
Hyperbola [noun]: Meaning over-thrown or excessive. A type of curve. It is one of the conic sections and looks like two bows. The plural is hyperbolas.[10]

Rectangular hyperbola [noun]: A hyperbola whose asymptotes are perpendicular; at right angles.

Hyperbolic [adjective]: In the shape of a hyperbola, e.g. hyperbolic trajectory.



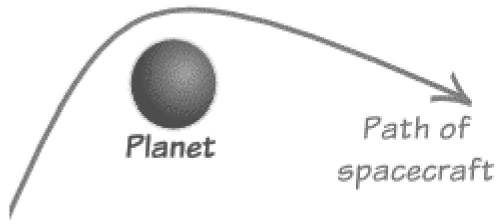
In this standard, we will be looking solely at rectangular hyperbolas whose asymptotes are parallel to the x and y-axes.



[10] Or hyperbolae if you want to sound clever.

Gravity slingshots

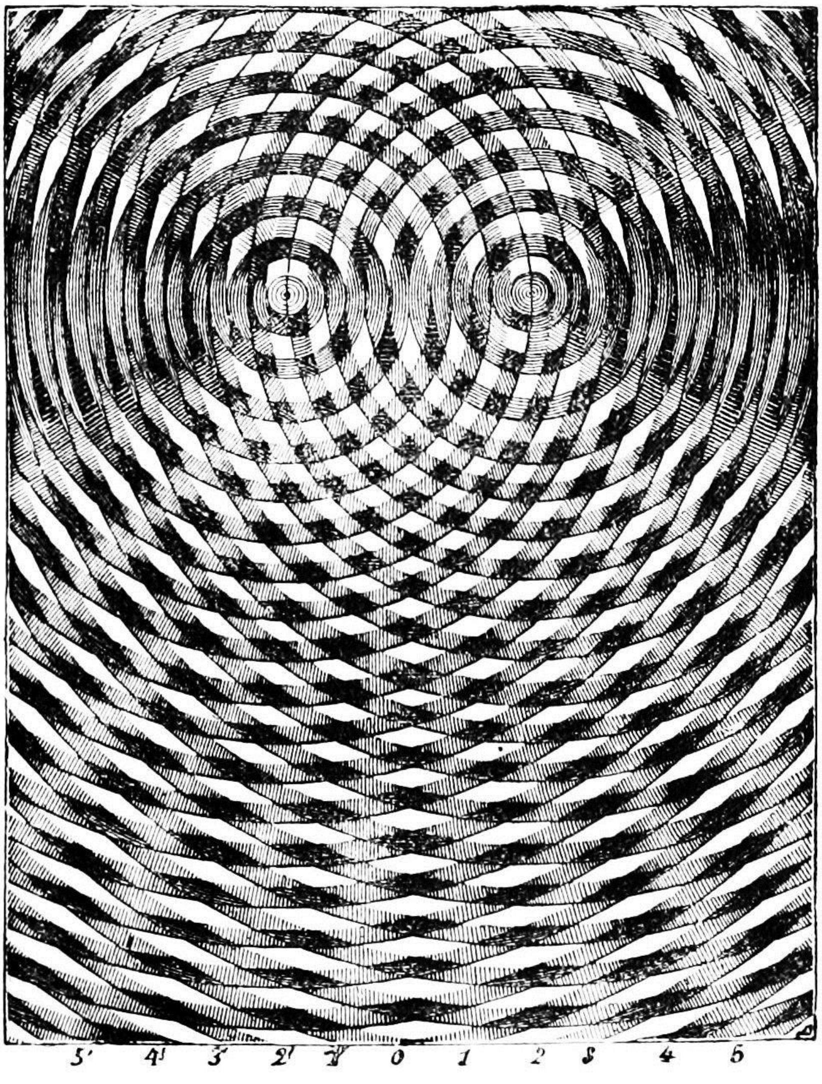
Hyperbolas appear in many places. A spacecraft or other mass approaching a larger mass, e.g. a planet will either get trapped in orbit [an elliptical path[11]] or follow a hyperbolic path in what's known as a gravity slingshot.[12]



Try hyperbolic paths yourself by searching "gravity freeplay" (http://www.mathsisfun.com/numbers/gravity-freeplay.html)

Wave interference

For reasons I don't yet understand, the interference pattern of waves forms hyperbolas.

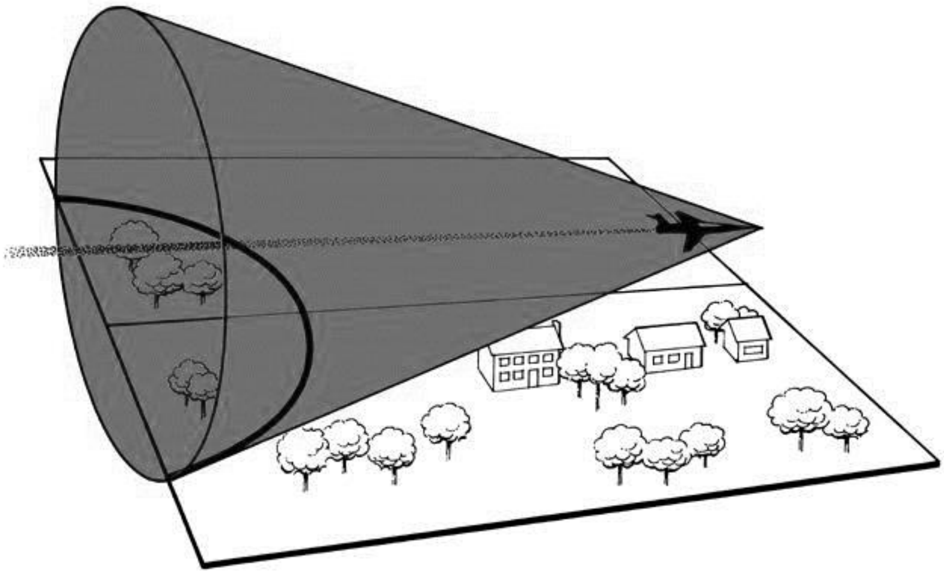


[11] Ellipses are another conic section.

[12] Surely, one of the coolest two-word phrases in the English language.

Sonic booms

A cone of sound [and therefore pressure] forms behind aircraft travelling faster than the speed of sound. Where that cone intersects with the ground, a hyperbola forms. For more information, see (http://www.pleacher.com/mp/mlessons/calculus/apphyper.html)



Equations of hyperbolas

Basic form:

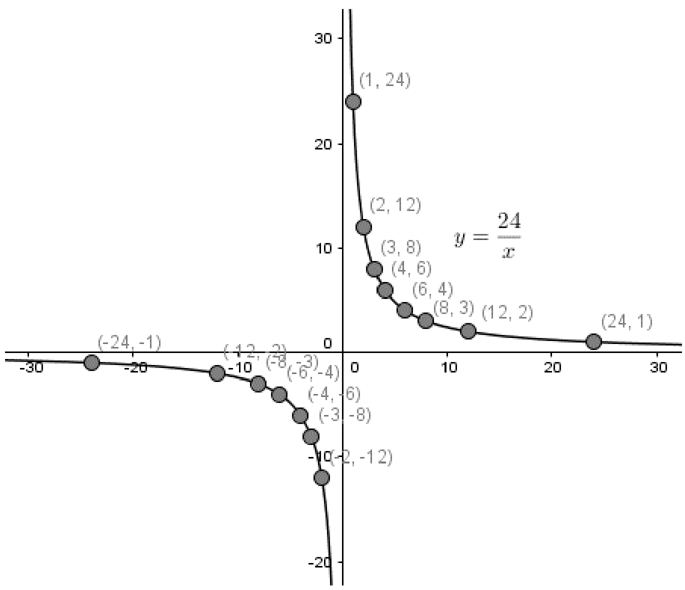
$$y = \frac{a}{x}$$

Full form:

$$y = \frac{a}{(x-b)} + c$$

where:
a =scale factor
b =horizontal shift
c =vertical shift

This is the most common way of expressing a hyperbola. Another way is $xy = a$ which shows a property of the hyperbola: it is the set of all factors of a constant, e.g. if $xy = 24$ then points on this hyperbola include (1,24), (2,12), (3,8), (4,6), (6,4), (8, 3), (12, 2), and (24, 1).



Some general hyperbola information

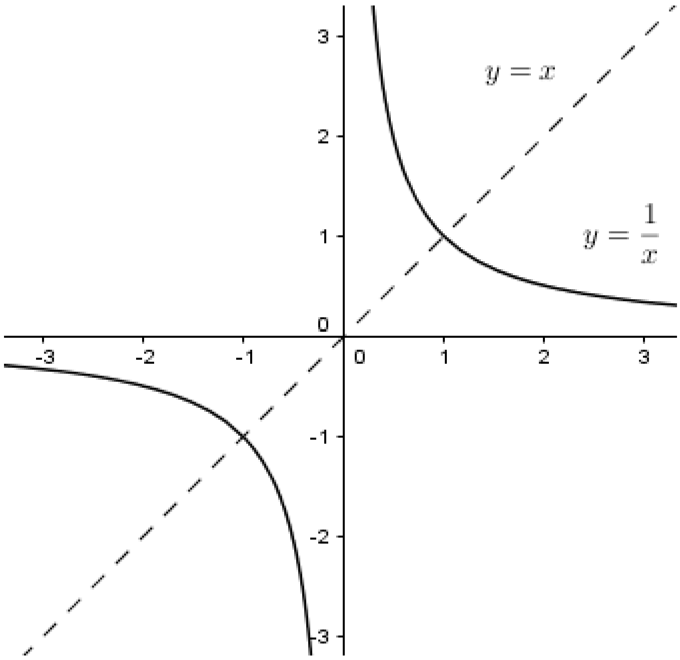
- Asymptotes: Hyperbolas have two asymptotes. These are lines that the hyperbola approaches but never touches. In rectangular hyperbolas, they are perpendicular. In the rectangular hyperbolas we'll look at, they'll also be parallel with the x and y-axes though they don't have to be the axes themselves, e.g. $y = \frac{1}{(x-1)} + 1$.
- y-intercept: The point at which the hyperbola crosses the y-axis, i.e. when $x = 0$. Uniquely, the hyperbola will not have a y-intercept if one of the asymptotes is the line $x = 0$, i.e. the y-axis itself.

Finding asymptotes

The vertical asymptote can be found by finding where the graph is undefined, i.e. there is no y value. The horizontal asymptote can be found by taking a very large value of x, say 1 000 000, and seeing what y-value it is approaching.

Inverse hyperbolas

An inverse hyperbola is the mirror image of a hyperbola in the mirror line $y = x$ and it looks like this:



It looks the same. Weird. Why is that?

The hyperbola is its own inverse.

Lesson Five: Exponential and logarithmic functions

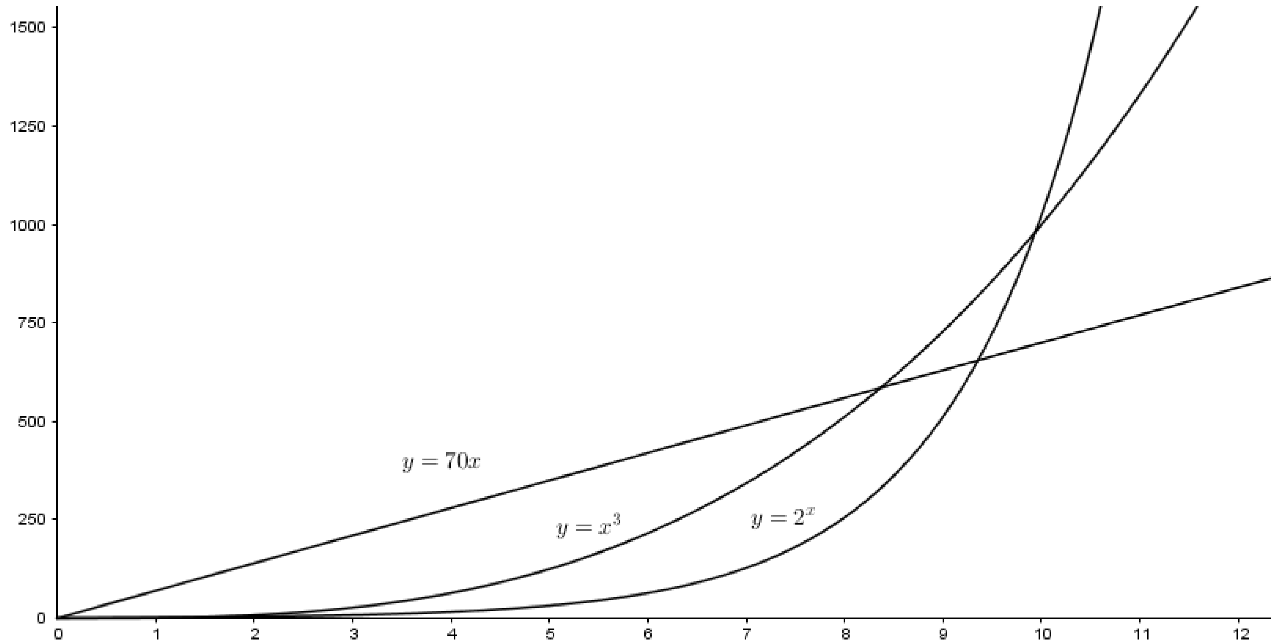
Exponential function [noun]: An equation where the x-variable is an exponent.

Exponential growth [noun]: When a quantity grows by a constant percentage for every unit of time, e.g. doubling every hour.

Exponential decay [noun]: When a quantity shrinks by a constant percentage for every unit of time, e.g. halving every 5,730 years like for carbon-14.[13]

Exponential functions [and their inverse] are weeeeird. I learnt exponential functions in high school and I labelled it "thing that I already know" and I put it to one side. I used it often, did calculus with it, but I begin to suspect that I don't really understand it. At least not fully. Yet.

Exponential growth will always exceed linear or polynomial growth, no matter what coefficients or degrees of the polynomial you choose. Given enough time, exponential growth will become larger.



Consider a pond in a school. It contains a species of water lily that, in defiance of all negative feedback[14], grows exponentially. Specifically, it follows the equation $y = 2^x$, i.e. the lilies double in size each day. At day 30, the lilies have taken over the entire pond, killing all the aquatic life inside the pond. The students decide to cut back the lilies when the pond is half covered. What day does this occur?

---

[13] Carbon-14 is used in carbon dating. If a sample has half the usual quantity of carbon-14, it's around 5,730 years old. If it has a quarter of the usual quantity of carbon-14, it's around $5\,730 \times 2 = 11\,460$ years old. An eighth of the usual amount means it's around $5\,730 \times 3 = 17\,190$ years old. And so on.

[14] Negative feedback is what eventually slows down exponential growth. It includes things such as lack of space or lack of food. Or in the case of the exponential spread of spam emails, people not forwarding on such nonsense.

Equations of exponential functions

Basic form:

$$y = a^x$$

Full form:

$$y = ka^x + c$$
$$y = a^{(x-b)} + c$$

where:
k =scale factor
a =the 'base'. It's also the y-intercept if $k = 1$
b =horizontal shift
c =vertical shift

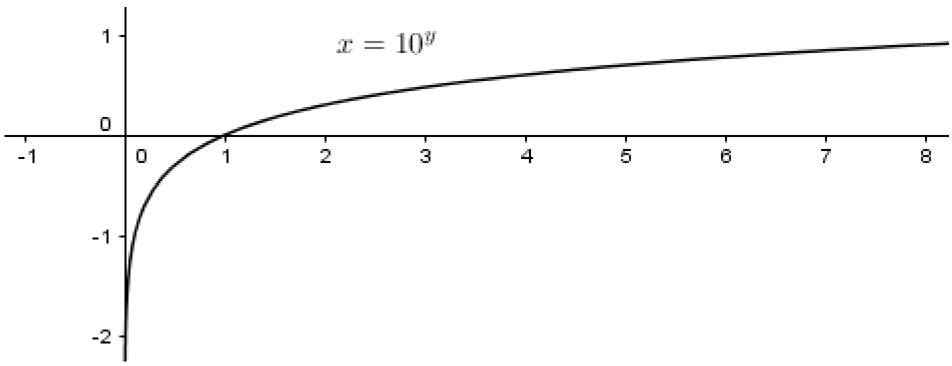
Note that there two versions of the full form. Why? Let's take the equation $y = a^{(x-b)}$. This can be rewritten as $y = a^x a^b$. But because a and b are constants, a^b is also a constant. Therefore it is exactly the same as $y = ka^x$.

Some general exponential information

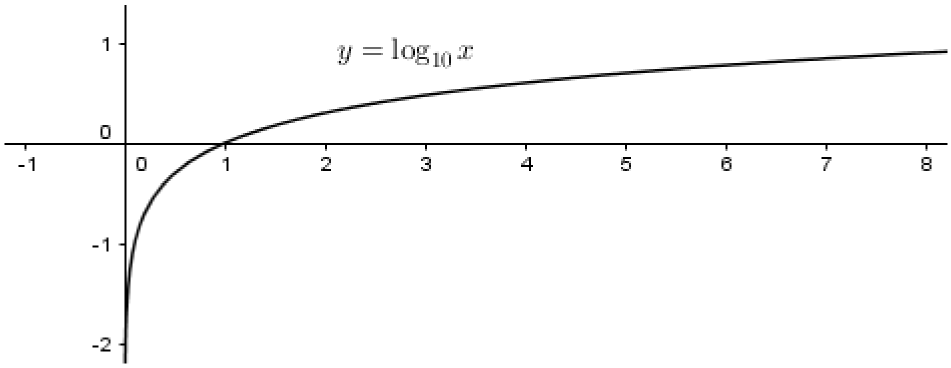
- Base: If $a > 1$ the curve is a growth curve. If $a < 1$ the curve is a decay curve. What happens when $a = 1$? Furthermore, the y-intercept will always be 1 because $a^0 \equiv 1$ [excluding scaling and shifting from k or c].

Inverse exponential functions

The inverse of an exponential function is flipped in the mirror line $y = x$ and looks like this:



It has the general equation $x = a^y$. Incredibly, this function is the graph of a seemingly unrelated concept: logarithms. I hope you can get a sense of how bizarre this is. In logarithmic notation, we can graph the equation $y = \log_a x$. Here's an example with $a = 10$:



It's exactly the same as $x = 10^y$. This works for any value of a. Find out more about logarithms in appendix X.

Lesson Six: Trigonometric functions

Trigonometry [noun]: the measuring of triangles.

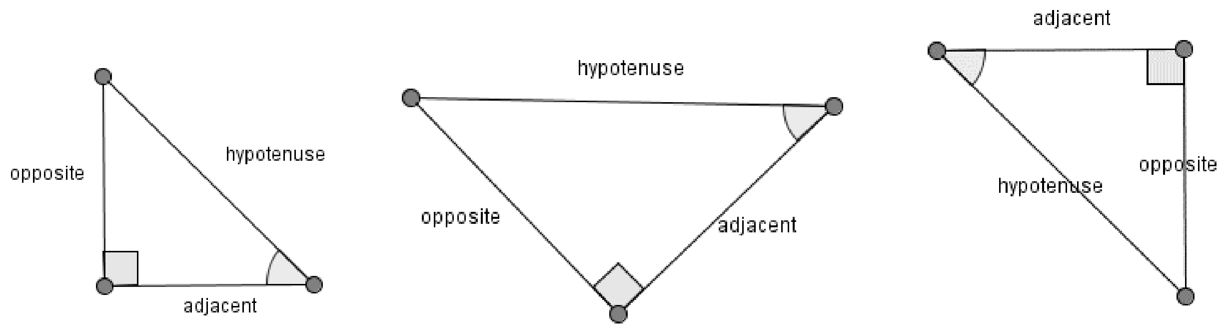
Trigonometric [adjective]: relating to trigonometry.

We've come a long way since then. Trigonometry has expanded from humble beginnings with triangles to encompass complex numbers, circles, waves, and simple harmonic motion. It is unexpectedly useful is modelling real world contexts such as height above ground on a Ferris wheel, predator and prey populations over time, sunrise and sunset times over the year, etc. Remind yourself how sine, cosine, and tangent connect to triangles.

Triangle sides

In a right-angled triangle [the only kind we look at in this standard], the sides of the triangle are defined as follows:

- Hypotenuse- the longest side. It means stretched or under tension. It's always opposite the right angle.
- Opposite- the side opposite the angle we're interested in. The opposite side is always shorter than the hypotenuse.
- Adjacent- the side adjacent to the angle we're interested in. Adjacent means next to. The adjacent side is always shorter than the hypotenuse.

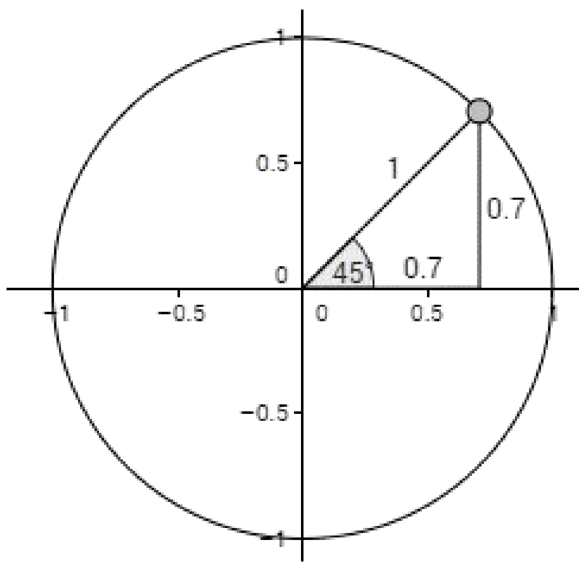


Radians

In this standard, we use radians to measure angles instead of degrees. For more information, see appendix C.

Unit circle

This is the unit circle. It will probably come in handy over the next few pages.

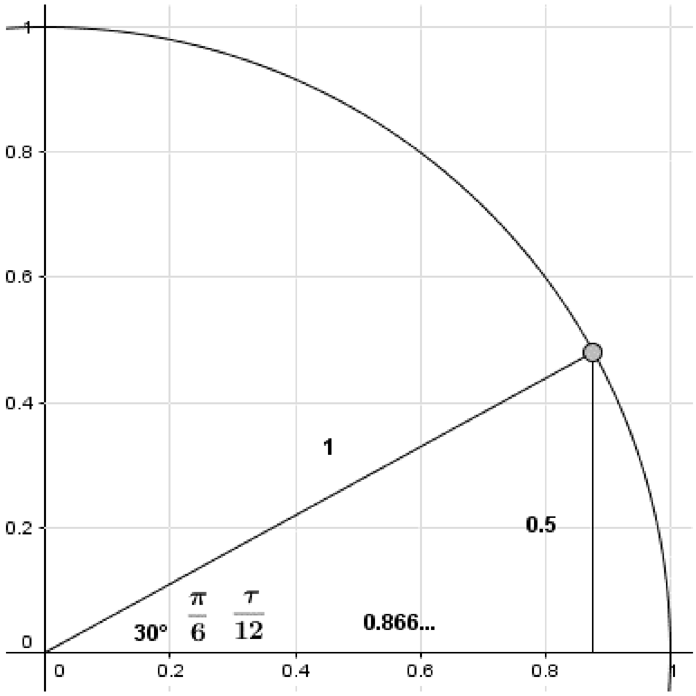


Sine

The sine of an angle is the ratio of the opposite side to the hypotenuse.

What does this mean? In the diagram below, the opposite side is half as long as the hypotenuse, i.e. the ratio of the opposite side to the hypotenuse is 0.5.

Now calculate sin(30°) [or $\sin(\frac{\pi}{6})$ or $\sin(\frac{\tau}{12})$] on a calculator. You're asking the calculator to find out how the opposite side compares to the hypotenuse when the angle is 30° which is half as long; 0.5.



Use the diagram above to visualise what sin(90°), $\sin(\frac{\pi}{2})$, $\sin(\frac{\tau}{4})$ is.

What about sin(0)?

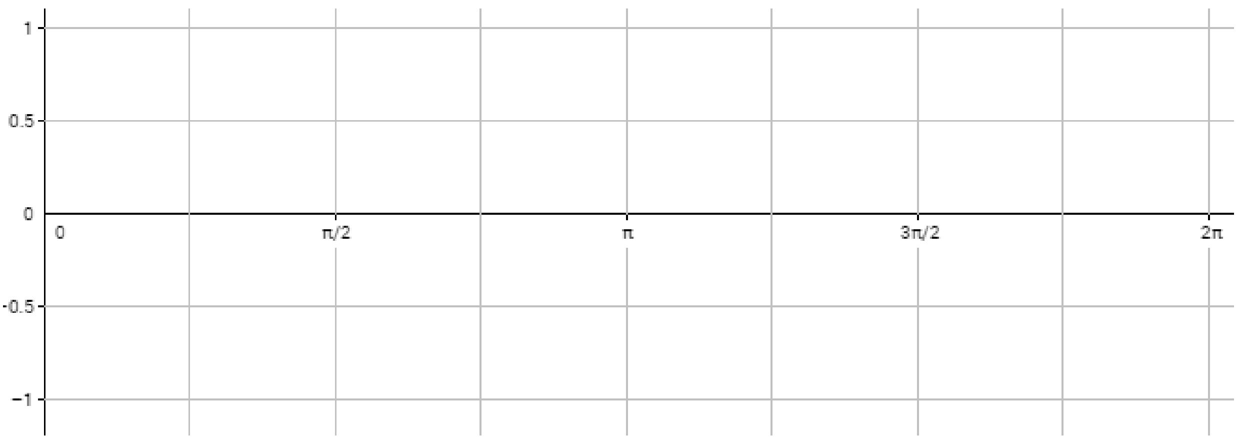
What about sin(45°), $\sin(\frac{\pi}{4})$, $\sin(\frac{\tau}{8})$ [roughly]?

Now use a calculator to calculate sin(45°), $\sin(\frac{\pi}{4})$, $\sin(\frac{\tau}{8})$.

Note that the sine ratio can never exceed 1. Why?

Sine graph

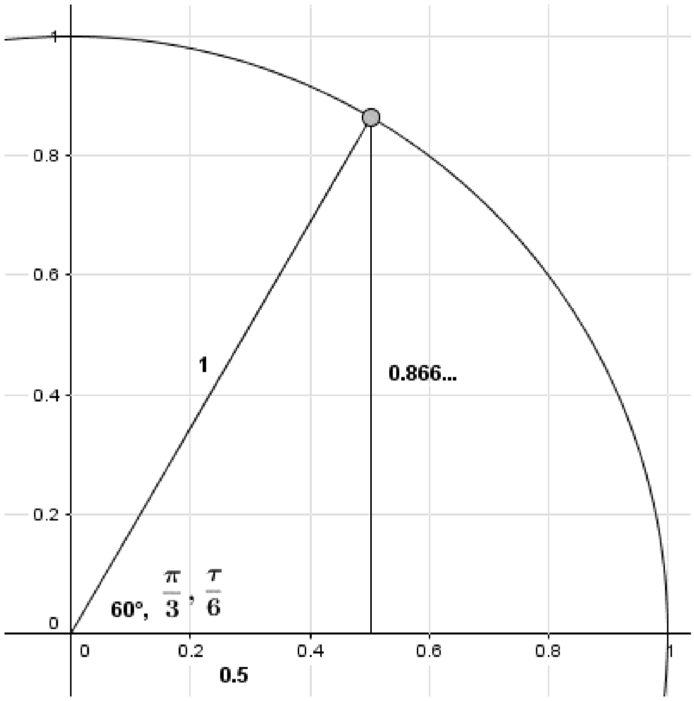
What happens to the sine ratio as the angle increases beyond $\frac{\pi}{2}$ radians? Use this information to plot the basic sine graph: $y = \sin(x)$ from $0 \leq x \leq 2\pi$



Cosine

The cosine of an angle is the ratio of the adjacent side to the hypotenuse. In the above diagram, we can see that cos(30°), cos($\frac{\pi}{6}$), cos($\frac{\tau}{12}$)=0.866…

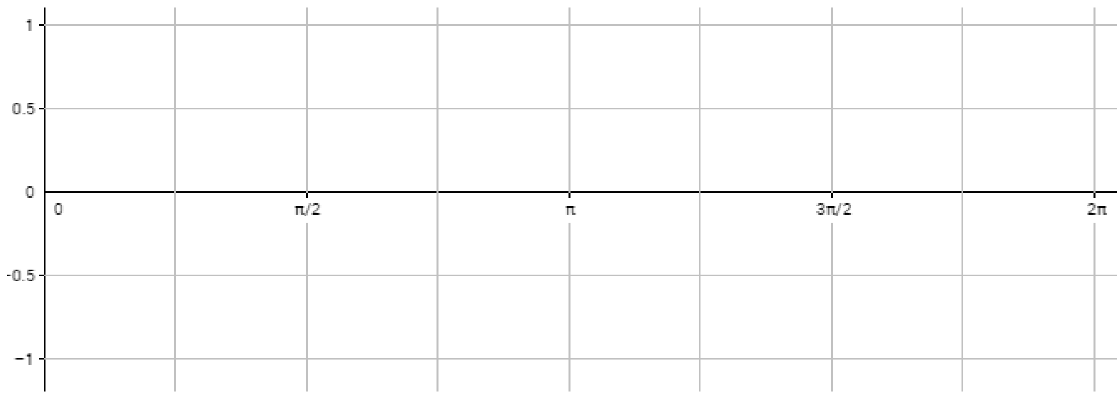
The angle required for the adjacent side to be half as long as the hypotenuse is 60°, $\frac{\pi}{3}$ radians, $\frac{\tau}{6}$ radians.



Use the diagram above to visualise what cos(90°), cos($\frac{\pi}{2}$), cos($\frac{\tau}{4}$) is.
What about cos(0)?
What about cos(45°), cos($\frac{\pi}{4}$), cos($\frac{\tau}{8}$) [roughly]?
Now use a calculator to calculate cos(45°), cos($\frac{\pi}{4}$), cos($\frac{\tau}{8}$).
What is sin(60°), sin($\frac{\pi}{3}$), sin($\frac{\tau}{6}$)?
Note that the cosine ratio can never exceed 1. Why?

Cosine graph

What happens to the cosine ratio as the angle increases beyond $\frac{\pi}{2}$ radians? Use this information to plot the basic cosine graph: $y = \cos(x)$ from $0 \leq x \leq 2\pi$



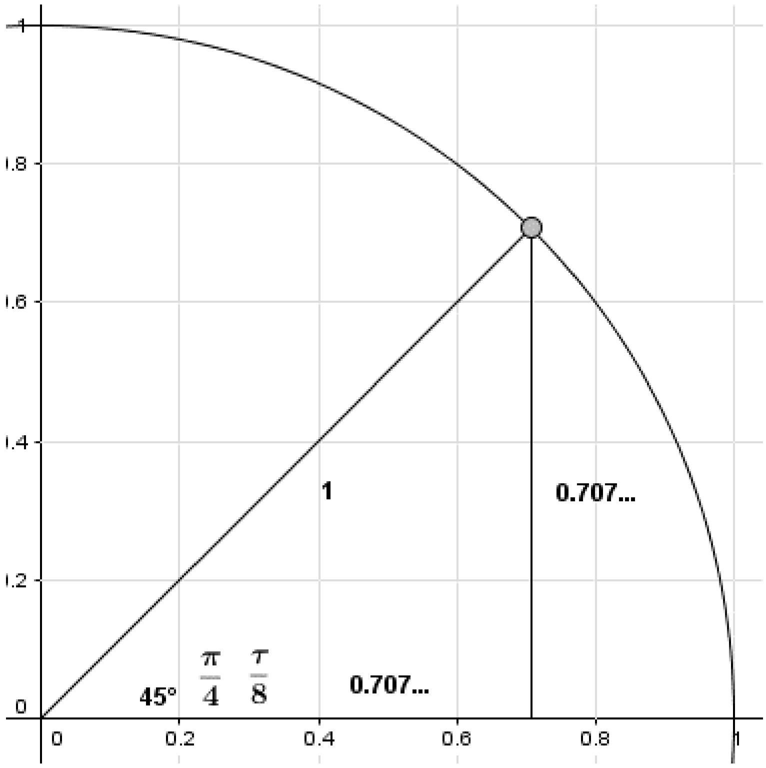
Tangent

Tangent is the ratio between the opposite side and the adjacent side. In the first diagram, we can see the ratio between the opposite and the adjacent is $\frac{0.5}{0.866\ldots} = 0.577\ldots$

This is what you're asking the calculator when you type tan(30°), tan($\frac{\pi}{6}$), tan($\frac{\tau}{12}$).

What is tan(60°), tan($\frac{\pi}{3}$), tan($\frac{\tau}{6}$)? [Look at the previous diagrams]

Use the diagram below to calculate tan(45°), tan($\frac{\pi}{4}$), tan($\frac{\tau}{8}$).



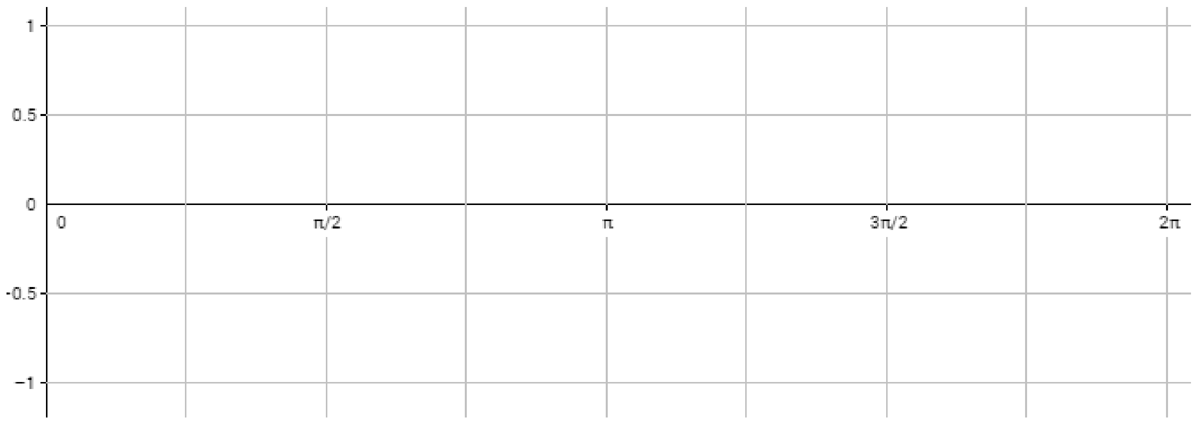
What is sin(45°), sin($\frac{\pi}{4}$), sin($\frac{\tau}{8}$)?

What is cos(45°), cos($\frac{\pi}{4}$), cos($\frac{\tau}{8}$)?

Note that the tangent ratio CAN exceed 1. Why?

Tangent graph

What happens to the tangent ratio as the angle increases beyond $\frac{\pi}{2}$ radians? Use this information to plot the basic tangent graph: $y = \tan(x)$ from $0 \le x \le 2\pi$



Lesson Seven: Trigonometric graph transformations

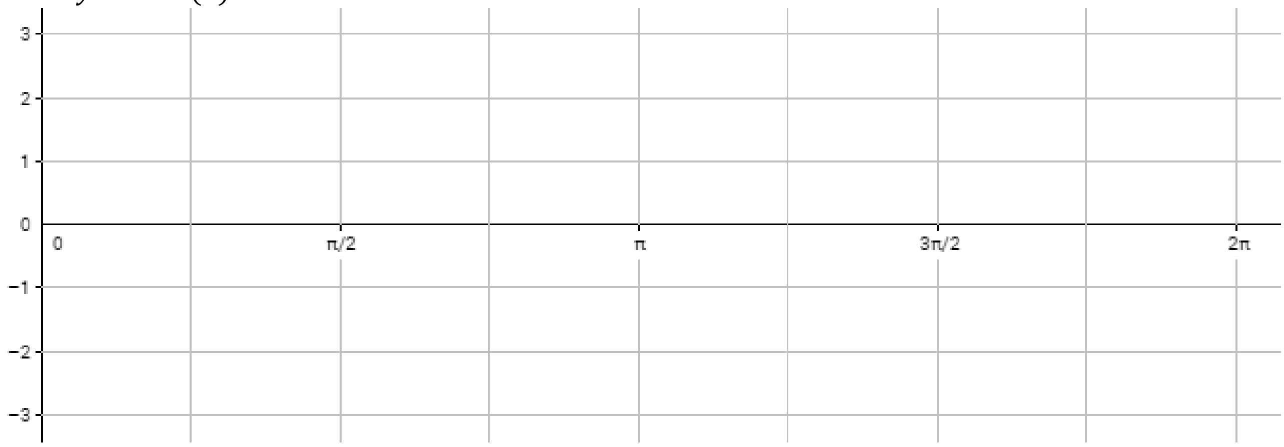
Now that you have the basic trigonometric functions under your belt, take a look at how you can transform them.

$y = A\sin(Bx - C) + D$ $\qquad$ $y = A\cos(Bx - C) + D$ $\qquad$ $y = A\tan(Bx - C) + D$

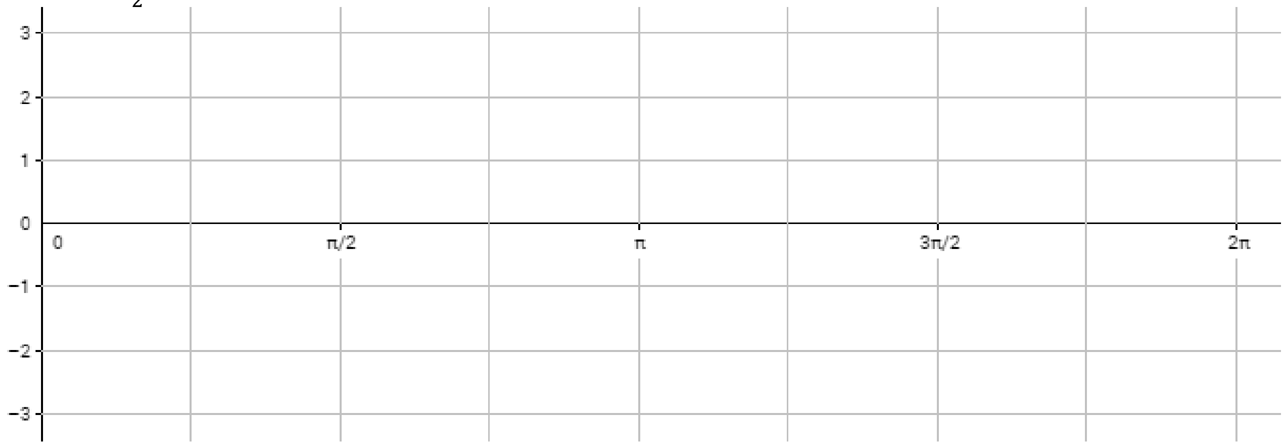
where:
A =vertical scale factor
B =horizontal scale factor
C =horizontal shift
D =vertical shift

A=vertical scale factor $\qquad$ $\boldsymbol{y = A\sin(x)}$
By multiplying the sine function by a number, we can stretch it vertically.

Draw $y = 3\sin(x)$



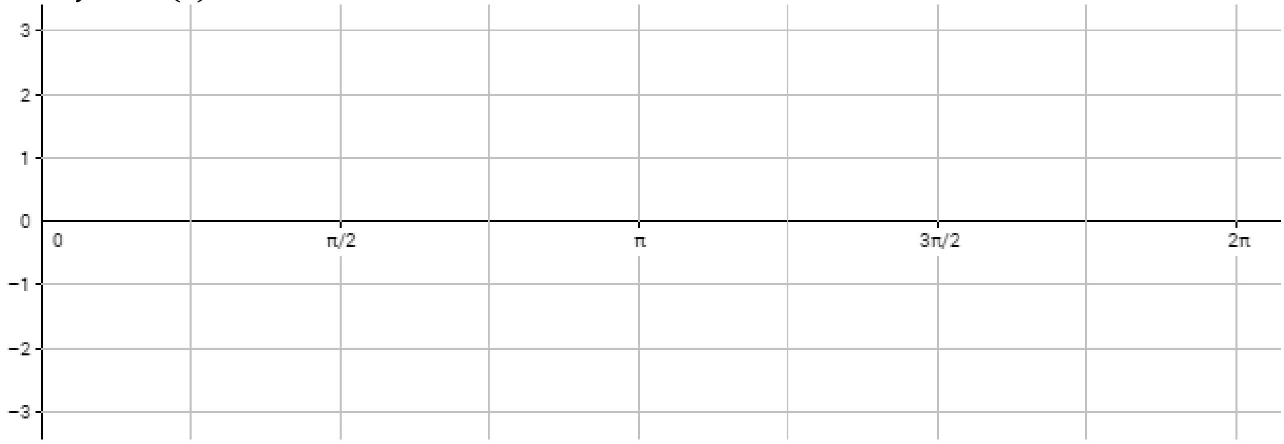
Draw $y = \frac{1}{2}\cos(x)$



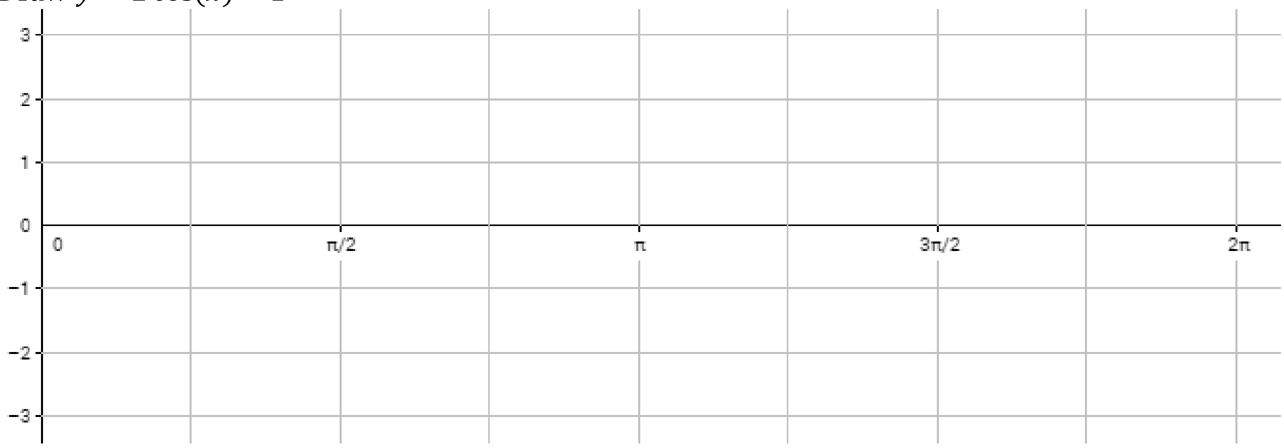
D=vertical shift $\boldsymbol{y = \sin(x) + D}$

Yes, we're skipping letters. That's because A and D are both vertical so I grouped them together. Adding a number to the sine function shifts the function up or down.

Draw $y = \sin(x) + 1$



Draw $y = 2\cos(x) - 1$

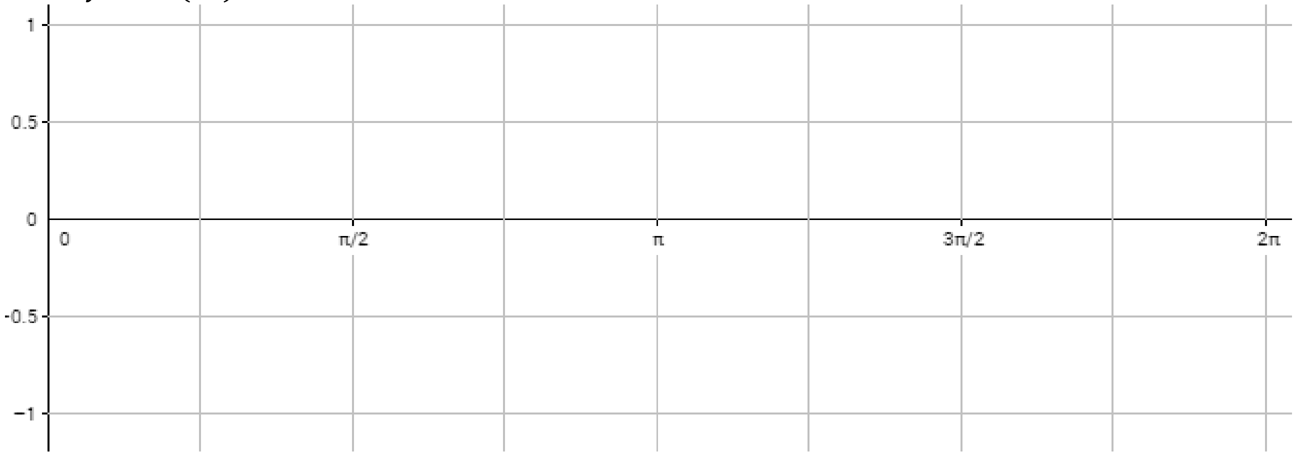


B=horizontal scale factor $\boldsymbol{y = \sin(Bx)}$

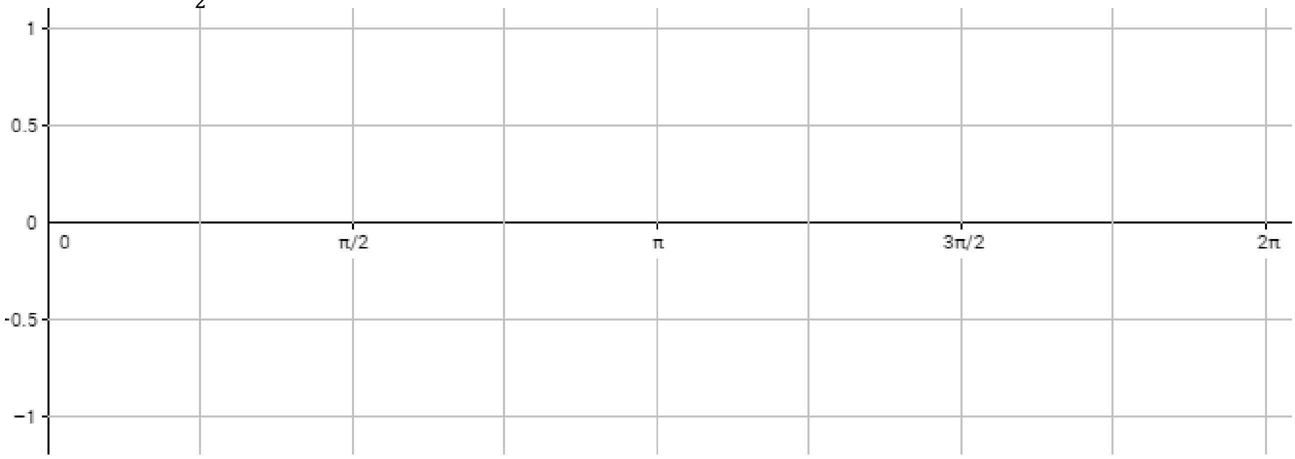
Multiplying the x variable by a number before applying the trigonometric function stretches the graph horizontally.

Note that multiplying by a number >1 compresses the function and multiplying by a number <1 stretches the function.

Draw $y = \sin(2x)$



Draw $y = \cos(\frac{1}{2}x)$

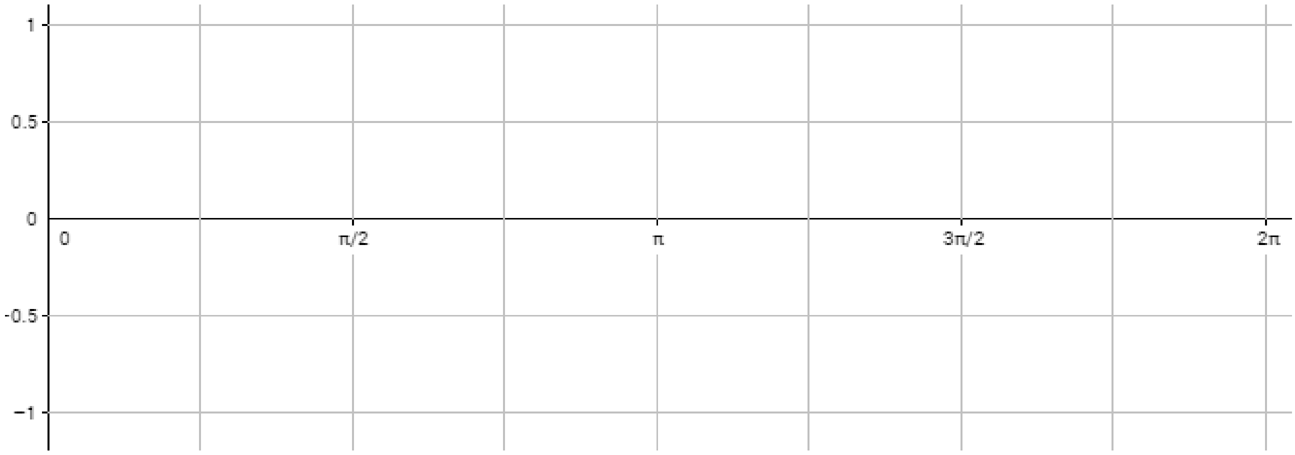


C=horizontal shift $\qquad\qquad$ $\boldsymbol{y = \sin(x - c)}$

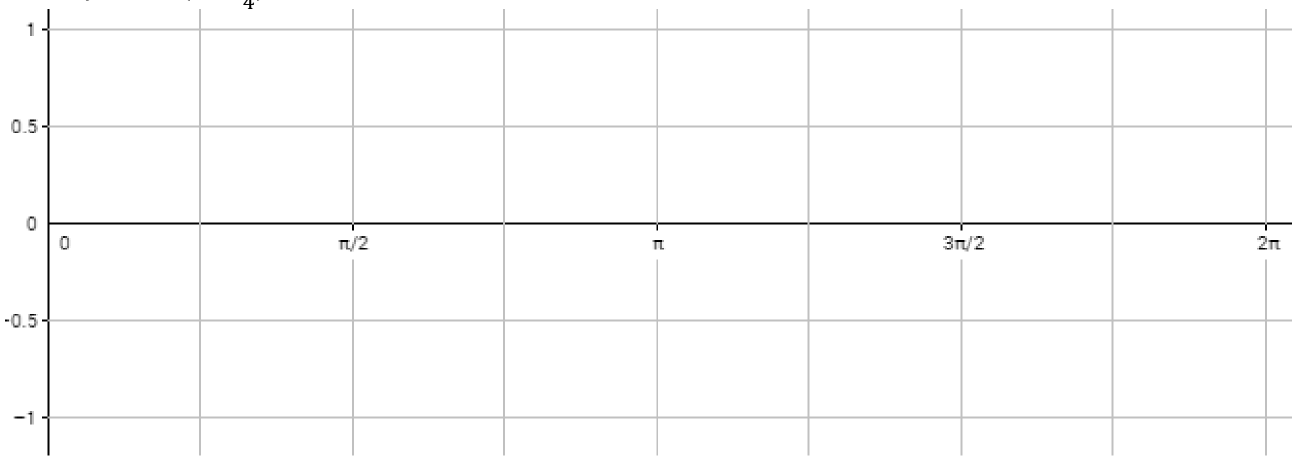
Adding a number to the x variable before applying the trigonometric function shifts the graph horizontally so that the graph begins at $x = c$ rather than $x = 0$

Note that the graph shifts to the left when a positive number is added.

Draw $y = \sin(x + \frac{\pi}{4})$



Draw $y = \sin(x - \frac{\pi}{4})$



The inverse trigonometric functions

The inverse trigonometric functions are:

$$x = \sin(y) \qquad\qquad x = \cos(y) \qquad\qquad x = \tan(y)$$

or in another form:

$$y = \sin^{-1}(x) \qquad\qquad y = \cos^{-1}(x) \qquad\qquad y = \tan^{-1}(x)$$

They look like the original functions but, again, mirrored around the line $y = x$.

Lesson Eight: Trigonometric modelling

To model a situation using a trigonometric function the situation needs to be periodic [repeating]. I've never yet seen a situation that is best modelled by a tangent function so we'll be using only sine and cosine.[15] There are four key pieces of information you need to create a trigonometric model:

A=Vertical scale
The vertical scale is the difference between the highest points and the lowest point divided by two, i.e.

$$A = \frac{(\max - min)}{2}$$

D=Vertical shift
The vertical shift is the average height of the graphs, i.e.

$$D = \frac{(max + min)}{2}$$

B=Horizontal stretch
The horizontal stretch is how many cycles [periods] fit into 2π, i.e.

$$B = \frac{2\pi}{period}$$

C=Horizontal shift
The horizontal shift is the horizontal distance between the origin and where the curve starts, i.e. for a sine graph:

$$C = -shift$$
$$C = -\left(time\ to\ maximum - \frac{1}{4}period\right)$$
$$C = -time\ to\ maximum + \frac{1}{4}period)$$

and for a cosine graph:

$$C = -shift$$
$$C = -time\ to\ maximum$$

[15] Which you've just seen are the same thing.

Lesson Nine: Graphical models

This is it. This is what the standard is all about: modelling a real situation with graphical models. You'll use a combination[16] of polynomials, hyperbolas, exponential functions, logarithmic functions, and trigonometric functions. You won't require inverse polynomials, inverse trigonometric functions, or the tangent function.

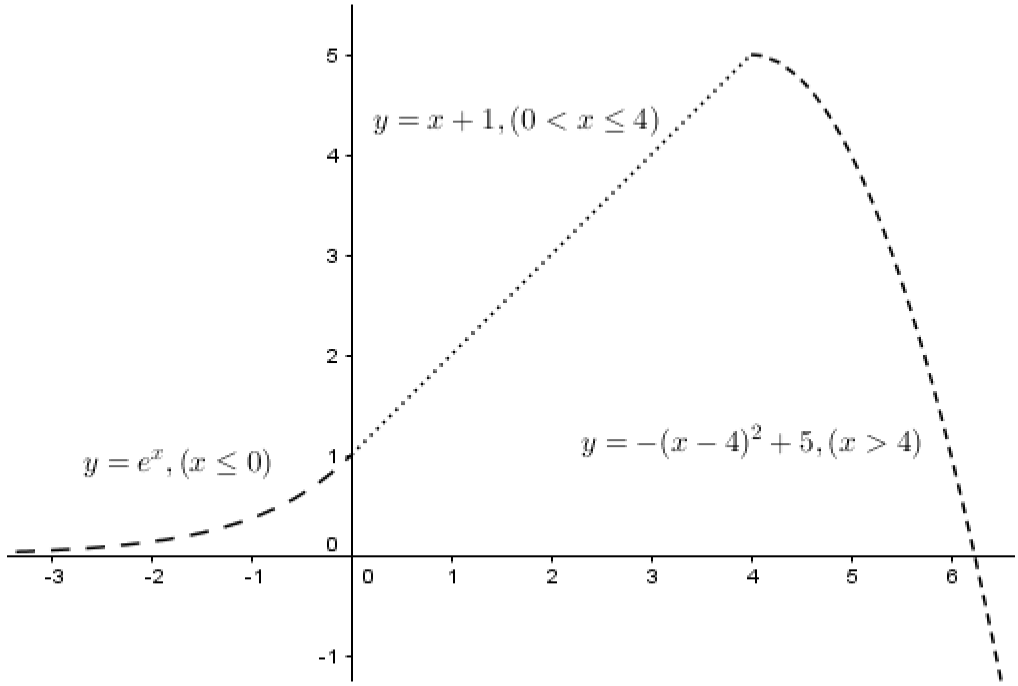
Piecewise functions

To combine more than one graph into a model, we create a piecewise function. Piecewise functions have different functions in different domains.

Domain [noun]: The range of x values for which the function applies.

For example, the following piecewise function has an exponential function for $x \leq 0$, a straight line for $0 < x \leq 4$, and a hyperbola for $x > 4$. This could be written like this:

$$y = \begin{cases} e^x, & x \leq 0 \\ x+1, & 0 < x \leq 4 \\ -(x-4)^2 + 5, & x > 4 \end{cases}$$



Domains

No x value can be defined more than once. I was very careful to make sure the two intersections [$x = 0$ and $x = 4$] belonged to only one function. Be aware of what each symbol means:

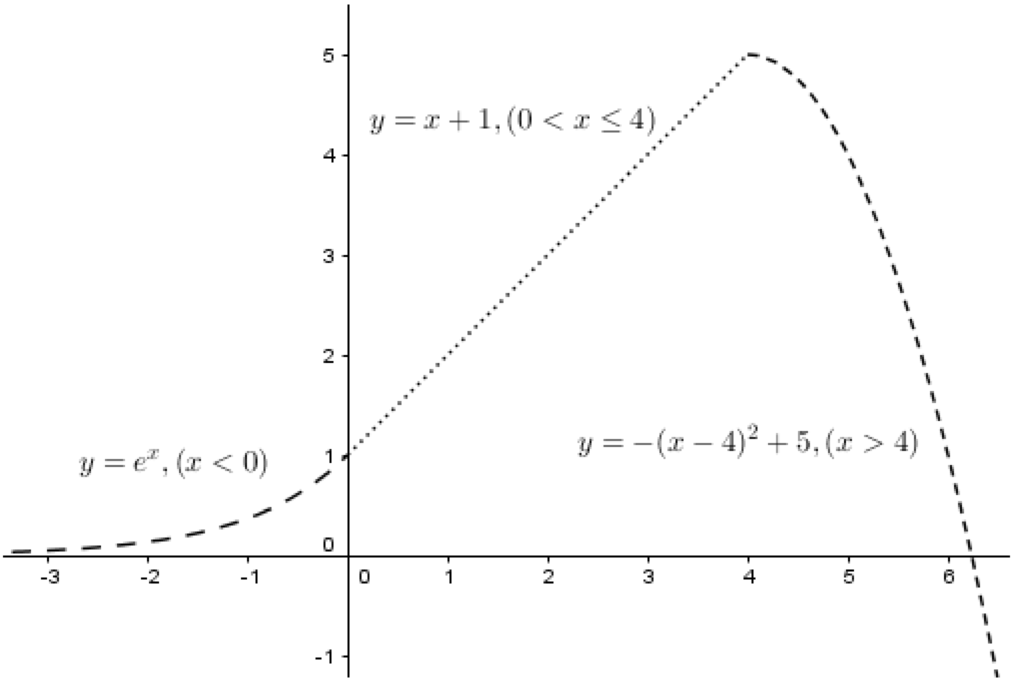
- less than [<]
- greater than [>]
- less than or equal to [≤]
- greater than or equal to [≥]

[16] Yes, that means more than one.

If any x-values are undefined, they need to be at the edges of the graph and have to make sense, e.g. can't have time less than 0, can't have height less than 0, etc. Don't introduce holes into your graph by not defining x at the intersections.

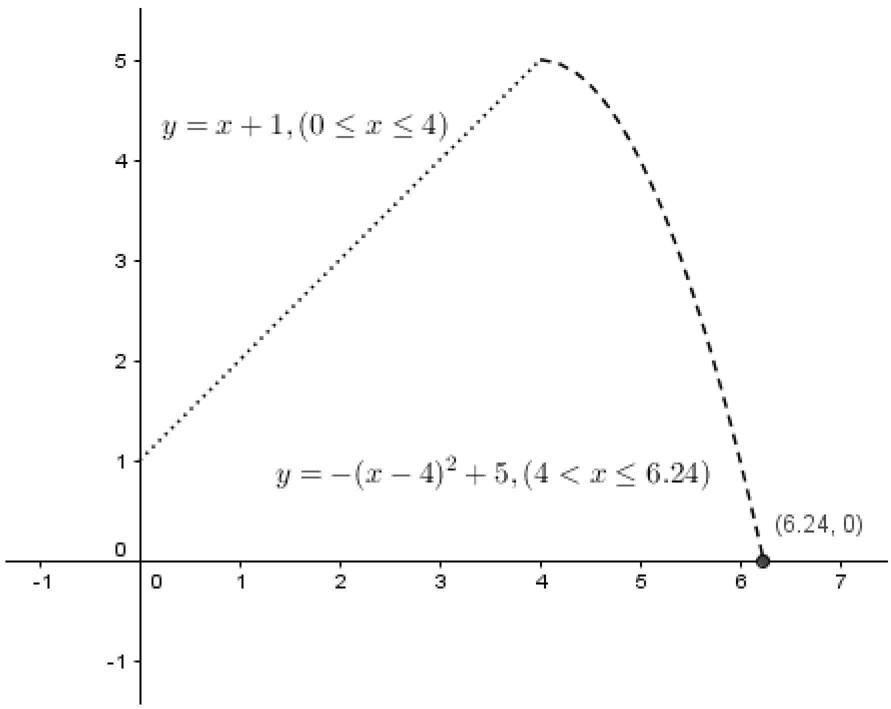
For example, in this piecewise function $x = 0$ is not defined which is not okay.

$$y = \begin{cases} e^x, & x < 0 \\ x + 1, & 0 < x \leq 4 \\ -(x-4)^2 + 5, & x > 4 \end{cases}$$



But this is okay:

$$y = \begin{cases} x + 1, & 0 \leq x \leq 4 \\ -(x-4)^2 + 5, & 4 < x \leq 6.24 \end{cases}$$



Practice internal

I've made some fictional numbers for the fur seal population living around New Zealand.[17] Time is measured in years and begins in the middle of summer and the population is measured in thousands.

The seal population fluctuates with the seasons at around 5 000 seals. The seal population peaks in summer [$t = 0, 1, 2$, etc.] with around 5 250 seals and reaches the minimum value in winter [$t = 0.5, 1.5, 2.5$, etc.] with around 4 750 seals.

In the eighth winter [$t = 7.5$] since recording began, seal hunters started hunting the seals for their pelts [fur]. At first they didn't make much difference to the seal population but when the seal numbers didn't increase in the spring as expected, something was noticeably wrong.

The seal population continued to decline until the conservationists got involved and petitioned the government to ban seal hunting. In the tenth summer since recording began, the government banned seal hunting. At that point there were around 1 000 seals left. The seal population immediately slowed down in its decline but continued to decline nonetheless.

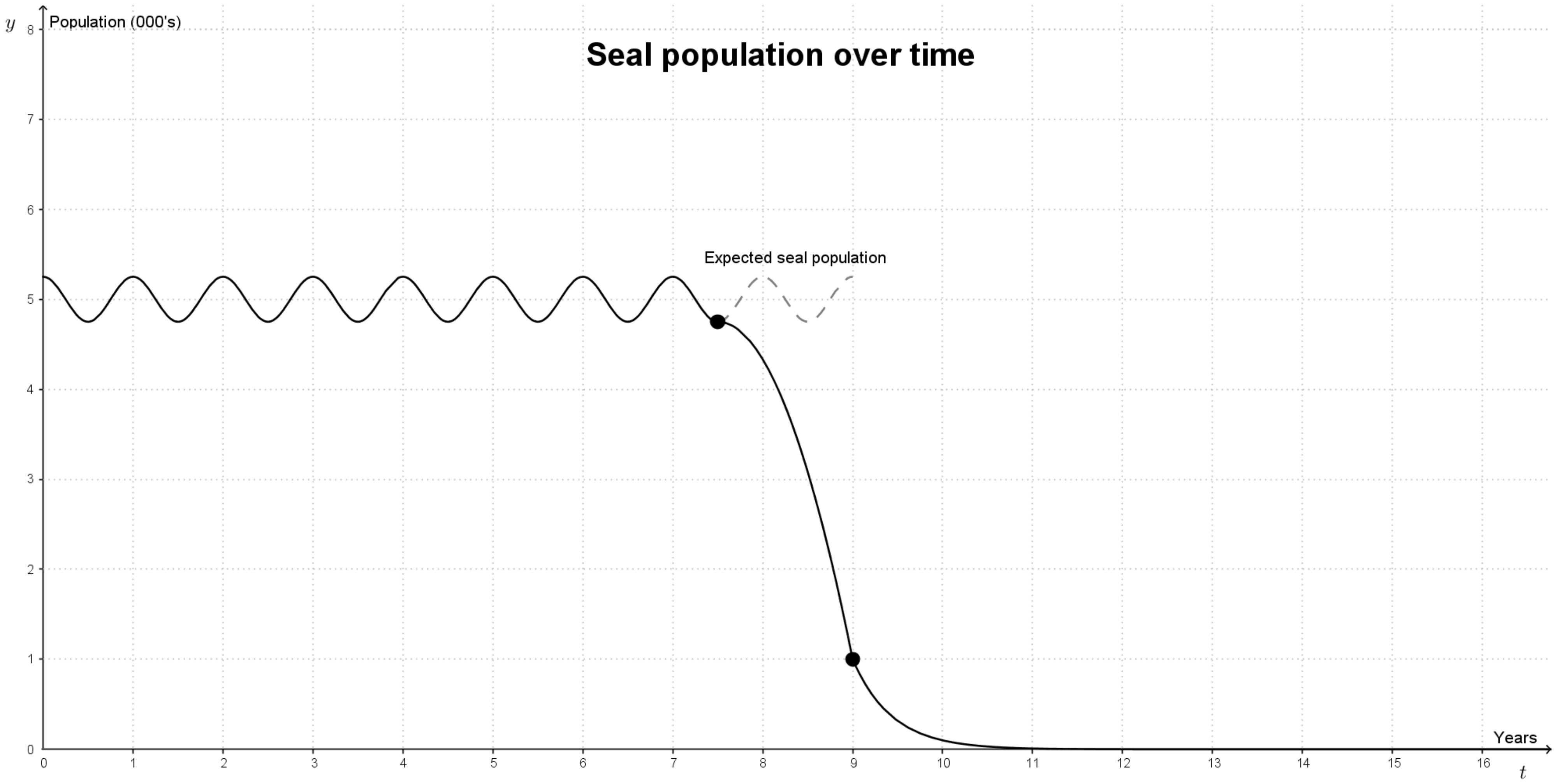
From that point on, the seal population fell to 10% of the previous year's population each year and continued to do so until they were extinct.

The graph on the following page shows the same information in a visual way.

Use this information to create a graphical model of the situation. You will need to:

- find the points of intersection
- find the functions for all three pieces of the piecewise model
- define the domain for which each function applies
- make comments about the appropriateness of your model by describing the features of the functions you selected
- calculate when the fur seal went extinct

[17] Which are vaguely believable.


Seal population over time
y
Population (000's)
8
7
6
5
4
3
2
1
0
Expected seal population
Years
t
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16

Grade requirements

These are the requirements for each grade for the practice internal. The actual requirements for the internal will be slightly different from these.

<table>
<tr><th></th><th>Achieved</th><th>Merit</th><th>Excellence</th></tr>
<tr><td>Equations</td><td>Write the correct equation for at least one of those functions. Name appropriate functions to model the other pieces [without writing their equations]</td><td colspan="2">Write the correct equations for all three pieces of the piecewise function.</td></tr>
<tr><td>Domains</td><td>State the domain for at least one of the functions.</td><td colspan="2">State the domain for all functions.</td></tr>
<tr><td>Extra</td><td></td><td colspan="2">Comment on the appropriateness [validity] of your model by describing features of the functions you selected.

Calculate when the fur seal went extinct.</td></tr>
</table>

Homework tracking

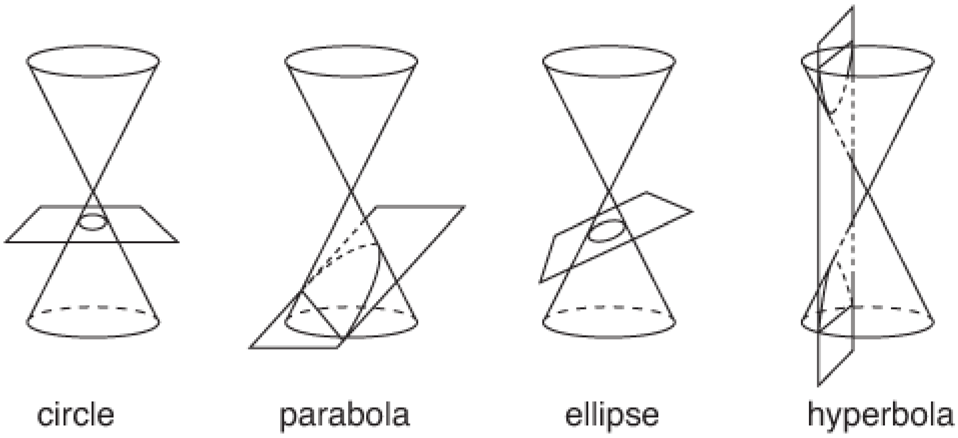
As you do homework for this standard, keep track of it all right here. Do around 30 minutes for each class time.

Week	Day	Length of time	What did you do?
Term 1, Week 8			
Term 1, Week 9			
Term 1, Week 10			
Term 1, Week 11			

Appendix A: Conic sections

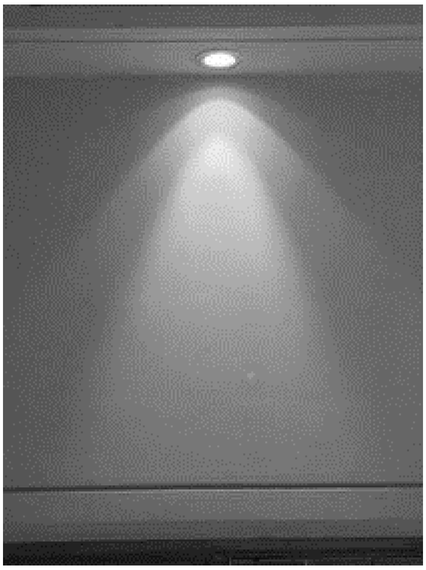
You've read that both parabolas and hyperbolas are conic sections. But what is a conic section?

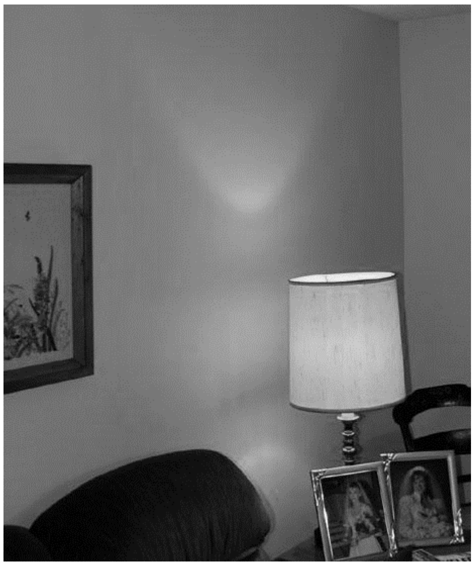
Take a double cone and slice into it on an angle to make a section. This is a conic section. Slicing at different angles make different conic sections.



There are three conic sections [the circle being a special case of the ellipse].[18]

Orbits are elliptical. And hyperbolas can be formed with lights and walls.



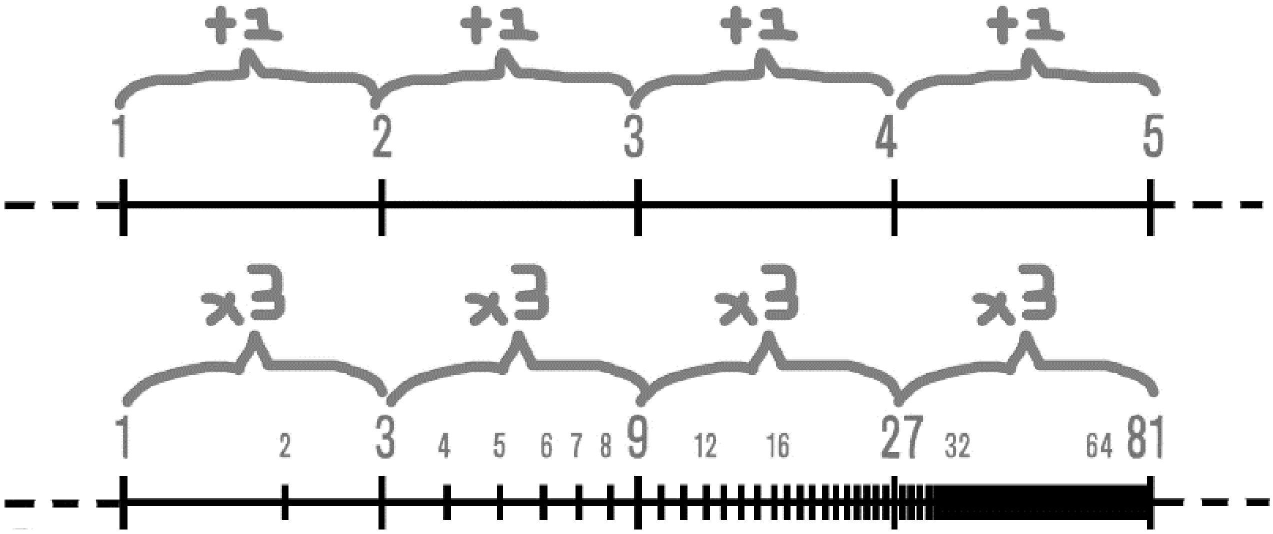


[18] You could argue there's four conic sections.

Appendix B: Logarithms

Logarithms turn multiplication into addition. A seemingly simple switch but profound when you see the consequences.

One consequence relates to how we count. We're used to counting additively but with logarithms, we can count multiplicatively where each number is the same multiple of the number before it.



In this example we used 3 as the base:

$$\log_3 1 = 0$$
$$\log_3 3 = 1$$
$$\log_3 9 = 2$$
$$\log_3 27 = 3$$
$$\log_3 81 = 4$$

But we can use other bases. Base 10 is useful because it describes the number of digits in a quantity.[19] For example, "6-figure salary" is using multiplicative counting, i.e. logarithms, rather than additive counting.[20]

$$\log_{10} 1 = 0$$
$$\log_{10} 10 = 1$$
$$\log_{10} 100 = 2$$
$$\log_{10} 1\,000 = 3$$
$$\log_{10} 10\,000 = 4$$

[19] It does this because it matches with our system of writing numbers: base-10 arithmetic.

[20] A 6-figure salary isn't 1 more than a 5-figure salary, it's 10 more. And it's 100 more than a 4-figure salary, etc.

We can use decimals too. For example:

$\log_{10} 1 = 0$
$\log_{10} 2 = 0.3010$
$\log_{10} 3 = 0.4771$
$\log_{10} 4 = 0.6021$
$\log_{10} 5 = 0.6990$
$\log_{10} 6 = 0.7782$
$\log_{10} 7 = 0.8451$
$\log_{10} 8 = 0.9031$
$\log_{10} 9 = 0.9542$
$\log_{10} 10 = 1$

$\log_{10} 11 = 1.0414$
$\log_{10} 12 = 1.0792$
$\log_{10} 13 = 1.1139$
$\log_{10} 14 = 1.1461$
$\log_{10} 15 = 1.1761$
$\log_{10} 16 = 1.2041$
$\log_{10} 17 = 1.2304$
$\log_{10} 18 = 1.2553$
$\log_{10} 19 = 1.2788$
$\log_{10} 20 = 1.3010$

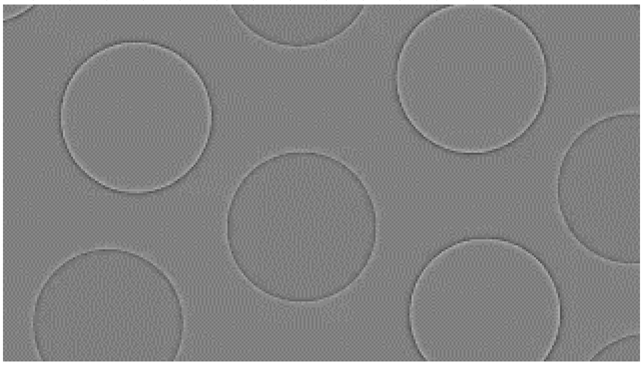
Have a closer look at the difference between:

- $\log_{10} 1$ and $\log_{10} 2$
- $\log_{10} 10$ and $\log_{10} 20$

This is how log tables are made.

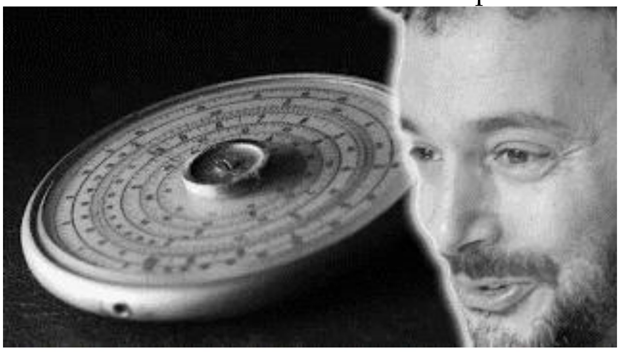
For more explanations, search:

1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,
21,22,23,24,25,26,27,28,29,30,31,32,33,34,35...



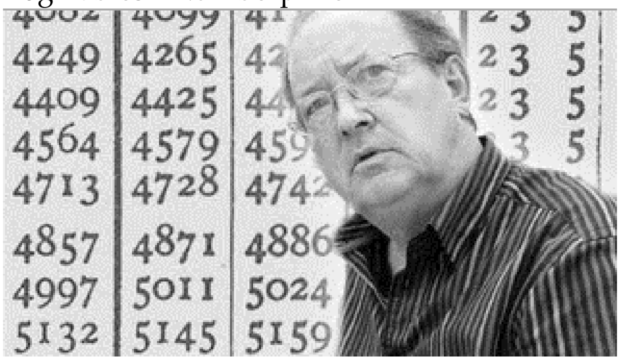
https://youtu.be/Pxb5lSPLy9c

The iPhone of Slide Rules – Numberphile



https://youtu.be/xRpR1rmPbJE

Log Tables - Numberphile



https://youtu.be/VRzH4xB0GdM

Number 1 and Benford's Law - Numberphile

https://youtu.be/XXjlR2OK1kM

Appendix C: Radians

Circles are useful but whole circles aren't everything, just like whole turns aren't everything. We're often interested in part-circles or part-turns, e.g. a half turn, a quarter turn, a third turn, etc. But what total shall we assign to the circle so that we can divide it? A total of 10 units?

Defining the circle as 10 units means we get a nice value for a half circle: 5 units. But dividing the circle into quarters or thirds doesn't produce nice values.

What about defining the circle to have 100 units? We still get a nice value for a half circle: 50 and we now get a nice value for a quarter circle: 25. But we still don't have a nice value for a third circle.

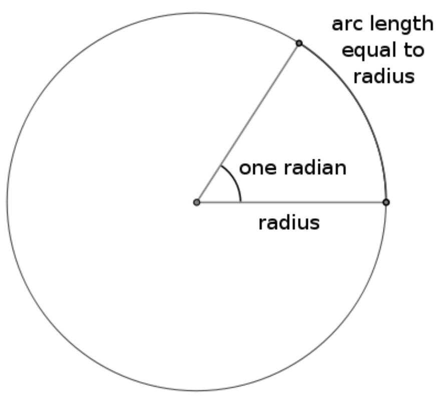
The Babylonians preferred to count using base-60 rather than base-10 because 60 has more factors than 10. Similarly, 360 has more factors than 100. And for our present situation, factors is what it's all about.

If we define the circle to have 360 units, we get nice values for half circles=180, third circles=120, quarter circles=90, sixth circles=60, and eighth circles=45. At least these are the ones we use commonly. There are others. We define this unit to be a degree.

But 360 isn't uniquely special. Sure, it has a lot of factors. But so does 3600 and 7200. We could keep increasing the number of units in a circle ad infinitum. Can we define a unit that is somehow more special than all others, somehow linked to the circle itself?

Well, what numbers are linked with circles? π comes to mind. Can we define the circle to have a total of π units? Hmmmm, I can't imagine what 1 π^{th} of a circle would look like. It's confusing. Let's try something else.

A circle is defined by two variables: the position and the radius. The position isn't a number [it's two numbers] but the radius is a number [technically, it's a variable but a variable that takes a single number value]. Could we use r to define the unit?



Let's say the angle required to make an arc length equal to the radius is one unit. We call these units radians because it sounds like radius. Radians, unlike degrees, are intimately linked with the circle itself.

How many radians are in a full circle? Recall that:

$$C = 2\pi r$$

So how many r's are in the circumference? 2π of them.

2π radians = full circle

π radians = half circle

$\frac{\pi}{2}$ radians = quarter circle

$\frac{\pi}{3}$ radians = sixth circle

$\frac{\pi}{4}$ radians = eighth circle

$\frac{\pi}{6}$ radians = twelfth circle

Appendix D: Tau $\boldsymbol{\tau}$

A full turn is 2π radians. And a half circles is π radians. And a quarter circle is $\frac{\pi}{2}$ radians.

Is that confusing for anyone else? Why is there always this factor of 2? You've seen why in appendix A, but is there a way make a full turn = 1, a half turn = $\frac{1}{2}$, etc.

There is! Let's make a new number and make it equal to 2π. Simple genius!

One of this new number is a full turn, half of this new number is a half turn, etc. All we need to do is give this new number a name. Mathematicians have decided to call it "tau" because it's Greek letter [τ] looks like π.

You may choose to learn trigonometry using τ because it's more intuitive than π. Feel free to. However, π will remain common at university and research-level maths so you should learn τ and π alongside each other to be best prepared.

For more information on τ, search for:

Tau replaces Pi - Numberphile

https://youtu.be/83ofi_L6eAo

Tau vs Pi Smackdown – Numberphile

https://youtu.be/ZPv1UV0rD8U

Image credits

Sample question

London Bridge	http://www.hdwallpapersnew.net/london-bridge-wallpaper-for-desktop/
Auckland Bridge	http://www.wikiwand.com/en/Auckland_Harbour_Bridge

Lesson One: Introduction to parabolas

Basketball	http://blog.mrmeyer.com/2010/wcydwt-will-it-hit-the-hoop/
Headlights	http://www.everythingmaths.co.za/science/grade-11/05-geometrical-optics/05-geometrical-optics-04.cnxmlplus
Whisper dishes	Unknown. If you own this picture, please let me know at david@waora.school.nz
McDonalds arches	http://omega-unlimited.blogspot.co.nz/2011/10/golden-arches-exposed.html

Lesson Four: Hyperbolas

Hyperbola 1	http://www.mathsisfun.com/geometry/hyperbola.html
Hyperbola 2	http://intmstat.com/plane-analytic-geometry/xyis1.gif
Gravity slingshots	http://www.mathsisfun.com/geometry/hyperbola.html
Wave interference	https://en.wikipedia.org/wiki/Hyperbola
Sonic booms	http://www.pleacher.com/mp/mlessons/calculus/apphyper.html

Appendix A: Conic Sections

Conic sections	http://www.cliffsnotes.com/study-guides/algebra/algebra-ii/conic-sections/the-four-conic-sections
Halogen lamp	https://en.wikipedia.org/wiki/Hyperbola
Lamp shade	http://www.pleacher.com/mp/mlessons/calculus/apphyper.html

Appendix B: Logarithms

Number lines	https://youtu.be/Pxb5lSPLy9c