Level 3 – AS91578

Standard 3.6

6 credits – External

Apply differentiation methods in solving problems

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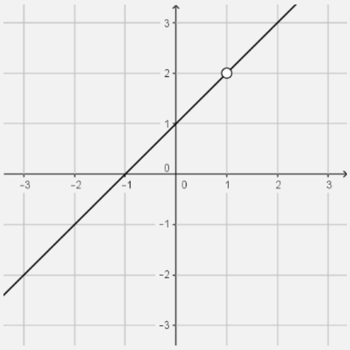
Lesson One: Limits

Introduction to limits

Consider the function:

$$f(x) = \frac{x^2 - 1}{x - 1}$$



Why is there a hole in the graph when $x = 1$?

Even though $f(x)$ is undefined when $x = 1$, it clearly approaches a value. This is called the limit of $f(x)$ and it is written as:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

In other words, "the limit of the function as x approaches 1 is 2."

There are two methods to calculating a limit: algebra and numerical methods.

Method One: Algebra

Sometimes, you'll be able to simplify the expression to the point where you can let x approach the limiting value; in this case, 1. For example:

$$\begin{aligned}\lim_{x \to 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \to 1} \frac{(x + 1)(x - 1)}{x - 1} \\ &= \lim_{x \to 1} x + 1 \\ &= 1 + 1 \\ &= 2\end{aligned}$$

So the limit as x approaches 1 is 2.

Calculate the limit:

$$\lim_{x \to -1} \frac{4x^2 - 4}{x + 1}$$

Method Two: Numerical methods

Sometimes, it's just not possible to algebraically rearrange a function to calculate its limit. In these cases, you can substitute a value just above and just below the limiting value to find the limit. For example:

$$\lim_{x \to 0} \frac{\sin x}{x}$$

Substituting $x = 0.001$ and $x = -0.001$ gives:

$$\frac{\sin 0.001}{0.001} = 0.9999998333$$

$$\frac{\sin(-0.001)}{-0.001} = 0.9999998333$$

Therefore:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

It is worthwhile graphing this on your graphics calculator to see what this function looks like.

Infinity

A similar case is the limit as x approaches infinity. In these cases, substitute a large number to find the limit. For example:

$$\lim_{x \to \infty} \frac{x}{e^x}$$

Substituting $x = 100$ gives:

$$\frac{100}{e^{100}} = 3.72 \times 10^{-42}$$

which is a very, very small number. Therefore:

$$\lim_{x \to \infty} \frac{x}{e^x} = 0$$

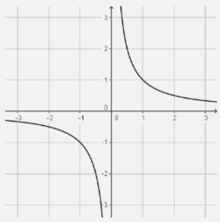
Calculate the limit:

$$\lim_{x\to\infty} \frac{3x+1}{x+2}$$

Cases where there is no limit

If the graph approaches two different values (depending on which direction you approach from), then **there is no limit** at that point. For example, consider the following function:

$$f(x) = \frac{1}{x}$$



Clearly, the function is undefined when $x = 0$. But what is the limit as x approaches 0?

If 0 is being approached from the positive direction, then:

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

And if 0 is being approached from the negative direction, then:

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

As these two values are different, there is no limit as x approaches 0.

When you're ready, answer the following questions.

Lesson One questions

Calculate the following limits, if possible.

Q1.

$$\lim_{x\to 4} \frac{(x+5)(x-4)}{(x-4)}$$

Q2.

$$\lim_{x\to 0} \frac{x(x+1)}{x}$$

Q3.

$$\lim_{x\to 5} \frac{1}{5-x}$$

Q4.

$$\lim_{x\to\infty} \frac{1}{x}$$

Q5.

$$\lim_{x \to -2} \frac{x^2 + 2x}{x + 2}$$

Q6.

$$\lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 3}$$

Q7.

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$$

Q8.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x^2 + 5x + 6}$$

Q9.

$$\lim_{x \to 0} \ln x$$

Q10.

$$\lim_{x \to \infty} \ln x$$

Q11.

$$\lim_{x \to 3} \frac{(x + 4)(x^2 - 6x + 9)}{x^2 - x - 6}$$

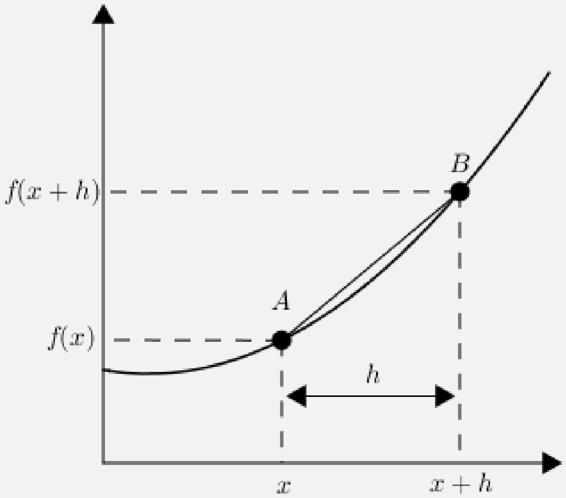
Q12.

$$\lim_{x \to 0} \frac{\cos x}{x}$$

Lesson Two: Differentiation from first principles

Differentiation from first principles

Consider a function $f(x)$. Take any two points on that function, A and B. The two points are separated by a distance h along the x-axis.



The gradient between them is:

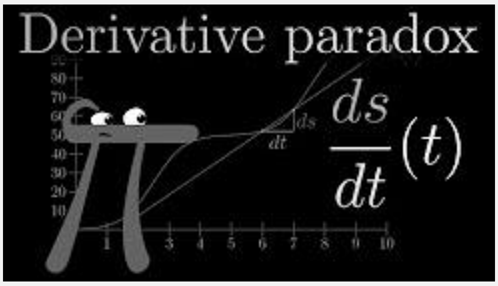
$$\text{Gradient} = \frac{\text{rise}}{\text{run}} = \frac{f(x+h) - f(x)}{x + h - x}$$

If you let h approach zero, the points A and B approach the same position and the gradient between them becomes the gradient of the function at that point. This is differentiation from first principles.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Watch the following video on the paradox of the derivative.

The paradox of the derivative



https://youtu.be/9vKqVkMQHKk

When you're ready, answer the following questions.

Lesson Two questions

Differentiate the following functions from first principles.

Q1. $f(x) = ax^2$

Q2. $f(x) = x^3$

Q3. $f(x) = x^2 + 4x + 6$

Q4. $f(x) = 3x^2 - 5x - 7$

Q5. $f(x) = (x - 3)(x - 5)$

Q6. $f(x) = (x + 4)^2 - 3$

Q7. $f(x) = \frac{1}{x}$

Q8. $f(x) = \sqrt{x}$

Lesson Three: Differentiating polynomials and simple trigonometric functions

Differentiating polynomials

After differentiating enough polynomials from first principles, you'll have noticed a pattern. Specifically, if $f(x) = ax^n$, then:

$$f'(x) = anx^{n-1}$$

In other words, reduce the power by 1 and multiply the by the old power. This is also known as the power rule

Note well: To be differentiated in this way, polynomials need to be in expanded form, e.g.

$$f(x) = ax^n + bx^{n-1} + \cdots + c$$

Derivatives of sums

If a function is a sum of other functions, i.e. it has several terms, the terms can each be differentiated separately. For example:

$$f(x) = x^3 + 5x^2 - x + 4$$
$$f'(x) = 3x^2 + 10x - 1$$

In general:

$$(f(x) + g(x))' = f'(x) + g'(x)$$

Differentiate the following function:

$$f(x) = 2x^4 - 5x^3 + 0.5x + 6$$

Negative and fractional powers

Functions such as $\frac{1}{x}$ and $\sqrt{x}$ can be differentiated using the fact that:

$$\frac{1}{x} = x^{-1} \qquad \text{and} \qquad \sqrt[q]{x^p} = x^{\frac{p}{q}}$$

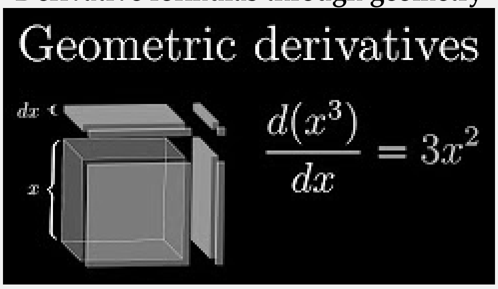
Differentiate the following functions:

$$f(x) = \frac{3}{4x}$$

$$f(x) = 3\sqrt[5]{x^2}$$

Watch the following video on how to intuitively understand common derivatives.

Derivative formulas through geometry



https://youtu.be/S0_qX4VJhMQ

Derivatives of simple trigonometric functions

As you saw from the video, the derivative of $\sin x$ is $\cos x$. In general:

$$(\sin x)' = \cos x$$
$$(\cos x)' = -\sin x$$
$$(-\sin x)' = -\cos x$$
$$(-\cos x)' = \sin x$$

When you're ready, answer the following questions.

Lesson Three questions

Differentiate the following functions.

Q1. $f(x) = (3x + 3)(2x - 4)$

Q2. $f(x) = (0.5x + 5)(4x + 10)$

Q3. $f(x) = x(x + 2)(x - 5)$

Q4. $f(x) = x(2x - 1)(x + 6)$

Q5. $f(x) = (x^2 - 9)(x + 3)$

Q6. $f(x) = 5x^2(x + 1)(x - 2)$

Q7.

$$f(x) = \frac{x^3}{3} + \frac{x^2}{4} - \frac{x}{5} + \frac{2}{3}$$

Q8.

$$f(x) = \frac{4x^3}{5} + \frac{x}{2} + 5$$

Q9.

$$f(x) = 3x^2 - x + 5 + \frac{1}{x}$$

Q10.

$$f(x) = \frac{2x^2}{3} + \frac{x}{4} + 2 - \frac{1}{x} + \frac{1}{x^2}$$

Q11.

$$f(x) = \frac{5x^2 + 2x + 2}{x}$$

Q12.

$$f(x) = \frac{(x+5)(x-4)}{x^2}$$

Q13.

$$f(x) = 4x + \sqrt{x}$$

Q14.

$$f(x) = 3\sqrt{x} + \sqrt[3]{x}$$

Q15.

$$f(x) = \frac{1}{x} + \frac{4}{\sqrt{x}} - \frac{2}{\sqrt[3]{x}}$$

Q16.

$$f(x) = \frac{1}{\sqrt[3]{x^2}} - \frac{3}{\sqrt[4]{x^4}} + \frac{x}{\sqrt{x}} + \frac{4}{\sqrt[3]{x^6}}$$

A cubic is defined by $f(x) = ax^3$. Find a given that $f'(-1) = 4$.

A hyperbola is defined by $f(x) = \frac{a}{x}$. Find a given that $f'(10) = -\frac{1}{5}$.

A function is defined by $f(x) = \sqrt[3]{ax^4}$. Find a given than $f'(1) = 2$.

Find the value of *k* so that f(x)= (X+k)^2 has a gradient of 0 at x=-3.

Find the value of *k* so that f(x)= k/x has a gradient of -1 at x=2.

Lesson Four: Differentiating products

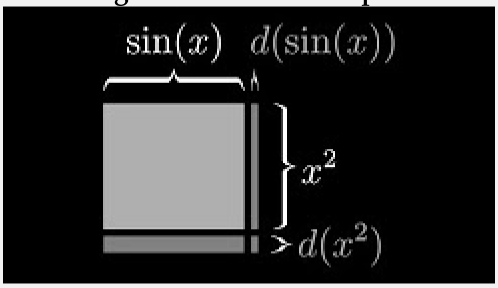
Products of functions
Products of functions are functions that are multiplied together, e.g.

$(x+2)(x-3)$ $\sqrt{x}(x^3 - x)$ $x^2 \sin x$

Sometimes, it will be easy to expand the product before differentiating and at other times it will be difficult or impossible.

It is possible to differentiate the product of two (or more) functions. The method for doing so is called the product rule. Watch the following video up to 8:40.

Visualizing the chain rule and product rule



https://youtu.be/YG15m2VwSjA

Therefore, the product rule is:

$$(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$$

This is in your formula sheet.

Differentiate the following functions using the product rule:

$$f(x) = \sqrt{x}\cos x$$

$$f(x) = \cos x \cdot x^{\frac{3}{2}}$$

Products of more than two functions
You can use the product rule for more than two functions.

Consider three functions: $f(x), g(x),$ and $h(x)$.

$$f(x) = 2x^2 \cdot x^3 \cdot \cos x$$

When you're ready, read the proof of the product rule or answer the following questions.

Extension: Proving the product rule

It is possible to prove the product rule is true for products of all functions by using first principles.

Using first principles, you can say that:

$$(f(x)g(x))' = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \quad (1)$$

In order to simplify this, you need an expression for $f(x+h)$ and $g(x+h)$. Note that:

$$f'(x) = \frac{f(x+h)-f(x)}{h}$$
$$hf'(x) = f(x+h)-f(x)$$
$$f(x+h) = hf'(x)+f(x)$$

Similarly, $g(x+h) = hg'(x)+g(x)$. You can substitute these into (1) to obtain:

$$\begin{aligned}(f(x)g(x))' &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{[hf'(x)+f(x)][hg'(x)+g(x)]-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{h^2f'(x)g'(x)+hf'(x)g(x)+hg'(x)f(x)+f(x)g(x)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{h^2f'(x)g'(x)+hf'(x)g(x)+hg'(x)f(x)}{h}\\ &= \lim_{h\to 0} hf'(x)g'(x)+f'(x)g(x)+g'(x)f(x)\\ &= f'(x)g(x)+g'(x)f(x)\end{aligned}$$

which is the product rule.

Lesson Four questions

Differentiate the following products using whichever method you prefer.

Q1. $x^3 \cdot x^2$

Q2. $(x+4)(2x-1)$

Q3. $(x^2-3)(x-4)$

Q4. $x\sin x$

Q5. $\sqrt{x}(x^4 - 5x^3 + 2x)$

Q6. $\frac{1}{x}\cos x$

Q7. $x^{\frac{4}{3}}(12x + 5)$

Q8. $x^2(\sin x - \cos x)$

Q9. $-\sin x\,(2x^2 \cdot 3x^3)$

Q10. $\sin x \cos x$

Q11. $\sin^2 x$

Q12. $\cos^2 x$

Q13. $(1 + \sqrt{x})(2 - \sqrt[3]{x})$

Q14. $x^3 \sin x \cos x$

Lesson Five: Differentiating composite functions (the chain rule)

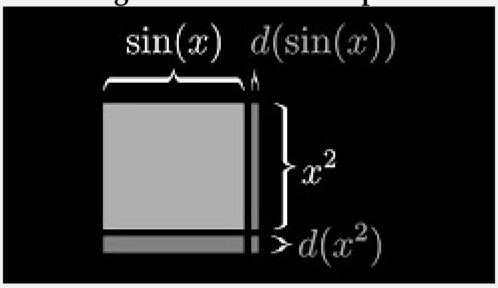
Composite functions

Composite functions are functions 'inside' other functions. For example:

$$\sin(\cos(x)) \qquad \frac{1}{\sin x} \qquad (x+6)^5$$

Sometimes, composite functions can be expanded (with great difficulty). But mostly, they are impossible to expand. How should you consider the derivative of such functions? Watch the following video.

Visualizing the chain rule and product rule



https://youtu.be/YG15m2VwSjA?t=8m42s

Differentiate $f(x) = \cos\left(\frac{1}{x}\right)$

Differentiate $f(x) = (x+2)^7$

Extension: Proof of the chain rule

You can use first principles to prove the chain rule.

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

Let $k = g(x+h) - g(x)$. Then:

$$g(x+h) = g(x) + k$$

and as $h \to 0$, $k \to 0$.

$$\begin{aligned}
[f(g(x))]' &= \lim_{h,k\to 0} \frac{f(g(x)+k) - f(g(x))}{h} \\
&= \lim_{h,k\to 0} \frac{f(g(x)+k) - f(g(x))}{k} \cdot \frac{k}{h} \\
&= \lim_{h,k\to 0} \frac{f(g(x)+k) - f(g(x))}{k} \\
&\quad \cdot \frac{g(x+h) - g(x)}{h} \\
&= f'(g(x)) \cdot g'(x)
\end{aligned}$$

When you're ready, answer the following questions.

Lesson Five questions

Differentiate the following functions using the chain rule.

Q1.

$$f(x) = (10x + 3)^4$$

Q2.

$$f(x) = (3x^2 - 6x + 4)^5$$

Q3.

$$f(x) = \sin^2 x$$

Q4.

$$f(x) = \cos^2 x$$

Q5.

$$f(x) = \frac{1}{\sin x}$$

Q6.

$$f(x) = \sin(2x)$$

Q7.

$$f(x) = \cos(x^2)$$

Q8.

$$f(x) = (x^3 - x + 4)^{-\frac{1}{2}}$$

Q9. Prove the chain rule using first principles.

Lesson Six: Differentiating quotients

The unnecessary quotient rule

There is no need for the quotient rule. In fact, the quotient rule is really just a combination of the product rule and the chain rule. Just like multiplication and division are really the same operation, the quotient rule is a rehashed version of the product rule.

Deriving the quotient rule from the product rule and chain rule

You can derive the quotient rule using the product rule and the chain rule. Consider the quotient:

$$\frac{f(x)}{g(x)}$$

It can be rearranged as:

$$f(x) \cdot \frac{1}{g(x)}$$

which can be derived using the product rule:

$$\begin{aligned}\left(\frac{f(x)}{g(x)}\right)' &= \left(f(x) \cdot \frac{1}{g(x)}\right)' \\ &= f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \left(\frac{1}{g(x)}\right)'\end{aligned}$$

Deriving $\frac{1}{g(x)}$ requires the chain rule, where $g(x)$ is the inside function and $\frac{1}{x}$ is the outside function. Applying the chain rule:

$$\begin{aligned}\left(\frac{f(x)}{g(x)}\right)' &= f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \frac{-1}{g(x)^2} \cdot g'(x) \\ &= \frac{f'(x)}{g(x)} - \frac{g'(x)f(x)}{g(x)^2} \\ &= \frac{f'(x)g(x)}{g(x)^2} - \frac{g'(x)f(x)}{g(x)^2} \\ &= \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}\end{aligned}$$

This result is the quotient rule.

Use the quotient rule to derive $\tan x$. You could check your formula sheet to see what the correct derivative is before you begin.

Use the quotient rule to derive:

$$f(x) = \frac{x^3}{\cos x}$$

On the next page is a proof of the quotient rule from first principles.

Extension: Proving the quotient rule
You can prove the quotient rule using first principles. You can say that:

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0}\frac{\left(\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}\right)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)}$$

As with the product rule, recall that $f(x+h) = hf'(x) + f(x)$ and $g(x+h) = hg'(x) + g(x)$. Therefore:

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)}$$

$$= \lim_{h\to 0}\frac{[hf'(x) + f(x)]g(x) - f(x)[hg'(x) + g(x)]}{hg(x)[hg'(x) + g(x)]}$$

$$= \lim_{h\to 0}\frac{hf'(x)g(x) + f(x)g(x) - hf(x)g'(x) - f(x)g(x)}{hg(x)[hg'(x) + g(x)]}$$

$$= \lim_{h\to 0}\frac{hf'(x)g(x) - hf(x)g'(x)}{hg(x)[hg'(x) + g(x)]}$$

$$= \lim_{h\to 0}\frac{f'(x)g(x) - f(x)g'(x)}{g(x)[hg'(x) + g(x)]}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)[g(x)]}$$

$$= \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

which is the quotient rule.

When you're ready, answer the following questions.

Lesson Six questions
Derive the following functions using whichever method you prefer.

Q1.

$$f(x) = \frac{x^5}{x^2}$$

Q2.

$$f(x) = \frac{\sqrt{x}}{\sin x}$$

Q3.

$$f(x) = \frac{-3x + 6}{x^2 - 4}$$

Q4.

$$f(x) = \frac{x^2 + 5x + 6}{x^2 - 9}$$

Q5.

$$f(x) = \frac{\sin x}{x^3}$$

Q6.

$$f(x) = \frac{\cos x}{\sin x}$$

Q7.

$$f(x) = -\cos x \cdot x^{-2}$$

Q8.

$$f(x) = x^{-\frac{1}{2}} \cdot \sin x$$

Q9.

$$f(x) = \frac{(x + 4)(2x - 5)}{4\sqrt[3]{x^2}}$$

Q10.

$$f(x) = \frac{x^2 + 2x - 8}{(x^2 - 4)(x - 3)}$$

Q11. Use the product and chain rules to prove the quotient rule

Q12. Prove the quotient rule from first principles.

Lesson Seven: Differentiating the (natural) exponential function

Exponential functions

Exponential functions are different from polynomials in that exponential functions are some constant raised to a variable. For example:

Polynomial	Exponential
$f(x) = x^2$	$f(x) = 2^x$

There are many different exponential functions such as:

$\left(\frac{1}{2}\right)^x$ $\quad 3^x \quad 3.5^x \quad 10^x$

However, we give special importance to one exponential function in particular. We call it the (natural) exponential function.

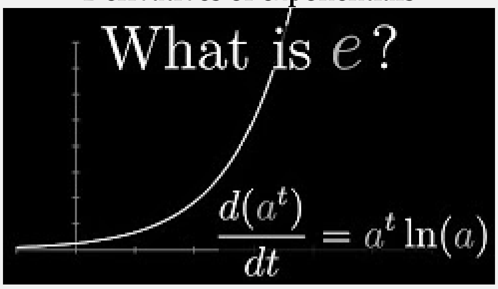
***The* exponential function**

The exponential function is the exponent to the base e. In other words:

$$f(x) = e^x$$

Watch the following video for why e is unique among the exponential functions and how to derive exponential functions in general.

Derivatives of exponentials



https://youtu.be/m2MIpDrF7Es

Explain what the growth rate of the pi creatures **over an entire day** would be if they tripled in mass each day, i.e. $M(t) = 3^t$

Given that $e^{\ln 2} = 2$, or in other words, $e^{0.6931\ldots} = 2$, rewrite 2^x in terms of e^x

Similarly, rewrite 3^x and 4^x in terms of e^x

Why do we say that e is the "natural" base to use for continuous growth?

Derivative of e^x from first principles

To prove e^x from first principles, you need to know that:

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

You can use this to prove the derivative of e^x

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} \\ &= \lim_{h \to 0} e^x \left(\frac{e^h - 1}{h}\right) \\ &= e^x \times 1 \\ &= e^x \end{aligned}$$

Answer the following questions.

Lesson Seven questions

Differentiate these functions.

Q1.

$$f(x) = e^{9x}$$

Q2.

$$f(x) = e^{-3x}$$

Q3.

$$f(x) = 2e^{4x-5}$$

Q4.

$$f(x) = e^{2-7x}$$

Q5.

$$f(x) = e^{x^2}$$

Q6.

$$f(x) = 4e^{x^2-3x+5}$$

Q7.

$$f(x) = e^{\sqrt{x}}$$

Q8.

$$f(x) = e^{\frac{1}{x}}$$

Q9.

$$f(x) = x^2 e^{5x-2}$$

Q10.

$$f(x) = \frac{\cos x}{e^{4x}}$$

Q11.

$$f(x) = \sin\left(e^{x^3}\right)$$

Q12. Use the result that $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$ to differentiate e^x from first principles.

Lesson Eight: Differentiating the natural logarithm function

The natural logarithm function

The natural logarithm function is a logarithm with base e.

The function $y = \ln x$ can be rewritten as:

$$y = \log_e x$$

Recall that, using logarithms, this can be rewritten as:

$$e^y = x$$

In other words, e to the power of what equals x?

Deriving the natural logarithm function

Consider the function $e^y = x$ from above. Deriving this with respect to y gives:

$$e^y = \frac{dx}{dy}$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$\frac{dy}{dx} = \frac{1}{x}$$

Differentiate $f(x) = \ln x$

Differentiate $f(x) = \ln(2x)$

Differentiate $f(x) = 3\ln x$

Differentiate $f(x) = \ln(x^2)$

Derivative of $\ln x$ from first principles

As usual, begin with the definition of first principles and use the properties of logarithms to rearrange:

$$f'(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h}$$

$$= \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \ln\left(1 + \frac{h}{x}\right)$$

$$= \lim_{h\to 0} \ln\left(1 + \frac{h}{x}\right)^{\frac{1}{h}}$$

Let $\frac{h}{x} = \frac{1}{n}$. Therefore $\frac{1}{h} = \frac{n}{x}$ and $n = \frac{x}{h}$. And as $h \to 0$, $n \to \infty$. Making these substitutions gives:

$$f'(x) = \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right)^{\frac{n}{x}}$$

$$= \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right)^{n \cdot \frac{1}{x}}$$

$$= \frac{1}{x} \cdot \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right)^{n}$$

You can recognise that $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ is defined as e. Therefore:

$$f'(x) = \frac{1}{x} \cdot \ln(e)$$

$$= \frac{1}{x} \times 1$$

$$= \frac{1}{x}$$

When you're ready, attempt the following questions.

Lesson Eight questions

Differentiate these functions.

Q1. $f(x) = \ln(4x)$

Q2. $f(x) = \ln(9x - 6)$

Q3. $f(x) = 4\ln x$

Q4. $f(x) = 9\ln x$

Q5. $f(x) = \ln(x^2 - 4)$

Q6. $f(x) = \ln(5x^2)$

Q7.
$$f(x) = \frac{\ln x}{4x^3}$$

Q8.
$$f(x) = \sin x \ln x$$

Q9.
$$f(x) = \ln(e^x)$$

Q10.
$$f(x) = \ln\left(e^{x^2}\right)$$

Q11.
$$f(x) = (4x + 2)\ln(2x)$$

Q12.
$$f(x) = \ln\left(\sqrt{x}\right)$$

Q13.
$$f(x) = \ln\left(\frac{1}{x}\right)$$

Q14.
$$f(x) = (3x^2 - 4x + 2)\ln(\sin x)$$

Q15.
$$f(x) = \frac{\ln(\cos x)}{x^2}$$

Q16.
$$f(x) = \ln(\ln x)$$

Lesson Nine: Differentiating trigonometric functions

Review of previous results

You already know that:

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(-\sin x)' = -\cos x$$

$$(-\cos x)' = \sin x$$

Now you will see algebraic proofs for these using first principles or methods such as the product rule, chain rule, or quotient rule.

Note well: All of these derivatives can be found in the formula sheet.

Deriving $\sin x$ from first principles

You can derive $\sin x$ from first principles using the result that $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$.

As usual, begin with the definition of first principles and use trigonometric identities to rearrange:

$$f'(x) = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h}$$

Using $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$ where $C = x + h$ and $D = x$ gives:

$$f'(x) = \lim_{h\to 0}\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

$$= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\cdot\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

As $h \to 0$, $\frac{h}{2} \to 0$, so:

$$f'(x) = \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\cdot\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

$$= \cos x \times 1$$

$$= \cos x$$

Deriving $\cos x$ from first principles is similar.

Deriving $\tan x$ from the quotient rule

You can use the quotient rule to derive $\tan x$ by recalling that $\tan x = \frac{\sin x}{\cos x}$.

$$f(x) = \tan x$$

$$= \frac{\sin x}{\cos x}$$

Applying the quotient rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

$$\left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x\cos x + \sin x\sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

Deriving the reciprocal trigonometric functions

The reciprocal trigonometric functions (secant, cosecant, and cotangent) can be differentiated by using the chain rule.

These are left as exercises.

When you're ready, answer the following questions.

Lesson Nine questions

Q1. Use $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ to prove that the derivative of $\cos x$ is $-\sin x$.

Q2. Prove the derivative of $\csc x$ is $-\csc x \cot x$ by rewriting $\csc x$ as $(\sin x)^{-1}$.

Q3. Similarly, prove that the derivative of $\sec x$ is $\sec x \tan x$.

Q4. Use the quotient rule to prove that the derivative of $\cot x$ is $-\csc^2 x$.

Differentiate these functions. Have the formula sheet handy.

Q5.

$$f(x) = 2\cos 5x$$

Q6.

$$f(x) = -3\tan\left(\frac{x}{2}\right)$$

Q7.

$$f(x) = 4\sin\sqrt{x}$$

Q8.

$$f(x) = \csc 4x$$

Q9.

$$f(x) = \ln(\sin x)$$

Q10.

$$f(x) = \sin(\ln x)$$

Q11.

$$f(x) = 6\sin^2 x$$

Q12.

$$f(x) = \csc^2 x$$

Q13.

$$f(x) = x^2 \tan(x^2)$$

Q14.

$$f(x) = \cos\left(\frac{2}{x}\right)$$

Q15.

$$f(x) = \frac{x^3}{\sin x}$$

Q16.

$$f(x) = \sqrt{\sin x}$$

Q17.

$$f(x) = e^{\sin x}$$

Q18.

$$f(x) = \ln\left(\sin\left(\sqrt{x}\right)\right)$$

Q19.

$$f(x) = \frac{(3x-4)(2x+5)}{\cot 3x}$$

Q20.

$$f(x) = \cos\left(e^{x^2}\right) \sin x$$

Lesson Ten: Parametric differentiation

Parametrically defined curves

Up until now, you have been working with equations that have a dependent variable (usually y) defined in terms of an independent variable (usually x). For example:

$$y = x^3 + \sin x$$

Parametric equations have their variables (x and y) defined in terms of another variable (also called a parameter). Often, t is used. In other words:

$$x = f(t) \text{ and } y = f(t)$$

The parameter, t, is often thought of as time and x and y being the horizontal and vertical positions of some object. Often, t begins at 0 but it can have negative values too.

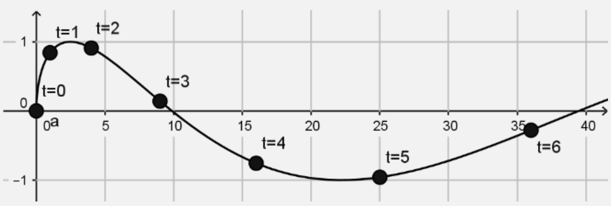
For example, consider the parametric equations:

$$x = t^2$$
$$y = \sin t$$

At different times, the x and y values of the function can be calculated.

t	x	y
0	0	0
1	1	0.84
2	4	0.91
3	9	0.14
4	16	-0.76
5	25	-0.96
6	36	-0.28

And the graph of it looks like this:

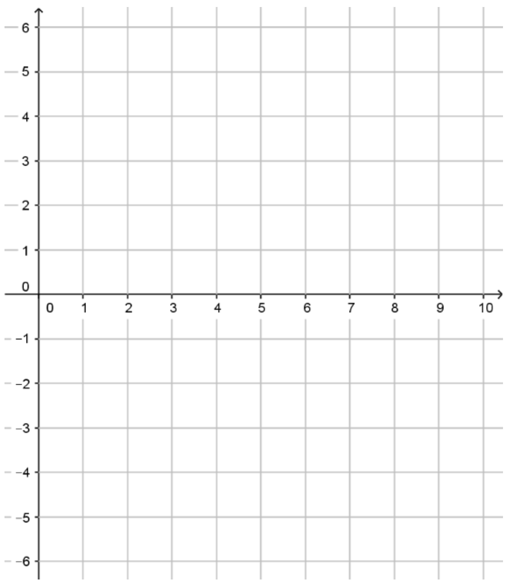


Note well: parametric functions, unlike explicit functions, can have more than one y value for each x value.

Complete the table and sketch the graph for the following parametric equation:

$$x = t^2$$
$$y = 2t$$

t	x	y
-3		
-2		
-1		
0		
1		
2		
3		



Derivatives of parametric functions

The derivative of a parametric function is the same essential idea: a tiny change in y compared to a tiny change in x, i.e. $\frac{dy}{dx}$.

Given that x isn't the independent variable (the variable you're 'nudging'), you can't calculate this directly. However, you can calculate a tiny change in x and y compared to a tiny change in t, i.e. $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

And using your knowledge of the chain rule, you know that:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

For example, to find the derivative of the following parametric function:

$$x = 3t^2$$
$$y = -2t$$

First calculate the derivative of x and y:

$$\frac{dx}{dt} = 6t \qquad \frac{dy}{dt} = -2$$

Then, you know that the derivative is:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= -2 \times \frac{1}{6t} \\ &= \frac{-1}{3t}\end{aligned}$$

Note that $\frac{dx}{dt}$ needs to be inverted as you want $\frac{dt}{dx}$.

When you're ready, attempt the following questions.

Lesson Ten questions
Differentiate the following parametric functions.

Q1.

$$x = t^2$$
$$y = 2t$$

Q2.

$$x = \cos t$$
$$y = \sin t$$

Q3.

$$x = \frac{1}{\cos t}$$
$$y = (t + 1)(2t - 5)$$

Q4.

$$x = e^{t^2+4t}$$
$$y = t^{\frac{3}{4}}$$

Q5.

$$x = \sqrt{t}\tan t$$
$$y = \ln(t^2)$$

Q6.

$$x = \frac{t^3}{\sin t}$$
$$y = \sec\left(\sqrt{t}\right)$$

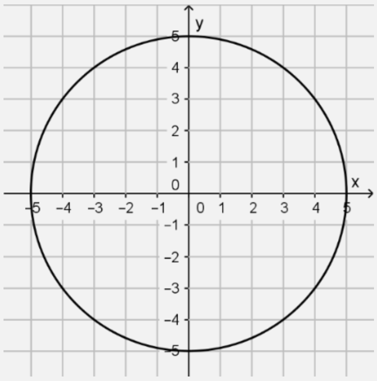
Lesson Eleven: Implicit differentiation

Implicit relationships
Up until now, you have been dealing with functions where a dependent variable is defined explicitly in terms of an independent variable.

But now consider an implicit relationship between x and y:

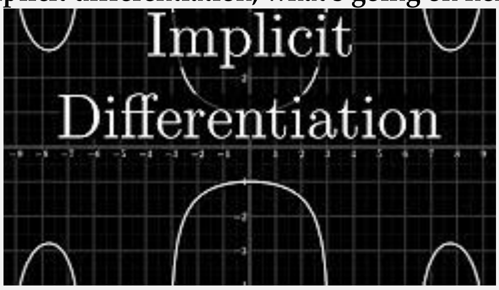
$$x^2 + y^2 = 25$$

The graph of this relationship is:



Sometimes, it's possible to rearrange the relationship into an explicit function but most times, it'll be impossible. Watch the following video to see how to differentiate implicit relationships.

Implicit differentiation, what's going on here?



https://youtu.be/qb40J4N1fa4

Using the chain rule to differentiate implicit relationships
As an alternative to considering an implicit relationship as a multivariable function, you could use the chain rule to differentiate.

For example, to differentiate $x^2 + y^2 = 25$, you could differentiate x^2 and 25 normally:

$$2x + (y^2)' = 0$$

But how do you differentiate y^2? Notice that there are two functions here: y is inside a squared function. So use the chain rule to differentiate the outside function and multiply by the derivative on the inside function.

$$2x + 2y \cdot y' = 0$$

But what is the derivative of y? Well, this is just $\frac{dy}{dx}$. This is, literally, what $\frac{dy}{dx}$ means; it's the derivative of y.

$$2x + 2y \cdot \frac{dy}{dx} = 0$$
$$2y \cdot \frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = \frac{-2x}{2y}$$
$$\frac{dy}{dx} = \frac{-x}{y}$$

In general, you can differentiate as you would normally expect to but every time you differentiate a function of y, multiply by $\frac{dy}{dx}$.

Differentiate $x^2 + y^2 = 25$ using the method shown in the video.

Differentiate $x^2 + y^2 = 25$ using the chain rule method explained above.

Lesson Eleven questions

Q1. Use implicit differentiation to show that the derivative of $\ln x$ is $\frac{1}{x}$.

Q2. Find the gradient of the following function at the given point: $y^2 - 2x - 1 = 0$ at $(-5, 3)$

Q3. Find the gradient of the following function at the given point: $xy + y^2 = 8$ at $(2, 2)$

Q4. Find the gradient of the following function at the given point: $4x^2 - 3y^2 = 10$ at $(3, 2.94)$

Differentiate the following relationships.

Q5. $x^2 + 4xy + 3y^2 = 0$

Q6. $xy^2 = x^3 + y^3$

Q7. $x^3 - y^3 = -2xy - 1$

Q8. $e^y + x^2 = 1$

Q9. $\ln y - x^3 - 4x = 2$

Q10. $\ln y + x^2 y = 2$

Lesson Twelve: Tangents and normals

Lesson Twelve questions

Find the equation of the tangent or normal (as specified) at the given point.

Q1. Tangent at $x = 3$

$$y = x^2 - 5$$

Q2. Tangent at $(1, 5)$

$$y = 2x^3 + 6x^2 - 7x + 4$$

Q3. Normal at $(-2, -14)$

$$y = (x + 4)(x - 5)$$

Q4. Normal at $x = 4$

$$y = x^6 + \frac{1}{2}x^3 - \frac{1}{x} + \frac{3}{x^2}$$

Q5. Tangent at $x = -1$

$$f(x) = x^3 \sin x$$

Q6. Normal at $x = 2$

$$f(x) = \ln(x^2)$$

Q7. Normal at $x = 0$

$$y = \sin x \cos x$$

Q8. Tangent at $x = 2.5$

$$f(x) = \frac{e^{\frac{x}{2}}}{\sqrt{x}}$$

Find the

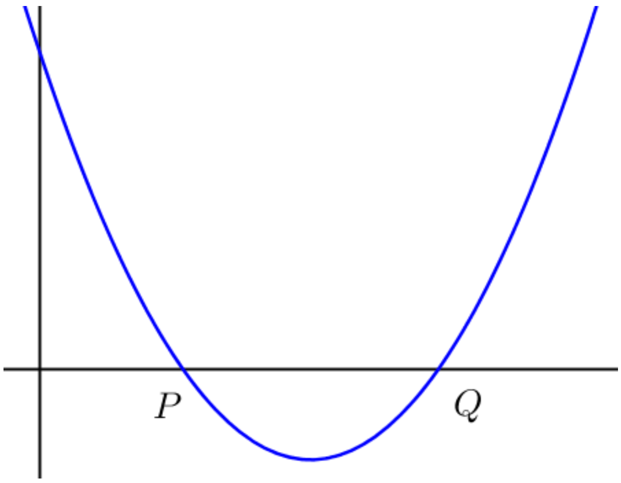
Where does the tangent to $f(x)x^3 + 4x^2 - 2$ at the point $(1, 3)$ cut the curve again?

Write down the coordinates of the point of intersection between the tangents to $f(x) = x^2 - 5x - 6$ at the points where the parabola cuts the x-axis.

1. A tangent to $f(x) = x^2 - 2x + 3$ passes through the point $(4, 7)$. Find the point(s) where the tangent touches the curve.

2. One of the points on the function $f(x) = 3x^3 + 4$ can be written as (a, b), where $a > 0$. The tangent at (a, b) makes an angle of 45° with each axis. Find where the tangent cuts the x-axis.

3. $f(x) = ax^2 + bx + c$ has real roots at points P and Q, where $P = (p, 0)$ and $Q = (q, 0)$.



Show that the normals to the parabola at P and Q intersect on the line $x = \frac{p+q}{2}$

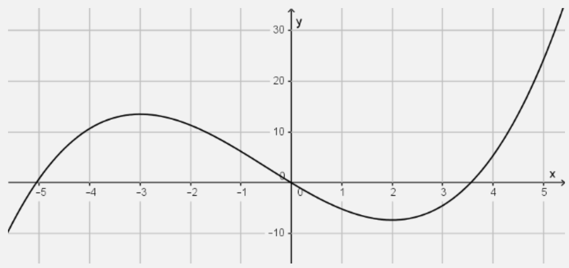
4. A tangent to the curve $f(x) = x^2 + 4$ passes through the point $(-2, 7)$. Find the equation(s) of this tangent.

Lesson Thirteen: Turning points and stationary points

The first derivative
The first derivative of a curve tells you the instantaneous gradient at any point on that curve.

You can track the gradient at different points. For example, consider the following curve:

$$f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 6x$$



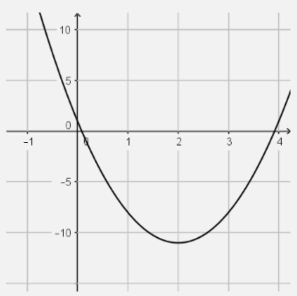
Calculate the gradient of the curve at the specified points:

$f'(x) =$

$f'(-5) =$	$f'(0) =$
$f'(-4) =$	$f'(1) =$
$f'(-3) =$	$f'(2) =$
$f'(-2) =$	$f'(3) =$
$f'(-1) =$	$f'(4) =$

Finding turning points
Turning points are where the gradient changes from positive to negative or vice versa. You can find turning points by differentiating the function then setting the derivative to 0. For example, find the turning point of the following function:

$f(x) = 3x^2 - 12x + 1$



$$\begin{aligned} f(x) &= 3x^2 - 12x + 1 \\ f'(x) &= 6x - 12 \\ 0 &= 6x - 12 \\ x &= 2 \end{aligned}$$

So the turning point is at $x = 2$. Now you need to know the y value of the function at this point. Do this by substituting into the original function.

$$\begin{aligned} f(2) &= 3 \times 2^2 - 12 \times 2 + 1 \\ &= -11 \end{aligned}$$

So the turning point is at $(2, -11)$.

Find the turning point of $f(x) = -x^2 + 4x - 4$

Determining the nature of a turning point
There are two types of turning points: maximums and minimums.

Maximum

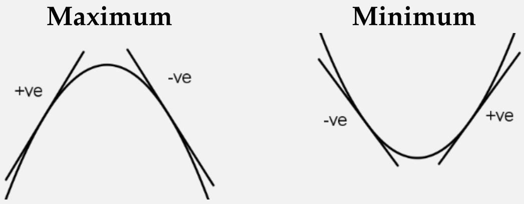
Minimum

There are several methods for determining the type of turning points. You will look at two now and the third in the next lesson.

1. Knowing the shape of the function
You'll often know the shape of the function. For example, a positive parabola can only have a minimum turning point and a negative parabola can only have a maximum turning point. Similarly, a positive cubic will have a maximum and a minimum (in that order) and a negative cubic will have a minimum and a maximum (in that order). **Note well: It's possible for a cubic to have neither maximums nor minimums if it has a point of inflection. This is coming up in lesson fourteen.**

2. Testing the gradient on either side

A maximum, by definition, goes from a positive gradient to a negative gradient and a minimum goes from a negative gradient to a positive gradient.



Testing the gradient on either side of a turning point will tell you what type of turning point it is.

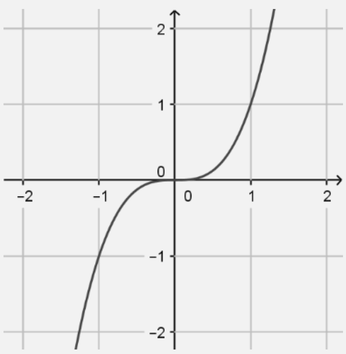
The function $f(x) = x^3 - 4x^2 + 2x + 3$ has a turning point at $(2.39, -1.42)$. Test the gradient on either side to determine whether it is a maximum or a minimum.

Stationary points

Stationary points are all points on the graph where the instantaneous gradient is 0, i.e. for that instant, the graph is not changing. All turning points are stationary points. But there are stationary points that aren't turning points. These are called points of inflection.

Points of inflection

Consider $y = x^3$:



Clearly, it has a derivative of 0 at the origin but it is neither a maximum nor a minimum at that point. We call it a point of inflection.

A point of inflection is a stationary point but it is not a turning point.

Points of inflection have the same sign gradient on either side, i.e. they go from a positive gradient to a positive gradient or vice versa.

The function $f(x) = -x^3 + 9x^2 - 27x + 29$ has a stationary point at $(3, 2)$. Test the gradient on either side to prove that the stationary point is a point of inflection.

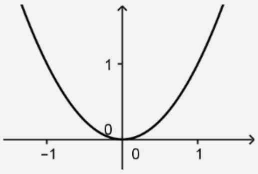
When you're ready, answer the following questions.

Lesson Fourteen: Concavity, points of inflection, and the second derivative

Concavity
The concavity of a graph relates to whether the rate of change (or gradient) is changing positively or negatively.

Concave up
Consider the following graph that is concave up:

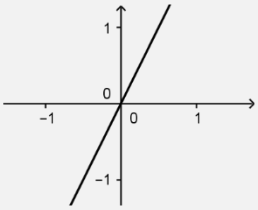
$y = x^2$



Its rate of change (gradient) is negative at first, then zero, then positive. Notice that at all points, its rate of change is becoming *more* positive. In other words, the rate of change of the rate of change is increasing.

The graph below shows the derivative of x^2

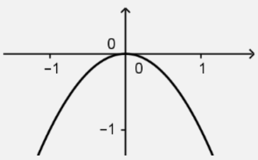
$y = 2x$



This graph shows the rate of change of the parabola at each x value. Notice that the rate of change of the rate of change is increasing, in other words the derivative has a positive gradient.

Concave down
A graph that is concave down has a rate of change that is decreasing, for example:

$y = -x^2$

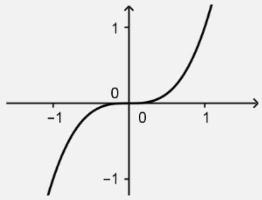


The gradient is getting more negative as x increases.

What is the concavity of $\sin x$ between $0 < x < \pi$?

Points of inflection
Points of inflection are where the graph changes concavity. Consider the following cubic:

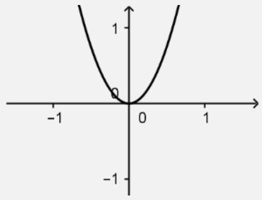
$y = x^3$



Notice that when $x < 0$, the graph is concave down and when $x > 0$, the graph is concave up. There is a point of inflection at $x = 0$ where the graph changes concavity.

Consider the derivative of x^3 below:

$y = 3x^2$

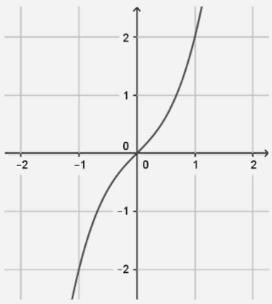


Remember that this graph shows the gradients for each point on the original graph. Notice that for $x < 0$, the graph has a negative gradient, i.e. the rate of change of the rate of change is negative. Similarly for $x > 0$.

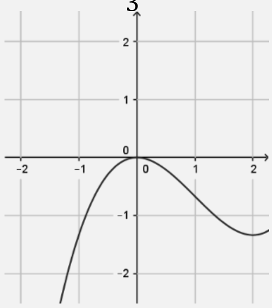
For $y = x^3$, the point of inflection occurs when the gradient is zero. **This is not always the case.**

Consider the cubics below.

$y = x^3 + x$



$y = \frac{x^3}{3} - x^2$



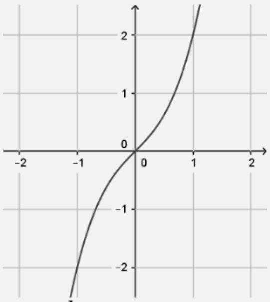
They have points of inflection at $x = 0$ and $x = 1$ respectively but clearly the gradient is not 0 at either location.

Finding points of inflection

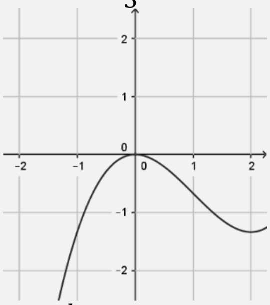
How can you find points of inflection? They are sometimes, but not always, at turning points (when $\frac{dy}{dx} = 0$). What is a guaranteed test for points of inflection?

Notice that they occur when the concavity changes, i.e. the derivative goes from increasing to decreasing or vice versa. Consider the cubics from before with their derivatives:

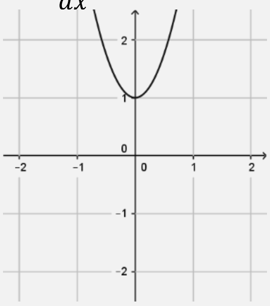
$y = x^3 + x$



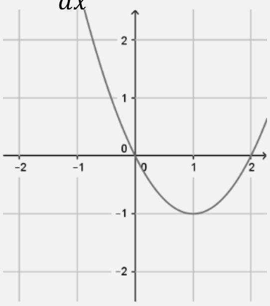
$y = \frac{x^3}{3} - x^2$



$\frac{dy}{dx} = 3x^2 + 1$



$\frac{dy}{dx} = x^2 - 2x$



What about $y = \sqrt[3]{x}$? Or $y = x^4$. There are exceptions but you can assume that if the second derivative is 0 that there is a POI.

Lesson Fifteen: Features of graphs

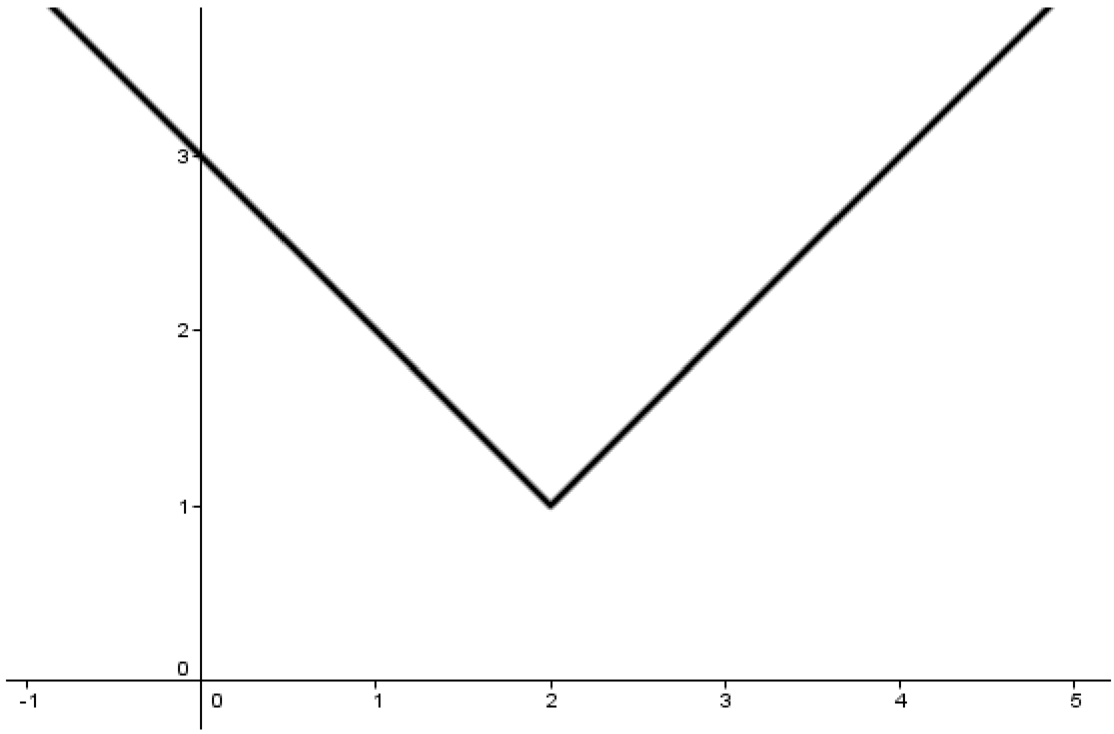
A function is continuous if you don't need to lift your pen off the paper to draw it.

More specifically, a function, $f(x)$, is continuous at a point $x = a$ if it is defined at that point and the limit is the defined value of $f(x)$, i.e. $\lim_{x \to a} f(x) = f(a)$

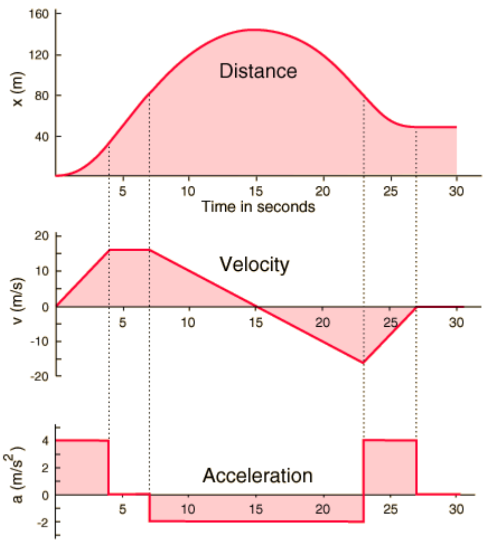
Differentiable if it's continuous and doesn't have any kinks in it.

More specifically, a function is differentiable at a point if it is continuous and the limit of the gradient $\lim_{h \to 0} \frac{rise}{run}$ is the same when approached from both sides.

Take the graph of $f(x) = |x - 2| + 1$



Lesson XX: Kinematics?



Lesson Sixteen: Sketching polynomials

Lesson Seventeen: Related rates of change

Lesson Seventeen questions

Calculate the following rates using the given information.

Q1. An inverted cone contains water but it is flowing out from the bottom (the tip of the cone). The cone has a radius of 2cm (at the top) and a height of 3cm. The water is flowing out at a rate of 5cm³/ minute. Find the rate at which the level of the water is falling when the depth is 2cm. Hint: Let the depth of the liquid after t minutes be x. The radius of the surface of the water is $\frac{2x}{3}$cm and the volume of a cone is $\frac{1}{3}\pi r^2 h$.	Q2. A circular ink-blot is formed by ink dripping from a pen. The radius of the ink-blot is increasing by 0.3mm/s. Find the rate at which the rea of the ink-blot is increasing: a) When the radius is 5mm b) After 3 seconds.

Lesson Eighteen: Optimisation

Introduction
Optimisation is the process of finding the maximum or minimum value given some constraints, e.g.

- The maximum area of a paddock with a limited amount of fencing
- The minimum time taken to travel across a field with areas of tall and short grass

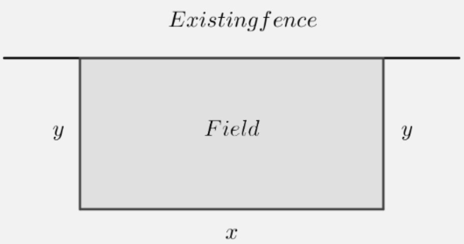
Given that the process involves finding a maximum or minimum, derivatives will be involved to find turning points of the function.

Best practice
Optimising a function always involves finding a turning point but it will also involve other complicating factors. Because of this, there are some best practice strategies you can employ:

- Always draw a diagram
- Identify the function you are asked to optimise
- Define that function explicitly in terms of a single variable if it isn't already

An example
A farmer has 120m of fencing that she wants to use to construct a rectangular field against an already existing fence. What is the maximum area of the field?

First, draw a diagram and assign variables to unknown values.



Second, identify the function you are asked to optimise:

$$Area = xy$$

Third, define that function explicitly in terms of a single variable. To do this, you can use a second equation that links x to y.

In this case, the length of the fence is 120m:

$$x + 2y = 120$$
$$2y = 120 - x$$
$$y = 60 - \frac{x}{2}$$

$$\begin{aligned} Area &= xy \\ &= x\left(60 - \frac{x}{2}\right) \\ &= -0.5x^2 + 60x \end{aligned}$$

Now you can differentiate to find the turning point:

$$\begin{aligned} Area' &= -x + 60 \\ 0 &= -x + 60 \\ x &= 60\text{m} \end{aligned}$$

So length x is 60m. And length y is:

$$\begin{aligned} y &= 60 - \frac{x}{2} \\ &= 60 - \frac{60}{2} \\ &= 30\text{m} \end{aligned}$$

Therefore, the largest field that can be made with 120m of fencing against an already existing fence is 60m long and 30m wide. The area of this field is:

$$\begin{aligned} Area &= 60\text{m} \times 30\text{m} \\ &= 1\,800\text{m}^2 \end{aligned}$$

Lesson Eighteen questions

Q1. A rectangular sheet of cardboard is to have four equal square corners removed and the sides turning up to form an open box. If the cardboard measures 18cm x 10cm, find the maximum volume of the box.

Q2. A farmer needs to have a water storage container in order to fight forest fires. The water storage container is a cuboid, has an open top, and is to have a volume of 20 000 litres ($20m^3$). The length of its base is twice the width. Construction of the base costs \$75 / m^2. Construction of the sides costs \$40 / m^2. Find the minimum cost of materials.

Q3. A 250L cylindrical hot-water tank is to be designed to minimise heat loss which is proportional to surface area. Calculate the dimensions of the optimum water tank so that the surface area is minimised.

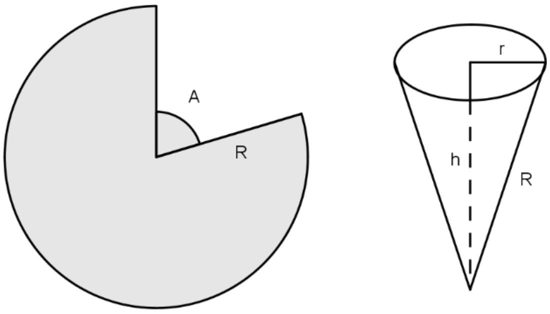
Q4. The vertical height of a model aeroplane is described by the equation:

$$h = 15 + 12t - 3t^2$$

where t is in seconds and h is in metres.

a) Find the maximum height of the plane
b) Find the vertical velocity when the plane hits the ground

Q5. A circle can be made into a cone by cutting out a sector and joining the cut edges. Find the maximum capacity of such a cup and the angle A that should be removed to make this optimum cup.



Q6. A triathlete is running up a river bank and needs to swim across the 32m wide river in order to finish. The athlete can run at 6.0m/s and swim at 1.8m/s. At which point should he enter the water to minimise the time taken to complete the race?

Practice exam

As good mathematical practice, candidates should be encouraged to show intermediate steps clearly and logically, communicating what is being calculated. Candidates who give the correct response only may lose the opportunity to provide evidence for other grades or to have minor errors ignored.

QUESTION ONE

a) Differentiate: $y = \sqrt{x^3 - 2x}$. You do not need to simplify your answer.

b) Find the equation of the tangent to the curve $y = \tan x$ at the point $\left(\frac{\pi}{4}, 1\right)$.

c) A curve is defined by the parameters $x = 4e^{0.5t}$ and $y = t^2 - 4t$. Find the gradient of the tangent to the curve at $t = 6$.

d) The point P, $(\pi, 0)$, lies on the curve $y = x \sin x$. Find the equation of the tangent to the curve at point P.

e) A spherical balloon is deflating so that its volume is decreasing at a constant rate of 12 cm^3 per hour. Find the rate at which the surface area of the balloon is decreasing when the volume of the balloon is 2 500 cm^3.

QUESTION TWO

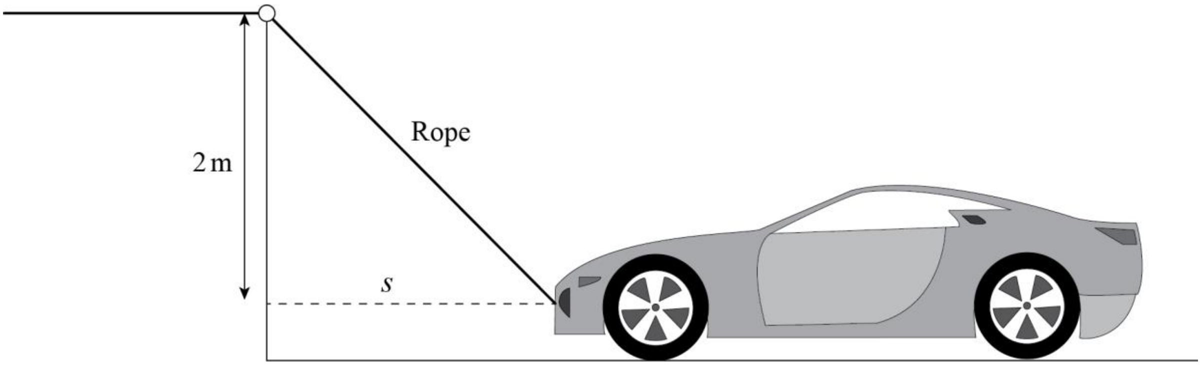
a) Differentiate the following: $y = e^{2x} + \frac{\pi}{x}$. You do not need to simplify your answer.

b) A car engine is started and is left to idle. The temperature of the water in the car's radiator increases. The temperature of this water is given by the equation: $T(m) = 20\sqrt{m} + 5\ln m, \quad 0 < m < 10$ where T is the temperature of the water (°C) and m is the number of minutes since the car was turned on. Find the rate at which the temperature of the water is changing 6 minutes after the car engine is turned on.

c) Differentiate the following function. You do not need to simplify your answer.

$$y = \frac{3x^2 + \cos x}{e^{4x}}$$

d) A tow truck is using a winch to move an illegally parked car. The winch pulls a rope that runs through a pulley and is tied to the front of the car. The pulley is 2m higher than the point on the car where the rope is attached. The rope is being pulled through the pulley at a rate of 6m per minute. The front wheels of the car do not leave the ground.



At what rate will the car be moving when there is 8m of rope between the car and the pulley?

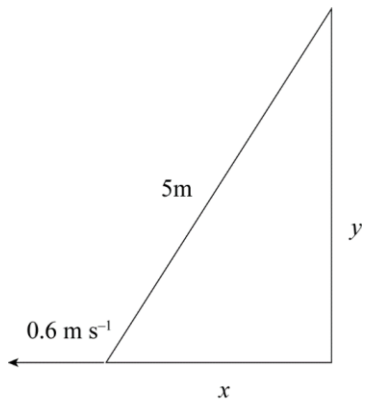
e) Find the values of t for which the function defined by $x = 2t^3 - 5$ and $y = 4t^2 - 5t$ is concave up.

QUESTION THREE

a) Differentiate the following: $f(x) = \sin(2x)\ln x$. You do not need to simplify your answer.

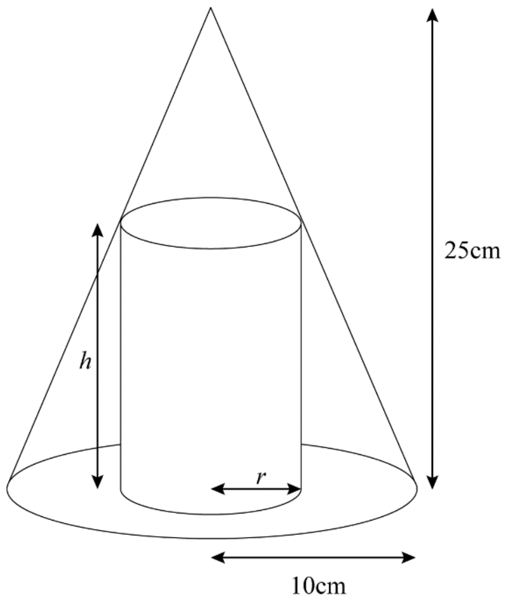
b) Find the equation of the normal to $y = \frac{1}{x+2}$ at the point $\left(2, \frac{1}{4}\right)$.

c) A 5 m long ladder is placed against a vertical wall. The bottom of the ladder slides out away from the wall, moving at 0.6 m per second. At what rate will the top of the ladder be moving downwards when it is 3 m above the ground?



d) Find the x value of the point on the curve $y^3 - 3x - e^{2x-1} = 0$ where the gradient is 7.

a. A cone has a height of 25 cm and a radius of 10 cm. A cylinder is inscribed in the cone. The base of the cylinder is in the same plane as the base of the cone. The cylinder base has the same centre as the base of the cone. Find the radius of the cylinder so that it has the maximum volume. You may assume that $\frac{d^2V}{dr^2} < 0$. Show any derivatives that you need to find when solving this problem.



Answers

Lesson One answers

Calculate the following limits, if possible.

Q1.

$$\lim_{x\to4}\frac{(x+5)(x-4)}{(x-4)}=\lim_{x\to4}x+5$$
$$=4+5$$
$$=9$$

Q2.

$$\lim_{x\to0}\frac{x(x+1)}{x}=\lim_{x\to0}x+1$$
$$=0+1$$
$$=1$$

Q3.

$$\lim_{x\to5}\frac{1}{5-x}$$

DNE as approaching from positive and negative direction gives different limits.

Q4.

$$\lim_{x\to\infty}\frac{1}{x}$$

Substituting $x = 1\,000\,000$

$$\frac{1}{1\,000\,000}\approx 0$$
$$\lim_{x\to\infty}\frac{1}{x}=0$$

Q5.

$$\lim_{x\to-2}\frac{x^2+2x}{x+2}=\lim_{x\to-2}\frac{x(x+2)}{x+2}$$
$$=\lim_{x\to-2}x$$
$$=-2$$

Q6.

$$\lim_{x\to3}\frac{2x^2-5x-3}{x-3}=\lim_{x\to3}\frac{2(x-3)(x+0.5)}{x-3}$$
$$=\lim_{x\to3}2(x+0.5)$$
$$=7$$

Q7.

$$\lim_{x\to4}\frac{x^2-16}{x-4}=\lim_{x\to4}\frac{(x-4)(x+4)}{x-4}$$
$$=\lim_{x\to4}x+4$$
$$=8$$

Q8.

$$\lim_{x\to-3}\frac{x^2+x-6}{x^2+5x+6}=\lim_{x\to-3}\frac{(x+3)(x-2)}{(x+3)(x+2)}$$
$$=\lim_{x\to-3}\frac{x-2}{x+2}$$
$$=\frac{-5}{-1}$$
$$=5$$

Q9.

$$\lim_{x\to0}\ln x$$

Because the function isn't defined for negative values, there is no limit as x approaches 0.

Q10.

$$\lim_{x\to\infty}\ln x$$

Substituting different large values of x gives:

$$\ln(1\,000)=6.9$$
$$\ln(1\,000\,000)=13.8$$
$$\ln(1\,000\,000\,000)=20.7$$

These values aren't approaching a limit so there is no limit.

Q11.

$$\lim_{x\to 3}\frac{(x+4)(x^2-6x+9)}{x^2-x-6}$$

$$=\lim_{x\to 3}\frac{(x+4)(x-3)(x-3)}{(x-3)(x+2)}$$

$$=\lim_{x\to 3}\frac{(x+4)(x-3)}{(x+2)}$$

$$=\frac{(3+4)(3-3)}{(3+2)}$$

$$=0$$

Q12.

$$\lim_{x\to 0}\frac{\cos x}{x}$$

Because the limit is different depending on which direction you approach from, there is no limit as x approaches 0.

Lesson Two answers

Differentiate the following functions from first principles.

Q1. $f(x) = ax^2$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{a(x+h)^2-ax^2}{h}$$

$$=\lim_{h\to 0}\frac{a(x^2+2xh+h^2)-ax^2}{h}$$

$$=\lim_{h\to 0}\frac{ax^2+2axh+ah^2-ax^2}{h}$$

$$=\lim_{h\to 0}\frac{2axh+ah^2}{h}$$

$$=\lim_{h\to 0}2ax+ah$$

$$=2ax$$

Q2. $f(x) = x^3$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^3-x^2}{h}$$

$$=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$

$$=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$$

$$=\lim_{h\to 0}3x^2+3xh+h^2$$

$$=3x^2$$

Q3. $f(x) = x^2+4x+6$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^2+4(x+h)+6-(x^2+4x+6)}{h}$$

$$=\lim_{h\to 0}\frac{x^2+2xh+h^2+4x+4h+6-x^2-4x-6}{h}$$

$$=\lim_{h\to 0}\frac{2xh+h^2+4h}{h}$$

$$=\lim_{h\to 0}2x+4+3h$$

$$=2x+4$$

Q4. $f(x) = 3x^2-5x-7$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{3(x+h)^2-5(x+h)-7-(3x^2-5x-7)}{h}$$

$$=\lim_{h\to 0}\frac{3x^2+6xh+h^2-5x-5h-7-3x^2+5x+7}{h}$$

$$=\lim_{h\to 0}\frac{6xh+h^2-5h}{h}$$

$$=\lim_{h\to 0}6x-5+h$$

$$=6x-5$$

Q5. $f(x) = (x-3)(x-5)$

$$f(x) = x^2 - 8x + 15$$
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{(x+h)^2 - 8(x+h) + 15 - (x^2 - 8x + 15)}{h}$$
$$= \lim_{h\to 0}\frac{x^2 + 2xh + h^2 - 8x - 8h + 15 - x^2 + 8x - 15}{h}$$
$$= \lim_{h\to 0}\frac{2xh + h^2 - 8h}{h}$$
$$= \lim_{h\to 0} 2x - 8 + h$$
$$= 2x - 8$$

Q6. $f(x) = (x+4)^2 - 3$

$$f(x) = x^2 + 8x + 13$$
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{(x+h)^2 + 8(x+h) + 13 - (x^2 + 8x + 13)}{h}$$
$$= \lim_{h\to 0}\frac{x^2 + 2xh + h^2 + 8x + 8h + 13 - x^2 - 8x - 13}{h}$$
$$= \lim_{h\to 0}\frac{2xh + h^2 + 8h}{h}$$
$$= \lim_{h\to 0} 2x + 8 + h$$
$$= 2x + 8$$

Q7. $f(x) = x^4$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{(x+h)^4 - x^4}{h}$$
$$= \lim_{h\to 0}\frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$
$$= \lim_{h\to 0}\frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$
$$= \lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2$$
$$= 4x^3$$

Q8. $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{\left(\frac{1}{x+h} - \frac{1}{x}\right)}{h}$$
$$= \lim_{h\to 0}\frac{\left(\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}\right)}{h}$$
$$= \lim_{h\to 0}\frac{\left(\frac{x-x-h}{x(x+h)}\right)}{h}$$
$$= \lim_{h\to 0}\frac{\left(\frac{-h}{x(x+h)}\right)}{h}$$
$$= \lim_{h\to 0}\frac{-h}{xh(x+h)}$$
$$= \lim_{h\to 0}\frac{-1}{x(x+h)}$$
$$= \frac{-1}{x^2}$$

Lesson Five answers

From the notes / examples

Differentiate $f(x) = \cos\left(\frac{1}{x}\right)$

$$f(x) = \cos\left(\frac{1}{x}\right)$$
$$f'(x) = -\sin\left(\frac{1}{x}\right) \times \frac{-1}{x^2}$$
$$= \frac{\sin\left(\frac{1}{x}\right)}{x^2}$$

Differentiate $f(x) = (x+2)^7$

$$f(x) = (x+2)^7$$

Q7.

$$f(x) = \cos(x^2)$$
$$f'(x) = -\sin(x^2)\cdot 2x$$
$$= -2x\sin(x^2)$$

Q8.

$$f(x) = (x^3 - x + 4)^{-\frac{1}{2}}$$
$$f'(x) = \frac{-1}{2}(x^3 - x + 4)^{-\frac{3}{2}}\cdot(3x^2 - 1)$$
$$= \frac{1 - 3x^2}{2\sqrt{(x^2 - x + 4)^3}}$$

Q9. Prove the chain rule using first principles.

$$f'(x) = 7(x+2)^6 \times 1$$
$$= 7(x+2)^6$$

See the lesson five notes.

Lesson Five questions

Differentiate the following functions using the chain rule.

Q1.

$$f(x) = (10x+3)^4$$
$$f'(x) = 4(10x+3)^3 \times 10$$
$$= 40(10x+3)^3$$

Q2.

$$f(x) = (3x^2 - 6x + 4)^5$$
$$f'(x) = 5(3x^2 - 6x + 4)^4 \times (6x - 6)$$
$$= 30(x-1)(3x^2 - 6x + 4)^4$$

Q3.

$$f(x) = \sin^2 x$$
$$f'(x) = 2 \sin x \cos x$$

Q4.

$$f(x) = \cos^2 x$$
$$f'(x) = 2 \cos x \cdot - \sin x$$
$$= -2 \sin x \cos x$$

Q5.

$$f(x) = \frac{1}{\sin x}$$
$$f'(x) = \frac{-1}{\sin^2 x} \times \cos x$$
$$= \frac{-\cos x}{\sin^2 x}$$

Q6.

$$f(x) = \sin(2x)$$
$$f'(x) = 2\cos(2x)$$

Lesson Six answers

From the notes / examples

Use the quotient rule to derive $\tan x$. You could check your formula sheet to see what the correct derivative is before you begin.

$$f(x) = \tan x$$
$$= \frac{\sin x}{\cos x}$$
$$f'(x) = \frac{f'g - g'f}{g^2}$$
$$= \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x}$$
$$= \sec^2 x$$

Use the quotient rule to derive:

$$f(x) \frac{x^3}{\cos x}$$

$$f'(x) = \frac{f'g - g'f}{g^2}$$
$$= \frac{3x^2 \cos x + \sin x \cdot x^3}{\cos^2 x}$$

Lesson Six answers

Q1.

$$f(x) = \frac{x^5}{x^2}$$
$$f(x) = x^3$$
$$f'(x) = 3x^2$$

Q2.

$$f(x) = \frac{\sqrt{x}}{\sin x}$$
$$f'(x) = \frac{f'g - g'f}{g^2}$$
$$= \frac{\frac{1}{2\sqrt{x}} \sin x - \cos x \cdot \sqrt{x}}{\sin^2 x}$$

Q3.

$$f(x) = \frac{-3x + 6}{x^2 - 4}$$
$$f'(x) = \frac{f'g - g'f}{g^2}$$
$$= \frac{-3(x^2 - 4) - (2x)(-3x + 6)}{(x^2 - 4)^2}$$

OR

$$f(x) = \frac{-3x + 6}{x^2 - 4}$$
$$= \frac{-3(x - 2)}{(x + 2)(x - 2)}$$
$$= \frac{-3}{x + 2}, x \neq 2$$
$$f'(x) = \frac{f'g - g'f}{g^2}$$
$$= \frac{3}{(x + 2)^2}, x \neq 2$$

Q4.

$$f(x) = \frac{x^2 + 5x + 6}{x^2 - 9}$$
$$f'(x) = \frac{f'g - g'f}{g^2}$$
$$= \frac{(2x + 5)(x^2 - 9) - 2x(x^2 + 5x + 6)}{(x^2 - 9)^2}$$

OR

$$f(x) = \frac{x^2 + 5x + 6}{x^2 - 9}$$
$$= \frac{(x + 2)(x + 3)}{(x + 3)(x - 3)}$$
$$= \frac{x + 2}{x - 3}, x \neq -3$$
$$f'(x) = \frac{x - 3 - (x + 2)}{(x - 3)^2}$$
$$= \frac{-5}{(x - 3)^2}, x \neq -3$$

Q5.

$$f(x)=\frac{\sin x}{x^3}$$
$$f'(x)=\frac{f'g-g'f}{g^2}$$
$$=\frac{\cos x\cdot x^3-3x^2\sin x}{x^6}$$
$$=\frac{x\cos x-3\sin x}{x^4}$$

Q6.

$$f(x)=\frac{\cos x}{\sin x}$$
$$f'(x)=\frac{f'g-g'f}{g^2}$$
$$=\frac{-\sin x\sin x-\cos x\cos x}{\sin^2 x}$$
$$=\frac{-(\sin^2 x+\cos^2 x)}{\sin^2 x}$$
$$=\frac{-1}{\sin^2 x}$$
$$=-\csc^2 x$$

Q7.

$$f(x)=-\cos x\cdot x^{-2}$$
$$f(x)=\frac{-\cos x}{x^2}$$
$$f'(x)=\frac{f'g-g'f}{g^2}$$
$$=\frac{\sin x\cdot x^2-2x\times-\cos x}{x^4}$$
$$=\frac{x\sin x+2\cos x}{x^3}$$

OR

$$f(x)=-\cos x\cdot x^{-2}$$
$$f'(x)=f'g+g'f$$
$$=\sin x\cdot x^{-2}+(x^{-2})'\times-\cos x$$
$$=\frac{\sin x}{x^2}-\cos x\times-2x^{-3}$$
$$=\frac{\sin x}{x^2}+\frac{2\cos x}{x^3}$$
$$=\frac{x\sin x}{x^3}+\frac{2\cos x}{x^3}$$
$$=\frac{x\sin x+2\cos x}{x^3}$$

Q8.

$$f(x)=x^{-\frac{1}{2}}\cdot\sin x$$
$$=\frac{\sin x}{\sqrt{x}}$$
$$f'(x)=\frac{f'g-g'f}{g^2}$$
$$=\frac{\cos x\cdot\sqrt{x}-\frac{1}{2\sqrt{x}}\sin x}{x}$$

Q9.

$$f(x)=\frac{(x+4)(2x-5)}{4\sqrt[3]{x^2}}$$
$$=\frac{2x^2+3x-20}{4x^{\frac{2}{3}}}$$
$$f'(x)=\frac{f'g-g'f}{g^2}$$
$$=\frac{(4x+3)\cdot 4x^{\frac{2}{3}}-\frac{8}{3}x^{-\frac{1}{3}}\cdot(2x^2+3x-20)}{4x^{\frac{4}{3}}}$$

Q10.

$$f(x)=\frac{x^2+2x-8}{(x^2-4)(x-3)}$$
$$=\frac{(x+4)(x-2)}{(x+2)(x-2)(x-3)}$$
$$=\frac{x+4}{(x+2)(x-3)},x\neq 2$$
$$=\frac{x+4}{x^2-x-6}$$
$$f'(x)=\frac{f'g-g'f}{g^2}$$
$$=\frac{x^2-x-6-(2x-1)(x+4)}{(x^2-x-6)^2}$$
$$=\frac{x^2-x-6-2x^2-7x+4}{(x^2-x-6)^2}$$
$$=\frac{-x^2-8x-2}{(x^2-x-6)^2},x\neq 2$$

For questions 11 and 12, see the lesson six notes.

Lesson Seven answers

From the notes/ examples

Explain what the growth rate of the pi creatures **over an entire day** would be if they tripled in mass each day, i.e. $M(t) = 3^t$

Their growth rate over an entire day would be twice their population at the start of the day, i.e. 2×3^t

Given that $e^{\ln 2} = 2$, or in other words, $e^{0.6931\ldots} = 2$, rewrite 2^x in terms of e^x

$$\begin{aligned} 2^x &= \left(e^{\ln 2}\right)^x \\ &= e^{\ln 2x} \end{aligned}$$

Similarly, rewrite 3^x and 4^x in terms of e^x

$$3^x = e^{\ln 3x}$$
$$4^x = e^{\ln 4x}$$

Why do we say that e is the "natural" base to use for continuous growth?

Something along the lines of: "Because e is the base which, when raised to a power, has a growth rate of exactly its size at that point, i.e. the 'proportionality constant' is 1."

And that: "Every exponential is a scaled version of e^x*"*

Lesson Seven questions

Lesson Eight answers

From the notes

Differentiate $f(x) = \ln x$

$$f'(x) = \frac{1}{x}$$

Differentiate $f(x) = \ln(2x)$

$$f'(x) = \frac{1}{2x} \times 2$$
$$= \frac{1}{x}$$

Differentiate $f(x) = 3\ln x$

$$f'(x) = 3 \times \frac{1}{x}$$
$$= \frac{3}{x}$$

Differentiate $f(x) = \ln(x^2)$

$$f'(x) = \frac{1}{x^2} \times x$$
$$= \frac{x}{x^2}$$
$$= \frac{1}{x}$$

Lesson Eight questions

Q1. $f(x) = \ln(4x)$

$$f'(x) = \frac{1}{4x} \times 4$$
$$= \frac{1}{x}$$

Q2. Q2. $f(x) = \ln(9x - 6)$

$$f'(x) = \frac{1}{9x - 6} \times 9$$
$$= \frac{9}{9x - 6}$$

Q3. $f(x) = 4\ln x$

$$f'(x) = \frac{4}{x}$$

Q4. $f(x) = 9\ln x$

$$f'(x) = \frac{9}{x}$$

Q5. $f(x) = \ln(x^2 - 4)$

$$f'(x) = \frac{1}{x^2 - 4} \times 2x$$
$$= \frac{2x}{x^2 - 4}$$

Q6. $f(x) = \ln(5x^2)$

$$f'(x) = \frac{1}{5x^2} \times 5x$$
$$= \frac{5x}{5x^2}$$
$$= \frac{1}{x}$$

Q7.

$$f(x) = \frac{\ln x}{4x^3}$$

$$f'(x) = \frac{f'g - g'f}{g^2}$$
$$= \frac{\frac{1}{x} \times 4x^3 - 12x^2 \ln x}{16x^6}$$

This is sufficient but you may choose to simplify your answer:

$$f'(x) = \frac{4x^2 - 12x^2 \ln x}{16x^6}$$
$$= \frac{4 - 12\ln x}{16x^4}$$
$$= \frac{1 - 3\ln x}{4x^4}$$

Q8.

$$f(x) = \sin x \ln x$$

$$f'(x) = f'g + g'f$$
$$= \cos x \ln x + \frac{1}{x}\sin x$$
$$= \cos x \ln x + \frac{\sin x}{x}$$

Q9.

$$f(x) = \ln(e^x)$$

$$f(x) = x$$
$$f'(x) = 1$$

This is the ideal method for finding the solution. However, you could also show it like this:

$$f'(x) = \frac{1}{e^x} \cdot e^x$$
$$= \frac{e^x}{e^x}$$
$$= 1$$

Q10.

$$f(x) = \ln\left(e^{x^2}\right)$$

$$f(x) = x^2$$
$$f'(x) = 2x$$

Again, this is the ideal method for finding the solution. However, you could also show it like this:

$$f'(x) = \frac{1}{e^{x^2}} \cdot e^{x^2} \cdot 2x$$
$$= \frac{e^{x^2}}{e^{x^2}} \cdot 2x$$
$$= 2x$$

Q11.

$$f(x) = (4x + 2)\ln(2x)$$

$$f'(x) = f'g + g'f$$
$$= 4\ln(2x) + \frac{1}{x}(4x + 2)$$

Q12.

$$f(x) = \ln\left(\sqrt{x}\right)$$

$$f'(x) = \frac{1}{\sqrt{x}} \times \frac{1}{2\sqrt{x}}$$
$$= \frac{1}{2x}$$

Q13.

$$f(x) = \ln\left(\frac{1}{x}\right)$$

$$f'(x) = \frac{1}{\left(\frac{1}{x}\right)} \cdot \frac{-1}{x^2}$$
$$= x \cdot \frac{-1}{x^2}$$
$$= \frac{-1}{x}$$

Q14.

$$f(x) = (3x^2 - 4x + 2)\ln(\sin x)$$

$$(fg)' = f'g + g'f$$
$$f'(x) = (6x - 4)\ln(\sin x) + \frac{\cos x}{\sin x}(3x^2 - 4x + 2)$$

Q15.

$$f(x) = \frac{\ln(\cos x)}{x^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$
$$f'(x) = \frac{\frac{-\sin x}{\cos x} \cdot x^2 - 2x \cdot \ln(\cos x)}{x^4}$$

Q16.

$$f(x) = \ln(\ln x)$$

$$f'(x) = \frac{1}{\ln x} \cdot \frac{1}{x}$$
$$= \frac{1}{x \ln x}$$

Lesson Nine answers

Lesson Nine questions

Q1. Use $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ to prove that the derivative of $\cos x$ is $-\sin x$.

$$\begin{aligned} f(x) &= \cos x \\ &= \sin\left(x + \frac{\pi}{2}\right) \\ f'(x) &= \cos\left(x + \frac{\pi}{2}\right) \times 1 \\ &= -\sin x \times 1 \\ &= -\sin x \end{aligned}$$

Q2. Prove the derivative of $\csc x$ is $-\csc x \cot x$ by rewriting $\csc x$ as $(\sin x)^{-1}$.

$$\begin{aligned} f(x) &= \csc x \\ &= \frac{1}{\sin x} \\ &= (\sin x)^{-1} \\ f'(x) &= \frac{-1}{\sin^2 x} \cdot \cos x \\ &= \frac{-\cos x}{\sin^2 x} \\ &= -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \\ &= -\csc x \cot x \end{aligned}$$

Q3. Similarly, prove that the derivative of $\sec x$ is $\sec x \tan x$.

$$\begin{aligned} f(x) &= \sec x \\ &= (\cos x)^{-1} \\ f'(x) &= \frac{-1}{\cos^2 x} \cdot -\sin x \\ &= \frac{\sin x}{\cos^2 x} \\ &= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \\ &= \tan x \sec x \end{aligned}$$

Q4. Use the quotient rule to prove that the derivative of $\cot x$ is $-\csc^2 x$.

$$\begin{aligned} f(x) &= \cot x \\ &= \frac{\cos x}{\sin x} \\ f'(x) &= \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} \\ &= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} \\ &= -\frac{1}{\sin^2 x} \\ &= -\csc^2 x \end{aligned}$$

Differentiate these functions. Have the formula sheet handy.

Q5.

$$f(x) = 2\cos 5x$$

$$f'(x) = -10\sin 5x$$

Q6.

$$f(x) = -3\tan\left(\frac{x}{2}\right)$$

$$f'(x) = -\frac{3}{2}\sec^2\left(\frac{x}{2}\right)$$

Q7.

$$f(x) = 4\sin\sqrt{x}$$

$$\begin{aligned} f'(x) &= 4\cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{2\cos\sqrt{x}}{\sqrt{x}} \end{aligned}$$

Q8.

$$f(x) = \csc 4x$$

$$f'(x) = -\csc 4x \cot 4x$$

Q9.

$$f(x) = \ln(\sin x)$$

$$\begin{aligned} f'(x) &= \frac{1}{\sin x} \cdot \cos x \\ &= \frac{\cos x}{\sin x} \\ &= \cot x \end{aligned}$$

Q10.

$$f(x) = \sin(\ln x)$$

$$f'(x) = \cos(\ln x) \cdot \frac{1}{x}$$
$$= \frac{\cos(\ln x)}{x}$$

Q11.

$$f(x) = 6\sin^2 x$$

$$f'(x) = 12\sin x \cdot \cos x$$

Q12.

$$f(x) = \csc^2 x$$

$$f'(x) = 2\csc x \cdot -\csc x \cot x$$
$$= -2\csc^2 x \cot x$$

Q13.

$$f(x) = x^2 \tan(x^2)$$

$$f'(x) = 2x\tan(x^2) + \sec^2(x^2) \cdot 2x \cdot x^2$$
$$= 2x\tan(x^2) + 2x^3\sec^2(x^2)$$

Q14.

$$f(x) = \cos\left(\frac{2}{x}\right)$$

$$f'(x) = -\sin\left(\frac{2}{x}\right) \cdot -\frac{2}{x^2}$$
$$= \frac{2\sin\left(\frac{2}{x}\right)}{x^2}$$

Q15.

$$f(x) = \frac{x^3}{\sin x}$$

$$f'(x) = \frac{3x^2\sin x - x^3\cos x}{\sin^2 x}$$

Q16.

$$f(x) = \sqrt{\sin x}$$

$$f'(x) = \frac{1}{2\sqrt{\sin x}} \cdot \cos x$$
$$= \frac{\cos x}{2\sqrt{\sin x}}$$

Q17.

$$f(x) = e^{\sin x}$$

$$f'(x) = e^{\sin x} \cdot \cos x$$

Q18.

$$f(x) = \ln\left(\sin\left(\sqrt{x}\right)\right)$$

$$f'(x) = \frac{1}{\sin\left(\sqrt{x}\right)} \cdot \cos\left(\sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}}$$
$$= \frac{\cos\left(\sqrt{x}\right)}{2\sqrt{x}\sin\left(\sqrt{x}\right)}$$

Q19.

$$f(x) = \frac{(3x-4)(2x+5)}{\cot 3x}$$

$$f(x) = \frac{6x^2 + 7x - 20}{\cot 3x}$$
$$f'(x) = \frac{(12x+7)\cot 3x + 3\csc^2 3x\,(6x^2 + 7x - 20)}{\cot^2 3x}$$

Q20.

$$f(x) = \cos\left(e^{x^2}\right)\sin x$$

$$f'(x) = -\sin\left(e^{x^2}\right) \cdot e^{x^2} \cdot 2x \cdot \sin x + \cos x \cos\left(e^{x^2}\right)$$

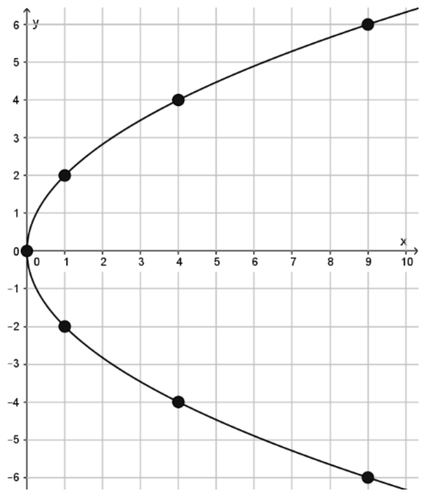
Lesson Ten: Parametric differentiation answers

From the notes

Complete the table and sketch the graph for the following parametric equation:

$$x = t^2$$
$$y = 2t$$

t	x	y
-3	9	-6
-2	4	-4
-1	1	-2
0	0	0
1	1	2
2	4	4
3	9	6



Lesson Ten questions

Q1.

$$x = t^2$$
$$y = 2t$$

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$
$$= 2 \cdot \frac{1}{2t}$$
$$= \frac{1}{t}$$

Q2.

$$x = \cos t$$
$$y = \sin t$$

$$\frac{dx}{dt} = -\sin t \qquad \frac{dy}{dt} = \cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$
$$= \cos t \cdot \frac{-1}{\sin t}$$
$$= -\cot t$$

Q3.

$$x = \frac{1}{\cos t}$$
$$y = (t+1)(2t-5)$$

$$x = \sec t \qquad y = 2t^2 - 3t - 5$$
$$\frac{dx}{dt} = \sec t \tan t \qquad \frac{dy}{dt} = 4t - 3$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$
$$= (4t-3) \cdot \frac{1}{\sec t \tan t}$$
$$= \frac{4t-3}{\sec t \tan t}$$

Q4.

$$x = e^{t^2+4t}$$
$$y = t^{\frac{3}{4}}$$

$$\frac{dx}{dt} = (2t+4)e^{t^2+4t} \qquad \frac{dy}{dt} = \frac{3}{4}t^{-\frac{1}{4}}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$
$$= \frac{3}{4\sqrt[4]{t}} \cdot \frac{1}{(2t+4)e^{t^2+4t}}$$
$$= \frac{3}{4\sqrt[4]{t} \cdot (2t+4) \cdot e^{t^2+4t}}$$

Q5.

Lesson Eleven: Implicit differentiation answers

From the notes
Differentiate $x^2 + y^2 = 25$ using the method shown in the video.

$$2x \cdot dx + 2y \cdot dy = 0$$
$$2x + 2y \cdot \frac{dy}{dx} = 0$$
$$2y \cdot \frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = \frac{-2x}{2y}$$
$$= -\frac{x}{y}$$

Differentiate $x^2 + y^2 = 25$ using the chain rule method explained above.

$$2x + 2y \cdot \frac{dy}{dx} = 0$$
$$2y \cdot \frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = \frac{-2x}{2y}$$
$$= -\frac{x}{y}$$

Lesson Eleven answers
Q1. Use implicit differentiation to show that the derivative of $\ln x$ is $\frac{1}{x}$.

$$y = \ln x$$
$$e^y = x$$
$$\frac{dy}{dx} \cdot e^y = 1$$
$$\frac{dy}{dx} = \frac{1}{e^y}$$
$$= \frac{1}{x}$$

Q2. Find the gradient of the following function at the given point: $y^2 - 2x - 1 = 0$ at $(-5, 3)$

$$y^2 - 2x - 1 = 0$$
$$2y \cdot \frac{dy}{dx} - 2 = 0$$
$$2y \cdot \frac{dy}{dx} = 2$$
$$\frac{dy}{dx} = \frac{2}{2y}$$
$$= \frac{1}{y}$$
$$= \frac{1}{3}$$

Q3. Find the gradient of the following function at the given point: $xy + y^2 = 8$ at $(2, 2)$

$$y + x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = 0$$
$$x\frac{dy}{dx} + 2y\frac{dy}{dx} = -y$$
$$\frac{dy}{dx}(x + 2y) = -y$$
$$\frac{dy}{dx} = \frac{-y}{x + 2y}$$
$$= \frac{-2}{2 + 4}$$
$$= -\frac{1}{3}$$

Q4. Find the gradient of the following function at the given point: $4x^2 - 3y^2 = 10$ at $(3, 2.94)$

$$8x - 6y\frac{dy}{dx} = 0$$
$$-6y\frac{dy}{dx} = -8x$$
$$\frac{dy}{dx} = \frac{-8x}{-6y}$$
$$= \frac{4x}{3y}$$
$$= \frac{4 \times 3}{3 \times 2.94}$$
$$= 1.36 \text{ (2 d.p.)}$$

Differentiate the following relationships.
Q5. $x^2 + 4xy + 3y^2 = 0$

Q9. $\ln y - x^3 - 4x = 2$

$$2x + 4y + 4x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$$
$$4x\frac{dy}{dx} + 6y\frac{dy}{dx} = -2x - 4y$$
$$\frac{dy}{dx}(4x + 6y) = -2x - 4y$$
$$\frac{dy}{dx} = \frac{-2x - 4y}{4x + 6y}$$
$$= -\frac{x + 2y}{2x + 3y}$$

Q6. $xy^2 = x^3 + y^3$

$$y^2 + 2xy\frac{dy}{dx} = 3x^2 + 3y^2\frac{dy}{dx}$$
$$2xy\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 3x^2 - y^2$$
$$\frac{dy}{dx}(2xy - 3y^2) = 3x^2 - y^2$$
$$\frac{dy}{dx} = \frac{3x^2 - y^2}{2xy - 3y^2}$$

Q7. $x^3 - y^3 = -2xy - 1$

$$3x^2 - 3y^2\frac{dy}{dx} = -2\left(y + x\frac{dy}{dx}\right)$$
$$3x^2 - 3y^2\frac{dy}{dx} = -2y - 2x\frac{dy}{dx}$$
$$2x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = -3x^2 - 2y$$
$$\frac{dy}{dx}(2x - 3y^2) = -3x^2 - 2y$$
$$\frac{dy}{dx} = \frac{-3x^2 - 2y}{2x - 3y^2}$$

Q8. $e^y + x^2 = 1$

$$e^y\frac{dy}{dx} + 2x = 0$$
$$e^y\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = -\frac{2x}{e^y}$$

Q10. $\ln y + x^2y = 2$

Lesson Eleven questions

Lesson Twelve answers

Find the equation of the tangent or normal (as specified) at the given point

Q1. Tangent at $x = 3$ $y = x^2 - 5$ $\frac{dy}{dx} = 2x$ Q4. Normal at $x = 4$	

$y = x^6 + \frac{1}{2}x^3 - \frac{1}{x} + \frac{3}{x^2}$ $\frac{dy}{dx} = 6x^5 + \frac{3x^2}{2} + \frac{1}{x^2} - \frac{6}{x^3}$ $\frac{dy}{d(4)} = 6 \times 4^5 + \frac{3 \times 4^2}{2} + \frac{1}{4^2} - \frac{6}{4^3}$ $= 6168\ (0\ \mathrm{d.p.})$ $\mathrm{Normal} = \frac{1}{6168}$ $y = \frac{x}{6168} + c$ $4128 = \frac{4}{6168} + c$ $c = 4128$ $y = \frac{x}{6168} + 4128$	

Q2. Tangent at $(1, 5)$

$$y = 2x^3 + 6x^2 - 7x + 4$$

Q3. Normal at $(-2, -14)$

$$y = (x + 4)(x - 5)$$

Q5. Tangent at $x = -1$

$$f(x) = x^3 \sin x$$

Q6. Normal at $x = 2$

$$f(x) = \ln(x^2)$$

Q7. Normal at $x = 0$

$$y = \sin x \cos x$$

Q8. Tangent at $x = 2.5$

	$f(x) = \frac{e^{\frac{x}{2}}}{\sqrt{x}}$

Lesson 17: Related rates of change answers

<table>
<tr>
<td>
Q1.
We want to find the rate at which the level of the water is falling, e.g. $\frac{dx}{dt}$.

The related rates equation looks like:
$\frac{dx}{dt} = \frac{dx}{\quad} \times \frac{\quad}{dt}$

We already know that:
$\frac{dV}{dt} = -5\text{cm}^3/\text{minute}$

Therefore:
$\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt}$
$= \frac{dx}{dV} \times -5$

We need an equation linking x and V:
$V = \frac{1}{3}\pi r^2 h$
$= \frac{1}{3}\pi \left(\frac{2x}{3}\right)^2 x$
$= \frac{4\pi x^3}{27}$

Differentiating:
$\frac{dV}{dx} = \frac{4\pi x^2}{9}$
$\frac{dx}{dV} = \frac{9}{4\pi x^2}$

Substituting:
$\frac{dx}{dt} = \frac{dx}{dV} \times -5$
$= \frac{9}{4\pi x^2} \times -5$
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Q2.
We want to find the rate at which the area of the ink-blot is increasing, e.g. $\frac{dA}{dt}$

The related rates equation looks like:
$\frac{dA}{dt} = \frac{dA}{\quad} \times \frac{\quad}{dt}$

We already know that:
$\frac{dr}{dt} = 0.3\text{mm/s}$

Therefore:
$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$
$= \frac{dA}{dr} \times 0.3$

We need an equation linking A and r:
$A = \pi r^2$

Differentiating:
$\frac{dA}{dr} = 2\pi r$

Substituting:
$\frac{dA}{dt} = \frac{dA}{dr} \times 0.3$
$= 2\pi r \times 0.3$
$= 0.6\pi r$

a)
$\frac{dA}{d(5)} = 0.6\pi \times 5$
$= 3\pi$
$= 9.4\text{mm}^2/\text{s (2 s.f.)}$

b)
When $t = 3, r = 0.9$
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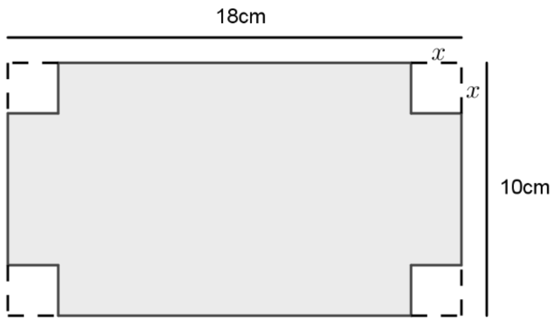
$= \frac{-45}{4\pi x^2}$ $= \frac{-45}{4\pi(2)^2}$ $= -0.895\text{cm/min (3 s.f.)}$	$\frac{dA}{d(0.9)} = 0.6\pi \times 0.9$ $= 0.54\pi$ $= 1.7\text{mm/s (2 s.f.)}$

Lesson Eighteen answers

Q1. A rectangular sheet of cardboard is to have four equal square corners removed and the sides turning up to form an open box. If the cardboard measures 18cm x 10cm, find the maximum volume of the box.

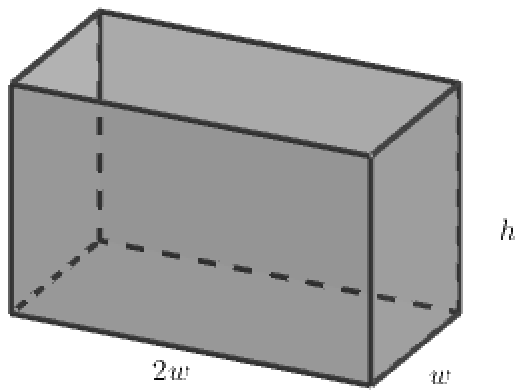


$$\begin{aligned} Volume &= wlh \\ &= (18-2x)(10-2x)(x) \\ &= (180-36x-20x+4x^2)(x) \\ &= 4x^3-56x^2+180x \\ Volume' &= 12x^2-112x+180 \\ 0 &= 12x^2-112x+180 \\ x &= 2.06 \text{ or } 7.27 \text{ (3 s.f.)} \end{aligned}$$

Clearly $x = 7.27$ is nonsense as x can't be bigger than 5cm (otherwise the width would be negative). So the maximum volume occurs when $x = 2.06$ and the volume is:

$$\begin{aligned} Volume &= 4(2.06)^3 - 56(2.06)^2 + 180(2.06) \\ &= 168.1\text{cm}^3 \end{aligned}$$

Q2. A farmer needs to have a water storage container in order to fight forest fires. The water storage container is a cuboid, has an open top, and is to have a volume of 20 000 litres (20m^3). The length of its base is twice the width. Construction of the base costs \$75 / m^2. Construction of the sides costs \$40 / m^2. Find the minimum cost of materials.



$$\begin{aligned} Cost &= \$75 \times (2w^2) + \$40 \times \big(2(2wh) + 2(wh)\big) \\ &= 150w^2 + \$40(4wh + 2wh) \\ &= 150w^2 + 240wh \end{aligned}$$

$$\begin{aligned} Volume &= 2w^2h \\ 20 &= 2w^2h \\ h &= \frac{10}{w^2} \end{aligned}$$

$$\begin{aligned} Cost &= 150w^2 + 240wh \\ &= 150w^2 + 240w\left(\frac{10}{w^2}\right) \\ &= 150w^2 + \frac{2\,400}{w} \\ Cost' &= 300w - \frac{2\,400}{w^2} \\ 0 &= 300w - \frac{2\,400}{w^2} \\ 0 &= 300w^3 - 2\,400 \\ 0 &= w^3 - 8 \\ w &= 2\text{m} \end{aligned}$$

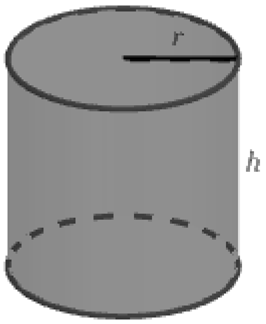
Therefore the width is 2m, the length is 4m, and the height is:

$$\begin{aligned} h &= \frac{10}{2^2} \\ &= 2.5\text{m} \end{aligned}$$

And the cost of materials is:

$$\begin{aligned} Cost &= 150 \times 2^2 + \frac{2\,400}{2} \\ &= \$1\,800 \end{aligned}$$

Q3. A 250L cylindrical hot-water tank is to be designed to minimise heat loss which is proportional to surface area. Calculate the dimensions of the optimum water tank so that the surface area is minimised.



$$Surface\ area = 2\pi r^2 + 2\pi rh$$

$$Volume = \pi r^2 h$$
$$0.25\text{m}^3 = \pi r^2 h$$
$$h = \frac{0.25}{\pi r^2}$$

$$Surface\ area = 2\pi r^2 + 2\pi r\left(\frac{0.25}{\pi r^2}\right)$$
$$= 2\pi r^2 + \frac{1}{2r}$$
$$SA' = 4\pi r - \frac{1}{2r^2}$$
$$0 = 4\pi r - \frac{1}{2r^2}$$
$$0 = 4\pi r^3 - 0.5$$
$$r^3 = \frac{1}{8\pi}$$
$$r = 0.341\text{m (3 s.f.)}$$

$$h = \frac{0.25}{\pi \times 0.341^2}$$
$$= 0.683\text{m (3 s.f.)}$$

Q4. The vertical height of a model aeroplane is described by the equation:

$$h = 15 + 12t - 3t^2$$

where t is in seconds and h is in metres.

a) Find the maximum height of the plane
b) Find the vertical velocity when the plane hits the ground

a)
$$h' = 12 - 6t$$
$$0 = 12 - 6t$$
$$6t = 12$$
$$t = 2$$

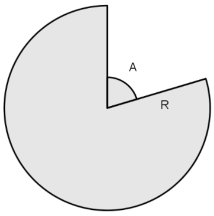
$$h = 15 + 12 \times 2 - 3 \times 2^2$$
$$= 27\text{m}$$

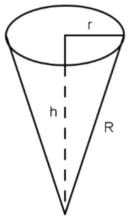
b)
$$h = 15 + 12t - 3t^2$$
$$0 = 15 + 12t - 3t^2$$
$$t = 5\ or - 1$$
$$t = 5$$

$$h'(5) = 12 - 6 \times 5$$
$$= -18$$

The vertical velocity is 18m/s down.

Q5. A circle can be made into a cone by cutting out a sector and joining the cut edges. Find the maximum volume of such a cone and the angle A that should be removed to make this optimum cup.





$$Volume = \frac{1}{3}\pi r^2 h$$

$$R^2 = r^2 + h^2$$
$$r^2 = R^2 - h^2$$

$$\begin{aligned} Volume &= \frac{1}{3}\pi(R^2 - h^2)h \\ &= \frac{1}{3}\pi R^2 h - \frac{1}{3}\pi h^3 \\ Volume' &= \frac{1}{3}\pi R^2 - \pi h^2 \\ 0 &= \frac{1}{3}\pi R^2 - \pi h^2 \\ \pi h^2 &= \frac{1}{3}\pi R^2 \\ h^2 &= \frac{R^2}{3} \\ h &= \frac{R}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} r^2 &= R^2 - h^2 \\ &= R^2 - \frac{R^2}{3} \\ &= \frac{2R^2}{3} \\ r &= \frac{\sqrt{2}R}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} 2\pi r &= (2\pi - A)R \\ 2\pi\frac{\sqrt{2}R}{\sqrt{3}} &= (2\pi - A)R \\ 2\pi\frac{\sqrt{2}}{\sqrt{3}} &= 2\pi - A \\ A &= 2\pi - 2\pi\frac{\sqrt{2}}{\sqrt{3}} \\ &= 2\pi\left(1 - \frac{\sqrt{2}}{\sqrt{3}}\right) \\ &= 1.15 \text{ radians } (3 \text{ s.f.}) \end{aligned}$$

$$\begin{aligned} Volume_{max} &= \frac{1}{3}\pi R^2 h - \frac{1}{3}\pi h^3 \\ &= \frac{1}{3}\pi R^2\left(\frac{R}{\sqrt{3}}\right) - \frac{1}{3}\pi\left(\frac{R}{\sqrt{3}}\right)^3 \\ &= \frac{\pi}{3}\left(\frac{R^3}{\sqrt{3}} - \frac{R^3}{\sqrt{3}^3}\right) \\ &= \frac{\pi}{3\sqrt{3}}R^3\left(1 - \frac{1}{\sqrt{3}^2}\right) \\ &= \frac{\pi}{3\sqrt{3}}R^3\left(1 - \frac{1}{3}\right) \\ &= \frac{2\pi}{9\sqrt{3}}R^3 \end{aligned}$$

Q6. A triathlete is running up a river bank and needs to swim across the 32m wide river in order to finish. The athlete can run at 6.0m/s and swim at 1.8m/s. At which point should he enter the water to minimise the time taken to complete the race?

Practice exam answers

Question One

a) Differentiate: $y = \sqrt{x^3 - 2x}$. You do not need to simplify your answer.

$$y = (x^3 - 2x)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2}(x^3 - 2x)^{-\frac{1}{2}}(3x^2 - 2)$$

b) Find the equation of the tangent to the curve $y = \tan x$ at the point $(\frac{\pi}{4}, 1)$. Show any derivatives that you need to find when solving this problem.

$$\frac{dy}{dx} = \sec^2 x$$
$$\frac{dy}{d\left(\frac{\pi}{4}\right)} = \sec^2\left(\frac{\pi}{4}\right)$$
$$= 2$$

$y - y_1 = m(x - x_1)$	$y = mx + c$
$y - 1 = 2\left(x - \frac{\pi}{4}\right)$	$1 = 2 \times \frac{\pi}{4} + c$
$y = 2x - \frac{\pi}{2} + 1$	$c = 1 - \frac{\pi}{2}$
	$y = 2x - \frac{\pi}{2} + 1$

c) A curve is defined by the parameters $x = 4e^{0.5t}$ and $y = t^2 - 4t$. Find the gradient of the tangent to the curve at t = 6. Show any derivatives that you need to find when solving this problem.

$$\frac{dx}{dt} = 2e^{0.5t} \qquad \frac{dy}{dt} = 2t - 4$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$
$$= (2t - 4) \cdot \frac{1}{2e^{0.5t}}$$
$$= \frac{2t - 4}{2e^{0.5t}}$$
$$= \frac{2(6) - 4}{2e^{0.5(6)}}$$
$$= 0.199 \text{ (3 s.f.)}$$

d) The point P $(\pi, 0)$ lies on the curve $y = x \sin x$. Find the equation of the tangent to the curve at point P.

$$\frac{dy}{dx} = \sin x + x \cos x$$
$$\frac{dy}{d(\pi)} = \sin \pi + \pi \cos \pi$$
$$= 0 + \pi \times -1$$
$$= -\pi$$

$y - y_1 = m(x - x_1)$	$y = mx + c$
$y - 0 = -\pi(x - \pi)$	$0 = -\pi \times \pi + c$
$y = -\pi x + \pi^2$	$c = \pi^2$
	$y = -\pi x + \pi^2$

e) A spherical balloon is deflating so that its volume is decreasing at a constant rate of 12 cm³ per hour. Find the rate at which the surface area of the balloon is decreasing when the volume of the balloon is 2 500 cm³. Show any derivatives that you need to find when solving this problem.

Setting up the related rates:

$$\frac{dSA}{dt} = \frac{dSA}{} \cdot \frac{}{dt}$$

$$\frac{dV}{dt} = -12$$

$$\frac{dSA}{dt} = \frac{dSA}{dV} \cdot \frac{dV}{dt}$$

Finding an equation that links SA and V. There are two options (the right is preferable):

$SA = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$
$r^2 = \frac{SA}{4\pi}$	$r^3 = \frac{3V}{4\pi}$
$r = \left(\frac{SA}{4\pi}\right)^{\frac{1}{2}}$	$r = \sqrt[3]{\frac{3V}{4\pi}}$
$V = \frac{4}{3}\pi r^3$	$SA = 4\pi r^2$
$V = \frac{4}{3}\pi \cdot \left(\frac{SA}{4\pi}\right)^{\frac{3}{2}}$	$SA = 4\pi\left(\frac{3V}{4\pi}\right)^{\frac{2}{3}}$

Differentiating:

$$V = \frac{4}{3}\pi \cdot \left(\frac{SA}{4\pi}\right)^{\frac{3}{2}}$$

$$\frac{dV}{dSA} = \frac{4\pi}{3} \cdot \frac{3}{2}\left(\frac{SA}{4\pi}\right)^{\frac{1}{2}} \cdot \frac{1}{4\pi}$$

$$= \frac{1}{2}\left(\frac{SA}{4\pi}\right)^{\frac{1}{2}}$$

$$\frac{dSA}{dV} = 2\left(\frac{4\pi}{SA}\right)^{\frac{1}{2}}$$

$$SA = 4\pi\left(\frac{3V}{4\pi}\right)^{\frac{2}{3}}$$

$$\frac{dSA}{dV} = \frac{8\pi}{3}\left(\frac{3V}{4\pi}\right)^{-\frac{1}{3}} \cdot \frac{3}{4\pi}$$

$$= 2\left(\frac{3V}{4\pi}\right)^{-\frac{1}{3}}$$

Calculating the rate:

$$\frac{dSA}{dt} = \frac{dSA}{dV} \cdot \frac{dV}{dt}$$

$$= 2\left(\frac{4\pi}{SA}\right)^{\frac{1}{2}} \cdot -12$$

$$= -24\left(\frac{4\pi}{SA}\right)^{\frac{1}{2}}$$

$$\frac{dSA}{dt} = \frac{dSA}{dV} \cdot \frac{dV}{dt}$$

$$= 2\left(\frac{3V}{4\pi}\right)^{-\frac{1}{3}} \cdot -12$$

$$= -24\left(\frac{3V}{4\pi}\right)^{-\frac{1}{3}}$$

When the volume is 2 500:

$$V = \frac{4}{3}\pi \cdot \left(\frac{SA}{4\pi}\right)^{\frac{3}{2}}$$

$$2\,500 = \frac{4\pi}{3}\left(\frac{SA}{4\pi}\right)^{\frac{3}{2}}$$

$$\frac{1875}{\pi} = \left(\frac{SA}{4\pi}\right)^{\frac{3}{2}}$$

$$\left(\frac{1875}{\pi}\right)^{\frac{2}{3}} = \frac{SA}{4\pi}$$

$$SA = 4\pi\left(\frac{1875}{\pi}\right)^{\frac{2}{3}}$$

$$SA = 890.794\text{cm}^2$$

$$\frac{dSA}{dt} = -24\left(\frac{4\pi}{SA}\right)^{\frac{1}{2}}$$

$$= -24\left(\frac{4\pi}{890.794}\right)^{\frac{1}{2}}$$

$$= -2.85\text{cm}^2\ (3\text{ s.f.})$$

$$\frac{dSA}{dt} = -24\left(\frac{3 \times 2\,500}{4\pi}\right)^{-\frac{1}{3}}$$

$$= -2.85\text{cm}^2\ (3\text{ s.f.})$$

Question Two

a) Differentiate the following: $y = e^{2x} + \frac{\pi}{x}$. You do not need to simplify your answer.

$$y = e^{2x} + \pi x^{-1}$$

$$\frac{dy}{dx} = e^{2x} - \pi x^{-2}$$

$$\frac{dy}{dx} = 2e^{2x} - \frac{\pi}{x^2}$$

b) A car engine is started and is left to idle. The temperature of the water in the car's radiator increases. The temperature of this water is given by the equation: $T(m) = 20\sqrt{m} + 5\ln m,\ \ 0 < m < 10,$ where T is the temperature of the water (°C) and m is the number of minutes since the car was turned on. Find the rate at which the temperature of the water is changing 6 minutes after the car engine is turned on. Show any derivatives that you need to find when solving this problem.

$$T(m) = 20m^{\frac{1}{2}} + 5\ln m$$

$$T'(m) = 10m^{-\frac{1}{2}} + \frac{5}{m}$$

$$= \frac{10}{\sqrt{m}} + \frac{5}{m}$$

$$= \frac{10}{\sqrt{6}} + \frac{5}{6}$$

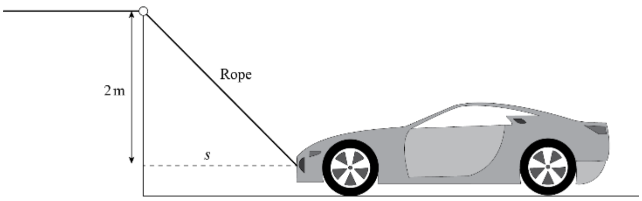
$$= 4.92°\text{C}/\min(3\text{ s.f.})$$

c) Differentiate the following: $y = \frac{3x^2+\cos x}{e^{4x}}$. You do not need to simplify your answer.

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

$$\frac{dy}{dx} = \frac{(6x - \sin x)e^{4x} - 4e^{4x}(3x^2 + \cos x)}{e^{8x}}$$

d) A tow truck is using a winch to move an illegally parked car. The winch pulls a rope that runs through a pulley and is tied to the front of the car. The pulley is 2m higher than the point on the car where the rope is attached. The rope is being pulled through the pulley at a rate of 6 m per minute. The front wheels of the car do not leave the ground.



At what rate will the car be moving when there is 8 m of rope between the car and the pulley? Show any derivatives that you need to find when solving this problem.

Setting up the related rates:

$$\frac{dS}{dt} = \frac{dS}{\quad} \cdot \frac{\quad}{dt}$$

$$\frac{dR}{dt} = -6\text{m/min}$$

$$\frac{dS}{dt} = \frac{dS}{dR} \cdot \frac{dR}{dt}$$

Finding an equation that links S and R:

$$R^2 = S^2 + 2^2$$
$$S^2 = R^2 - 4$$
$$S = \sqrt{R^2 - 4}$$

Differentiating:

$$\frac{dS}{dR} = \frac{1}{2\sqrt{R^2 - 4}} \cdot 2R$$
$$= \frac{2R}{2\sqrt{R^2 - 4}}$$

Calculating the rate:

$$\frac{dS}{dt} = \frac{dS}{dR} \cdot \frac{dR}{dt}$$
$$= \frac{2R}{2\sqrt{R^2 - 4}} \cdot -6$$
$$= \frac{-12R}{2\sqrt{R^2 - 4}}$$

When $R = 8$:

$$\frac{dS}{dt} = \frac{-12(8)}{2\sqrt{(8)^2 - 4}}$$
$$= -6.20\text{m/min (3 s.f.)}$$

e) Find the values of t for which the function defined by $x = 2t^3 - 5$ and $y = 4t^2 - 5t$ is concave up. Show any derivatives that you need to find when solving this problem.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dx}{dt} = 6t^2 \qquad \frac{dy}{dt} = 8t - 5$$

$$\frac{dy}{dx} = (8t - 5) \cdot \frac{1}{6t^2}$$
$$= \frac{8t - 5}{6t^2}$$

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}}{dx} = \frac{d\frac{dy}{dx}}{dt} \cdot \frac{dt}{dx}$$

$$\frac{d\frac{dy}{dx}}{dt} = \frac{8 \times 6t^2 - 12t \times (8t - 5)}{36t^4}$$
$$= \frac{48t^2 - 96t^2 + 60t}{36t^4}$$
$$= \frac{60t - 48t^2}{36t^4}$$
$$= \frac{5 - 4t}{3t^3}$$

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}}{dt} \cdot \frac{dt}{dx}$$
$$= \frac{5 - 4t}{3t^3} \cdot \frac{1}{6t^2}$$
$$= \frac{5 - 4t}{18t^5}$$
$$0 = \frac{5 - 4t}{18t^5}$$
$$0 = 5 - 4t$$
$$4t = 5$$
$$t = 1.25$$

There is a point of inflection when $t = 1.25$. When $t < 0$, $\frac{d^2y}{dx^2}$ is +ve so it is concave up. By looking at the graph, there is another point of inflection at $t = 0$ so the function is concave up when $0 < t < 1.25$

Question Three

a) Differentiate the following: $y = \sin(2x)\ln x$. You do not need to simplify your answer.

$$(f.g)' = f'g + g'f$$
$$\frac{dy}{dx} = 2\cos(2x)\ln x + \frac{1}{x}\sin(2x)$$

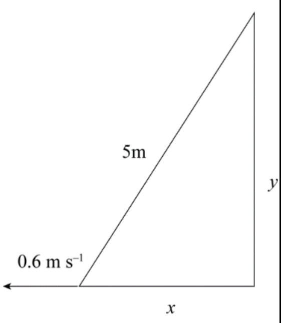
b) Find the equation of the tangent to $y = \frac{1}{x+2}$ at the point $\left(2, \frac{1}{4}\right)$. Show any derivatives that you need to find when solving this problem.

$$y = (x+2)^{-1}$$
$$\frac{dy}{dx} = -(x+2)^{-2} \times 1$$
$$= -\frac{1}{(x+2)^2}$$
$$\frac{dy}{d(2)} = -\frac{1}{(2+2)^2}$$
$$= -\frac{1}{16}$$

$$y = mx + c$$
$$\frac{1}{4} = -\frac{1}{16} \times 2 + c$$
$$c = \frac{3}{8}$$
$$y = -\frac{1}{16}x + \frac{3}{8}$$

$$y - y_1 = m(x - x_1)$$
$$y - \frac{1}{4} = -\frac{1}{16}(x-2)$$
$$= -\frac{1}{16}x + \frac{1}{8} + \frac{1}{4}$$
$$y = -\frac{1}{16}x + \frac{3}{8}$$

c) A 5 m long ladder is placed against a vertical wall. The bottom of the ladder slides out away from the wall, moving at 0.6 m per second. At what rate will the top of the ladder be moving downwards when it is 3 m above the ground? **Show any derivatives that you need to find when solving this problem.**



$$\frac{dy}{dt} = \frac{dy}{} \cdot \frac{}{dt}$$
$$\frac{dx}{dt} = 0.6\text{ms}^{-1}$$
$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$x^2 + y^2 = 5^2$$
$$y^2 = 5^2 - x^2$$
$$y = \sqrt{5^2 - x^2}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{5^2 - x^2}} \cdot -2x$$
$$= \frac{-x}{\sqrt{5^2 - x^2}}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$
$$= \frac{-x}{\sqrt{5^2 - x^2}} \cdot 0.6$$
$$= \frac{-0.6x}{\sqrt{5^2 - x^2}}$$

When $y = 3$:

$$x^2 + y^2 = 5^2$$
$$x^2 + 3^2 = 5^2$$
$$x^2 = 16$$
$$x = 4$$

$$\frac{dy}{dt} = \frac{-0.6x}{\sqrt{5^2 - x^2}}$$
$$= \frac{-0.6(4)}{\sqrt{5^2 - (4)^2}}$$
$$= -0.8\text{ms}^{-1}$$

The ladder is moving down at a rate of 0.8ms^{-1}.

d) Find the gradient of the tangent to the following curve when $x = 2$.

$$y^3 - 3x - e^{2x-1} = 0$$

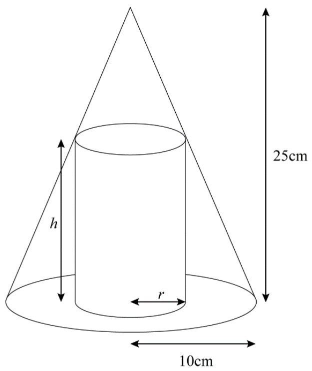
$$3y^2\frac{dy}{dx} - 3 - 2e^{2x-1} = 0$$
$$3y^2\frac{dy}{dx} = 3 + 2e^{2x-1}$$
$$\frac{dy}{dx} = \frac{3 + 2e^{2x-1}}{3y^2}$$

When $x = 2$:

$$y^3 - 3(2) - e^{2(2)-1} = 0$$
$$y^3 = e^3 + 6$$
$$y = \sqrt[3]{e^3 + 6}$$
$$y = 2.9657 \ldots$$

$$\frac{dy}{dx} = \frac{3 + 2e^{2(2)-1}}{3(2.9657)^2}$$
$$= 2.45 \text{ (3 s.f.)}$$

e) A cone has a height of 25 cm and a radius of 10 cm. A cylinder is inscribed in the cone. The base of the cylinder is in the same plane as the base of the cone. The cylinder base has the same centre as the base of the cone. Find the maximum volume of the cylinder.



$$h = -2.5r + 25$$

$$\begin{aligned} V_{cylinder} &= \pi r^2 h \\ &= \pi r^2(-2.5r + 25) \\ &= 25\pi r^2 - 2.5\pi r^3 \\ V'_{cylinder} &= 50\pi r - 7.5\pi r^2 \\ 0 &= 50\pi r - 7.5\pi r^2 \\ 0 &= 2.5\pi r(20 - 3r) \end{aligned}$$

Either $r = 0$cm or $r = \frac{20}{3} = 6.7$cm. I can assume that the second value is a maximum.