Level 3 – AS 91575– 3.3 – 4 Credits – Internal

Apply trigonometric methods in solving problems

Context

Level 1	Level 2	Level 3 calculus
	2.2 Graphs 4 credits [Int] →	3.3 Trigonometry 4 credits [Int]

Sample question: Ferris wheel

Maths End amusement park has two Ferris wheels: the Kiddy-wheel, a small wheel that reaches a maximum height of 10 m above the ground; and the Flying-high, a large wheel that reaches a maximum height of 50 m above the ground.

While they are at the amusement park, Manu has a ride on the Flying-high wheel and his little sister Jade goes for a ride on the Kiddy-wheel. Manu and Jade go on the rides at the same time. Because of trees and buildings between the two rides, Jade can only see Manu some of the time.

Both Ferris wheels load passengers from ramps at their lowest point. The seat is at the same level as the ramp at this loading point. Jade and Manu start their rides at the same time.

The Kiddy-wheel reaches a maximum height of 10 m and its ramp is 1.5 m above the ground. The ride makes two revolutions each minute.

The Flying-high Ferris wheel reaches a maximum height of 50 m and its ramp is 2 m above the ground. The ride makes three revolutions in two minutes.

Draw a diagram to show Jade's and Manu's heights above ground over time.

Passport

Use this to track your progress through this workbook.

Trigonometric graphs	First time	Revision
Sine, cosine, and tangent		
Amplitude, frequency, and period		
Transformations of trigonometric graphs		

Solutions to trigonometric equations

Exact solutions		
Particular solutions		
General solutions		

Modelling using trigonometric functions

Modelling using trigonometric functions		

Grade requirements

The requirements for each grade for the practice internal at the end of the workbook are below. The actual requirements for the internal will be slightly different from these.

Achieved	Merit	Excellence
Determine the amplitude of a trigonometric function for either Ferris wheel. Determine the period of a trigonometric function for either Ferris wheel. Determine the horizontal and vertical transformation of a trigonometric function for either Ferris wheel. Solve a trigonometric equation to determine an interval when a Ferris wheel is above 5m.	Form a model for each of the two Ferris wheels. Use the models to find any interval when Jade can see Manu.	As for merit, plus: Use the models to find **all intervals** when Jade can see Manu. Use the intervals to calculate the percentage of the time that Jade can see Manu.

Lesson One: Sine, cosine, and tangent

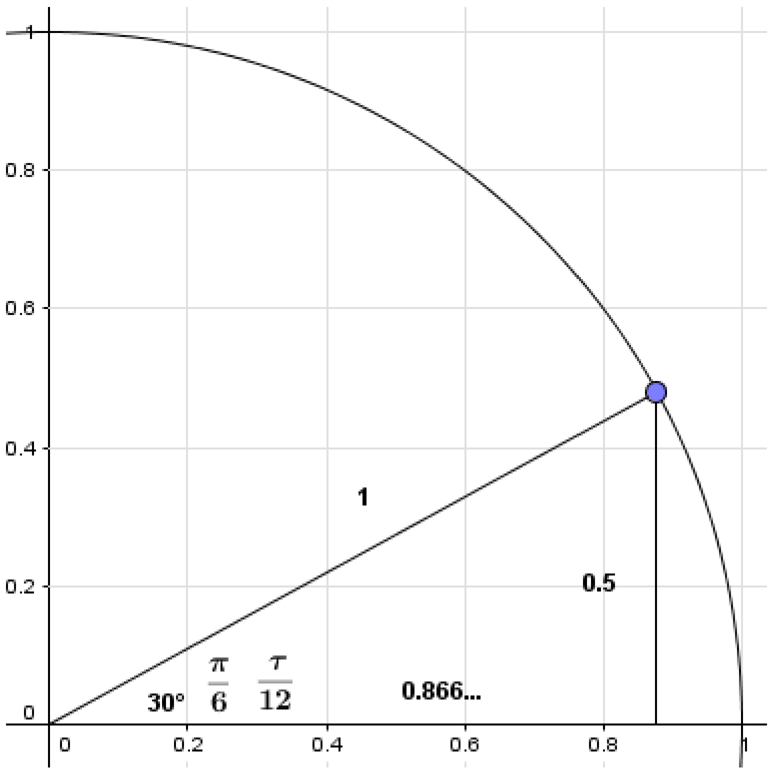
Trigonometry [noun]: the measuring of triangles.

We've come a long way since then. Trigonometry has expanded from humble beginnings with triangles to encompass complex numbers, circles, waves, and simple harmonic motion. It is unexpectedly useful is modelling real world contexts such as height above ground on a Ferris wheel, predator and prey populations over time, sunrise and sunset times over the year, etc. Remind yourself how sine, cosine, and tangent connect to triangles.

Sine
The sine of an angle is the ratio of the opposite side to the hypotenuse.

What does this mean? In the diagram below, the opposite side is half as long as the hypotenuse, i.e. the ratio of the opposite side to the hypotenuse is 0.5.

Now calculate sin(30°) [or $\sin(\frac{\pi}{6})$ or $\sin(\frac{\tau}{12})$] on a calculator. You're asking the calculator to find out how the opposite side compares to the hypotenuse when the angle is 30° which is half as long; 0.5.



Use the diagram above to visualise what sin(90°), $\sin(\frac{\pi}{2})$, $\sin(\frac{\tau}{4})$ is.

What about sin(0)?

What about sin(45°), $\sin(\frac{\pi}{4})$, $\sin(\frac{\tau}{8})$ [roughly]?

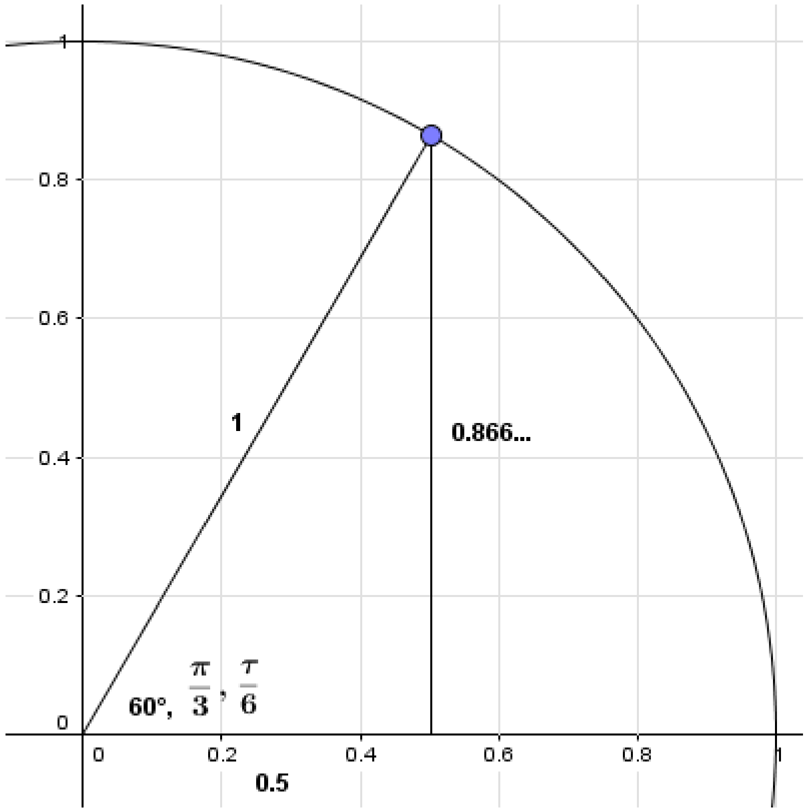
Now use a calculator to calculate sin(45°), $\sin(\frac{\pi}{4})$, $\sin(\frac{\tau}{8})$.

Note that the sine ratio can never exceed 1. Why?

Cosine

The cosine of an angle is the ratio of the adjacent side to the hypotenuse. In the above diagram, we can see that $\cos(30°)$, $\cos(\frac{\pi}{6})$, $\cos(\frac{\tau}{12})$=0.866…

The angle required for the adjacent side to be half as long as the hypotenuse is 60°, $\frac{\pi}{3}$ radians, $\frac{\tau}{6}$ radians.



Use the diagram above to visualise what $\cos(90°)$, $\cos(\frac{\pi}{2})$, $\cos(\frac{\tau}{4})$ is.

What about $\cos(0)$?

What about $\cos(45°)$, $\cos(\frac{\pi}{4})$, $\cos(\frac{\tau}{8})$ [roughly]?

Now use a calculator to calculate $\cos(45°)$, $\cos(\frac{\pi}{4})$, $\cos(\frac{\tau}{8})$.

What is $\sin(60°)$, $\sin(\frac{\pi}{3})$, $\sin(\frac{\tau}{6})$?

Note that the cosine ratio can never exceed 1. Why?

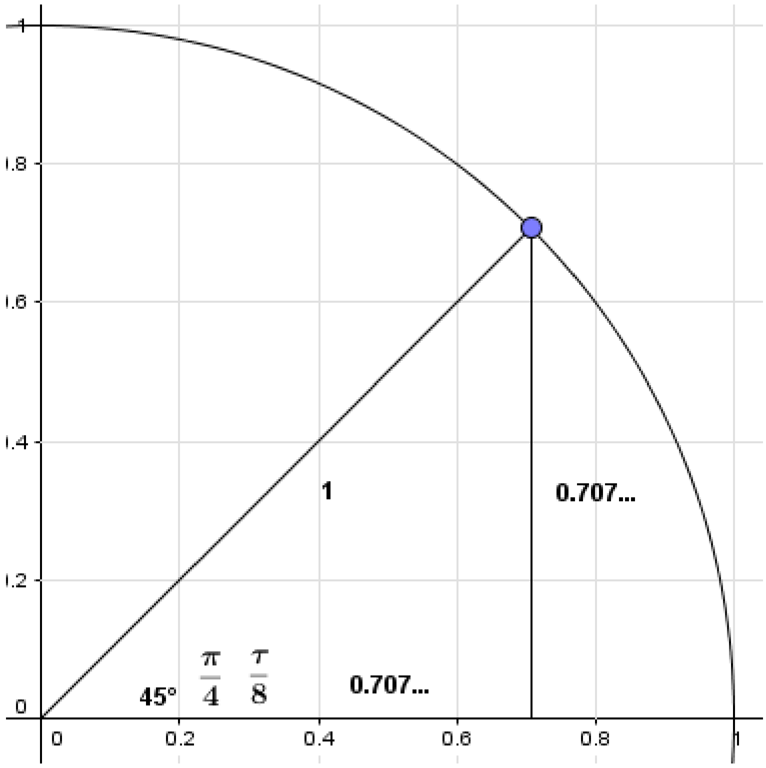
Tangent

Tangent is the ratio between the opposite side and the adjacent side. In the first diagram, we can see the ratio between the opposite and the adjacent is $\frac{0.5}{0.866\ldots} = 0.577\ldots$

This is what you're asking the calculator when you type tan(30°), tan($\frac{\pi}{6}$), tan($\frac{\tau}{12}$).

What is tan(60°), tan($\frac{\pi}{3}$), tan($\frac{\tau}{6}$)? [Look at the second diagram]

Use the diagram below to calculate tan(45°), tan($\frac{\pi}{4}$), tan($\frac{\tau}{8}$).



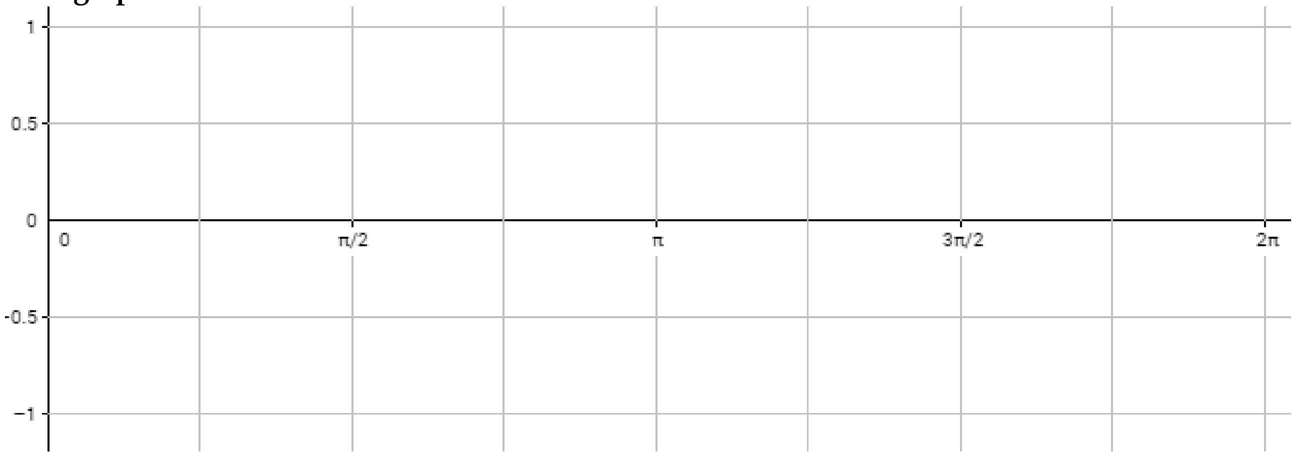
What is sin(45°), sin($\frac{\pi}{4}$), sin($\frac{\tau}{8}$)?

What is cos(45°), cos($\frac{\pi}{4}$), cos($\frac{\tau}{8}$)?

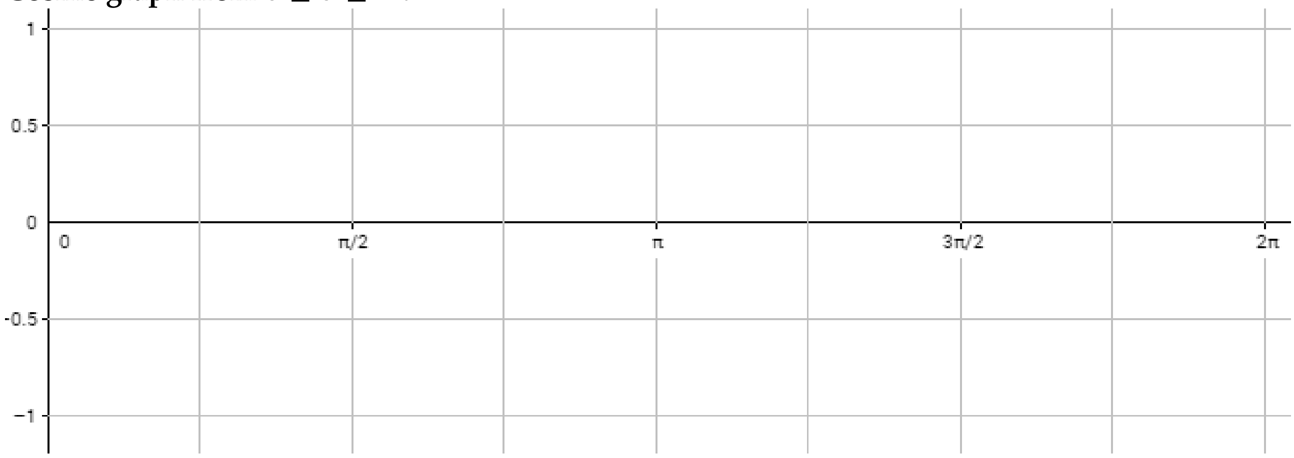
Note that the tangent ratio can exceed 1. Why?

Now, use the Unit circle- GeoGebra resource on the 3.3 Trigonometry page to explore the ratios at angles bigger than $\frac{\pi}{2}$ then complete the graphs on the next page.

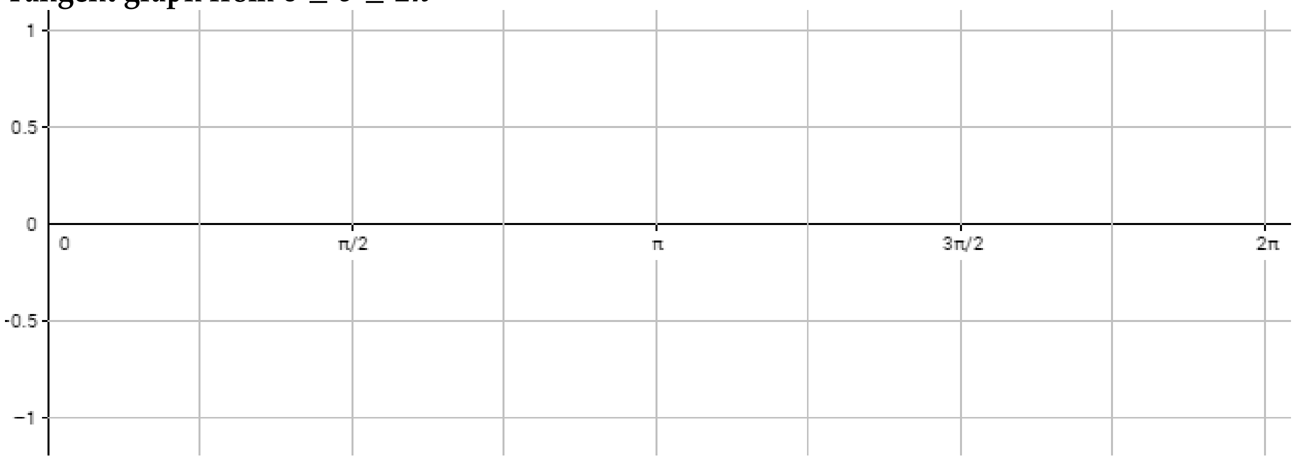
Sine graph from $0 \leq \theta \leq 2\pi$



Cosine graph from $0 \leq \theta \leq 2\pi$



Tangent graph from $0 \leq \theta \leq 2\pi$



Lesson Two: Amplitude, frequency, and period

From now on, the graphs and calculations will be expressed solely in terms of π radians rather than τ radians or degrees. However, feel free to include the τ value, you'll just need to write it in yourself.

Amplitude, frequency, and period are concepts relating to waves.

Amplitude
The amplitude is the height of the wave from the highest points [peaks] to the lowest points [troughs].

What is the amplitude of the sine and cosine graphs?

What is the amplitude of the tangent graph?

Frequency
The frequency is how many cycles occur in 2π radians.

What is the amplitude of the sine and cosine graphs?

What is the amplitude of the tangent graph?

Period
The period is the width of the wave. It is the distance before the graph starts repeating itself. It's best to measure it from peak to peak or trough to trough rather than using a midpoint. Why?

What is the period of the sine and cosine graphs?

What is the period of the tangent graph?

Lesson Three: Transformations of trigonometric graphs

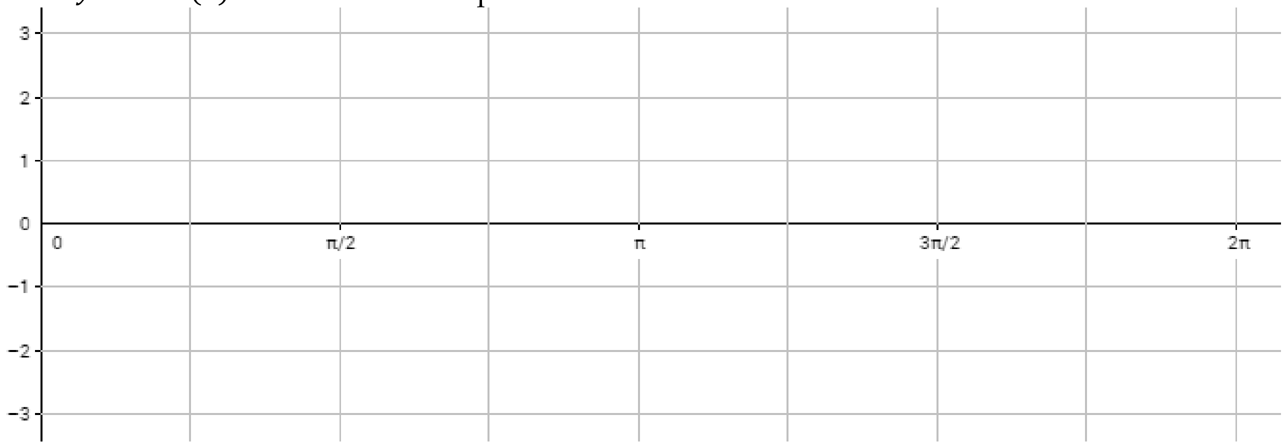
Recall our purpose for dealing with trigonometric graphs: they are used for modelling real world situations, e.g. height above ground on a Ferris wheel, predator and prey populations, sunrise and sunset times, etc. The sine graph is the best at modelling these. We will disregard the tangent graph because it's not an appropriate model for these situations. We will disregard the cosine graph because it's merely a horizontal translation of the sine graph [see below]. So from here on, it's sine graphs all the way.

Let's start with the basic sine graph: $y = \sin(x)$

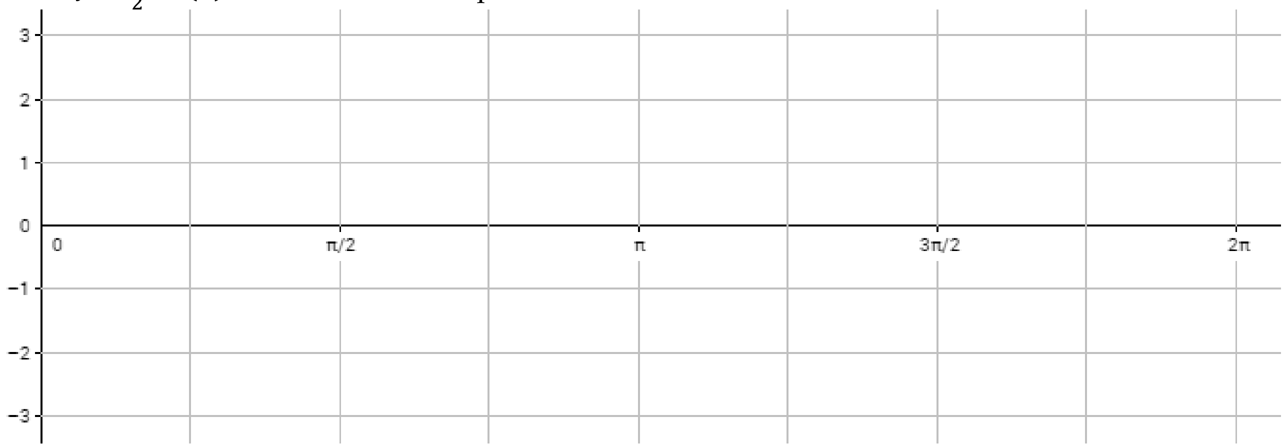
A=vertical stretch $\boldsymbol{y = A\sin(x)}$

By multiplying the sine function by a number, we can stretch it vertically, thereby changing the amplitude.

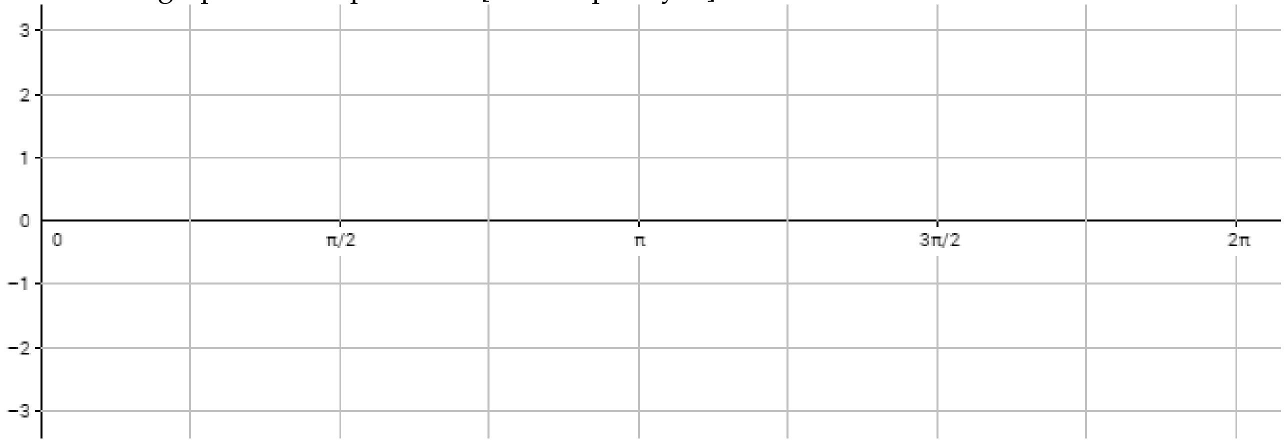
Draw $y = 3\sin(x)$ What is it's amplitude?



Draw $y = \frac{1}{2}\sin(x)$ What is it's amplitude?



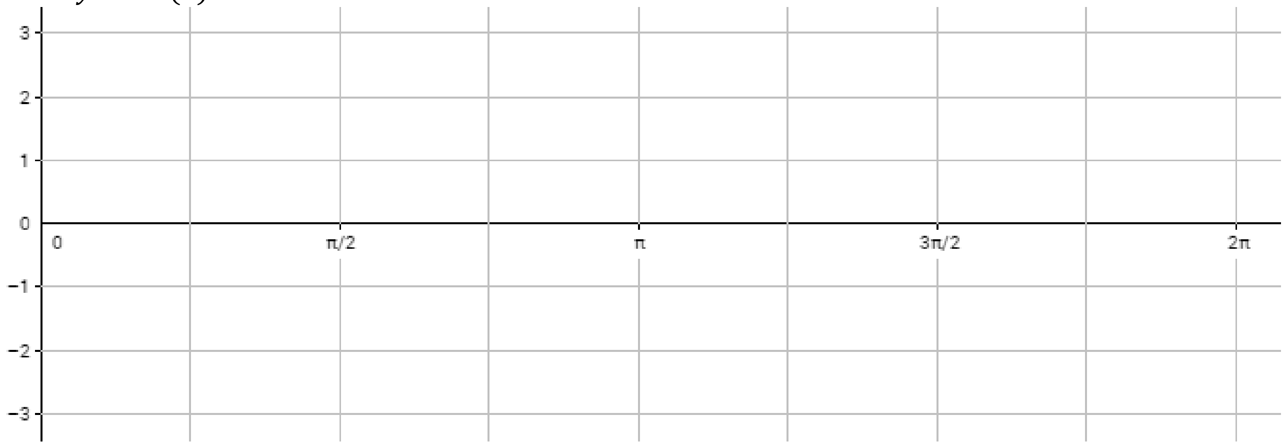
Draw a sine graph with amplitude=4 [and frequency=1].



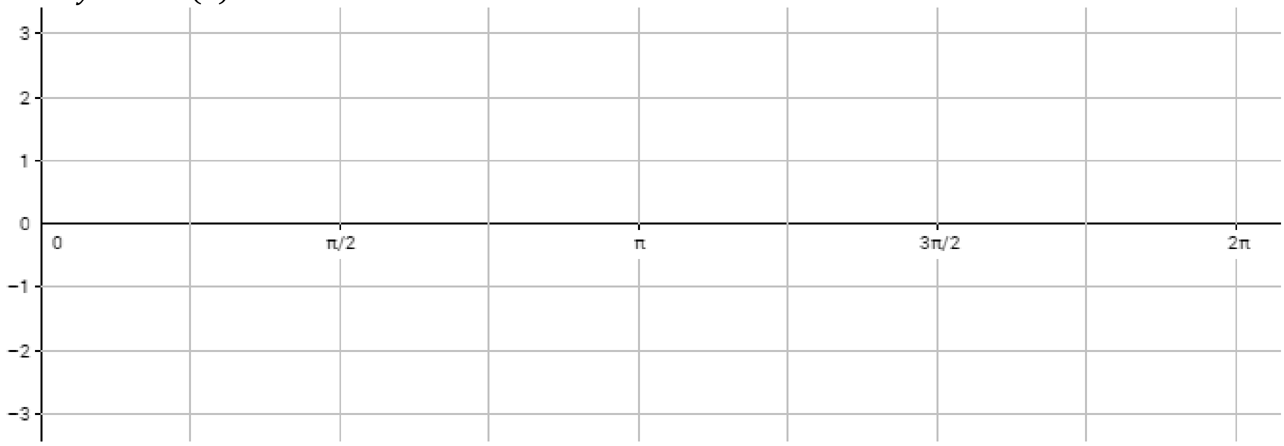
D=vertical shift $\boldsymbol{y = \sin(x) + D}$

Yes, we're skipping letters. That's because I want you to see the two vertical transformations next to each other. Adding a number to the sine function shifts the function up or down.

Draw $y = \sin(x) + 1$



Draw $y = 2\sin(x) - 1$

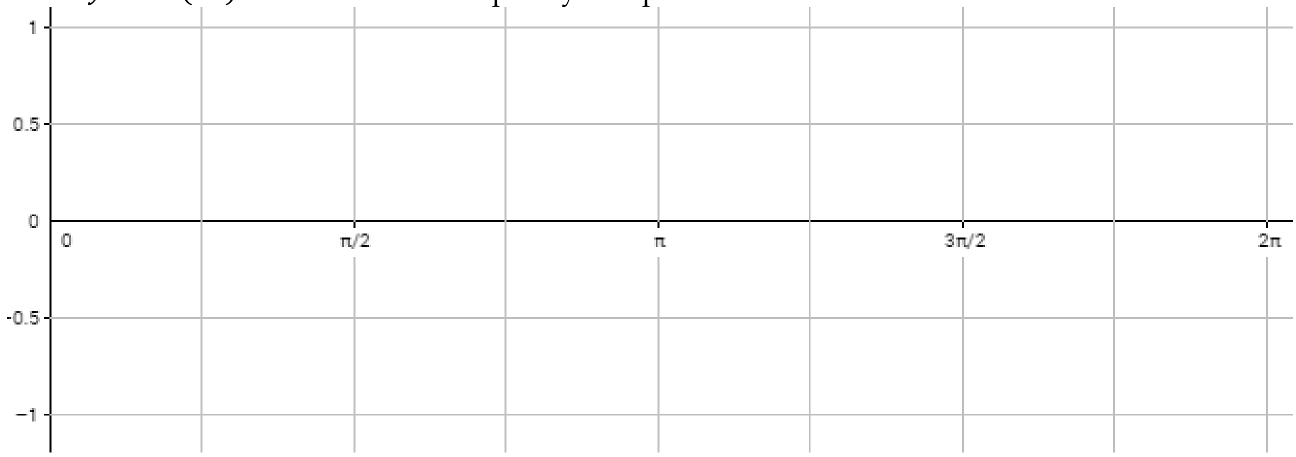


B=horizontal stretch $\boldsymbol{y = \sin(Bx)}$

Multiplying the x variable by a number before applying the sine function stretches the graph horizontally. The frequency of the new graph is B. And the period of the new graph is $\frac{2\pi}{B}$

Note that multiplying by a number >1 compresses the function and multiplying by a number <1 stretches the function.

Draw $y = \sin(2x)$ What is it's frequency and period?



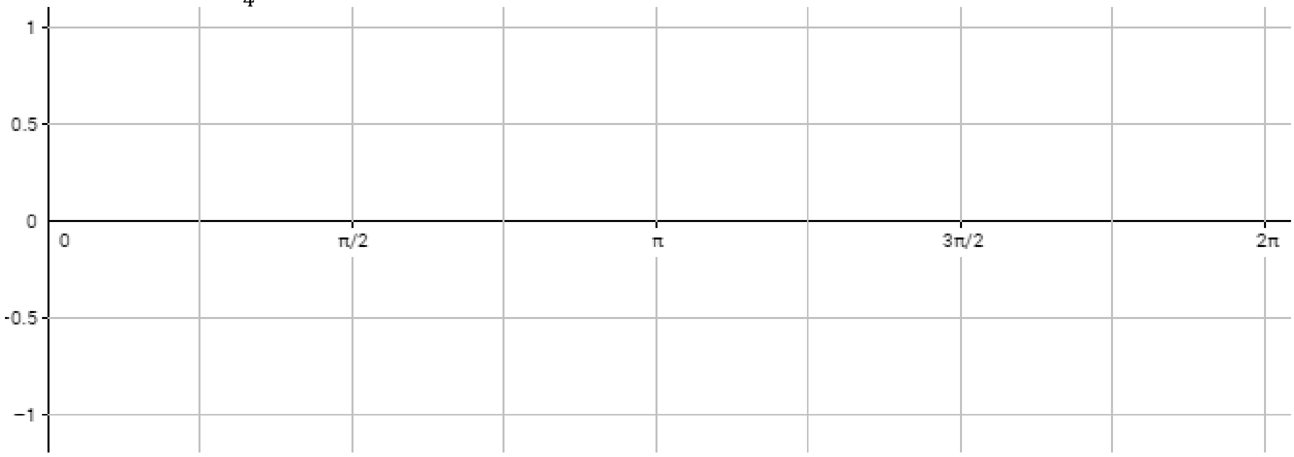
Draw $y = \sin(\frac{1}{2}x)$ What is it's frequency and period?

C=horizontal shift $\boldsymbol{y = \sin(x + c)}$

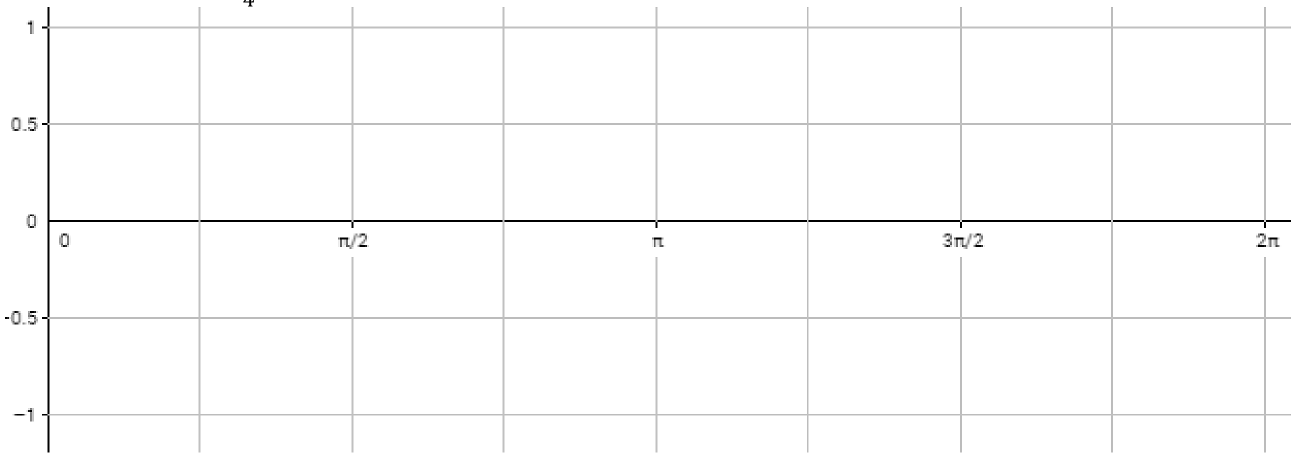
Adding a number to the x variable before applying the sine function shifts the graph horizontally so that the graph begins at $x = -c$ rather than $x = 0$

Note very carefully that the graph shifts to the left when a positive number is added.

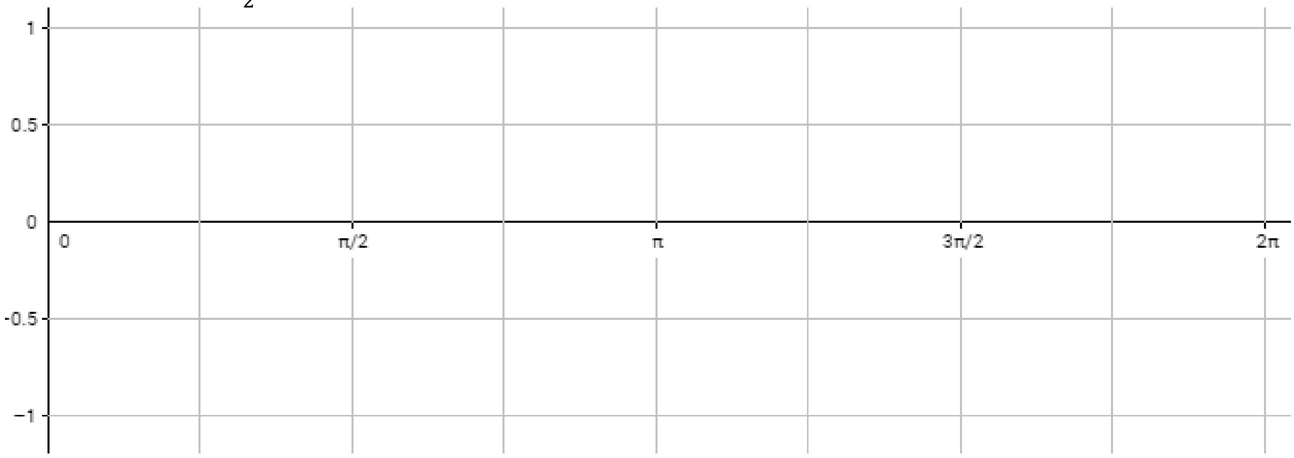
Draw $y = \sin(x + \frac{\pi}{4})$



Draw $y = \sin(x - \frac{\pi}{4})$



Draw $y = \sin(x + \frac{\pi}{2})$

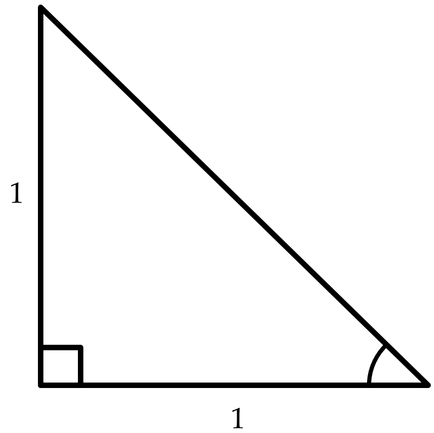


And this is why we disregard the cosine graph. It's just a transformed sine graph.

Lesson Four: Exact solutions

Let's revisit some old friends. Look at the diagrams on pages 5-7 and note the values 0.866… and 0.707…

I purposely didn't round them because there's a better way to simplify them. First, take a right-angle triangle with short sides both of length 1:



What must the hypotenuse length be?

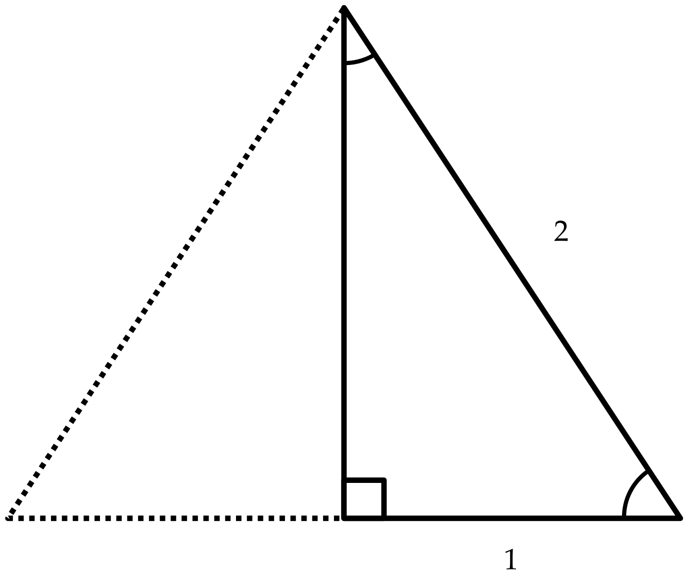
What must the angle be?

Using $\sin\theta = \frac{o}{h}$ and $\cos\theta = \frac{a}{h}$, calculate:

$\sin(\frac{\pi}{4}) =$

$\cos(\frac{\pi}{4}) =$

Now take an equilateral triangle of length 2 and cut it in half:



What must the height of the triangle be?

What must the bottom-right angle be?

What must the top angle be?

Using $\sin\theta = \frac{o}{h}$ and $\cos\theta = \frac{a}{h}$, calculate:

$\sin(\frac{\pi}{6}) =$

$\cos(\frac{\pi}{6}) =$

$\sin(\frac{\pi}{3}) =$

$\cos(\frac{\pi}{3}) =$

The standard doesn't require you to use exact solutions unless specifically asked for. However, I expect you to use exact solutions when you can, i.e. when $\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$

There are cases when there is no exact value, e.g. $\sin(1) = 0.8414709848\ldots$ and there's no way to be exact with that.

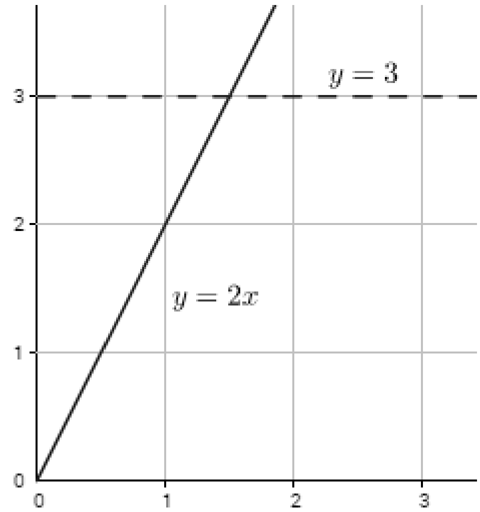
Lesson Four: Particular solutions

You will revise what it means to find a solution of an equation.

Find when the equation $y = 2x$ has y=3, i.e. solve $3 = 2x$

$$3 = 2x$$
$$x = 1.5$$

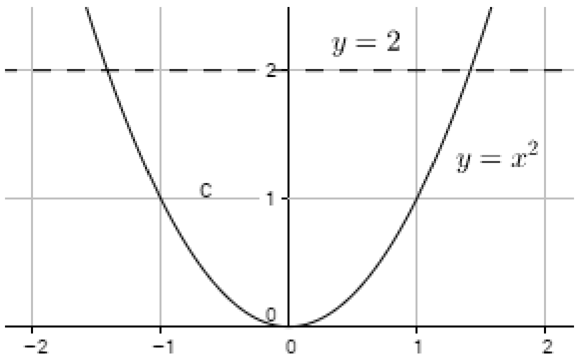
There is one solution for this equation. We had to use division to 'undo' the multiplication by two applied to the x variable.



Solve $2 = x^2$

$$x^2 = 2$$
$$x = \pm\sqrt{2}$$

There are two solutions for this equation. We had to use a square root function to 'undo' the square function applied to the x variable.



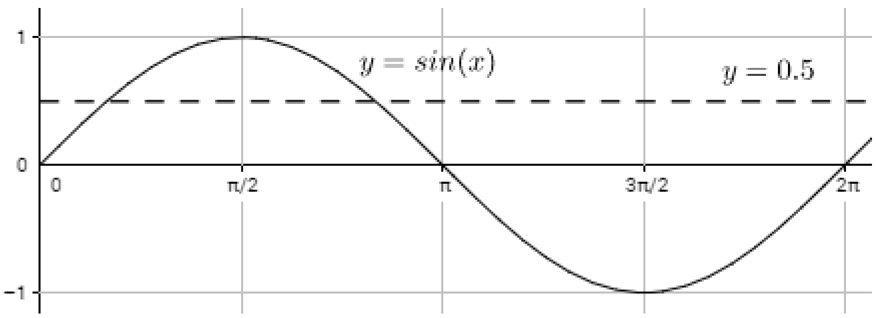
To solve trigonometric equations we need to 'undo' the sine function applied to the x variable. The inverse of sine is, the uncreatively named, inverse sine which is denoted $\sin^{-1}(x)$

Solve $0.5 = \sin(x)$

$$\sin(x) = 0.5$$
$$x = \sin^{-1}(0.5)$$
$$x = 0.523 \ldots$$

or more accurately:

$$x = \frac{\pi}{6}$$



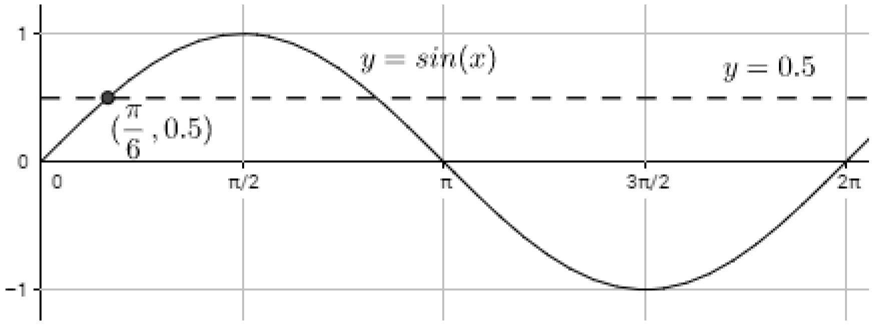
But there are two solutions between 0 and 2π [and infinite solutions from $-\infty$ to $+\infty$]. How do we find the others? One option is to graph $\sin(x)$ on a graphics calculator and use G-Solve to find all the solutions in the view window. Or you could use general solution which you'll learn about next lesson.

Lesson Five: General solutions

What is a general solution? It is a way of describing all solutions to a trigonometric equation without listing all infinitely many of them.

You can make use of symmetry to find general solutions.

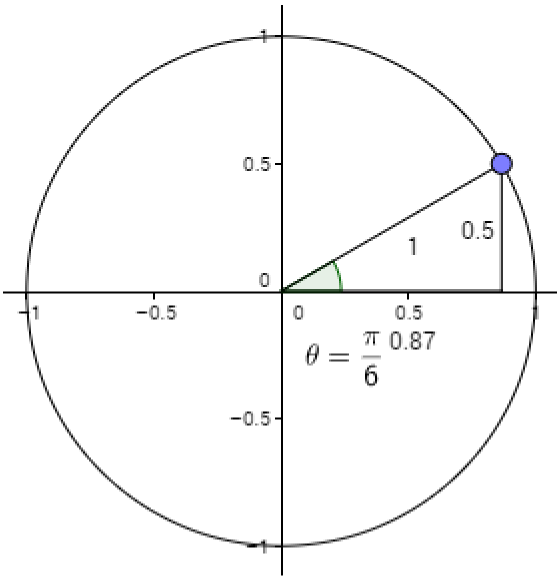
Recall that one solution for $0.5 = \sin(x)$ is $x = \frac{\pi}{6}$

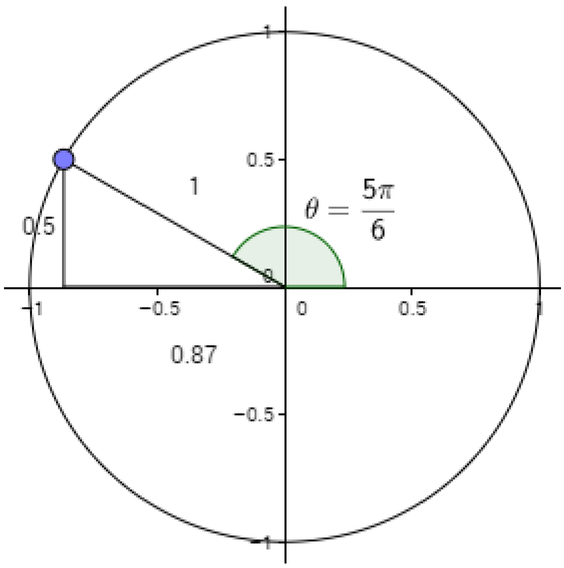


One solution is $\frac{\pi}{6}$ to the right of the origin so, using symmetry, you can notice that another solution is $\frac{\pi}{6}$ to the left of π, i.e. $\frac{5\pi}{6}$

Try it. Calculate $\sin(\frac{\pi}{6})$ and $\sin(\frac{5\pi}{6})$ on a calculator.

We can visualise this on the unit circle too.





There are two angles between 0 and 2π that cause the opposite side to be half as long as the hypotenuse.

But we can be even more general than this. Because each of these solutions yields more solutions when we add or subtract multiples of 2π, e.g. $\sin(\frac{13\pi}{12}) = 0.5$

So how do we describe all of these solutions succinctly?

If $\sin(x) = \sin(\alpha)$, i.e. a particular solution, then

$$x = n\pi + (-1)^n\alpha$$

In other words, $\frac{\pi}{6}$ is a solution and we say that this corresponds to $n = 0$ [because we've gone ahead from 0π, the origin].

$$x = 0\pi + (-1)^0\frac{\pi}{6}$$
$$x = 0 + \frac{\pi}{6}$$
$$x = \frac{\pi}{6}$$

The next solution corresponds to $n = 1$:

$$x = 1\pi + (-1)^1\frac{\pi}{6}$$
$$x = \pi - \frac{\pi}{6}$$
$$x = \frac{5\pi}{6}$$

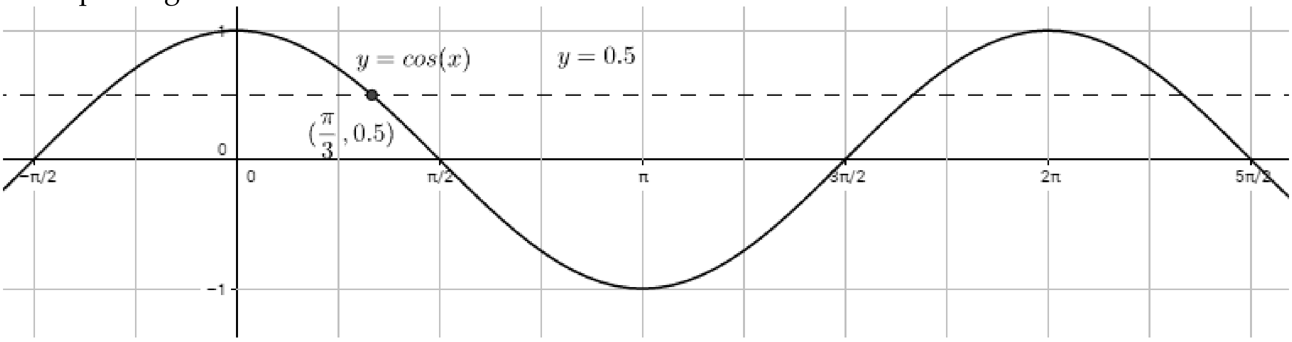
Another solution corresponds to $n = 2$:

$$x = 2\pi + (-1)^2\frac{\pi}{6}$$
$$x = 2\pi + \frac{\pi}{6}$$
$$x = \frac{13\pi}{6}$$

More solutions can be found when $n < 0$, e.g. $n = -1$:

$$x = -1\pi + (-1)^{-1}\frac{\pi}{6}$$
$$x = -\pi - \frac{\pi}{6}$$
$$x = -\frac{7\pi}{6}$$

Perhaps the general solution for the cosine function is more intuitive.



One solution for $0.5 = \cos(x)$ is:

$$\cos(x) = 0.5$$
$$x = \cos^{-1}(0.5)$$
$$x = 1.047 \ldots$$
$$x = \frac{\pi}{3}$$

Using symmetry, can you see that since $\frac{\pi}{3}$ is a solution, so must $-\frac{\pi}{3}$ also be a solution?

And that if we add 2π, we can repeat the same trick?

If $\cos(x) = \cos(\alpha)$, e.g. a particular solution, then the general solution for the cosine function is:

$$x = 2n\pi \pm \alpha$$

In other words, $\pm\frac{\pi}{3}$ are the two solutions that correspond to $n = 0$ [because we've gone $\pm$ from 0π, the origin].

$$x = 2 \times 0\pi \pm \frac{\pi}{3}$$
$$x = \pm\frac{\pi}{3}$$

The next two solutions correspond to $n = 1$:

$$x = 2\pi \pm \frac{\pi}{3}$$
$$x = \frac{5\pi}{3}, \frac{7\pi}{3}$$

More solutions can be found when $n < 0$, e.g. $n = -1$:

$$x = -2\pi \pm \frac{\pi}{3}$$
$$x = -\frac{7\pi}{3}, -\frac{5\pi}{3}$$

General solutions can get very, very tricky when combined with transformed graphs for reasons I don't currently fully understand. Practice is crucial to master these situations.

Lesson Six: Modelling using trigonometric functions

Trigonometry came from triangles but is unexpectedly useful in modelling real-world contexts such as height above ground on a Ferris wheel, predator and prey populations over time, sunrise and sunset times over the year, etc.

Any situation that is periodic [i.e. repeats itself] lends itself towards a trigonometric model. Key information you'll need to produce a model is:

- The highest and lowest points of the context to calculate the vertical stretch and vertical shift
- How often the cycle repeats to calculate the horizontal stretch
- What the initial value is [or any specific value, e.g. the location of the first peak above $x = 0$] to calculate the horizontal shift

A=Vertical stretch

The vertical stretch is the difference between the highest points and the lowest point divided by two, i.e.

$$A = \frac{(\max - min)}{2}$$

Alternatively, you could think of the vertical stretch as:

D=Vertical shift

The vertical shift is the average height of the graphs, i.e.

$$D = \frac{(max + min)}{2}$$

Alternatively, you could think of the vertical shift as:

B=Horizontal stretch
The horizontal stretch is how many cycles [periods] fit into 2π, i.e.

$$B = \frac{2\pi}{period}$$

Alternatively, you could think of the horizontal stretch as:

C=Horizontal shift
The horizontal shift is the horizontal distance between the origin and where the sine curve starts, i.e.

$$C = -shift$$
$$C = -\left(time\ to\ maximum - \frac{1}{4}period\right)$$
$$C = -time\ to\ maximum + \frac{1}{4}period)$$

Alternatively, you could think of the horizontal shift as:

Practice internal

Maths End amusement park has two Ferris wheels: the Kiddy-wheel, a small wheel that reaches a maximum height of 10 m above the ground; and the Flying-high, a large wheel that reaches a maximum height of 50 m above the ground.

While they are at the amusement park, Manu has a ride on the Flying-high wheel and his little sister Jade goes for a ride on the Kiddy-wheel. Manu and Jade go on the rides at the same time. Because of trees and buildings between the two rides, Jade can only see Manu some of the time.

Both Ferris wheels load passengers from ramps at their lowest point. The seat is at the same level as the ramp at this loading point. Jade and Manu start their rides at the same time.

The Kiddy-wheel reaches a maximum height of 10 m and its ramp is 1.5 m above the ground. The ride makes two revolutions each minute.

The Flying-high Ferris wheel reaches a maximum height of 50 m and its ramp is 2 m above the ground. The ride makes three revolutions in two minutes.

Due to trees, buildings, and the positions of the rides, Jade can only see Manu some of the time. For Jade to see Manu she needs to be more than 5 m above the ground. Manu needs to be going up and more than 5 m above the ground but less than 25 m above the ground.

Calculate for what percentage of the time Jade could see Manu.

Homework tracking

As you do homework for this standard, keep track of it all right here. Do around 30 minutes for each class time.

Week	Day of the week	Length of time	What did you do?
Term 1, Week 8			
Term 1, Week 9			
Term 1, Week 10			
Term 1, Week 11			

Appendix A: Radians

Circles are useful but whole circles aren't everything, just like whole turns aren't everything. We're often interested in part-circles or part-turns, e.g. a half turn, a quarter turn, a third turn, etc. But what total shall we assign to the circle so that we can divide it? A total of 10 units?

Defining the circle as 10 units means we get a nice value for a half circle: 5 units. But dividing the circle into quarters or thirds doesn't produce nice values.

What about defining the circle to have 100 units? We still get a nice value for a half circle: 50 and we now get a nice value for a quarter circle: 25. But we still don't have a nice value for a third circle.

The Babylonians preferred to count using base-60 rather than base-10 because 60 has more factors than 10. Similarly, 360 has more factors than 100. And for our present situation, factors is what it's all about.

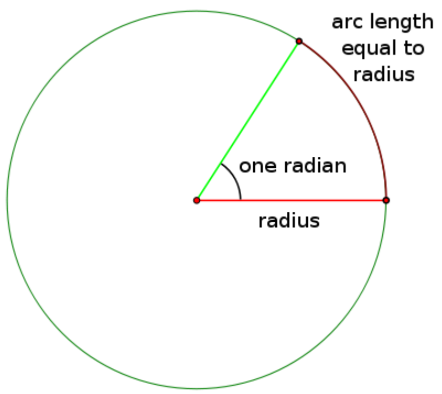
If we define the circle to have 360 units, we get nice values for half circles=180, third circles=120, quarter circles=90, sixth circles=60, and eighth circles=45. At least these are the ones we use commonly. There are others. We define this unit to be a degree.

But 360 isn't uniquely special. Sure, it has a lot of factors. But so does 3600 and 7200. We could keep increasing the number of units in a circle ad infinitum. Can we define a unit that is somehow more special than all others, somehow linked to the circle itself?

Well, what numbers are linked with circles? π comes to mind. Can we define the circle to have a total of π units? Hmmmm, I can't imagine what 1 π^{th} of a circle would look like. It's confusing. Let's try something else.

A circle is defined by two variables: the position and the radius. The position isn't a number [it's two numbers] but the radius is a number [technically, it's a variable but a variable that takes a single number value]. Could we use r to define the unit?

Let's say the angle required to make an arc length equal to the radius is one unit. We call these units radians because it sounds like radius. Radians, unlike degrees, are intimately linked with the circle itself.



How many radians are in a full circle? Recall that:

$$C = 2\pi r$$

So how many r's are in the circumference? 2π of them.

2π radians = full circle	$\frac{\pi}{3}$ radians = sixth circle
π radians = half circle	$\frac{\pi}{4}$ radians = eighth circle
$\frac{\pi}{2}$ radians = quarter circle	$\frac{\pi}{6}$ radians = twelfth circle

Appendix B: Tau τ

A full turn is 2π radians. And a half circles is π radians. And a quarter circle is $\frac{\pi}{2}$ radians.

Is that confusing for anyone else? Why is there always this factor of 2? You've seen why in appendix A, but is there a way make a full turn = 1, a half turn = $\frac{1}{2}$, etc.

There is! Let's make a new number and make it equal to 2π. Simple genius!

One of this new number is a full turn, half of this new number is a half turn, etc. All we need to do is give this new number a name. Mathematicians have decided to call it "tau" because it's Greek letter [τ] looks like π.

You may choose to learn trigonometry using τ because it's more intuitive than π. Feel free to. However, π will remain common at university and research-level maths so you should learn τ and π alongside each other to be best prepared.

For more information on τ, search for:

Tau vs Pi Smackdown – Numberphile

Tau replaces Pi - Numberphile