# A Collection of 110 Geometry Theorems and Their Machine Produced Proofs Using Full-Angles y

Article



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# A Collection of 110 Geometry Theorems and Their Machine Produced Proofs Using Full-Angles[†]

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This is a collection of 110 geometry theorems and their machine proofs produced by a prover using inference rules based on *full-angles*. These examples with difficulties range from moderate to those in the level of International Mathematical Olympiad and those proposed in the recent American Mathematics Monthly.

The prover is implemented using SB-Prolog on a SPARK-10 workstation. Briefly speaking, the prover works as follows.

**S1** The prover first builds a geometry information base (GIB) from the hypotheses by collecting information about parallel lines, perpendicular lines, and circles.

**S2** The prover tries to prove the theorem using backward chaining.

**S3** If the prover fails to find a proof in step S2, it further collects new geometry information related to equal-angles into the GIB and does S2 again.

**S4** If a geometry statement is true, the prover can automatically generate the *machine proof* in TeX typesetting.

Details of the method/prover can be found in the following papers.

1. S. C. Chou, X. S. Gao, & J. Z. Zhang, Automated Generation of Multiple and Shortest Proofs in Geometry with High Level Inference Rules, I. Multiple and Shortest Proofs Generation, CS-WSU-94-2, March, 1994.

2. S. C. Chou, X. S. Gao, & J. Z. Zhang, Automated Generation of Multiple and Shortest Proofs in Geometry with High Level Inference Rules, II. Theorem Proving with Full-angles, CS-WSU-94-3, March, 1994.

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This collection is organized as follows. Section 1 introduces some notations. Section 2 is the collection of the 110 examples. Section 3 includes the time and proof length statistics.

# 1 Notation Convenience

A geometry statement consists of four parts

(Nm,Pts,Hs,Cs).

1. Nm is the name of the statement, which is used as the identification of the geometry statement.
2. Pts is a list of the points occurring in the statement. The order of the points in the list is used to guide the search in the proving process.
3. Hs is a list of predicates which are the hypotheses of the geometry statement.
4. Cs is a predicate or an equation of full-angles which is the conclusion of the statement.

The following predicates are allowed in the description of geometry statements

coll (collinear)
para (parallel)
perp (perpendicular)
cong (congruent)
eqangle (equal-angle)
midpoint
cyclic
foot
circumcenter
orthocenter
similar (two triangles are similar)
incenter pbisector (perp-bisector).

In all the examples, we use lower case English letter to denote points. This is in accordance with the Prolog rule that capital letters are used as variables.

# 2 A Collection of 110 Geometry Theorems

## 2.1 Examples Whose Conclusions Are in the Database

For the following five examples, their proofs are already in the GIB. We thus proved these examples by forward chaining.

**Example 1** *In triangle $ABC$, let $F$ the midpoint of the side $BC$, $D$ and $E$ the feet of the altitudes on $AB$ and $AC$, respectively. $FG$ is perpendicular to $DE$ at $G$. Show that $G$ is the midpoint of $DE$.*

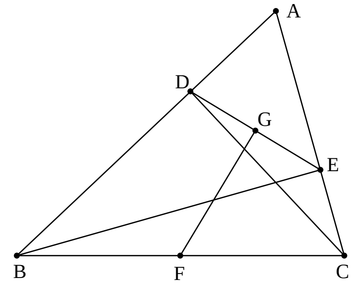

Figure 1

Point order: $a, b, c, d, e, f, g$.

Hypotheses: foot$(d, c, a, b)$, foot$(e, b, a, c)$, midpoint$(f, b, c)$, midpoint$(g, d, e)$.

Conclusion: perp$(f, g, d, e)$.

The Machine Proof

The result is in the database.

$gf \perp ed$, because midpoint$(g, d, e)$; and $df = ef$.

$df = ef$, because $d, e, b, c$ are on the circle with $f$ as center.

$d, e, b, c$ are on the circle with $f$ as center, because midpoint$(f, b, c)$; $bd \perp cd$; $be \perp ce$.

**Example 2** *Prove that the circumcenter of a triangle is the orthocenter of its medial triangle.*

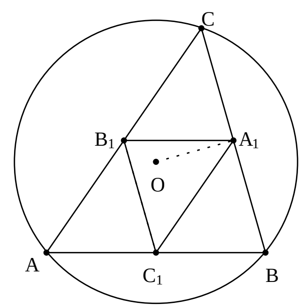

Figure 2

Point order: $a, b, c, a1, b1, c1, o$.

Hypotheses: midpoint$(a1, b, c)$, midpoint$(b1, a, c)$, midpoint$(c1, a, b)$, circumcenter$(o, a, b, c)$.

Conclusion: perp$(o, a1, b1, c1)$.

The Machine Proof

The result is in the data base.

$oa1 \perp c1b1$, because $oa1 \perp bc$, and $bc \parallel b1c1$.

$bc \parallel b1c1$, because midpoint$(b1, a, c)$, midpoint$(c1, a, b)$.

$oa1 \perp bc$, because midpoint$(a1, b, c)$; $ob = oc$.

**Example 3** *Show that the triangle formed by the foot of the altitude to the base of a triangle and the midpoints of the altitudes to the lateral sides is similar to the given triangle; its circumcircle passes through the orthocenter of the given triangle and through the midpoint of its base.*

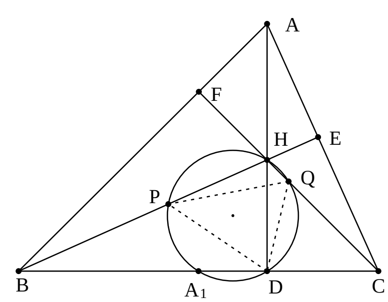

Figure 3

Point order: $a, b, c, d, e, f, h, a1, p, q$.

Hypotheses: foot$(d, a, b, c)$, foot$(e, b, a, c)$, foot$(f, c, a, b)$, coll$(h, a, d)$, coll$(h, b, e)$, coll$(h, c, f)$, midpoint$(a1, b, c)$, midpoint$(p, b, e)$, midpoint$(q, c, f)$.

Conclusion: cyclic$(p, q, h, d)$.

The Machine Proof

The conclusion is in the database

$q$, $d$, $a1$, $h$, $p$ are cyclic, because $d$, $a1$, $h$, $q$ are cyclic and $d$, $a1$, $h$, $p$ are cyclic.
$d$, $a1$, $h$, $q$ are cyclic, because $da1 \perp dh$; $qa1 \perp qh$.
$d$, $a1$, $h$, $p$ are cyclic, because $da1 \perp dh$; $pa1 \perp ph$.
$qa1 \perp qh$, because $qa1 \parallel ab$; $ab \perp qh$.
$qa1 \parallel ab$, because midpoint($a1, b, c$); midpoint($q, c, f$).
$pa1 \perp ph$, because $pa1 \parallel ac$; $ac \perp ph$.
$pa1 \parallel ac$, because midpoint($a1, b, c$); midpoint($p, b, e$).

**Example 4** *Show that the anticenter of a cyclic quadrilateral is the orthocenter of the triangle having for vertices the midpoints of the diagonals and the point of intersection of those two lines.*

Point order: $a, b, c, d, o, q, s, j, m$.

Hypotheses: circumcenter$(o, a, b, c, d)$, coll$(i, a, d)$, coll$(i, b, c)$, midpoint$(q, b, c)$, midpoint$(s, a, d)$, midpoint$(j, s, q)$, midpoint$(j, o, m)$.

Conclusion: perp$(s, m, b, c)$.

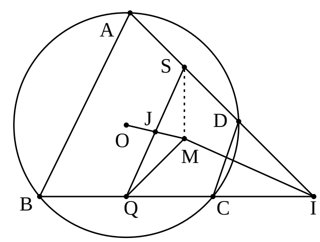


Figure 4

The Machine Proof

The conclusion is in the database.
$ms \perp cb$, because $sm \parallel oq$; $oq \perp bc$.
$sm \parallel oq$, because midpoint$(j, o, m)$; midpoint$(j, s, q)$.
$oq \perp bc$, because midpoint$(q, b, c)$; $ob = oc$.

**Example 5** *Let $H$ be the orthocenter of triangle $ABC$. Then the circumcenters of the four triangles $ABC$, $ABH$, $ACH$, and $BCH$ are such that each is the orthocenter of the triangle formed by the remaining three.*

Point order: $a, b, c, h, o, p, q, r$.

Hypotheses: orthocenter$(h, a, b, c)$, circumcenter$(o, a, b, c)$, circumcenter$(p, a, c, h)$, circumcenter$(q, a, b, h)$.

Conclusion: $[para, p, q, b, c]$.

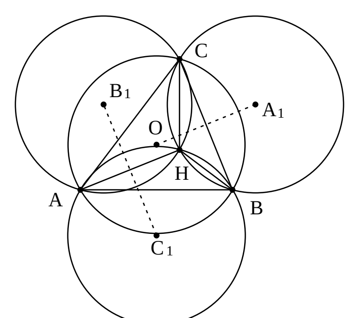


Figure 5

The Machine Proof

The result is in the database.
$pq \parallel bc$, because $pq \perp ah$, $bc \perp ah$.
$pq \perp ah$, because $ah$ is the axis of circle$(p, a, c, h)$ and circle$(q, a, b, h)$.

## 2.2 Examples Proved by Backward Chaining

The proof for the examples in this subsection are obtained by a backward chaining search based on the GIB. Generally speaking, the information in GIB is obvious and does not need further explanation.

**Example 6 (The Orthocenter Theorem)** *The three altitudes of a triangle are concurrent.*

Point order: $a, b, c, e, f, h$.

Hypotheses: foot$(e, b, a, c)$, foot$(f, a, b, c)$, coll$(h, a, f)$, coll$(h, b, e)$.

Conclusion: perp$(a, b, c, h)$.

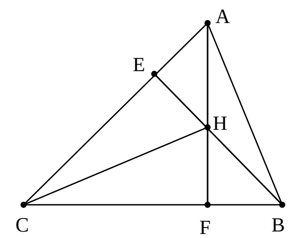


Figure 6

The Machine Proof

$-[hc, ba] + 1$

(Since $a$, $f$, $h$ are collinear; $h$, $c$, $f$, $e$ are cyclic; $[hc, ba] = [hc, hf] + [fa, ba] = -[fe, ec] + [fa, ba]$.)

$= [fe, ec] - [fa, ba] + 1$

(Since $a$, $c$, $e$ are collinear; $e$, $f$, $a$, $b$ are cyclic; $[fe, ec] = [fb, ba]$.)

$= [fb, ba] - [fa, ba] + 1$

(Since $b$, $c$, $f$ are collinear; $[fb, ba] = [cb, ba]$.)

$= -[fa, ba] + [cb, ba] + 1$

(Since $fa \perp bc$; $[fa, ba] = [cb, ba] + 1$.)

$= 0$

**Example 7 (Simson's Theorem)** *Let $D$ be a point on the circumcircle of triangle $ABC$. From $D$ three perpendiculars are drawn to the three sides $BC$, $AC$, and $AB$ of triangle $ABC$. Let $E$, $F$, and $G$ be the three feet respectively. Show that $E$, $F$ and $G$ are collinear.*

Point order: $o, a, b, c, d, e, f, g$.

Hypotheses: cong$(o, a, o, b)$, cong$(o, a, o, c)$, cong$(o, a, o, d)$, coll$(e, b, c)$, perp$(e, d, b, c)$, coll$(f, a, c)$, perp$(f, d, a, c)$, coll$(g, a, b)$, perp$(g, d, a, b)$.

Conclusion: coll$(g, f, e)$.

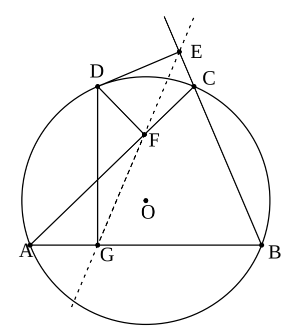


Figure 7

The Machine Proof

$[gf, ge]$

(Since $f$, $g$, $d$, $a$ are cyclic; $e$, $g$, $d$, $b$ are cyclic; $[gf, ge] = [gf, gd] + [gd, ge] = [fa, da] - [eb, db]$.)

$= [fa, da] - [eb, db]$

(Since $b$, $c$, $e$ are collinear; $[eb, db] = -[db, cb]$.)

$= [fa, da] + [db, cb]$

(Since $a$, $c$, $f$ are collinear; $[fa, da] = -[da, ca]$.)

$= [db, cb] - [da, ca]$

(Since $b$, $d$, $c$, $a$ are cyclic; $[db, cb] = [da, ca]$.)

$= 0$

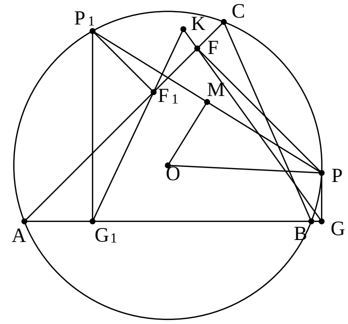


Figure 8

**Example 8 (Simson Line, Extension 1)** *The angle formed by the Simson lines of two points for the same triangle is measured by half the are between the two points.*

Point order: $a, b, c, p, p_1, g, g_1, f, f_1, k$.

Hypotheses: cyclic$(a, b, c, p, p_1)$, foot$(g, p, a, b)$, foot$(f, p, a, c)$, foot$(g_1, p_1, a, b)$, foot$(f_1, p_1, a, c)$, coll$(k, f, g)$, coll$(k, f_1, g_1)$.

Conclusion: eqangle$(p, a, p_1, f_1, k, f)$.

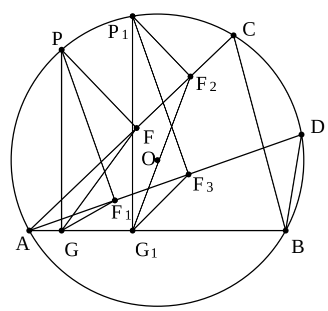


Figure 9

The Machine Proof

$-[kf_1, kf] - [p_1a, pa]$

(Since $f_1$, $g_1$, $k$ are collinear; $f$, $g$, $k$ are collinear; $[kf_1, kf] = [f_1g_1, fg]$.)

$= -[f_1g_1, fg] - [p_1a, pa]$

(Since $a$, $f$, $f_1$ are collinear; $f_1$, $g_1$, $a$, $p_1$ are cyclic; $[f_1g_1, fg] = [f_1g_1, f_1a] + [af, fg] = -[fg, fa] + [g_1p_1, p_1a]$.)

$= [fg, fa] - [g_1p_1, p_1a] - [p_1a, pa]$

(Since $-[g_1p_1, p_1a] - [p_1a, pa] = -[g_1p_1, pa]$.)

$= [fg, fa] - [g_1p_1, pa]$

(Since $f$, $g$, $a$, $p$ are cyclic; $[fg, fa] = [gp, pa]$.)

$= -[g_1p_1, pa] + [gp, pa]$

(Since $g_1p_1 \parallel pg$; $[g_1p_1, pa] = [gp, pa]$.)

$= 0$

**Example 9 (Simson Line, Extension 2)** *The angle formed by the Simson lines of the same points with respect to two triangles inscribed in the same circle is the same for all positions of the point on the circle.*

Point order: $a, b, c, d, p, p_1, g, f, f_1, g_1, f_2, f_3$.

Hypotheses: cyclic$(a, b, c, d, p, p_1)$, perp$(p, f, a, c)$, coll$(f, a, c)$, perp$(p, g, a, b)$, coll$(g, a, b)$, perp$(p, f_1, a, d)$, coll$(f_1, a, d)$, perp$(p_1, g_1, a, b)$, coll$(g_1, a, b)$, perp$(p_1, f_2, a, c)$, perp$(p_1, f_3, a, d)$, coll$(f_2, a, c)$, coll$(f_3, a, d)$.

Conclusion: eqangle$(f, g, f_1, f_2, g_1, f_3)$.

The Machine Proof

$[f_3g_1, f_2g_1] - [f_1g, fg]$

(Since $g_1$, $f_3$, $f_2$, $a$ are cyclic; $[f_3g_1, f_2g_1] = [f_3a, f_2a]$.)

$= [f_3a, f_2a] - [f_1g, fg]$

(Since $a$, $d$, $f_3$ are collinear; $[f_3a, f_2a] = -[f_2a, da]$.)

$= -[f_2a, da] - [f_1g, fg]$

(Since $a$, $c$, $f_2$ are collinear; $[f_2a, da] = -[da, ca]$.)

$= -[f_1g, fg] + [da, ca]$

(Since $g$, $f_1$, $f$, $p$ are cyclic; $[f_1g, fg] = [f_1p, fp]$.)

$= -[f_1p, fp] + [da, ca]$

(Since $f_1p \perp ad$ $fp \perp ac$ $[f_1p, fp] = [da, ca]$.)

$= 0$

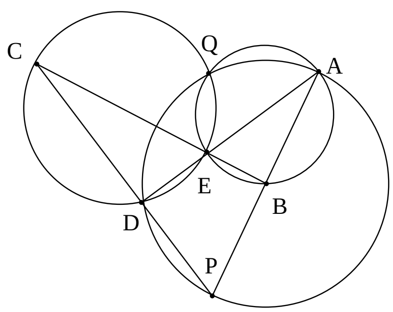


Figure 10

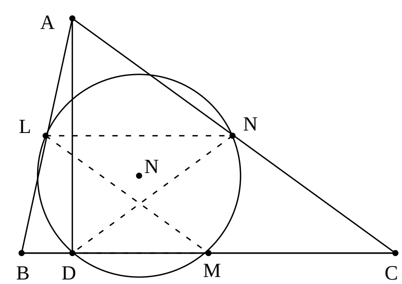


Figure 11

**Example 10 (Miquel Point Theorem)** *Four lines form four triangles. Show that the circumcircles of the four triangles passes through a common point.).*

Point order: $a, b, c, d, e, q, p$.

Hypotheses: cyclic$(a, b, e, q)$, cyclic$(c, d, e, q)$, coll$(a, b, p)$, coll$(c, d, p)$, coll$(b, c, e)$, coll$(a, d, e)$.

Conclusion: cyclic$(d, a, p, q)$.

The Machine Proof

$[pd, pa] - [qd, qa]$

(Since $c$, $d$, $p$ are collinear; $a$, $b$, $p$ are collinear; $[pd, pa] = [dc, ba]$.)

$= -[qd, qa] + [dc, ba]$

(Since $d$, $q$, $e$, $c$ are cyclic; $a$, $q$, $e$, $b$ are cyclic;

$[qd, qa] = [qd, qe] + [qe, qa] = -[ec, dc] + [eb, ba]$.)
$= [ec, dc] - [eb, ba] + [dc, ba]$
(Since $b$, $c$, $e$ are collinear; $[ec, dc] - [eb, ba] = -[dc, ba]$.)
$= 0$

**Example 11 (Nine Point Circle Theorem)** *Let the midpoints of the sides $AB, BC$, and $CA$ of $\triangle ABC$ be $L, M$, and $N$, and $AD$ the altitude on $BC$. Show that $L, M, N$, and $D$ are on the same circle.*

Point order: $a, b, c, d, l, m, n$.
Hypotheses: midpoint$(m, b, c)$, midpoint$(n, a, c)$, midpoint$(l, a, b)$, foot$(d, a, b, c)$.
Conclusion: cyclic$(l, d, m, n)$.

The Machine Proof
$-[nl, nd] + [ml, md]$
(Since $nl \parallel bc$, $[nl, nd] = -[nd, cb]$.)
$= [nd, cb] + [ml, md]$
(Since $d$, $c$, $a$ have equal distance from point $n$,
$[nd, cb] = [nd, dc] = [da, ca] + 1$.)
$= [ml, md] + [da, ca] + 1$
(Since $ml \parallel ca$, $[ml, md] + [da, ca] = -[md, da]$.)
$= -[md, da] + 1$
(Since $md \perp da$, $[md, da] = 1$.)
$= 0$

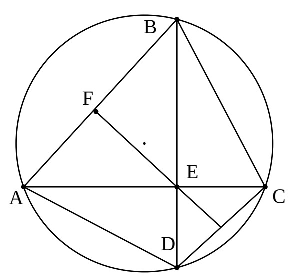

Figure 12

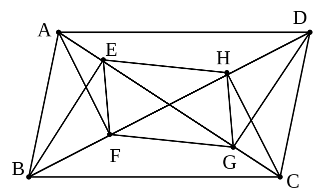

Figure 13

**Example 12 (Theorem of Brahmagupta)** *In a quadrilateral which is both orthodiagonal and cyclic the perpendicular from the point of intersection of the diagonals to a side bisects the side opposite.*

Point order: $a, b, c, d, e, f$.
Hypotheses: cyclic$(a, b, c, d)$, perp$(a, c, b, d)$, coll$(e, a, c)$, coll$(e, b, d)$, midpoint$(f, a, b)$.

Conclusion: perp$(e, f, c, d)$.

The Machine Proof

$[fe, dc] + 1$

(Since circumcenter($f$,$e$, $a$, $b$ ); $[fe, dc] = [fe, ea] + [ea, dc] = [eb, ba] + [ea, dc] + 1$.)

$= [eb, ba] + [ea, dc]$

(Since $b$, $d$, $e$ are collinear; $[eb, ba] = [db, ba]$.)

$= [ea, dc] + [db, ba]$

(Since $a$, $c$, $e$ are collinear; $[ea, dc] = -[dc, ca]$.)

$= -[dc, ca] + [db, ba]$

(Since $c$, $d$, $a$, $b$ are cyclic; $[dc, ca] = [db, ba]$.)

$= 0$

**Example 13** *Let $ABCD$ be a parallelogram. Then the feet from $A, B, C, D$ to the diagonals of the parallelogram form a parallelogram.*

Point order: $a, b, c, d, e, f, h, g$.

Hypotheses: para$(d, a, b, c)$, para$(d, c, a, b)$, foot$(e, b, a, c)$, foot$(f, a, b, d)$, foot$(h, c, b, d)$, foot$(g, d, a, c)$.

Conclusion: para$(e, f, h, g)$.

The Machine Proof

$-[gh, fe]$

(Since $c$, $e$, $g$ are collinear; $g$, $h$, $c$, $d$ are cyclic;
$[gh, fe] = [gh, gc] + [ce, fe] = [hd, dc] - [fe, ec]$.)

$= -[hd, dc] + [fe, ec]$

(Since $b$, $d$, $h$ are collinear; $[hd, dc] = -[dc, db]$.)

$= [fe, ec] + [dc, db]$

(Since $a$, $c$, $e$ are collinear; $e$, $f$, $a$, $b$ are cyclic; $[fe, ec] = [fb, ba]$.)

$= [fb, ba] + [dc, db]$

(Since $dc \parallel ab$; $[dc, db] = -[db, ba]$.)

$= [fb, ba] - [db, ba]$

(Since $b$, $d$, $f$ are collinear; $[fb, ba] = [db, ba]$.)

$= 0$

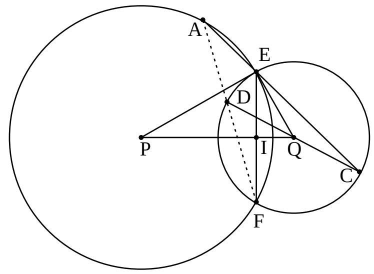


Figure 14

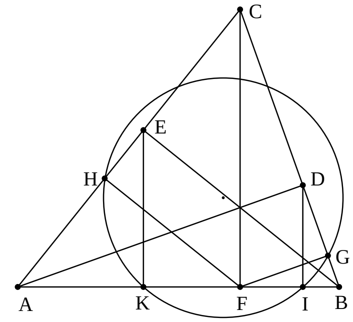


Figure 15

**Example 14** *The two lines joining the points of intersection of two orthogonal circles to a point on one of the circles meet the other circle in two diametrically opposite points.*

Point order: $c, e, f, q, d, i, p, a$.

Hypotheses: circumcenter$(q, c, e, f, d)$, midpoint$(q, c, d)$, midpoint$(i, e, f)$, coll$(p, i, q)$, perp$(p, e, e, q)$, coll$(a, c, e)$, cong$(a, p, p, e)$.

Conclusion: coll$(a, d, f)$.

The Machine Proof

$[af, df]$

(Since $df \perp cf$ $[af, df] = [af, fc] + 1$.)

$= [af, fc] + 1$

(Since $a$, $f$, $e$ have equal distance from point$p$; $a$, $c$, $e$ are collinear;
$i$ is the midpoint of $ef$; $[af, fc] = [af, ae] + [ec, fc] = -[pi, pf] - [fc, ec]$.)

$= -[pi, pf] - [fc, ec] + 1$

(Since $pi \perp ef$ $pf \perp fq$ $[pi, pf] = -[qf, fe]$.)

$= [qf, fe] - [fc, ec] + 1$

(Since $f$, $e$, $c$ have equal distance from point$q$; $[qf, fe] = [qf, fe] = [fc, ec] + 1$.)

$= 0$

**Example 15** *Let $ABC$ be a triangle. Show that the six feet obtained by drawing perpendiculars through the foot of each altitude upon the other two sides are co-circle.*

Point order: $a, b, c, f, d, e, g, i, h, k$.

Hypotheses: foot$(f, c, a, b)$, foot$(d, a, b, c)$, foot$(e, b, a, c)$, foot$(g, f, b, c)$, foot$(i, d, a, b)$, foot$(h, f, a, c)$, foot$(k, e, a, b)$.

Conclusion: cyclic$(h, i, g, k)$.

The Machine Proof

$-[kh, ki] + [hg, ig]$

(Since $ki \perp cf$ $[kh, ki] = [kh, fc] + 1$.)

$= -[kh, fc] + [hg, ig] - 1$

(Since $fc \perp ab$ $[kh, fc] = [kh, ba] + 1$.)

$= -[kh, ba] + [hg, ig]$

(Since $a$, $b$, $k$ are collinear; $a$, $b$, $f$ are collinear; $k$, $h$, $f$, $e$ are cyclic; $[kh, ba] = [kh, kf] = [he, ef]$.)

$= [hg, ig] - [he, ef]$

(Since $c$, $e$, $h$ are collinear; $h$, $g$, $c$, $f$ are cyclic; $[hg, ig] = [hg, hc] + [ce, ig] = -[ig, ec] + [gf, fc]$.)

$= -[he, ef] - [ig, ec] + [gf, fc]$

(Since $c$, $e$, $h$ are collinear; $-[he, ef] - [ig, ec] = -[ig, ef]$.)

$= -[ig, ef] + [gf, fc]$

(Since $a$, $f$, $i$ are collinear; $i$, $g$, $f$, $d$ are cyclic; $[ig, ef] = [ig, if] + [fa, ef] = [gd, df] - [ef, fa]$.)

$= -[gd, df] + [gf, fc] + [ef, fa]$

(Since $b$, $c$, $d$, $g$ are collinear; $[gd, df] = -[df, cb]$.)

$= [gf, fc] + [ef, fa] + [df, cb]$

(Since $gf \parallel ad$; $[gf, fc] = [da, fc]$.)

$= [ef, fa] + [df, cb] + [da, fc]$

(Since $a$, $b$, $f$ are collinear; $f$, $e$, $b$, $c$ are cyclic; $[ef, fa] = [ec, cb]$.)

$= [ec, cb] + [df, cb] + [da, fc]$

(Since $b$, $c$, $d$ are collinear; $d$, $f$, $c$, $a$ are cyclic; $[df, cb] = [df, dc] = [fa, ca]$.)

$= [ec, cb] + [da, fc] + [fa, ca]$

(Since $a$, $c$, $e$ are collinear; $[ec, cb] + [fa, ca] = [fa, cb]$.)

$= [da, fc] + [fa, cb]$

(Since $a$, $b$, $f$ are collinear; $[fa, cb] = -[cb, ba]$.)

$= [da, fc] - [cb, ba]$

(Since $da \perp bc$ $fc \perp ab$ $[da, fc] = [cb, ba]$.)

$= 0$

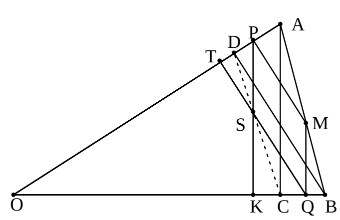

Figure 16

**Example 16** *$M$ is a point on side $AB$ of triangle $OAB$. $C$ and $D$ are the feet of the altitude $AC$ and $BD$. $P$ and $Q$ are the feet of the perpendiculars from $M$ to $OA$ and $OB$ respectively. $T$ and $K$ are the feet of the perpendiculars from $Q$ and $P$ to $OA$ and $OB$ respectively. $S = QP \cap PK$. Show that $OS \perp PQ$.*

Point order: $o, a, b, c, d, m, p, q, k, t, s$.

Hypotheses: foot$(c, a, o, b)$, foot$(d, b, o, a)$, coll$(m, a, b)$, foot$(p, m, a, o)$,

foot$(q, m, o, b)$, foot$(t, q, o, a)$, foot$(k, p, o, b)$, coll$(s, q, t)$, coll$(s, p, k)$.
Conclusion: perp$(o, s, p, q)$.

The Machine Proof

$[so, qp] + 1$

(Since $q$, $s$, $t$ are collinear; $s$, $o$, $t$, $k$ are cyclic; $[so, qp] = [so, st] + [tq, qp] = -[tk, ko] + [tq, qp]$.)

$= -[tk, ko] + [tq, qp] + 1$

(Since $ko \perp ac$; $[tk, ko] = [tk, ca] + 1$.)

$= -[tk, ca] + [tq, qp]$

(Since $ca \perp ob$; $[tk, ca] = [tk, bo] + 1$.)

$= -[tk, bo] + [tq, qp] - 1$

(Since $d$, $p$, $t$ are collinear; $t$, $k$, $p$, $q$ are cyclic; $[tk, bo] = [tk, tp] + [pd, bo] = [kq, qp] + [pd, bo]$.)

$= [tq, qp] - [kq, qp] - [pd, bo] - 1$

(Since $kq \parallel bo$; $-[kq, qp] - [pd, bo] = [qp, pd]$.)

$= [tq, qp] + [qp, pd] - 1$

(Since $qp \parallel qp$; $[tq, qp] + [qp, pd] = [tq, pd]$.)

$= [tq, pd] - 1$

(Since $tq \perp pd$; $[tq, pd] = 1$.)

$= 0$

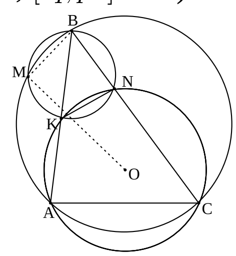

Figure 17

**Example 17** *$A$, $C$, $K$, and $N$ are four points on a circle. $B = AK \cap CN$. $M$ is the intersection of the circumcircles of triangles $BKN$ and $BAC$. Show that $K$, $O$, $C$, and $M$ are cyclic.*

Point order: $a, c, k, n, o, b, m$.
Hypotheses: circumcenter$(o, a, c, k, n)$, coll$(b, a, k)$, coll$(b, c, n)$, cyclic$(m, a, b, c)$, cyclic$(m, b, k, n)$.
Conclusion: cyclic$(k, o, c, m)$.

The Machine Proof

$-[mk, mc] + [ok, oc]$

(Since $k$, $m$, $b$, $n$ are cyclic; $c$, $m$, $b$, $a$ are cyclic;
$[mk, mc] = [mk, mb] + [mb, mc] = -[bn, nk] + [ba, ca]$.)

$= [bn, nk] - [ba, ca] + [ok, oc]$

(Since $b$, $c$, $n$ are collinear; $[bn, nk] = -[nk, nc]$.)

$= -[ba, ca] + [ok, oc] - [nk, nc]$

(Since $a$, $b$, $k$ are collinear; $[ba, ca] = [ka, ca]$.)

$= [ok, oc] - [nk, nc] - [ka, ca]$

(Since circumcenter($o$,$k$, $a$, $c$ ); $[ok, oc] = [ok, ka] + [ka, oc] = -[oc, ka] + [kc, ca] + 1$.)

$= -[oc, ka] - [nk, nc] + [kc, ca] - [ka, ca] + 1$

(Since $ka \parallel ka$; $-[oc, ka] - [ka, ca] = -[oc, ca]$.)

$= -[oc, ca] + [kc, ca] - [nk, nc] + 1$

(Since circumcenter($o$,$c$, $a$, $n$ ); $[oc, ca] = [oc, ca] = [nc, na] + 1$.)

$= -[nc, na] + [kc, ca] - [nk, nc]$

(Since $nc \parallel nc$; $-[nc, na] - [nk, nc] = -[nk, na]$.)

$= -[nk, na] + [kc, ca]$

(Since $n$, $k$, $a$, $c$ are cyclic; $[nk, na] = [kc, ca]$.)

$= 0$

**Example 18** [1] *The same as Example 17. Show that $BM \perp MO$.*

Point order: $a, c, k, n, o, b, m$.

Hypotheses: circumcenter$(o, a, c, k, n)$, coll$(b, a, k)$, coll$(b, c, n)$, cyclic$(k, o, c, m)$, cyclic$(m, a, b, c)$, cyclic$(m, b, k, n)$.

Conclusion: perp$(b, m, m, o)$.

The Machine Proof

$[mb, mo] + 1$

(Since $b$, $m$, $c$, $a$ are cyclic; $o$, $m$, $c$, $k$ are cyclic;
$[mb, mo] = [mb, mc] + [mc, mo] = [ba, ca] - [ok, kc]$.)

$= [ba, ca] - [ok, kc] + 1$

(Since $a$, $b$, $k$ are collinear; $[ba, ca] = [ka, ca]$.)

$= -[ok, kc] + [ka, ca] + 1$

(Since circumcenter($o$,$k$, $c$, $a$ ); $[ok, kc] = [ok, kc] = [ka, ca] + 1$.)

$= 0$

**Example 19** *Two circles $O$ and $Q$ meet in two points $A$ and $B$. A line passing through $A$ meets circles $O$ and $Q$ in $C$ and $E$. A line passing through $B$ meets circles $O$ and $Q$ in $D$ and $F$. Show that $CD \parallel EF$.*

Point order: $a, b, c, d, e, f$.

Hypotheses: cyclic$(a, b, c, d)$, cyclic$(a, b, e, f)$, coll$(a, c, e)$, coll$(b, d, f)$.

Conclusion: para$(c, d, e, f)$.

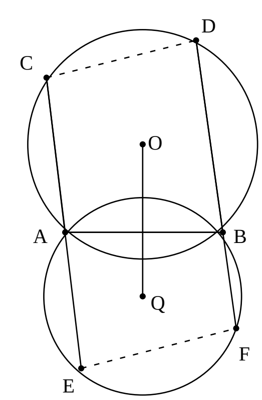

Figure 19

The Machine Proof

$-[fe, dc]$

(Since $b$, $d$, $f$ are collinear; $f$, $e$, $b$, $a$ are cyclic;
$[fe, dc] = [fe, fb] + [bd, dc] = [ea, ba] - [dc, db]$.)

$= -[ea, ba] + [dc, db]$

(Since $a$, $c$, $e$ are collinear; $[ea, ba] = [ca, ba]$.)

[1] This example is from the 1985 International Mathematical Olympiad

$= [dc, db] - [ca, ba]$

(Since $d$, $c$, $b$, $a$ are cyclic; $[dc, db] = [ca, ba]$.)

$= 0$

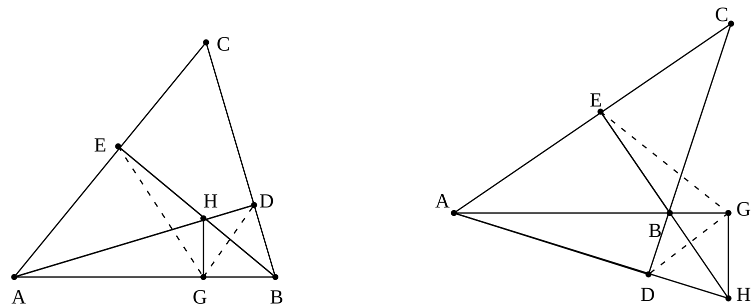


Figure 20

**Example 20** *In triangle $ABC$, two altitudes $AD$ and $BE$ meet in $H$. $G$ is the foot of the perpendicular from point $H$ to $AB$. Show that $\angle[DG, GH] = \angle[HG, GE]$.*

Point order: $a, b, c, d, e, h, g$.

Hypotheses: foot$(e, b, a, c)$, foot$(d, a, b, c)$, coll$(h, e, b)$, coll$(h, a, d)$, foot$(g, h, a, b)$.

Conclusion: eqangle$(d, g, h, h, g, e)$.

The Machine Proof

$-[gh, ge] - [gh, gd]$

(Since $gh \perp ab$; $[gh, ge] = -[ge, ba] + 1$.)

$= -[gh, gd] + [ge, ba] - 1$

(Since $gh \perp ab$; $[gh, gd] = -[gd, ba] + 1$.)

$= [ge, ba] + [gd, ba]$

(Since $a$, $b$, $g$ are collinear; $g$, $e$, $a$, $h$ are cyclic;
$[ge, ba] = [ge, ga] = [he, ha]$.)

$= [gd, ba] + [he, ha]$

(Since $a$, $b$, $g$ are collinear; $g$, $d$, $b$, $h$ are cyclic;
$[gd, ba] = [gd, gb] = [hd, hb]$.)

$= [he, ha] + [hd, hb]$

(Since $b$, $e$, $h$ are collinear; $a$, $d$, $h$ are collinear; $[he, ha] = [eb, da]$.)

$= [hd, hb] + [eb, da]$

(Since $hb \parallel eb$; $[hd, hb] + [eb, da] = [hd, da]$.)

$= [hd, da]$

(Since $a$, $d$, $h$ are collinear; $[hd, da] = 0$.)

$= 0$

**Example 21** *The circumcenter of triangle $ABC$ is $O$. $AD$ is the altitude on side $BC$. Show that $\angle[AO, DA] = \angle[BA, BC] - \angle[BC, CA]$.*

Point order: $a, b, c, o, d$.

Hypotheses: cong$(o, a, o, b)$, cong$(o, a, o, c)$, coll$(d, b, c)$, perp$(a, d, b, c)$.

Conclusion: $[ao, ad] - [ab, bc] + [bc, ca] = 0$.

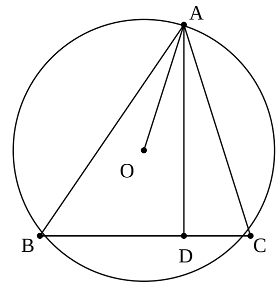


Figure 21

The Machine Proof

$-[da, oa] + [cb, ca] + [cb, ba]$

(Since $da \perp bc$; $[da, oa] = -[oa, cb] + 1$.)

$= [oa, cb] + [cb, ca] + [cb, ba] - 1$

(Since $cb \parallel cb$; $[oa, cb] + [cb, ca] = [oa, ca]$.)

$= [oa, ca] + [cb, ba] - 1$

(Since circumcenter($o$,$a$, $c$, $b$ ); $[oa, ca] = [oa, ac] = -[cb, ba] + 1$.)

$= 0$

**Example 22** *In a circle, the lines joining the midpoints of two arcs $AB$ and $AC$ meet line $AB$ and $AC$ at $D$ and $E$. Show that $AD = AE$.*

Point order: $a, m, n, o, p, q, d, e$.

Hypotheses: cong$(o, a, o, n)$, cong$(o, a, o, m)$, coll$(p, o, n)$, perp$(p, a, o, n)$, coll$(q, o, m)$, perp$(q, a, o, m)$, coll$(e, a, p)$, coll$(e, n, m)$, coll$(d, a, q)$, coll$(d, n, m)$.

Conclusion: $[ad, ed] + [ae, ed]$.

The Machine Proof

$-[ed, ea] - [ed, da]$

(Since $d$, $e$, $m$, $n$ are collinear; $a$, $e$, $p$ are collinear; $[ed, ea] = -[pa, nm]$.)

$= -[ed, da] + [pa, nm]$

(Since $d$, $e$, $m$, $n$ are collinear; $[ed, da] = -[da, nm]$.)

$= [da, nm] + [pa, nm]$

(Since $a$, $d$, $q$ are collinear; $[da, nm] = [qa, nm]$.)

$= [qa, nm] + [pa, nm]$

(Since $qa \perp mo$; $[qa, nm] = [om, nm] + 1$.)

$= [pa, nm] + [om, nm] + 1$

(Since $pa \perp no$; $[pa, nm] = [on, nm] + 1$.)

$= [on, nm] + [om, nm]$

(Since circumcenter($o$,$n$, $m$, $a$ ); $[on, nm] = [on, nm] = [na, ma] + 1$.)

$= [om, nm] + [na, ma] + 1$

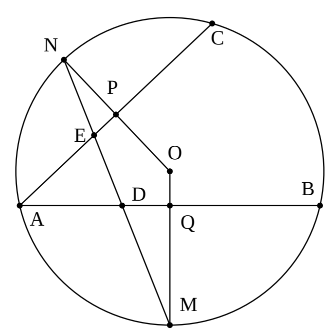


Figure 22

(Since circumcenter($o$,$m$, $n$, $a$ ); $[om, nm] = [om, mn] = -[na, ma] + 1$.)

$= 0$

**Example 23** *From the midpoint $C$ of arc $AB$ of a circle, two secants are drawn meeting line $AB$ at $F$, $G$, and the circle at $D$ and $E$. Show that $F$, $D$, $E$, and $G$ are on the same circle.*

Point order: $a, c, d, e, o, m, f, g$.

Hypotheses: cong$(o, a, o, c)$, cong$(o, a, o, d)$, cong$(o, a, o, e)$, coll$(m, c, o)$, perp$(m, a, c, o)$, coll$(f, a, m)$, coll$(f, c, d)$, coll$(g, a, m)$, coll$(g, c, e)$.

Conclusion: $[ce, fg] + [cd, de]$.

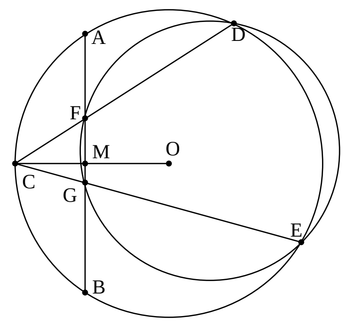


Figure 23

The Machine Proof

$-[gf, ec] - [ed, dc]$

(Since $a$, $f$, $g$, $m$ are collinear; $[gf, ec] = [ma, ec]$.)

$= -[ma, ec] - [ed, dc]$

(Since $ma \perp co$; $[ma, ec] = [oc, ec] + 1$.)

$= -[oc, ec] - [ed, dc] - 1$

(Since circumcenter($o$,$c$, $e$, $d$ ); $[oc, ec] = [oc, ce] = -[ed, dc] + 1$.)

$= 0$

**Example 24** *Let $Q$, $S$ and $Y$ be three collinear points and $(O, P)$ be a circle. Circles $SPQ$ and $YPQ$ meet circle $(O, P)$ again at points $R$ and $X$, respectively. Show that $XY$ and $RS$ meet on the circle $(O, P)$.*

Point order: $x, r, p, q, s, y, i$.

Hypotheses: cyclic$(r, p, q, s)$, coll$(y, q, s)$, cyclic$(y, p, q, x)$, coll$(i, x, y)$, coll$(i, r, s)$.

Conclusion: $[xi, ri] + [rp, xp]$.

The Machine Proof

$-[ir, ix] + [pr, px]$

(Since $i$, $r$, $s$ are collinear; $i$, $x$, $y$ are collinear; $[ir, ix] = -[yx, sr]$.)

$= [yx, sr] + [pr, px]$

(Since $q$, $s$, $y$ are collinear; $y$, $x$, $q$, $p$ are cyclic;
$[yx, sr] = [yx, yq] + [qs, sr] = [sq, sr] - [qp, px]$.)

$= [sq, sr] - [qp, px] + [pr, px]$

(Since $s$, $q$, $r$, $p$ are cyclic; $[sq, sr] = [qp, pr]$.)

$= [qp, pr] - [qp, px] + [pr, px]$

(Since $qp \parallel qp$; $[qp, pr] - [qp, px] = -[pr, px]$.)

$= 0$

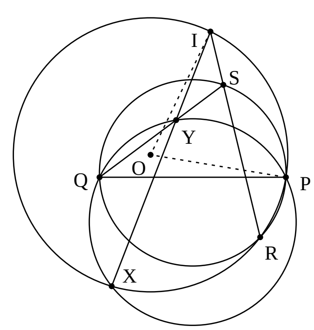


Figure 24

**Example 25** *The nine-point circle cuts the sides of the triangle at angles $|B-C|$, $|C-A|$, and $|A-B|$.*

Point order: $a, b, c, f, m, q, p, l, s, n$.

Hypotheses: foot$(f, c, a, b)$, midpoint$(m, b, c)$, midpoint$(q, a, c)$, midpoint$(p, a, b)$, midpoint$(l, f, p)$, midpoint$(s, p, q)$, perp$(n, l, a, b)$, perp$(n, s, p, q)$.

Conclusion: $[bc, ab] + [ac, ab] + [fn, ln]$.

The Machine Proof

$-[nl, nf] + [cb, ba] + [ca, ba]$

(Since $nl \parallel cf$; $[nl, nf] = -[nf, fc]$.)

$= [nf, fc] + [cb, ba] + [ca, ba]$

(Since $fc \perp ab$; $[nf, fc] = [nf, ba] + 1$.)

$= [nf, ba] + [cb, ba] + [ca, ba] + 1$

(Since circumcenter($n$,$f$, $p$, $q$ ); $[nf, ba] = [nf, fp] = -[pq, qf] + 1$.)

$= -[pq, qf] + [cb, ba] + [ca, ba]$

(Since $pq \parallel cb$; $-[pq, qf] + [cb, ba] = [qf, ba]$.)

$= [qf, ba] + [ca, ba]$

(Since circumcenter($q$,$f$, $a$, $c$ ); $[qf, ba] = [qf, fa] = [fc, ca] + 1$.)

$= [fc, ca] + [ca, ba] + 1$

(Since $ca \parallel ca$; $[fc, ca] + [ca, ba] = [fc, ba]$.)

$= [fc, ba] + 1$

(Since $fc \perp ba$; $[fc, ba] = 1$.)

$= 0$

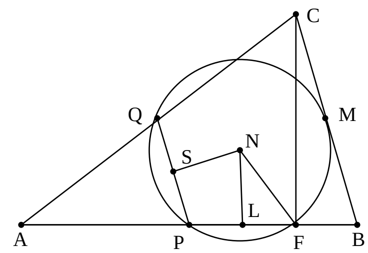


Figure 25

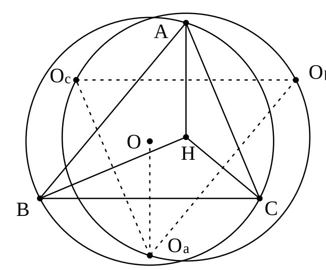


Figure 26

**Example 26** *Let $H$ be the orthocenter of triangle $ABC$. Then the circumcenters of the four triangles $ABC$, $ABH$, $ACH$, and $HBC$ form a triangle congruent to $ABC$; the sides of the two triangles are parallel.*

Point order: $a, b, c, h, o, p, q, r$.

Hypotheses: orthocenter$(h, a, b, c)$, circumcenter$(o, a, b, c)$, circumcenter$(r, b, c, h)$, circumcenter$(p, a, c, h)$, circumcenter$(q, a, b, h)$.

Conclusion: para$(p, c, q, b)$.

The Machine Proof

$-[qb, pc]$

(Since circumcenter($q$,$b$, $a$, $h$ ); $[qb, pc] = [qb, ba] + [ba, pc] = -[pc, ba] + [hb, ha] + 1$.)

$= [pc, ba] - [hb, ha] - 1$

(Since circumcenter($p$,$c$, $a$, $h$ ); $[pc, ba] = [pc, ca] + [ca, ba] = [hc, ha] + [ca, ba] + 1$.)

$= [hc, ha] - [hb, ha] + [ca, ba]$

(Since $hc \perp ab$; $ha \perp bc$; $[hc, ha] = -[cb, ba]$.)
$= -[hb, ha] - [cb, ba] + [ca, ba]$
(Since $hb \perp ac$; $ha \perp bc$; $[hb, ha] = -[cb, ca]$.)
$= [cb, ca] - [cb, ba] + [ca, ba]$
(Since $cb \parallel cb$; $[cb, ca] - [cb, ba] = -[ca, ba]$.)
$= 0$

**Example 27** *Continuing from Example 26, show that the point $H$ is the circumcenter of the triangle $O_aO_bO_c$.*

Point order: $a, b, c, h, o, p, q, r$.
Hypotheses: orthocenter$(h, a, b, c)$, circumcenter$(o, a, b, c)$, circumcenter$(r, b, c, h)$, circumcenter$(p, a, c, h)$, circumcenter$(q, a, b, h)$.
Conclusion: pbisector$(h, p, q)$.

The Machine Proof
$-[qp, qh] - [qp, ph]$
(Since $qp \parallel bc$; $[qp, qh] = -[qh, cb]$.)
$= -[qp, ph] + [qh, cb]$
(Since $qp \parallel bc$; $[qp, ph] = -[ph, cb]$.)
$= [qh, cb] + [ph, cb]$
(Since circumcenter($q$,$h$, $a$, $b$ ); $[qh, cb] = [qh, ha] + [ha, cb] = [hb, ba] + [ha, cb] + 1$.)
$= [ph, cb] + [hb, ba] + [ha, cb] + 1$
(Since circumcenter($p$,$h$, $a$, $c$ ); $[ph, cb] = [ph, ha] + [ha, cb] = [hc, ca] + [ha, cb] + 1$.)
$= [hc, ca] + [hb, ba] + 2[ha, cb]$
(Since $hc \perp ab$; $[hc, ca] = -[ca, ba] + 1$.)
$= [hb, ba] + 2[ha, cb] - [ca, ba] + 1$
(Since $hb \perp ac$; $[hb, ba] = [ca, ba] + 1$.)
$= 2[ha, cb]$
(Since $ha \perp cb$; $[ha, cb] = 1$.)
$= 0$

**Example 28** *The four projections of the foot of the altitude on a side of a triangle upon the other two sides and the other two altitudes are collinear.*

Point order: $a, b, c, f, h, p, q, t$.
Hypotheses: foot$(f, c, a, b)$, coll$(h, c, f)$, perp$(h, a, b, c)$, perp$(h, b, a, c)$, foot$(p, f, a, c)$, foot$(q, f, a, h)$, foot$(t, f, b, c)$.
Conclusion: coll$(p, q, t)$.

The Machine Proof

$-[tp, qp]$

(Since $b$, $c$, $t$ are collinear; $t$, $p$, $c$, $f$ are cyclic; $[tp, qp] = [tp, tc] + [cb, qp] = -[qp, cb] + [pf, fc]$.)

$= [qp, cb] - [pf, fc]$

(Since $a$, $h$, $q$ are collinear; $q$, $p$, $a$, $f$ are cyclic; $[qp, cb] = [qp, qa] + [ah, cb] = [pf, fa] + [ha, cb]$.)

$= -[pf, fc] + [pf, fa] + [ha, cb]$

(Since $pf \parallel pf$; $-[pf, fc] + [pf, fa] = [fc, fa]$.)

$= [ha, cb] + [fc, fa]$

(Since $ha \perp cb$; $[ha, cb] = 1$.)

$= [fc, fa] + 1$

(Since $fc \perp fa$; $[fc, fa] = 1$.)

$= 0$

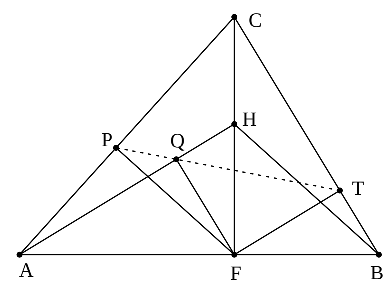


Figure 28

**Example 29** *$DP$, $DQ$ are the perpendiculars from the foot $D$ of the altitude $AD$ of the triangle $ABC$ upon the sides $AC$, $AB$. Prove that the points $B$, $C$, $P$, $Q$ are cyclic.*

Point order: $a, b, c, d, p, q$.
Hypotheses: foot$(d, a, b, c)$, foot$(p, d, a, c)$, foot$(q, d, a, b)$.
Conclusion: cyclic$(p, b, q, c)$.

The Machine Proof

$[qp, qb] - [pc, cb]$

(Since $a$, $b$, $q$ are collinear; $q$, $p$, $a$, $d$ are cyclic; $[qp, qb] = [pd, da]$.)

$= [pd, da] - [pc, cb]$

(Since $pd \perp ac$; $da \perp bc$; $[pd, da] = -[cb, ca]$.)

$= -[pc, cb] - [cb, ca]$

(Since $cb \parallel cb$; $-[pc, cb] - [cb, ca] = -[pc, ca]$.)

$= -[pc, ca]$

(Since $a$, $c$, $p$ are collinear; $[pc, ca] = 0$.)

$= 0$

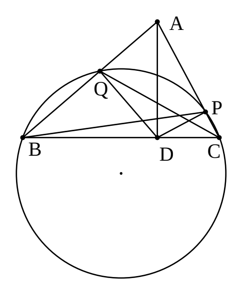


Figure 29

**Example 30** *The radii of the circumcircle passing through the vertices of a triangle are perpendicular to the corresponding sides of the orthic triangle.*

Point order: $a, b, c, e, f, o$.

Hypotheses: circumcenter$(o, a, b, c)$, foot$(e, b, a, c)$, foot$(f, c, a, b)$.

Conclusion: perp$(o, a, e, f)$.

The Machine Proof

$[oa, fe] + 1$

(Since circumcenter($o$,$a$, $b$, $c$ ); $[oa, fe] = [oa, ab] + [ab, fe] = -[fe, ba] - [cb, ca] + 1$.)

$= -[fe, ba] - [cb, ca]$

(Since $a$, $b$, $f$ are collinear; $f$, $e$, $b$, $c$ are cyclic; $[fe, ba] = [fe, fb] = [ec, cb]$.)

$= -[ec, cb] - [cb, ca]$

(Since $cb \parallel cb$; $-[ec, cb] - [cb, ca] = -[ec, ca]$.)

$= -[ec, ca]$

(Since $a$, $c$, $e$ are collinear; $[ec, ca] = 0$.)

$= 0$

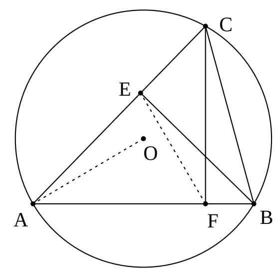


Figure 30

**Example 31** *The mediators of the sides $AC$, $AB$ of the triangle $ABC$ meet the sides $AB$, $AC$ in $P$ and $Q$. Prove that the points $B$, $C$, $P$, $Q$ lie on a circle.*

Point order: $a, b, c, o, b1, c1, p, q$.

Hypotheses: circumcenter$(o, a, b, c)$, midpoint$(c1, a, b)$, midpoint$(b1, a, c)$, coll$(p, o, c1)$, coll$(q, o, b1)$, coll$(p, a, c)$, coll$(q, a, b)$.

Conclusion: cyclic$(b, c, p, q)$.

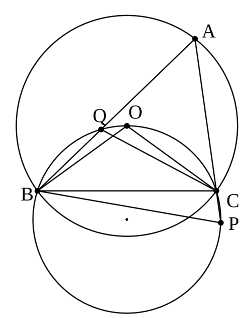


Figure 31

The Machine Proof

$[qc, qb] - [pc, pb]$

(Since $qb \perp oc1$; $[qc, qb] = [qc, c1o] + 1$.)

$= [qc, c1o] - [pc, pb] + 1$

(Since $c1o \perp ab$; $[qc, c1o] = [qc, ba] + 1$.)

$= [qc, ba] - [pc, pb]$

(Since $qc = qa$ $a$, $b$, $q$ are collinear; $[qc, ba] = [qc, ca] + [ca, ba] = 2[ca, ba]$.)

$= -[pc, pb] + 2[ca, ba]$

(Since $pc \perp ob1$; $[pc, pb] = -[pb, b1o] + 1$.)

$= [pb, b1o] + 2[ca, ba] - 1$

(Since $b1o \perp ac$; $[pb, b1o] = [pb, ca] + 1$.)

$= [pb, ca] + 2[ca, ba]$

(Since $pb = pa$ $a$, $c$, $p$ are collinear; $[pb, ca] = [pb, ba] + [ba, ca] = -2[ca, ba]$.)

$= 0$

**Example 32** *If two lines are antiparallel with respect to an angle, the perpendiculars dropped upon them from the vertex are isogonal in the angle considered.*

Point order: $b, c, r, s, o, a, m, n$.

Hypotheses: circumcenter$(o, b, c, r)$, cong$(o, b, o, s)$, coll$(a, r, b)$, coll$(a, s, c)$, foot$(m, a, r, s)$, foot$(n, a, b, c)$.

Conclusion: eqangle$(b, a, m, n, a, c)$.

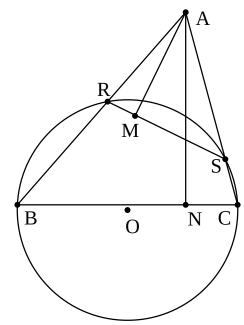


Figure 32

The Machine Proof

$-[na, ac] - [ma, ab]$

(Since $na \perp bc$; $[na, ac] = -[ac, cb] + 1$.)

$= -[ma, ab] + [ac, cb] - 1$

(Since $ma \perp rs$; $[ma, ab] = -[ab, sr] + 1$.)

$= [ac, cb] + [ab, sr]$

(Since $a$, $c$, $s$ are collinear; $[ac, cb] = [sc, cb]$.)

$= [ab, sr] + [sc, cb]$

(Since $a$, $b$, $r$ are collinear; $[ab, sr] = -[sr, rb]$.)

$= -[sr, rb] + [sc, cb]$

(Since $r$, $s$, $b$, $c$ are cyclic; $[sr, rb] = [sc, cb]$.)

$= 0$

**Example 33** *Show that the four perpendiculars to the sides of an angle at four concyclic points form a parallelogram whose opposite vertices lie on isogonal conjugate lines with respect to the given angle.*

Point order: $a, b, c, d, i, e, f$.

Hypotheses: cyclic$(a, b, c, d)$, coll$(i, a, b)$, coll$(i, c, d)$, perp$(e, d, d, c)$, perp$(e, a, a, b)$, perp$(f, b, b, a)$, perp$(f, c, c, d)$.

Conclusion: eqangle$(a, i, e, f, i, c)$.

The Machine Proof

$-[fi, ic] - [ei, ia]$

(Since $i$, $f$, $c$, $b$ are cyclic; $[fi, ic] = [fb, cb]$.)

$= -[fb, cb] - [ei, ia]$

(Since $fb \parallel ae$; $[fb, cb] = [ea, cb]$.)

$= -[ei, ia] - [ea, cb]$

(Since $i$, $e$, $a$, $d$ are cyclic; $[ei, ia] = [ed, da]$.)

$= -[ed, da] - [ea, cb]$

(Since $ed \perp cd$; $[ed, da] = [dc, da] + 1$.)

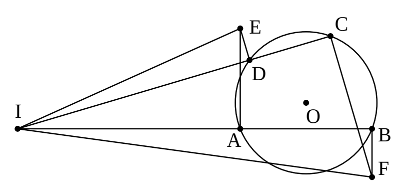


Figure 33

$= -[ea, cb] - [dc, da] - 1$

(Since $ea \perp ab$; $[ea, cb] = -[cb, ba] + 1$.)

$= -[dc, da] + [cb, ba]$

(Since $d$, $c$, $a$, $b$ are cyclic; $[dc, da] = [cb, ba]$.)

$= 0$

**Example 34** *The perpendicular at the orthocenter $H$ to the altitude $HC$ of the triangle $ABC$ meets the circumcircle of $HBC$ in $P$. Show that $ABPH$ is a parallelogram.*

Point order: $a, b, c, h, o, p$.

Hypotheses: orthocenter$(h, a, b, c)$, circumcenter$(o, b, c, h)$, perp$(p, h, h, c)$, cong$(p, o, o, b)$.

Conclusion: para$(a, h, b, p)$.

The Machine Proof

$-[pb, ha]$

(Since $ha \perp bc$; $[pb, ha] = [pb, cb] + 1$.)

$= -[pb, cb] - 1$

(Since $ph \perp hc$; $pb \perp bc$; $[pb, cb] = [pb, bc] + [bc, cb] = -[cb, cb] + 1$.)

$= [cb, cb]$

(Since $[cb, cb] = 0$.)

$= 0$

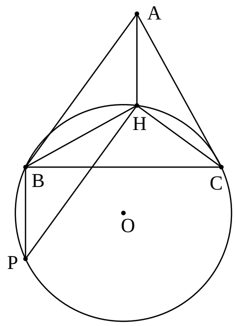


Figure 34

**Example 35** *The segment of the altitude extended between the orthocenter and the second point of intersection with the circumcircle is bisected by the corresponding side of the triangle.*

Point order: $a, b, c, o, d, e, h, k$.

Hypotheses: circumcenter$(o, a, b, c)$, foot$(d, c, a, b)$, foot$(e, b, a, c)$, coll$(h, b, e)$, coll$(h, c, d)$, coll$(k, c, d)$, cong$(k, o, a, o)$.

Conclusion: pbisector$(a, h, k)$.

The Machine Proof

$-[kh, ka] - [kh, ha]$

(Since $kh \perp ab$; $[kh, ka] = -[ka, ba] + 1$.)

$= -[kh, ha] + [ka, ba] - 1$

(Since $c$, $d$, $h$, $k$ are collinear; $[kh, ha] = -[ha, dc]$.)

$= [ka, ba] + [ha, dc] - 1$

(Since $c$, $d$, $k$ are collinear; $k$, $a$, $c$, $b$ are cyclic;
$[ka, ba] = [ka, kc] + [cd, ba] = [dc, ba] - [cb, ba]$.)

$= [ha, dc] + [dc, ba] - [cb, ba] - 1$

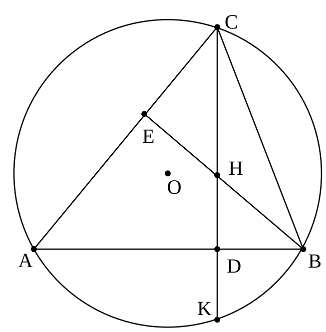


Figure 35

(Since $dc \parallel dc$; $[ha, dc] + [dc, ba] = [ha, ba]$.)
$= [ha, ba] - [cb, ba] - 1$
(Since $b$, $e$, $h$ are collinear; $h$, $a$, $e$, $d$ are cyclic;
$[ha, ba] = [ha, he] + [eb, ba] = -[ed, da] + [eb, ba]$.)
$= -[ed, da] + [eb, ba] - [cb, ba] - 1$
(Since $a$, $b$, $d$ are collinear; $d$, $e$, $b$, $c$ are cyclic; $[ed, da] = [ec, cb]$.)
$= -[ec, cb] + [eb, ba] - [cb, ba] - 1$
(Since $cb \parallel cb$; $-[ec, cb] - [cb, ba] = -[ec, ba]$.)
$= -[ec, ba] + [eb, ba] - 1$
(Since $a$, $c$, $e$ are collinear; $[ec, ba] = [ca, ba]$.)
$= [eb, ba] - [ca, ba] - 1$
(Since $eb \perp ac$; $[eb, ba] = [ca, ba] + 1$.)
$= 0$

**Example 36** *The circumcircle of the triangle formed by two vertices and the orthocenter of a given triangle is equal to the circumcircle of the given triangle.*

Point order: $a, b, c, o, d, e, h, o1$.
Hypotheses: circumcenter$(o, a, b, c)$, foot$(d, c, a, b)$, foot$(e, b, a, c)$, coll$(h, b, e)$, coll$(h, c, d)$, circumcenter$(o1, a, b, h)$.
Conclusion: pbisector$(a, o, o1)$.

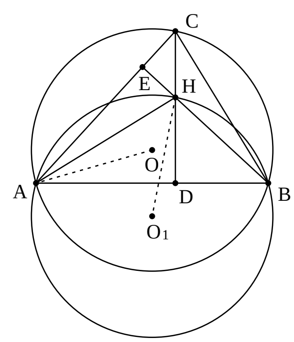

Figure 36

The Machine Proof
$-[o1o, o1a] - [o1o, oa]$
(Since $o1o \parallel cd$; $[o1o, o1a] = -[o1a, dc]$.)
$= -[o1o, oa] + [o1a, dc]$
(Since $o1o \parallel cd$; $[o1o, oa] = [dc, oa]$.)
$= [o1a, dc] - [dc, oa]$
(Since $dc \perp ab$; $[o1a, dc] = [o1a, ba] + 1$.)
$= [o1a, ba] - [dc, oa] + 1$
(Since circumcenter($o1$,$a$, $b$, $h$ ); $[o1a, ba] = [o1a, ab] = -[hb, ha] + 1$.)
$= -[hb, ha] - [dc, oa]$
(Since $hb \perp ac$; $[hb, ha] = -[ha, ca] + 1$.)
$= [ha, ca] - [dc, oa] - 1$
(Since $b$, $e$, $h$ are collinear; $h$, $a$, $e$, $d$ are cyclic;
$[ha, ca] = [ha, he] + [eb, ca] = -[ed, da] + [eb, ca]$.)
$= -[ed, da] + [eb, ca] - [dc, oa] - 1$
(Since $a$, $b$, $d$ are collinear; $d$, $e$, $b$, $c$ are cyclic; $[ed, da] = [ec, cb]$.)
$= -[ec, cb] + [eb, ca] - [dc, oa] - 1$
(Since $a$, $c$, $e$ are collinear; $[ec, cb] = -[cb, ca]$.)

$= [eb, ca] - [dc, oa] + [cb, ca] - 1$

(Since $eb \perp ca$; $[eb, ca] = 1$.)

$= -[dc, oa] + [cb, ca]$

(Since $dc \perp ab$; $[dc, oa] = -[oa, ba] + 1$.)

$= [oa, ba] + [cb, ca] - 1$

(Since circumcenter($o$,$a$, $b$, $c$ ); $[oa, ba] = [oa, ab] = -[cb, ca] + 1$.)

$= 0$

**Example 37** *A vertex of a triangle is the midpoint of the arc determined on its circumcircle by the two altitudes, produced, issued from the other two vertices.*

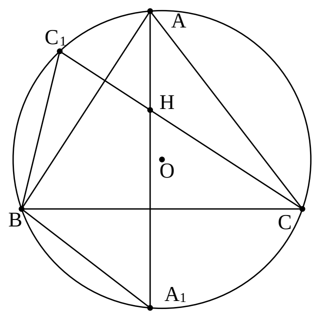

Figure 37

Point order: $a, b, c, o, h, a1, c1$.

Hypotheses: circumcenter$(o, a, b, c)$, orthocenter$(h, a, b, c)$, coll$(a1, a, h)$, cong$(o, a, o, a1)$, coll$(c1, c, h)$, cong$(o, a, o, c1)$.

Conclusion: pbisector$(b, a1, c1)$.

The Machine Proof

$-[c1a1, c1b] - [c1a1, a1b]$

(Since $c1$, $a1$, $b$, $a$ are cyclic; $[c1a1, c1b] = [a1a, ba]$.)

$= -[c1a1, a1b] - [a1a, ba]$

(Since $a1$, $c1$, $b$, $a$ are cyclic; $[c1a1, a1b] = [c1a, ba]$.)

$= -[c1a, ba] - [a1a, ba]$

(Since $c$, $c1$, $h$ are collinear; $c1$, $a$, $c$, $a1$ are cyclic; $[c1a, ba] = [c1a, c1c] + [ch, ba] = -[a1c, a1a] + [hc, ba]$.)

$= [a1c, a1a] - [a1a, ba] - [hc, ba]$

(Since $a1a \perp bc$; $[a1c, a1a] = [a1c, cb] + 1$.)

$= [a1c, cb] - [a1a, ba] - [hc, ba] + 1$

(Since $c$, $a1$, $b$, $a$ are cyclic; $[a1c, cb] = [a1a, ba]$.)

$= -[hc, ba] + 1$

(Since $hc \perp ba$; $[hc, ba] = 1$.)

$= 0$

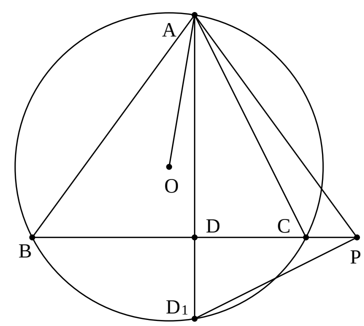

Figure 38

**Example 38** *If $D_1$ is the second point of intersection of the altitude $ADD_1$ of the triangle $ABC$ with the circumcircle, center $O$, and $P$ is the trace on $BC$ of the perpendicular from $D_1$ to $AC$, show that the lines $AP$, $AO$ make equal angles with the bisector of the angle $DAC$.*

Point order: $a, b, c, o, d, d1, p$.

Hypotheses: circumcenter$(o, a, b, c)$, foot$(d, a, b, c)$, coll$(d1, a, d)$,

cong$(d1, o, o, a)$, coll$(p, b, c)$, perp$(p, d1, a, c)$.
Conclusion: eqangle$(o, a, d, c, a, p)$.

The Machine Proof
$[pa, ca] - [da, oa]$
(Since $p$ is the orthocenter of triangle $c$, $d1$, $a$
$[pa, ca] = [pa, cd1] + [cd1, ca] = [d1c, ca] + 1$.)
$= [d1c, ca] - [da, oa] + 1$
(Since $c$, $d1$, $a$, $b$ are cyclic; $[d1c, ca] = [d1b, ba]$.)
$= [d1b, ba] - [da, oa] + 1$
(Since $a$, $d$, $d1$ are collinear; $d1$, $b$, $a$, $c$ are cyclic;
$[d1b, ba] = [d1b, d1a] + [ad, ba] = [da, ba] + [cb, ca]$.)
$= -[da, oa] + [da, ba] + [cb, ca] + 1$
(Since $da \parallel da$; $-[da, oa] + [da, ba] = [oa, ba]$.)
$= [oa, ba] + [cb, ca] + 1$
(Since circumcenter($o$,$a$, $b$, $c$ ); $[oa, ba] = [oa, ab] = -[cb, ca] + 1$.)
$= 0$

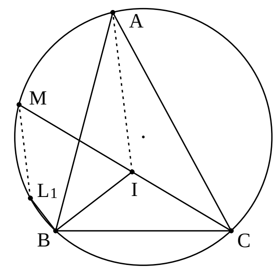

Figure 39

**Example 39** *Show that an external bisector of an angle of a triangle is parallel to the line joining the points where the circumcircle is met by the external (internal) bisectors of the other two angles of the triangle.*

Point order: $a, b, c, i, m, l_1$.
Hypotheses: incenter$(i, a, b, c)$, cyclic$(a, b, c, m, l_1)$, coll$(m, i, c)$, perp$(b, l_1, b, i)$.
Conclusion: para$(m, l_1, a, i)$.

The Machine Proof
$[l_1m, ia]$
(Since $l_1$, $m$, $b$, $c$ are cyclic; $l_1b \perp ib$;
$[l_1m, ia] = [l_1m, l_1b] + [l_1b, ia] = [mc, cb] + [ib, ia] + 1$.)
$= [mc, cb] + [ib, ia] + 1$
(Since $c$, $i$, $m$ are collinear; $[mc, cb] = [ic, cb]$.)
$= [ic, cb] + [ib, ia] + 1$
(Since $i$ is the incenter of triangle $c$, $b$, $a$ $[ic, cb] = -[ic, ca]$.)
$= -[ic, ca] + [ib, ia] + 1$
(Since $i$ is the incenter of triangle $c$, $b$, $a$ $[ic, ca] = [ib, ba] - [ia, ba] + 1$.)
$= [ib, ia] - [ib, ba] + [ia, ba]$
(Since $ib \parallel ib$; $[ib, ia] - [ib, ba] = -[ia, ba]$.)
$= 0$

**Example 40** *Show that a parallel through a tritangent center to a side of a triangle is equal to the sum, or difference, of the two segments on the other two sides of the triangle between the two parallel lines considered.*

Point order: $a, b, c, i, m.$

Hypotheses: incenter$(i, a, b, c)$, coll$(m, a, c)$, para$(i, m, a, b)$.

Conclusion: pbisector$(m, a, i)$.

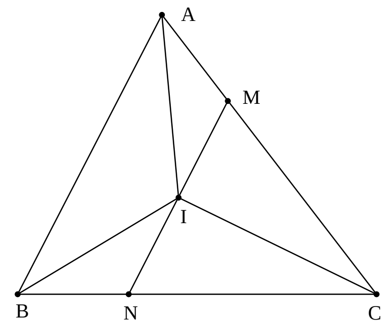

Figure 40

The Machine Proof

$[mi, ia] + [ma, ia]$

(Since $mi \parallel ab$; $[mi, ia] = -[ia, ba]$.)

$= [ma, ia] - [ia, ba]$

(Since $a$, $c$, $m$ are collinear; $[ma, ia] = -[ia, ca]$.)

$= -[ia, ca] - [ia, ba]$

(Since $i$ is the incenter of triangle $a$, $c$, $b$ $[ia, ca] = -[ia, ba]$.)

$= 0$

**Example 41** *The projection of the vertex $B$ of the triangle $ABC$ upon the internal bisector of the angle $A$ lies on the line joining the points of contact of the incircle with the sides $BC$ and $AC$. State and prove an analogous proposition for the external bisectors.*

Point order: $a, b, c, i, x, y, l.$

Hypotheses: incenter$(i, a, b, c)$, foot$(x, i, b, c)$, foot$(y, i, a, c)$, foot$(l, b, a, i)$.

Conclusion: coll$(x, y, l)$.

The Machine Proof

$-[lx, yx]$

(Since $a$, $i$, $l$ are collinear; $l$, $x$, $i$, $b$ are cyclic;

$[lx, yx] = [lx, li] + [ia, yx] = -[yx, ia] + [xb, ib]$.)

$= [yx, ia] - [xb, ib]$

(Since $a$, $c$, $y$ are collinear; $y$, $x$, $c$, $i$ are cyclic; $[yx, ia] = [yx, yc] + [ca, ia] = [xi, ic] - [ia, ca]$.)

$= [xi, ic] - [xb, ib] - [ia, ca]$

(Since $xi \perp bc$; $[xi, ic] = -[ic, cb] + 1$.)

$= -[xb, ib] - [ic, cb] - [ia, ca] + 1$

(Since $xb \parallel cb$; $-[xb, ib] - [ic, cb] = -[ic, ib]$.)

$= -[ic, ib] - [ia, ca] + 1$

(Since $i$ is the incenter of triangle $a$, $c$, $b$ $[ic, ib] = [ia, ba] + 1$.)

$= -[ia, ca] - [ia, ba]$

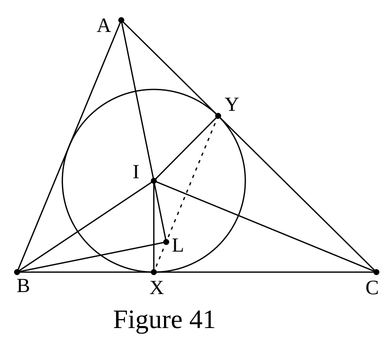

Figure 41

(Since $i$ is the incenter of triangle $a$, $c$, $b$ $[ia, ca] = -[ia, ba]$.)
$= 0$

**Example 42** *The midpoint of a side of a triangle, the foot of the altitude on this side, and the projections of the ends of this side upon the internal bisector of the opposite angle are four cyclic points.*

Point order: $a, b, c, i, a1, x, x1, y, d$.

Hypotheses: incenter$(i, a, b, c)$, midpoint$(a1, b, c)$, foot$(x, b, a, i)$, midpoint$(x, b, x1)$, coll$(x1, a, c)$, foot$(y, c, a, i)$, foot$(d, a, b, c)$.

Conclusion: cyclic$(x, y, d, a1)$.

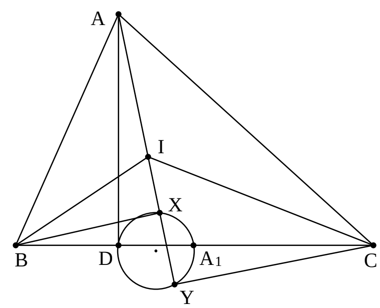

Figure 42

The Machine Proof

$-[dy, yx] + [da1, xa1]$

(Since $yx \perp bx$; $[dy, yx] = [dy, xb] + 1$.)

$= -[dy, xb] + [da1, xa1] - 1$

(Since $xb \perp ai$; $[dy, xb] = [dy, ia] + 1$.)

$= -[dy, ia] + [da1, xa1]$

(Since $b$, $c$, $d$ are collinear; $d$, $y$, $c$, $a$ are cyclic; $[dy, ia] = [dy, dc] + [cb, ia] = [ya, ca] - [ia, cb]$.)

$= [da1, xa1] - [ya, ca] + [ia, cb]$

(Since $ya \parallel ia$; $-[ya, ca] + [ia, cb] = -[cb, ca]$.)

$= [da1, xa1] - [cb, ca]$

(Since $xa1 \parallel ca$; $[da1, xa1] - [cb, ca] = [da1, cb]$.)

$= [da1, cb]$

(Since $da1 \parallel cb$; $[da1, cb] = 0$.)

$= 0$

**Example 43** *On the hypotenuse $AB$ of right triangle $ABC$ a square $ABFE$ is erected. Let $P$ be the intersection of the diagonals $AF$ and $BE$ of $ABFE$. Show that $\angle ACP = \angle PCB$.*

Point order: $a, b, c, p$.

Hypotheses: perp$(c, a, c, b)$, perp$(p, a, p, b)$, cong$(p, a, p, b)$.

Conclusion: eqangle$(a, c, p, p, c, b)$.

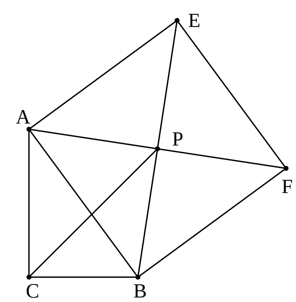

Figure 43

The Machine Proof

$-[pc, cb] - [pc, ca]$

(Since $c$, $p$, $b$, $a$ are cyclic; $[pc, cb] = [pa, ba]$.)

$= -[pc, ca] - [pa, ba]$

(Since $c$, $p$, $a$, $b$ are cyclic; $[pc, ca] = [pb, ba]$.)

$= -[pb, ba] - [pa, ba]$

(Since $pb = pa$ $[pb, ba] = -[pa, ba]$.)

$= 0$

**Example 44** *In the cyclic quadrilateral $ABCD$ the perpendicular to $AB$ at $A$ meets $CD$ in $A_1$, and the perpendicular to $CD$ at $C$ meets $AB$ in $C_1$. Show that the line $A_1C_1$ is parallel to the diagonal $BD$.*

Point order: $a, b, c, d, a1, c1$.

Hypotheses: cyclic$(a, b, c, d)$, coll$(a1, c, d)$, perp$(a1, a, a, b)$, coll$(c1, a, b)$, perp$(c1, c, c, d)$.

Conclusion: para$(a1, c1, b, d)$.

The Machine Proof

$[c1a1, db]$

(Since $a$, $b$, $c1$ are collinear; $c1$, $a1$, $a$, $c$ are cyclic; $[c1a1, db] = [c1a1, c1a] + [ab, db] = [a1c, ca] - [db, ba]$.)

$= [a1c, ca] - [db, ba]$

(Since $a1$, $c$, $d$ are collinear; $[a1c, ca] = [dc, ca]$.)

$= [dc, ca] - [db, ba]$

(Since $c$, $d$, $a$, $b$ are cyclic; $[dc, ca] = [db, ba]$.)

$= 0$

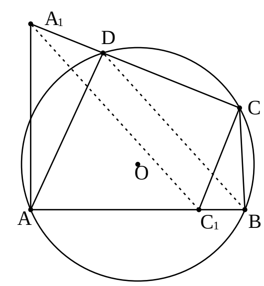


Figure 44

**Example 45** *Show that the perpendicular from the point of intersection of two opposite sides, produced, of a cyclic quadrilateral upon the line joining the midpoints of the two sides considered passes through the anticenter of the quadrilateral.*

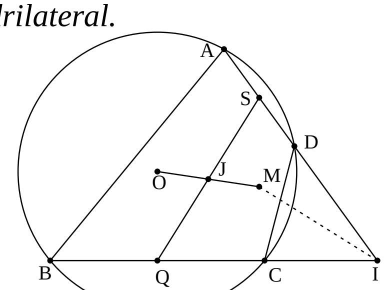

Figure 45

Point order: $a, b, c, d, o, i, q, s, j, m$.

Hypotheses: circumcenter$(o, a, b, c, d)$, coll$(i, a, d)$, coll$(i, b, c)$, midpoint$(q, b, c)$, midpoint$(s, a, d)$, midpoint$(j, s, q)$, midpoint$(j, o, m)$.

Conclusion: perp$(i, m, q, s)$.

The Machine Proof

$[mi, sq] + 1$

(Since $m$ is the orthocenter of triangle $q$, $s$, $i$ $[mi, sq] = [mi, qs] + [qs, sq] = -[sq, sq] + 1$.)

$= -[sq, sq]$

(Since $[sq, sq] = 0$.)

$= 0$

**Example 46** *Show that the product of the distances of two opposite sides of a cyclic quadrilateral from a point on the circumcircles is equal to the product of the distances of the other two sides from the same point.*

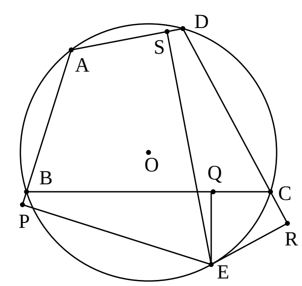

Figure 46

Point order: $a, b, c, d, e, p, q, s, r$.

Hypotheses: cyclic$(a, b, c, d, e)$, foot$(p, e, a, b)$, foot$(q, e, b, c)$, foot$(r, e, c, d)$, foot$(s, e, a, d)$.

Conclusion: eqangle$(e, p, s, e, q, r)$.

The Machine Proof

$[rq, qe] - [sp, pe]$

(Since $qe \perp bc$; $[rq, qe] = [rq, cb] + 1$.)

$= [rq, cb] - [sp, pe] + 1$

(Since $c$, $d$, $r$ are collinear; $r$, $q$, $c$, $e$ are cyclic; $[rq, cb] = [rq, rc] + [cd, cb] = [qe, ec] + [dc, cb]$.)

$= -[sp, pe] + [qe, ec] + [dc, cb] + 1$

(Since $pe \perp ab$; $[sp, pe] = [sp, ba] + 1$.)

$= -[sp, ba] + [qe, ec] + [dc, cb]$

(Since $a$, $d$, $s$ are collinear; $s$, $p$, $a$, $e$ are cyclic; $[sp, ba] = [sp, sa] + [ad, ba] = [pe, ea] + [da, ba]$.)

$= [qe, ec] - [pe, ea] + [dc, cb] - [da, ba]$

(Since $qe \perp bc$; $[qe, ec] = -[ec, cb] + 1$.)

$= -[pe, ea] - [ec, cb] + [dc, cb] - [da, ba] + 1$

(Since $pe \perp ab$; $[pe, ea] = -[ea, ba] + 1$.)

$= -[ec, cb] + [ea, ba] + [dc, cb] - [da, ba]$

(Since $c$, $e$, $b$, $a$ are cyclic; $[ec, cb] = [ea, ba]$.)

$= [dc, cb] - [da, ba]$

(Since $c$, $d$, $b$, $a$ are cyclic; $[dc, cb] = [da, ba]$.)

$= 0$

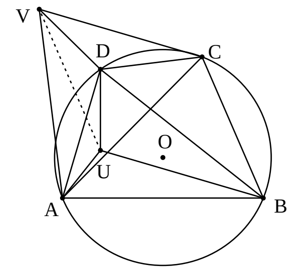


Figure 47

**Example 47** *A line $AD$ through the vertex $A$ meets the circumcircle of the triangle $ABC$ in $D$. If $U$, $V$ are the orthocenters of the triangle $ABD$, $ACD$, respectively, prove that $UV$ is equal and parallel to $BC$.*

Point order: $a, b, c, d, u, v$.

Hypotheses: cyclic$(a, b, c, d)$, orthocenter$(u, a, b, d)$, orthocenter$(v, a, c, d)$.

Conclusion: cyclic$(u, v, a, d)$.

The Machine Proof

$[vd, ud] - [va, ua]$

(Since $vd \perp ac$; $ud \perp ab$; $[vd, ud] = [ca, ba]$.)

$= -[va, ua] + [ca, ba]$

(Since $va \perp cd$; $ua \perp bd$; $[va, ua] = [dc, db]$.)

$= -[dc, db] + [ca, ba]$

(Since $d$, $c$, $b$, $a$ are cyclic; $[dc, db] = [ca, ba]$.)

$= 0$

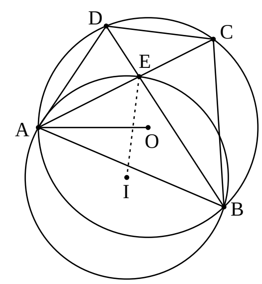


Figure 48

**Example 48** *Let $E$ be the intersection of the two diagonals $AC$ and $BD$ of cyclic quadrilateral $ABCD$. Let $I$ be the center of circumcircle of $ABE$. Show the $IE \perp DC$.*

Point order: $a, b, c, d, e, i$.

Hypotheses: cyclic$(a, b, c, d)$, coll$(e, a, c)$, coll$(e, b, d)$, circumcenter$(i, a, b, e)$.

Conclusion: perp$(i, e, c, d)$.

The Machine Proof

$[ie, dc] + 1$

(Since circumcenter($i$,$e$, $a$, $b$ ); $[ie, dc] = [ie, ea] + [ea, dc] = [eb, ba] + [ea, dc] + 1$.)

$= [eb, ba] + [ea, dc]$

(Since $b$, $d$, $e$ are collinear; $[eb, ba] = [db, ba]$.)

$= [ea, dc] + [db, ba]$

(Since $a$, $c$, $e$ are collinear; $[ea, dc] = -[dc, ca]$.)

$= -[dc, ca] + [db, ba]$

(Since $c$, $d$, $a$, $b$ are cyclic; $[dc, ca] = [db, ba]$.)

$= 0$

**Example 49** *In an orthodiagonal quadrilateral the midpoints of the sides lie on a circle having for center the centroid of the quadrilateral.*

Point order: $a, b, c, d, p, q, s, r, o$.

Hypotheses: perp$(a, c, b, d)$, midpoint$(p, a, b)$, midpoint$(q, b, c)$, midpoint$(s, a, d)$, midpoint$(r, c, d)$, coll$(o, p, r)$, coll$(o, q, s)$.

Conclusion: pbisector$(o, s, r)$.

The Machine Proof

$[or, rs] + [os, rs]$

(Since $o$, $p$, $r$ are collinear; $[or, rs] = -[rs, rp]$.)

$= [os, rs] - [rs, rp]$

(Since $o$, $q$, $s$ are collinear; $[os, rs] = -[rs, sq]$.)

$= -[rs, rp] - [rs, sq]$

(Since $rs \parallel ac$; $[rs, rp] = -[rp, ca]$.)

$= -[rs, sq] + [rp, ca]$

(Since $rs \parallel ac$; $[rs, sq] = -[sq, ca]$.)

$= [rp, ca] + [sq, ca]$

(Since $r$, $p$, $q$, $s$ are cyclic; $rq \perp ac$;
$[rp, ca] = [rp, rq] + [rq, ca] = -[sq, sp] - [ca, ca] + 1$.)

$= -[sq, sp] + [sq, ca] - [ca, ca] + 1$

(Since $sq \parallel sq$; $-[sq, sp] + [sq, ca] = [sp, ca]$.)

$= [sp, ca] - [ca, ca] + 1$

(Since $sp \perp ca$; $[sp, ca] = 1$.)

$= -[ca, ca]$

(Since $[ca, ca] = 0$.)

$= 0$

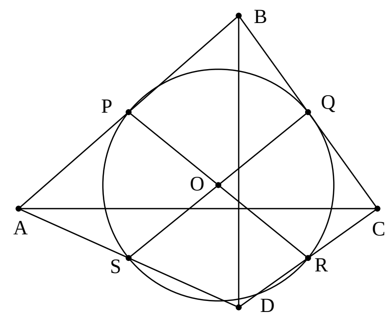

Figure 49

**Example 50** *If an orthodiagonal quadrilateral is cyclic, the anticenter coincides with the point of intersection of its diagonals.*

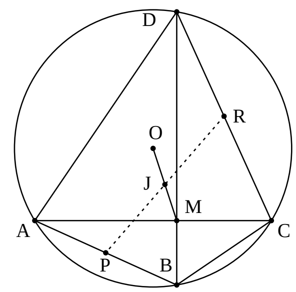

Figure 50

Point order: $a, b, c, d, o, m, p, r.$

Hypotheses: circumcenter$(o, a, b, c, d)$, perp$(a, c, b, d)$, coll$(m, a, c)$, coll$(m, b, d)$, midpoint$(p, a, b)$, midpoint$(r, c, d)$.

Conclusion: para$(o, p, r, m)$.

The Machine Proof

$-[rm, po]$

(Since $po \perp ab$; $[rm, po] = [rm, ba] + 1$.)

$= -[rm, ba] - 1$

(Since circumcenter($r$,$m$, $c$, $d$ ); $[rm, ba] = [rm, mc] + [mc, ba] = [md, dc] + [mc, ba] + 1$.)

$= -[md, dc] - [mc, ba]$

(Since $b$, $d$, $m$ are collinear; $[md, dc] = -[dc, db]$.)

$= -[mc, ba] + [dc, db]$

(Since $a$, $c$, $m$ are collinear; $[mc, ba] = [ca, ba]$.)

$= [dc, db] - [ca, ba]$

(Since $db \perp ac$; $[dc, db] = [dc, ca] + 1$.)

$= [dc, ca] - [ca, ba] + 1$

(Since $c$, $d$, $a$, $b$ are cyclic; $[dc, ca] = [db, ba]$.)

$= [db, ba] - [ca, ba] + 1$

(Since $db \perp ac$; $[db, ba] = [ca, ba] + 1$.)

$= 0$

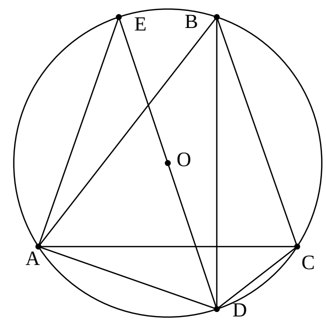

Figure 51

**Example 51** *If the diagonals of a cyclic quadrilateral $ABCD$ are orthogonal, and $E$ is the diametric opposite of $D$ on its circumcircle, show that $AE = CB$.*

Point order: $a, b, c, d, o, e.$

Hypotheses: circumcenter$(o, a, b, c, d, e)$, perp$(a, c, b, d)$, coll$(e, d, o)$.

Conclusion: para$(b, e, a, c)$.

The Machine Proof

$[eb, ca]$

(Since circumcenter($o$,$e$, $b$, $a$ ); $[eb, ca] = [eb, oa] + [ba, ca] = [ea, oa] - [ca, ba] + 1$.)

$= [ea, oa] - [ca, ba] + 1$

(Since $d$, $e$, $o$ are collinear; $e$, $a$, $d$, $b$ are cyclic;
$[ea, oa] = [ea, ed] + [do, oa] = [od, oa] - [db, ba]$.)

$= [od, oa] - [db, ba] - [ca, ba] + 1$

(Since circumcenter($o$,$d$, $a$, $b$ ); $[od, oa] = [od, da] + [da, oa] = -[oa, da] + [db, ba] + 1$.)

$= -[oa, da] - [ca, ba]$

(Since circumcenter($o$,$a$, $d$, $b$ ); $[oa, da] = [oa, ad] = -[db, ba] + 1$.)

$= [db, ba] - [ca, ba] - 1$

(Since $db \perp ac$; $[db, ba] = [ca, ba] + 1$.)

$= 0$

**Example 52** *Let $D$ be a point on the side $CB$ of a right triangle $ABC$ such that the circle $(O)$ with diameter $CD$ touches the hypotenuse $AB$ at $E$. Let $F = AC \cap DE$. Show that $AF = AE$.*

Point order: $c, d, e, o, a, f$.

Hypotheses: perp$(c, e, d, e)$, midpoint$(o, c, d)$, perp$(a, c, c, d)$, perp$(a, e, e, o)$, coll$(f, a, c)$, coll$(f, d, e)$.

Conclusion: pbisector$(a, e, f)$.

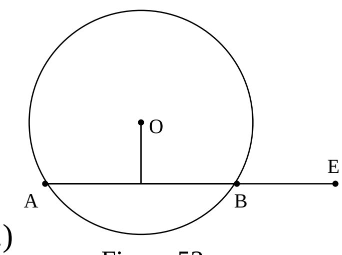


Figure 52

The Machine Proof

$[fa, fe] - [fe, ae]$

(Since $a$, $c$, $f$ are collinear; $d$, $e$, $f$ are collinear; $[fa, fe] = [ac, ed]$.)

$= -[fe, ae] + [ac, ed]$

(Since $d$, $e$, $f$ are collinear; $[fe, ae] = -[ae, ed]$.)

$= [ae, ed] + [ac, ed]$

(Since $ae \perp eo$; $[ae, ed] = [oe, ed] + 1$.)

$= [ac, ed] + [oe, ed] + 1$

(Since $ac \perp cd$; $[ac, ed] = -[ed, dc] + 1$.)

$= [oe, ed] - [ed, dc]$

(Since $oe = od$ $c$, $d$, $o$ are collinear; $[oe, ed] = [oe, ed] + [ed, ed] = -[ed, ed] + [ed, dc]$.)

$= -[ed, ed]$

(Since $[ed, ed] = 0$.)

$= 0$

**Example 53** *Let $C$ be a point on a chord $AB$ of circle $O$. Let $D$ and $E$ be the intersections of perpendicular of $OC$ through $C$ with the two tangents of the circle at $A$ and $B$, respectively. Show that $CE = CD$.*

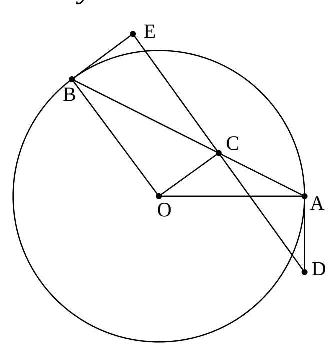


Figure 53

Point order: $o, a, b, c, e, d$.

Hypotheses: cong$(o, a, o, b)$, coll$(c, a, b)$, perp$(e, b, b, o)$, perp$(e, c, c, o)$, perp$(d, a, a, o)$, coll$(e, c, d)$.

Conclusion: pbisector$(o, e, d)$.

The Machine Proof

$-[de, do] - [de, eo]$

(Since $de \perp oc$; $[de, do] = -[do, co] + 1$.)
$= -[de, eo] + [do, co] - 1$
(Since $c$, $d$, $e$ are collinear; $[de, eo] = [ec, eo]$.)
$= [do, co] - [ec, eo] - 1$
(Since $c$, $d$, $e$ are collinear; $d$, $o$, $c$, $a$ are cyclic; $[do, co] = [do, dc] + [ce, co] = [ec, co] - [ca, ao]$.)
$= -[ec, eo] + [ec, co] - [ca, ao] - 1$
(Since $ec \parallel ec$; $-[ec, eo] + [ec, co] = [eo, co]$.)
$= [eo, co] - [ca, ao] - 1$
(Since $e$, $o$, $b$, $c$ are cyclic; $eb \perp bo$;
$[eo, co] = [eo, eb] + [eb, co] = -[cb, co] - [co, bo] + 1$.)
$= -[cb, co] - [ca, ao] - [co, bo]$
(Since $co \parallel co$; $-[cb, co] - [co, bo] = -[cb, bo]$.)
$= -[cb, bo] - [ca, ao]$
(Since $a$, $b$, $c$ are collinear; $[cb, bo] = [ba, bo]$.)
$= -[ca, ao] - [ba, bo]$
(Since $a$, $b$, $c$ are collinear; $[ca, ao] = [ba, ao]$.)
$= -[ba, bo] - [ba, ao]$
(Since $ob = oa$ $[ba, bo] = -[ba, ao]$.)
$= 0$

**Example 54** *Let $G$ be a point on the circle $(O)$ with diameter $BC$, $A$ be the midpoint of the arc $BG$. $AD \perp BC$. $E = AD \cap BG$ and $F = AC \cap BG$. Show that $AE = BE (= EF)$.*

Point order: $a, b, c, g, o, m, d, e, f$.
Hypotheses: perp$(b, a, a, c)$, midpoint$(o, b, c)$, foot$(d, a, b, c)$, foot$(m, b, a, o)$, coll$(e, b, m)$, coll$(e, a, d)$, coll$(f, a, c)$, coll$(f, b, m)$.
Conclusion: pbisector$(e, a, b)$.

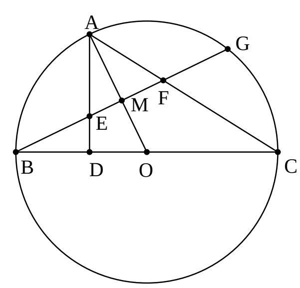

Figure 54

The Machine Proof
$[eb, ba] + [ea, ba]$
(Since $b$, $e$, $m$ are collinear; $[eb, ba] = [mb, ba]$.)
$= [ea, ba] + [mb, ba]$
(Since $a$, $d$, $e$ are collinear; $[ea, ba] = [da, ba]$.)
$= [da, ba] + [mb, ba]$
(Since $da \perp bc$; $[da, ba] = [cb, ba] + 1$.)
$= [mb, ba] + [cb, ba] + 1$
(Since $mb \perp ao$; $[mb, ba] = [oa, ba] + 1$.)
$= [oa, ba] + [cb, ba]$
(Since circumcenter($o$,$a$, $b$, $c$ ); $[oa, ba] = [oa, ab] = -[cb, ca] + 1$.)

$= -[cb, ca] + [cb, ba] + 1$

(Since $cb \parallel cb$; $-[cb, ca] + [cb, ba] = [ca, ba]$.)

$= [ca, ba] + 1$

(Since $ca \perp ba$; $[ca, ba] = 1$.)

$= 0$

**Example 55** *The same as the preceding example. Show that $EA = EM$.*

Point order: $a, b, c, g, o, m, d, e, f$.

Hypotheses: perp$(b, a, a, c)$, midpoint$(o, b, c)$, foot$(d, a, b, c)$,
foot$(m, b, a, o)$, coll$(e, b, m)$, coll$(e, a, d)$, coll$(f, a, c)$, coll$(f, b, m)$.

Conclusion: pbisector$(e, f, a)$.

The Machine Proof

$[fe, fa] - [fa, ea]$

(Since $b$, $e$, $f$, $m$ are collinear; $a$, $c$, $f$ are collinear; $[fe, fa] = [mb, ca]$.)

$= -[fa, ea] + [mb, ca]$

(Since $a$, $c$, $f$ are collinear; $[fa, ea] = -[ea, ca]$.)

$= [ea, ca] + [mb, ca]$

(Since $a$, $d$, $e$ are collinear; $[ea, ca] = [da, ca]$.)

$= [da, ca] + [mb, ca]$

(Since $da \perp bc$; $ca \perp ab$; $[da, ca] = [cb, ba]$.)

$= [mb, ca] + [cb, ba]$

(Since $mb \perp ao$; $ca \perp ab$; $[mb, ca] = [oa, ba]$.)

$= [oa, ba] + [cb, ba]$

(Since circumcenter($o$,$a$, $b$, $c$ ); $[oa, ba] = [oa, ab] = -[cb, ca] + 1$.)

$= -[cb, ca] + [cb, ba] + 1$

(Since $cb \parallel cb$; $-[cb, ca] + [cb, ba] = [ca, ba]$.)

$= [ca, ba] + 1$

(Since $ca \perp ba$; $[ca, ba] = 1$.)

$= 0$

**Example 56** *Let $M$ be the midpoint of the arc $AB$ of circle $(O)$, $D$ be the midpoint of $AB$. The perpendicular through $M$ is drawn to the tangent of the circle at $A$ meeting that tangent at $E$. Show $ME = MD$.*

Point order: $a, b, m, o, d, e$.

Hypotheses: cong$(m, a, m, b)$, cong$(o, a, o, m)$, cong$(o, a, o, b)$, midpoint$(d, a, b)$, coll$(m, o, d)$, perp$(e, a, a, o)$, para$(m, e, a, o)$.

Conclusion: pbisector$(m, e, d)$.

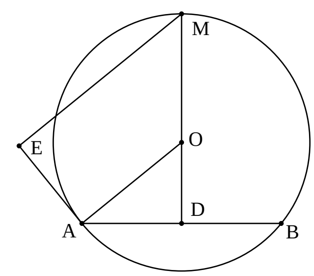


Figure 56

The Machine Proof

$-[ed, em] - [ed, dm]$

(Since $em \parallel ao$; $[ed, em] = [ed, oa]$.)

$= -[ed, dm] - [ed, oa]$

(Since $d$, $e$, $m$, $a$ are cyclic; $[ed, dm] = [ea, ma]$.)

$= -[ed, oa] - [ea, ma]$

(Since $e$, $d$, $a$, $m$ are cyclic; $ea \perp ao$; $[ed, oa] = [ed, ea] + [ea, oa] = [dm, ma] - [oa, oa] + 1$.)

$= -[ea, ma] - [dm, ma] + [oa, oa] - 1$

(Since $ea \perp ao$; $[ea, ma] = [oa, ma] + 1$.)

$= -[dm, ma] + [oa, oa] - [oa, ma]$

(Since $oa \parallel oa$; $[oa, oa] - [oa, ma] = -[oa, ma]$.)

$= -[dm, ma] - [oa, ma]$

(Since $dm \perp ab$; $[dm, ma] = -[ma, ba] + 1$.)

$= -[oa, ma] + [ma, ba] - 1$

(Since circumcenter($o$,$a$, $m$, $b$ ); $[oa, ma] = [oa, am] = -[mb, ba] + 1$.)

$= [mb, ba] + [ma, ba]$

(Since $mb = ma$ $[mb, ba] = -[ma, ba]$.)

$= 0$

**Example 57** *The circle with the altitude $AD$ of triangle $ABC$ as a diameter meets $AB$ and $AC$ at $E$ and $F$, respectively. Show that $B$, $C$, $E$ and $F$ are on the same circle.*

Point order: $a, b, c, d, o, e, f$.

Hypotheses: foot$(d, a, b, c)$, midpoint$(o, a, d)$, coll$(e, a, b)$, cong$(o, a, o, e)$, coll$(f, a, c)$, cong$(o, a, o, f)$.

Conclusion: cyclic$(b, c, e, f)$.

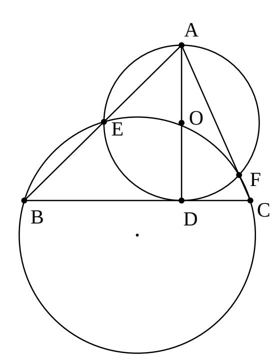


Figure 57

The Machine Proof

$[fe, eb] - [fc, cb]$

(Since $a$, $b$, $e$ are collinear; $e$, $f$, $a$, $d$ are cyclic; $[fe, eb] = [fd, da]$.)

$= [fd, da] - [fc, cb]$

(Since $fd \perp ac$; $da \perp bc$; $[fd, da] = -[cb, ca]$.)

$= -[fc, cb] - [cb, ca]$

(Since $cb \parallel cb$; $-[fc, cb] - [cb, ca] = -[fc, ca]$.)

$= -[fc, ca]$

(Since $a$, $c$, $f$ are collinear; $[fc, ca] = 0$.)

$= 0$

**Example 58** *Let $A$, $B$, $C$, $D$ be four points on circle $(O)$. $E = CD \cap AB$. $CB$ meets the line passing through $E$ and parallel to $AD$ at $F$. $GF$ is tangent to circle $(O)$ at $G$. Show that $FG = FE$.*

Point order: $a, b, c, d, e, f$.

Hypotheses: cyclic$(a, b, c, d)$, coll$(e, a, b)$, coll$(e, c, d)$, coll$(f, b, c)$, para$(f, e, a, d)$.

Conclusion: eqangle$(f, e, b, e, c, b)$.

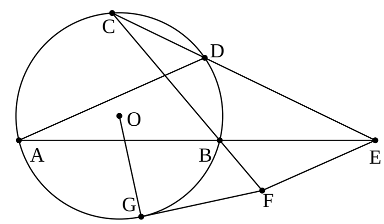


Figure 58

The Machine Proof

$[fe, eb] - [ec, cb]$

(Since $fe \parallel ad$; $[fe, eb] = -[eb, da]$.)

$= -[ec, cb] - [eb, da]$

(Since $c$, $d$, $e$ are collinear; $[ec, cb] = [dc, cb]$.)

$= -[eb, da] - [dc, cb]$

(Since $a$, $b$, $e$ are collinear; $[eb, da] = -[da, ba]$.)

$= -[dc, cb] + [da, ba]$

(Since $c$, $d$, $b$, $a$ are cyclic; $[dc, cb] = [da, ba]$.)

$= 0$

**Example 59** *The bisector of triangle $ABC$ at vertex $C$ bisects the arc $AB$ of the circumcircle of triangle $ABC$.*

Point order: $a, b, c, o, m, n$.

Hypotheses: circumcenter$(o, a, b, c)$, midpoint$(m, a, b)$, coll$(n, m, o)$, cong$(o, a, o, n)$.

Conclusion: eqangle$(a, c, n, n, c, b)$.

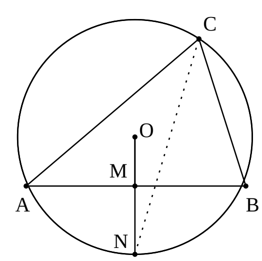


Figure 59

The Machine Proof

$-[nc, cb] - [nc, ca]$

(Since $c$, $n$, $b$, $a$ are cyclic; $[nc, cb] = [na, ba]$.)

$= -[nc, ca] - [na, ba]$

(Since $c$, $n$, $a$, $b$ are cyclic; $[nc, ca] = [nb, ba]$.)

$= -[nb, ba] - [na, ba]$

(Since $nb = na$ $[nb, ba] = -[na, ba]$.)

$= 0$

**Example 60** *Let $N$ be the traces of the internal bisectors of the triangle $ABC$ on the circumscribed circle $(O)$. Show that the Simson line of $N$ is the external bisector of the medial triangle of $ABC$.*

Point order: $a, b, c, o, d, e, f, n, k$.

Hypotheses: circumcenter$(o, a, b, c)$, midpoint$(d, b, c)$, midpoint$(e, a, c)$, midpoint$(f, a, b)$, coll$(n, o, f)$, cong$(o, a, o, n)$, foot$(k, n, a, c)$.

Conclusion: eqangle$(e, f, k, k, f, d)$.

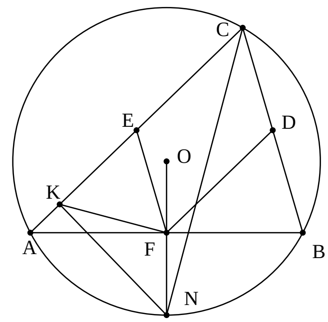


Figure 60

The Machine Proof

$-[kf, fe] - [kf, fd]$

(Since $fe \parallel bc$; $[kf, fe] = [kf, cb]$.)

$= -[kf, fd] - [kf, cb]$

(Since $fd \parallel ac$; $[kf, fd] = [kf, ca]$.)

$= -[kf, cb] - [kf, ca]$

(Since $a$, $e$, $k$ are collinear; $k$, $f$, $a$, $n$ are cyclic; $[kf, cb] = [kf, ka] + [ae, cb] = [nf, na] + [ea, cb]$.)

$= -[kf, ca] - [nf, na] - [ea, cb]$

(Since $a$, $c$, $k$ are collinear; $k$, $f$, $a$, $n$ are cyclic; $[kf, ca] = [kf, ka] = [nf, na]$.)

$= -2[nf, na] - [ea, cb]$

(Since $nf \perp ab$; $[nf, na] = -[na, ba] + 1$.)

$= 2[na, ba] - [ea, cb]$

(Since circumcenter($o$,$n$, $a$, $b$ ); $no \perp ab$; $[na, ba] = -[fo, oa] + 2[fo, ba]$.)

$= -[fo, oa] + 2[fo, ba] - [ea, cb]$

(Since $fo \perp ab$; $[fo, oa] = -[oa, ba] + 1$.)

$= 2[fo, ba] - [ea, cb] + [oa, ba] - 1$

(Since $fo \perp ba$; $[fo, ba] = 1$.)

$= -[ea, cb] + [oa, ba] - 1$

(Since $a$, $c$, $e$ are collinear; $[ea, cb] = -[cb, ca]$.)

$= [oa, ba] + [cb, ca] - 1$

(Since circumcenter($o$,$a$, $b$, $c$ ); $[oa, ba] = [oa, ab] = -[cb, ca] + 1$.)

$= 0$

**Example 61** *Let $A$ and $B$ the two common points of two circles $(O)$ and $(O_1)$. Through $B$ a line is drawn meeting the circles at $C$ and $D$ respectively. Show $AC/AD = OA/O_1A$.*

We need only to show that triangles $AOO_1$ and $ACD$ are similar.

Point order: $a, b, c, o, x, d, o1$.

Hypotheses: circumcenter$(o, a, b, c)$, midpoint$(x, a, b)$, cong$(o1, a, o1, b)$, coll$(o1, x, o)$, coll$(d, b, c)$, cong$(d, o1, o1, a)$.

Conclusion: eqangle$(a, o, o1, a, c, d)$.

The Machine Proof

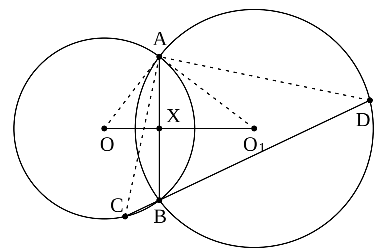

Figure 61

$-[o1o, oa] + [dc, ca]$

(Since $o$, $o1$, $x$ are collinear; $[o1o, oa] = [xo, oa]$.)

$= [dc, ca] - [xo, oa]$

(Since $b$, $c$, $d$ are collinear; $[dc, ca] = [cb, ca]$.)

$= -[xo, oa] + [cb, ca]$

(Since $xo \perp ab$; $[xo, oa] = -[oa, ba] + 1$.)

$= [oa, ba] + [cb, ca] - 1$

(Since circumcenter($o$,$a$, $b$, $c$ ); $[oa, ba] = [oa, ab] = -[cb, ca] + 1$.)

$= 0$

**Example 62** *If three chords drawn through a point of a circle are taken for diameters of three circles, these circles intersect, in pairs, in three new points, which are collinear.*

Point order: $d, a, b, c, e, f, g$.

Hypotheses: cyclic$(a, b, c, d)$, perp$(e, a, e, d)$, perp$(e, b, e, d)$, perp$(f, a, f, d)$, perp$(f, c, f, d)$, perp$(g, c, g, d)$, perp$(g, b, g, d)$.

Conclusion: coll$(e, f, g)$.

The Machine Proof

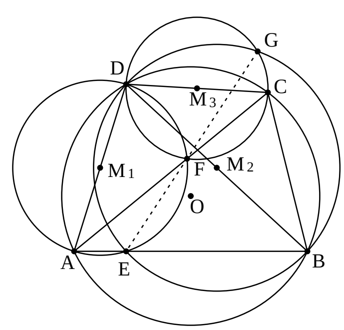

Figure 62

$-[ge, fe]$

(Since $b$, $c$, $g$ are collinear; $g$, $e$, $b$, $d$ are cyclic;
$[ge, fe] = [ge, gb] + [bc, fe] = -[fe, cb] + [ed, bd]$.)

$= [fe, cb] - [ed, bd]$

(Since $a$, $c$, $f$ are collinear; $f$, $e$, $a$, $d$ are cyclic;
$[fe, cb] = [fe, fa] + [ac, cb] = [ed, ad] - [cb, ca]$.)

$= -[ed, bd] + [ed, ad] - [cb, ca]$

(Since $ed \parallel ed$; $-[ed, bd] + [ed, ad] = [bd, ad]$.)

$= -[cb, ca] + [bd, ad]$

(Since $c$, $b$, $a$, $d$ are cyclic; $[cb, ca] = [bd, ad]$.)

$= 0$

**Example 63** *If three circles having a point in common intersect in pairs in three collinear points, their common point is concyclic with their centers.*

Point order: $p, a1, b1, c1, a, b, c$.

Hypotheses: circumcenter$(a, p, b1, c1)$, circumcenter$(b, p, a1, c1)$, circumcenter$(c, p, b1, a1)$, coll$(a1, b1, c1)$.

Conclusion: cyclic$(p, a, b, c)$.

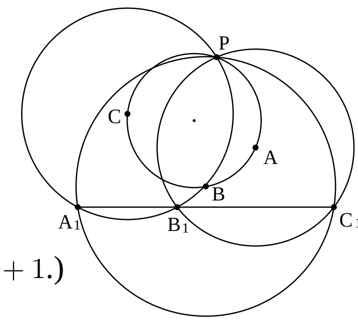


Figure 63

The Machine Proof

$[ca, cp] - [ba, bp]$

(Since $ca \perp pb1$; $[ca, cp] = -[cp, b1p] + 1$.)

$= -[cp, b1p] - [ba, bp] + 1$

(Since circumcenter($c$,$p$, $b1$, $a1$ ); $[cp, b1p] = [cp, pb1] = -[b1a1, a1p] + 1$.)

$= -[ba, bp] + [b1a1, a1p]$

(Since $ba \perp pc1$; $[ba, bp] = -[bp, c1p] + 1$.)

$= [bp, c1p] + [b1a1, a1p] - 1$

(Since circumcenter($b$,$p$, $c1$, $a1$ ); $[bp, c1p] = [bp, pc1] = -[c1a1, a1p] + 1$.)

$= -[c1a1, a1p] + [b1a1, a1p]$

(Since $a1$, $b1$, $c1$ are collinear; $[c1a1, a1p] = [b1a1, a1p]$.)

$= 0$

**Example 64** *Given two circles $(A)$, $(B)$ intersecting in $E$, $F$, show that the chord $E_1F_1$ determined in $(A)$ by the lines $MEE_1$, $MFF_1$ joining $E$, $F$ to any point $M$ of $(B)$ is perpendicular to $MB$.*

Point order: $e, f, m, b, d, a, e1, f1$.

Hypotheses: circumcenter$(b, e, f, m)$, midpoint$(d, e, f)$, coll$(a, d, b)$, coll$(e1, m, e)$, cong$(e1, a, e, a)$, coll$(f1, m, f)$, cong$(f1, a, e, a)$.

Conclusion: perp$(e1, f1, m, b)$.

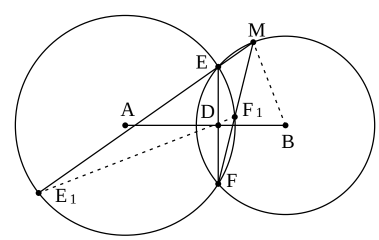


Figure 64

The Machine Proof

$[f1e1, bm] + 1$

(Since $f$, $f1$, $m$ are collinear; $f1$, $e1$, $f$, $e$ are cyclic;
$[f1e1, bm] = [f1e1, f1f] + [fm, bm] = [e1e, fe] - [bm, mf]$.)

$= [e1e, fe] - [bm, mf] + 1$

(Since $e$, $e1$, $m$ are collinear; $[e1e, fe] = [me, fe]$.)

$= -[bm, mf] + [me, fe] + 1$

(Since circumcenter($b$,$m$, $f$, $e$ ); $[bm, mf] = [bm, mf] = [me, fe] + 1$.)
$= 0$

**Example 65** *Let $D$ and $E$ be two points on sides $AB$ and $AC$ of triangle $ABC$ such that $DE \parallel BC$. Show that the circumcircles of triangle $ABC$ and $ADE$ are tangent.*

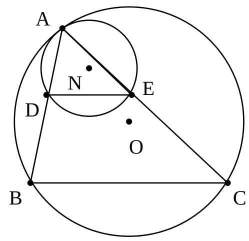

Figure 65

Point order: $a, b, c, o, d, e, n.$

Hypotheses: circumcenter$(o, a, b, c)$, coll$(d, a, b)$, coll$(e, a, c)$, para$(d, e, b, c)$, circumcenter$(n, a, d, e)$.

Conclusion: coll$(a, n, o)$.

The Machine Proof

$[na, oa]$
(Since circumcenter($n$,$a$, $d$, $e$ ); $[na, oa] = [na, ad] + [ad, oa] = -[ed, ea] + [da, oa] + 1$.)
$= -[ed, ea] + [da, oa] + 1$
(Since $ed \parallel bc$; $[ed, ea] = -[ea, cb]$.)
$= [ea, cb] + [da, oa] + 1$
(Since $a$, $c$, $e$ are collinear; $[ea, cb] = -[cb, ca]$.)
$= [da, oa] - [cb, ca] + 1$
(Since $a$, $b$, $d$ are collinear; $[da, oa] = -[oa, ba]$.)
$= -[oa, ba] - [cb, ca] + 1$
(Since circumcenter($o$,$a$, $b$, $c$ ); $[oa, ba] = [oa, ab] = -[cb, ca] + 1$.)
$= 0$

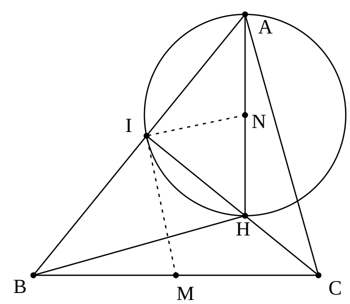

Figure 66

**Example 66** *Show that in a triangle $ABC$ the circles on $AH$ and $BC$ as diameters are orthogonal.*

Point order: $a, b, c, i, h, n, m.$

Hypotheses: foot$(i, c, a, b)$, coll$(h, c, i)$, perp$(h, a, b, c)$, midpoint$(m, c, b)$, midpoint$(n, a, h)$.

Conclusion: perp$(m, i, n, i)$.

The Machine Proof

$[mi, ni] + 1$
(Since circumcenter($m$,$i$, $b$, $c$ ); $[mi, ni] = [mi, ib] + [ib, ni] = -[ni, ib] + [ic, cb] + 1$.)
$= -[ni, ib] + [ic, cb]$
(Since circumcenter($n$,$i$, $a$, $h$ ); $[ni, ib] = [ni, ia] = [hi, ha] + 1$.)
$= -[hi, ha] + [ic, cb] - 1$
(Since $hi \parallel ic$; $-[hi, ha] + [ic, cb] = [ha, cb]$.)
$= [ha, cb] - 1$

(Since $ha \perp cb$; $[ha, cb] = 1$.)
$= 0$

**Example 67** *The circle $IBC$ is orthogonal to the circle on $I_bI_c$ as diameter.*

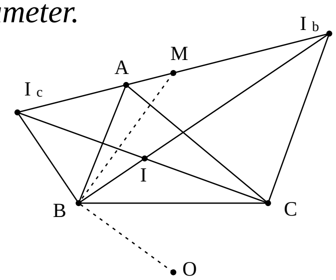

Figure 67

Point order: $a, b, c, i, o, b1, c1, m$.
Hypotheses: incenter$(i, a, b, c)$, circumcenter$(o, b, c, i)$, coll$(b1, b, i)$, perp$(b1, c, c, i)$, coll$(c1, c, i)$, perp$(c1, b, b, i)$, midpoint$(m, b1, c1)$.
Conclusion: perp$(m, b, o, b)$.

The Machine Proof
$[mb, ob] + 1$
(Since circumcenter($m$,$b$, $c$, $b1$ ); $[mb, ob] = [mb, bc] + [bc, ob] = -[b1c, b1b] - [ob, cb] + 1$.)
$= -[b1c, b1b] - [ob, cb]$
(Since $b1c \perp ci$; $[b1c, b1b] = -[b1b, ic] + 1$.)
$= [b1b, ic] - [ob, cb] - 1$
(Since $b$, $b1$, $i$ are collinear; $[b1b, ic] = -[ic, ib]$.)
$= -[ob, cb] - [ic, ib] - 1$
(Since circumcenter($o$,$b$, $c$, $i$ ); $[ob, cb] = [ob, bc] = -[ic, ib] + 1$.)
$= 0$

**Example 68** *Show that given two perpendicular diameters of two orthogonal circles, the lines joining an end of one of these diameters to the ends of the other pass through the points common to the two circles.*

Point order: $p, d, i, a, e, x, b, g, f$.
Hypotheses: circumcenter$(a, p, d, i, e)$, midpoint$(a, d, e)$, midpoint$(x, p, i)$, coll$(b, a, x)$, perp$(b, p, a, p)$, coll$(e, i, g)$, cong$(g, b, p, b)$, midpoint$(b, g, f)$.
Conclusion: perp$(f, g, d, e)$.

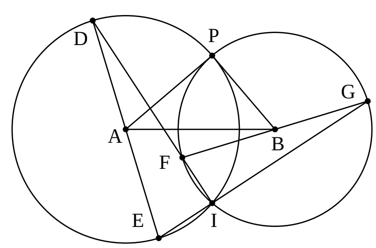

Figure 68

The Machine Proof
$[fg, ed] + 1$
(Since $b$, $f$, $g$ are collinear; $[fg, ed] = [gb, ed]$.)
$= [gb, ed] + 1$
(Since $e$, $g$, $i$ are collinear; $bg = bi$ $[gb, ed] = [gb, gi] + [ie, ed] = -[bi, ei] + [ei, ed]$.)
$= -[bi, ei] + [ei, ed] + 1$
(Since $bi \perp ia$; $ei \perp di$; $[bi, ei] = [ai, id]$.)
$= [ei, ed] - [ai, id] + 1$
(Since $ei \perp di$; $[ei, ed] = -[ed, id] + 1$.)
$= -[ed, id] - [ai, id]$

(Since $a$, $d$, $e$ are collinear; $[ed, id] = [ad, id]$.)

$= -[ai, id] - [ad, id]$

(Since circumcenter($a$,$i$, $d$, $p$ ); $[ai, id] = [ai, id] = [ip, dp] + 1$.)

$= -[ad, id] - [ip, dp] - 1$

(Since circumcenter($a$,$d$, $i$, $p$ ); $[ad, id] = [ad, di] = -[ip, dp] + 1$.)

$= 0$

**Example 69** *Show that if $AB$ is a diameter and $M$ any point of a circle, center $O$, the two circles $AMO$, $BMO$ are orthogonal.*

Point order: $a, b, m, o, i, j$.

Hypotheses: perp$(a, m, b, m)$, midpoint$(o, a, b)$, circumcenter$(i, o, a, m)$, circumcenter$(j, o, b, m)$.

Conclusion: perp$(i, m, m, j)$.

The Machine Proof

$-[jm, im] + 1$

(Since circumcenter($j$,$m$, $o$, $b$ ); $[jm, im] = [jm, mo] + [mo, im] = -[im, om] - [ob, mb] + 1$.)

$= [im, om] + [ob, mb]$

(Since circumcenter($i$,$m$, $o$, $a$ ); $[im, om] = [im, mo] = -[oa, ma] + 1$.)

$= [ob, mb] - [oa, ma] + 1$

(Since $ob \parallel oa$; $[ob, mb] - [oa, ma] = -[mb, ma]$.)

$= -[mb, ma] + 1$

(Since $mb \perp ma$; $[mb, ma] = 1$.)

$= 0$

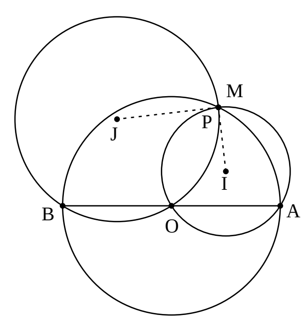


Figure 69

**Example 70** *If the line joining the ends $A$, $B$ of a diameter $AB$ of a given circle $(O)$ to a given point $P$ meets $(O)$ again in $A_1$, $B_1$, show that the circle $PA_1B_1$ is orthogonal to $(O)$.*

Point order: $a, b, a1, o, b1, p, o1$.

Hypotheses: circumcenter$(o, a, b, a1, b1)$, midpoint$(o, a, b)$, coll$(p, a, a1)$, coll$(p, b, b1)$, circumcenter$(o1, p, a1, b1)$.

Conclusion: perp$(o1, a1, o, a1)$.

The Machine Proof

$[o1a1, oa1] + 1$

(Since circumcenter($o1$,$a1$, $p$, $b1$ ); $[o1a1, oa1] = [o1a1, a1p] + [a1p, oa1] = -[pb1, b1a1] + [pa1, oa1] + 1$.)

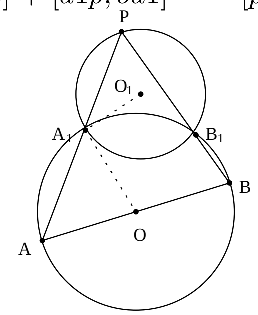

Figure 70

$= -[pb1, b1a1] + [pa1, oa1]$

(Since $b$, $b1$, $p$ are collinear; $[pb1, b1a1] = -[b1a1, b1b]$.)

$= [pa1, oa1] + [b1a1, b1b]$

(Since $a$, $a1$, $p$ are collinear; $[pa1, oa1] = -[oa1, a1a]$.)

$= [b1a1, b1b] - [oa1, a1a]$

(Since $b1$, $a1$, $b$, $a$ are cyclic; $[b1a1, b1b] = [a1a, ba]$.)

$= -[oa1, a1a] + [a1a, ba]$

(Since $oa1 = oa$; $a$, $b$, $o$ are collinear; $[oa1, a1a] = [oa1, a1a] + [a1a, a1a] = -[a1a, a1a] + [a1a, ba]$.)

$= [a1a, a1a]$

(Since $[a1a, a1a] = 0$.)

$= 0$

**Example 71** *Let $D$ be a point on the circumcircle of triangle $ABC$. If line $DA$ is parallel to $BC$, show that the Simson line $D(ABC)$ is parallel to the circumradius $OA$.*

Point order: $a, b, c, d, o, f, g$.

Hypotheses: circumcenter$(o, a, b, c, d)$, para$(d, a, b, c)$, foot$(g, d, a, b)$, foot$(f, d, a, c)$.

Conclusion: para$(g, f, o, a)$.

The Machine Proof

$[gf, oa]$

(Since $a$, $b$, $g$ are collinear; $g$, $f$, $a$, $d$ are cyclic; $[gf, oa] = [gf, ga] + [ab, oa] = [fd, da] - [oa, ba]$.)

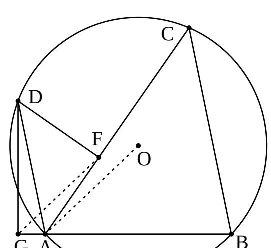

Figure 71

$= [fd, da] - [oa, ba]$

(Since $da \parallel bc$; $[fd, da] = [fd, cb]$.)

$= [fd, cb] - [oa, ba]$

(Since $fd \perp ac$; $[fd, cb] = -[cb, ca] + 1$.)

$= -[oa, ba] - [cb, ca] + 1$

(Since circumcenter($o$,$a$, $b$, $c$ ); $[oa, ba] = [oa, ab] = -[cb, ca] + 1$.)

$= 0$

**Example 72** *If $E$, $F$, $G$ are the feet of the perpendiculars from a point $D$ of the circumcircle of a triangle $ABC$ upon its sides $BC$, $CA$, $AB$, prove that the triangle $DFG$, $DBC$ are similar.*

Point order: $a, b, c, d, o, f, g$.

Hypotheses: circumcenter$(o, a, b, c, d)$, foot$(g, d, a, b)$, foot$(f, d, a, c)$.

Conclusion: similar$(d, f, g, d, c, b)$.

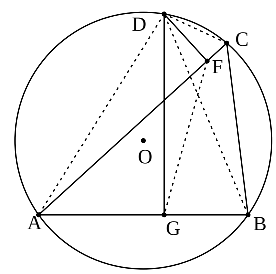

Figure 72

The Machine Proof

$-[gd, fd] - [dc, db]$

(Since $gd \perp ab$; $fd \perp ac$; $[gd, fd] = -[ca, ba]$.)

$= -[dc, db] + [ca, ba]$

(Since $d$, $c$, $b$, $a$ are cyclic; $[dc, db] = [ca, ba]$.)

$= 0$

**Example 73** *If the perpendicular from a point $D$ of the circumcircle $(O)$ of a triangle $ABC$ to the sides $BC$, $CA$, $AB$ meet $(O)$ again in the points $N$, $M$, $L$, the three lines $AN$, $BM$, $CL$ are parallel to the simson of $D$ for $ABC$.*

Point order: $a, b, c, d, o, f, g, l$.

Hypotheses: circumcenter$(o, a, b, c, d)$, foot$(g, d, a, b)$, foot$(f, d, a, c)$, coll$(l, d, g)$, cong$(l, o, o, a)$.

Conclusion: para$(c, l, f, g)$.

The Machine Proof

$[lc, gf]$

(Since $d$, $g$, $l$ are collinear; $l$, $c$, $d$, $a$ are cyclic;
$[lc, gf] = [lc, ld] + [dg, gf] = -[gf, gd] - [da, ca]$.)

$= -[gf, gd] - [da, ca]$

(Since $gd \perp ab$; $[gf, gd] = [gf, ba] + 1$.)

$= -[gf, ba] - [da, ca] - 1$

(Since $a$, $b$, $g$ are collinear; $g$, $f$, $a$, $d$ are cyclic;
$[gf, ba] = [gf, ga] = [fd, da]$.)

$= -[fd, da] - [da, ca] - 1$

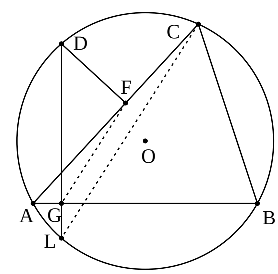

Figure 73

(Since $da \parallel da$; $-[fd, da] - [da, ca] = -[fd, ca]$.)
$= -[fd, ca] - 1$
(Since $fd \perp ca$; $[fd, ca] = 1$.)
$= 0$

**Example 74** *Show that the simson of the point where an altitude cuts the circumcircle again passes through the foot of the altitude and is antiparallel to the corresponding side of the triangle with respect to the other two sides.*

Point order: $a, b, c, o, g, d, e, f$.
Hypotheses: circumcenter$(o, a, b, c)$, foot$(g, c, a, b)$, coll$(d, c, g)$, cong$(d, o, o, a)$, foot$(e, d, a, c)$, foot$(f, d, b, c)$.
Conclusion: cyclic$(a, b, e, f)$.

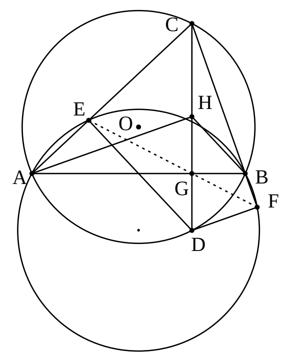

Figure 74

The Machine Proof
$[fe, ea] - [fb, ba]$
(Since $a$, $c$, $e$ are collinear; $e$, $f$, $c$, $d$ are cyclic; $[fe, ea] = [fd, dc]$.)
$= [fd, dc] - [fb, ba]$
(Since $fd \perp bc$; $dc \perp ab$; $[fd, dc] = [cb, ba]$.)
$= -[fb, ba] + [cb, ba]$
(Since $b$, $c$, $f$ are collinear; $[fb, ba] = [cb, ba]$.)
$= 0$

**Example 75** *Let $A$ and $B$ be the intersections of two circles $O_1$ and $O_2$. Through $A$ a secant is drawn meeting the two circles at $C$ and $D$, respectively. Show that angle $CBD$ is equal to the angle formed by lines $O_1C$ and $O_2D$.*

Point order: $a, b, c, o1, o, o2, d, i$.
Hypotheses: circumcenter$(o1, a, b, c)$, midpoint$(o, a, b)$, coll$(o1, o2, o)$, coll$(d, a, c)$, cong$(o2, a, o2, d)$, coll$(o1, c, i)$, coll$(o2, d, i)$.
Conclusion: eqangle$(c, b, d, o1, i, o2)$.

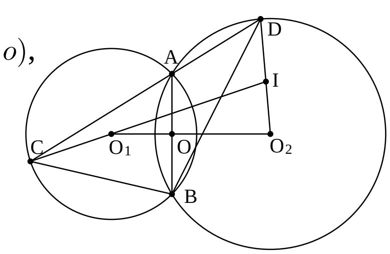

Figure 75

The Machine Proof
$[io2, io1] - [db, cb]$
(Since $d$, $i$, $o2$ are collinear; $c$, $i$, $o1$ are collinear; $[io2, io1] = [do2, o1c]$.)
$= [do2, o1c] - [db, cb]$
(Since $a$, $c$, $d$ are collinear; $o2d = o2a$ $[do2, o1c] = [do2, da] + [ac, o1c] = -[o2a, ca] - [o1c, ca]$.)
$= -[db, cb] - [o2a, ca] - [o1c, ca]$
(Since circumcenter($o2$,$d$, $b$, $a$ ); $[db, cb] = [db, o2a] + [ba, cb] = [da, o2a] - [cb, ba] + 1$.)

$= -[da, o2a] - [o2a, ca] - [o1c, ca] + [cb, ba] - 1$

(Since $o2a \parallel o2a$; $-[da, o2a] - [o2a, ca] = -[da, ca]$.)

$= -[da, ca] - [o1c, ca] + [cb, ba] - 1$

(Since $da \parallel ca$; $-[da, ca] - [o1c, ca] = -[o1c, ca]$.)

$= -[o1c, ca] + [cb, ba] - 1$

(Since circumcenter($o1$,$c$, $a$, $b$ ); $[o1c, ca] = [o1c, ca] = [cb, ba] + 1$.)

$= 0$

**Example 76** *Let $E$ be a point on the circumcircle of equilateral triangle $ABC$. $D = BC \cap AE$. Show that $BE \cdot CE = ED \cdot EA$.*

We need only to show that triangles $EDC$ and $EBA$ are similar.

Point order: $b, c, a, o, e, d$.

Hypotheses: cong$(a, b, a, c)$, cong$(b, c, b, a)$, pbisector$(o, a, b)$, pbisector$(o, c, b)$, pbisector$(o, e, b)$, coll$(a, e, d)$, coll$(b, c, d)$.

Conclusion: similar$(e, d, c, e, b, a)$.

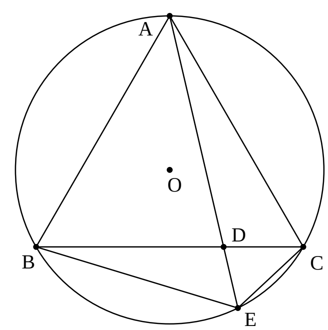

Figure 76

The Machine Proof

$[de, ec] + [ea, eb]$

(Since $a$, $d$, $e$ are collinear; $[de, ec] = [ea, ec]$.)

$= [ea, ec] + [ea, eb]$

(Since $e$, $a$, $c$, $b$ are cyclic; $[ea, ec] = [ab, cb]$.)

$= [ea, eb] + [ab, cb]$

(Since $e$, $a$, $b$, $c$ are cyclic; $[ea, eb] = [ac, cb]$.)

$= [ac, cb] + [ab, cb]$

(Since $ac = ab$ $[ac, cb] = -[ab, cb]$.)

$= 0$

**Example 77** *A circle touching $AB$ at $B$ and passing through the incenter $I$ (i.e., the center $I$ of the inscribed circle) of the triangle $ABC$ meets $AC$ in $H$, $K$. Prove that $IC$ bisects the angle $HIK$.*

Point order: $b, a, c, i, o, h, k$.

Hypotheses: incenter$(i, a, b, c)$, perp$(b, o, a, b)$, pbisector$(o, b, i)$, pbisector$(o, b, h)$, coll$(a, c, h)$, coll$(a, c, k)$, pbisector$(o, b, k)$.

Conclusion: eqangle$(h, i, c, c, i, k)$.

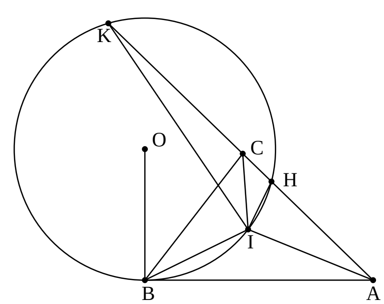

Figure 77

The Machine Proof

$[ki, ic] + [hi, ic]$

(Since $a$, $h$, $k$ are collinear; $k$, $i$, $h$, $b$ are cyclic;

$[ki, ic] = [ki, kh] + [ha, ic] = [ha, ic] - [hb, ib]$.)

$= [hi, ic] + [ha, ic] - [hb, ib]$

(Since circumcenter($o$,$h$, $i$, $b$ ); $[hi, ic] = [hi, ob] + [ib, ic] = [hb, ob] - [ic, ib] + 1$.)

$= [ha, ic] + [hb, ob] - [hb, ib] - [ic, ib] + 1$

(Since $hb \parallel hb$; $[hb, ob] - [hb, ib] = -[ob, ib]$.)

$= [ha, ic] - [ob, ib] - [ic, ib] + 1$

(Since $a$, $c$, $h$ are collinear; $[ha, ic] = -[ic, ca]$.)

$= -[ob, ib] - [ic, ib] - [ic, ca] + 1$

(Since $ob \perp ba$; $[ob, ib] = -[ib, ab] + 1$.)

$= -[ic, ib] - [ic, ca] + [ib, ab]$

(Since $i$ is the incenter of triangle $a$, $c$, $b$ $[ic, ib] = [ia, ab] + 1$.)

$= -[ic, ca] - [ia, ab] + [ib, ab] - 1$

(Since $i$ is the incenter of triangle $c$, $a$, $b$ $[ic, ca] = -[ic, cb]$.)

$= [ic, cb] - [ia, ab] + [ib, ab] - 1$

(Since $i$ is the incenter of triangle $c$, $a$, $b$ $[ic, cb] = [ia, ab] - [ib, ab] + 1$.)

$= 0$

**Example 78** *$ABC$ is triangle inscribed in a circle; $DE$ is the diameter bisecting $BC$ at $G$; from $E$ a perpendicular $EK$ is drawn to one of the sides, and the perpendicular from the vertex $A$ on $DE$ meets $DE$ in $H$. Show that $EK$ touches the circle $GHK$.*

Point order: $a, b, c, o, g, e, k, h, n$.

Hypotheses: circumcenter$(o, b, a, c)$, midpoint$(g, b, c)$, coll$(g, o, e)$, pbisector$(o, b, e)$, perp$(k, e, a, b)$, coll$(k, a, b)$, perp$(a, h, o, g)$, coll$(h, o, g)$, circumcenter$(n, g, h, k)$.

Conclusion: perp$(e, k, k, n)$.

The Machine Proof

$-[nk, ke] + 1$

(Since $ke \perp ab$; $[nk, ke] = [nk, ba] + 1$.)

$= -[nk, ba]$

(Since circumcenter($n$,$k$, $g$, $h$ ); $[nk, ba] = [nk, kg] + [kg, ba] = [hk, hg] + [kg, ba] + 1$.)

$= -[hk, hg] - [kg, ba] - 1$

(Since $hg \perp bc$; $[hk, hg] = [hk, cb] + 1$.)

$= -[hk, cb] - [kg, ba]$

(Since $e$, $g$, $h$ are collinear; $h$, $k$, $e$, $a$ are cyclic;
$[hk, cb] = [hk, he] + [eg, cb] = [ka, ea] + [eg, cb]$.)

$= -[kg, ba] - [ka, ea] - [eg, cb]$

(Since $a$, $b$, $k$ are collinear; $k$, $g$, $b$, $e$ are cyclic;
$[kg, ba] = [kg, kb] = [eg, eb]$.)

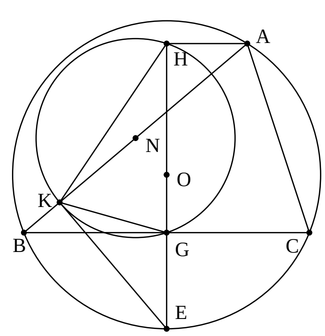


Figure 78

$= -[ka, ea] - [eg, eb] - [eg, cb]$

(Since $a$, $b$, $k$ are collinear; $[ka, ea] = -[ea, ba]$.)

$= -[eg, eb] - [eg, cb] + [ea, ba]$

(Since $eg \perp bc$; $[eg, eb] = -[eb, cb] + 1$.)

$= -[eg, cb] + [eb, cb] + [ea, ba] - 1$

(Since $eg \perp cb$; $[eg, cb] = 1$.)

$= [eb, cb] + [ea, ba]$

(Since $b$, $e$, $c$, $a$ are cyclic; $[eb, cb] = [ea, ca]$.)

$= [ea, ca] + [ea, ba]$

(Since circumcenter($o$,$a$, $e$, $c$, $b$ ); $oc \perp eb$; $[ea, ca] = -[ea, ba]$.)

$= 0$

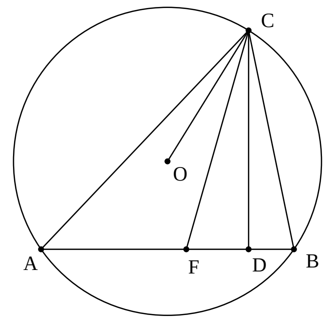


Figure 79

**Example 79** *The angle between the circumdiameter and the altitude issued from the same vertex of a triangle is bisected by the bisector of angle of the triangle at the vertex considered.*

Point order: $a, b, c, o, d, f$.

Hypotheses: pbisector$(o, a, b)$, pbisector$(o, a, c)$, coll$(a, b, d)$, perp$(a, b, c, d)$, coll$(f, a, b)$, eqangle$(a, c, f, f, c, b)$.

Conclusion: eqangle$(d, c, f, f, c, o)$.

The Machine Proof

$-[fc, dc] - [fc, oc]$

(Since $\angle[cf, cd] = \angle[ac, cf]$ ; $[fc, dc] = [fc, bc] + [bc, dc] = -[fc, ca] - [dc, cb]$.)

$= -[fc, oc] + [fc, ca] + [dc, cb]$

(Since $fc \parallel fc$; $-[fc, oc] + [fc, ca] = [oc, ca]$.)

$= [dc, cb] + [oc, ca]$

(Since $dc \perp ab$; $[dc, cb] = -[cb, ba] + 1$.)

$= [oc, ca] - [cb, ba] + 1$

(Since circumcenter($o$,$c$, $a$, $b$ ); $[oc, ca] = [oc, ca] = [cb, ba] + 1$.)

$= 0$

**Example 80** *Let $U$ and $U_1$ be the intersections of the bisectors of angle $A$ of a triangle $ABC$ with side $BC$. If the tangent at $A$ to the circumcircle meets $BC$ in $T$, we have $TA \equiv TU \equiv TU_1$.*

Point order: $a, b, c, o, u, t$.

Hypotheses: pbisector$(o, b, c)$, pbisector$(o, b, a)$, eqangle$(b, a, u, u, a, c)$, coll$(b, u, c)$, perp$(o, a, a, t)$, coll$(b, c, t)$.

Conclusion: pbisector$(t, a, u)$.

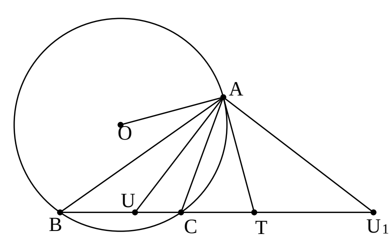


Figure 80

The Machine Proof

$[tu, ua] + [ta, ua]$

(Since $b$, $c$, $t$, $u$ are collinear; $[tu, ua] = -[ua, cb]$.)

$= [ta, ua] - [ua, cb]$

(Since $ta \perp ao$; $[ta, ua] = -[ua, oa] + 1$.)

$= -[ua, oa] - [ua, cb] + 1$

(Since $\angle[au, ao]$ = $\angle[ab, au]$ ; $[ua, oa] = [ua, ac] + [ac, oa] = -[ua, ba] - [oa, ca]$.)

$= -[ua, cb] + [ua, ba] + [oa, ca] + 1$

(Since $ua \parallel ua$; $-[ua, cb] + [ua, ba] = [cb, ba]$.)

$= [oa, ca] + [cb, ba] + 1$

(Since circumcenter($o$,$a$, $c$, $b$ ); $[oa, ca] = [oa, ac] = -[cb, ba] + 1$.)

$= 0$

**Example 81** *If the two bisectors of the angle $A$ of the triangle $ABC$ are equal, and the circle having $BC$ for diameter cuts the sides $AB$, $AC$ in the points $P$, $Q$, show that $CP \equiv CQ$.*

Point order: $u, v, a, b, c, o, p, q$.

Hypotheses: perp$(a, u, a, v)$, cong$(a, u, a, v)$, coll$(u, v, b)$, eqangle$(c, a, u, u, a, b)$, coll$(u, v, c)$, midpoint$(o, b, c)$, coll$(a, b, p)$, coll$(a, c, q)$, pbisector$(o, b, q)$, pbisector$(o, b, p)$.

Conclusion: pbisector$(c, p, q)$.

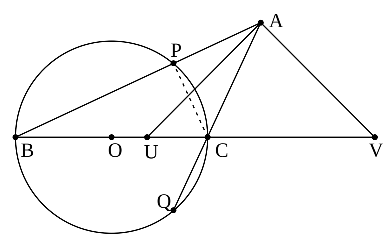

Figure 81

The Machine Proof

$-[qp, qc] - [qp, pc]$

(Since $q$, $p$, $c$, $b$ are cyclic; $[qp, qc] = [pb, cb]$.)

$= -[qp, pc] - [pb, cb]$

(Since $pc \perp ab$; $[qp, pc] = [qp, ba] + 1$.)

$= -[qp, ba] - [pb, cb] - 1$

(Since $a$, $c$, $q$ are collinear; $q$, $p$, $c$, $b$ are cyclic; $[qp, ba] = [qp, qc] + [ca, ba] = [pb, cb] + [ca, ba]$.)

$= -2[pb, cb] - [ca, ba] - 1$

(Since $a$, $b$, $p$ are collinear; $[pb, cb] = -[cb, ba]$.)

$= 2[cb, ba] - [ca, ba] - 1$

(Since $b$, $c$, $u$, $v$ are collinear; $[cb, ba] = -[ba, vu]$.)

$= -[ca, ba] - 2[ba, vu] - 1$

(Since $\angle[ac, ab]$ = $\angle[ua, ab]$ ; $[ca, ba] = [ca, ua] + [ua, ba] = -2[ba, au]$.)

$= 2[ba, au] - 2[ba, vu] - 1$

(Since $ba \parallel ba$; $2[ba, au] - 2[ba, vu] = -2[au, vu]$.)

$= -2[au, vu] - 1$

(Since $au = av$ $ua \perp av$; $[au, vu] =_1 423772$.)

$= 0$

**Example 82** *The feet of the four perpendiculars dropped from a vertex of a triangle upon the four bisectors of the other two angles are collinear.*

Point order: $a, b, c, o, i, e, j, l$.

Hypotheses: incenter$(o, a, b, c)$, foot$(i, c, a, o)$, perp$(e, a, o, a)$, foot$(j, c, a, e)$, foot$(l, c, b, o)$.

Conclusion: coll$(i, j, l)$.

The Machine Proof

$-[li, ji]$

(Since $b$, $l$, $o$ are collinear; $l$, $i$, $o$, $c$ are cyclic;
$[li, ji] = [li, lo] + [ob, ji] = -[ji, ob] + [ic, oc]$.)

$= [ji, ob] - [ic, oc]$

(Since $a$, $e$, $j$ are collinear; $j$, $i$, $a$, $c$ are cyclic;
$[ji, ob] = [ji, ja] + [ae, ob] = [ea, ob] + [ic, ca]$.)

$= [ea, ob] - [ic, oc] + [ic, ca]$

(Since $ea \parallel ic$; $[ea, ob] - [ic, oc] = [oc, ob]$.)

$= [ic, ca] + [oc, ob]$

(Since $ic \perp ao$; $[ic, ca] = [oa, ca] + 1$.)

$= [oc, ob] + [oa, ca] + 1$

(Since $o$ is the incenter of triangle $a$, $c$, $b$ $[oc, ob] = [oa, ba] + 1$.)

$= [oa, ca] + [oa, ba]$

(Since $o$ is the incenter of triangle $a$, $c$, $b$ $[oa, ca] = -[oa, ba]$.)

$= 0$

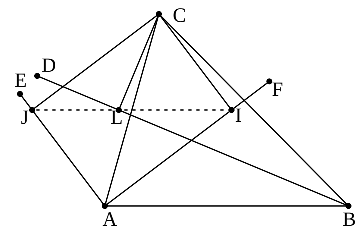

Figure 82

**Example 83** *The angle which a side of a triangle makes with the corresponding side of the orthic triangle is equal to the difference of the angles of the given triangle adjacent to the side considered.*

Point order: $a, b, c, d, e, x$.
Hypotheses: coll$(b, c, d)$, perp$(b, c, a, d)$, coll$(a, c, e)$, perp$(a, c, b, e)$, coll$(a, b, x)$, coll$(d, e, x)$.
Conclusion: $[dx, xa] - [ca, ab] - [cb, ba]$.

The Machine Proof
$[xd, xa] - [cb, ba] - [ca, ba]$
(Since $xa \parallel ba$; $[xd, xa] - [cb, ba] = [xd, cb]$.)
$= [xd, cb] - [ca, ba]$
(Since $d$, $e$, $x$ are collinear; $[xd, cb] = [ed, cb]$.)
$= [ed, cb] - [ca, ba]$
(Since $a$, $c$, $e$ are collinear; $e$, $d$, $a$, $b$ are cyclic; $[ed, cb] = [ed, ea] + [ac, cb] = [db, ba] - [cb, ca]$.)
$= [db, ba] - [cb, ca] - [ca, ba]$
(Since $db \parallel cb$; $[db, ba] - [cb, ca] = [ca, ba]$.)
$= 0$

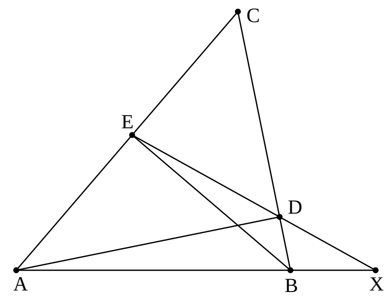

Figure 83

**Example 84** *The tangent to the nine-point circle at the midpoint of a side of the given triangle is antiparallel to this side with respect to the two other sides of the triangle.*

Point order: $b, c, a, a_1, b_1, c_1, n, k, j$.
Hypotheses: midpoint$(a_1, b, c)$, midpoint$(c_1, b, a)$, midpoint$(b_1, a, c)$, pbisector$(n, a_1, b_1)$, pbisector$(n, a_1, c_1)$, perp$(a_1, k, a_1, n)$, coll$(a, c, k)$, coll$(k, a_1, j)$, coll$(a, b, j)$.
Conclusion: cyclic$(k, j, b, c)$.

The Machine Proof
$[jk, jb] - [kc, cb]$
(Since $a_1$, $j$, $k$ are collinear; $a$, $b$, $j$ are collinear; $[jk, jb] = [ka_1, ab]$.)
$= [ka_1, ab] - [kc, cb]$
(Since $ka_1 \perp a_1n$; $[ka_1, ab] = [na_1, ab] + 1$.)
$= -[kc, cb] + [na_1, ab] + 1$
(Since $a$, $c$, $k$ are collinear; $[kc, cb] = [ac, cb]$.)
$= [na_1, ab] - [ac, cb] + 1$
(Since circumcenter($n$,$a_1$, $c_1$, $b_1$ );

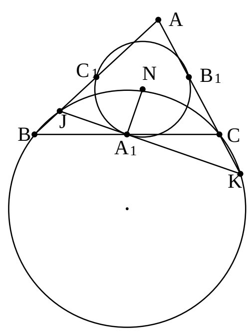

Figure 84

$[na_1, ab] = [na_1, a_1c_1] + [a_1c_1, ab] = -[c_1b_1, b_1a_1] + [c_1a_1, ab] + 1.$)
$= -[c_1b_1, b_1a_1] + [c_1a_1, ab] - [ac, cb]$
(Since $c_1a_1 \parallel ac$; $[c_1a_1, ab] - [ac, cb] = -[ab, cb]$.)
$= -[c_1b_1, b_1a_1] - [ab, cb]$
(Since $b_1a_1 \parallel ab$; $-[c_1b_1, b_1a_1] - [ab, cb] = -[c_1b_1, cb]$.)
$= -[c_1b_1, cb]$
(Since $c_1b_1 \parallel cb$; $[c_1b_1, cb] = 0$.)
$= 0$

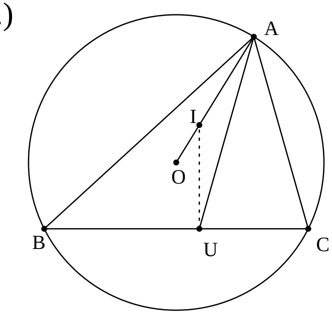

Figure 85

**Example 85** *Show that the mediator of the bisector $AU$ of the triangle $ABC$, the perpendicular to $BC$ at $U$, and the circumdiameter of $ABC$ passing through $A$ are concurrent.*

Point order: $a, b, c, o, u, i$.
Hypotheses: pbisector$(o, a, b)$, pbisector$(o, b, c)$, eqangle$(b, a, u, u, a, c)$, coll$(b, c, u)$, pbisector$(i, a, u)$, coll$(i, o, a)$.
Conclusion: perp$(i, u, b, c)$.

The Machine Proof
$[iu, cb] + 1$
(Since $iu = ia$ $a$, $i$, $o$ are collinear; $[iu, cb] = [iu, ua] + [ua, cb] = [ua, oa] + [ua, cb]$.)
$= [ua, oa] + [ua, cb] + 1$
(Since $\angle[au, ao] = \angle[ab, au]$ ; $[ua, oa] = [ua, ac] + [ac, oa] = -[ua, ba] - [oa, ca]$.)
$= [ua, cb] - [ua, ba] - [oa, ca] + 1$
(Since $ua \parallel ua$; $[ua, cb] - [ua, ba] = -[cb, ba]$.)
$= -[oa, ca] - [cb, ba] + 1$
(Since circumcenter($o$,$a$, $c$, $b$ ); $[oa, ca] = [oa, ac] = -[cb, ba] + 1$.)
$= 0$

**Example 86** *The internal bisector of the angle $B$ of the triangle $ABC$ meets the sides $B_1C_1$, $B_1A_1$ of the medial triangle in the points $A_2$, $C_2$. Prove that $AA_2$, $CC_2$ are perpendicular to the bisector, and that $B_1A_2$ = $B_1C_2$. Similarly for the external bisector.*

Point order: $a, b, c, a_1, b_1, c_1, a_2, c_2$.
Hypotheses: midpoint$(a, b_1, c)$, midpoint$(a, c_1, b)$, midpoint$(b, a_1, c)$, eqangle$(a, b, a_2, a_2, b, c)$, coll$(b_1, c_1, a_2)$, coll$(b, a_2, c_2)$, coll$(a_1, b_1, c_2)$.
Conclusion: pbisector$(b_1, a_2, c_2)$.

The Machine Proof
$-[c_2a_2, c_2b_1] - [c_2a_2, a_2b_1]$

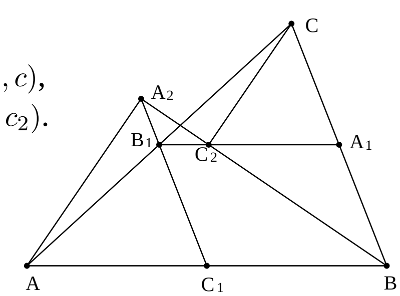

Figure 86

(Since $a_2$, $b$, $c_2$ are collinear; $a_1$, $b_1$, $c_2$ are collinear; $[c_2a_2, c_2b_1] = [a_2b, b_1a_1]$.)
$= -[c_2a_2, a_2b_1] - [a_2b, b_1a_1]$
(Since $a_2$, $b$, $c_2$ are collinear; $[c_2a_2, a_2b_1] = -[a_2b_1, a_2b]$.)
$= [a_2b_1, a_2b] - [a_2b, b_1a_1]$
(Since $a_2b_1 \parallel bc$; $[a_2b_1, a_2b] = -[a_2b, cb]$.)
$= -[a_2b, b_1a_1] - [a_2b, cb]$
(Since $\angle[ba_2, a_1b_1] = \angle[ab, ba_2]$ ; $[a_2b, b_1a_1] = [a_2b, bc] + [bc, b_1a_1] = -[a_2b, ba] - [b_1a_1, cb]$.)
$= -[a_2b, cb] + [a_2b, ba] + [b_1a_1, cb]$
(Since $a_2b \parallel a_2b$; $-[a_2b, cb] + [a_2b, ba] = [cb, ba]$.)
$= [b_1a_1, cb] + [cb, ba]$
(Since $cb \parallel cb$; $[b_1a_1, cb] + [cb, ba] = [b_1a_1, ba]$.)
$= [b_1a_1, ba]$
(Since $b_1a_1 \parallel ba$; $[b_1a_1, ba] = 0$.)
$= 0$

**Example 87** *With the usual notation, show that the angle formed by the lines $C_1E$, $B_1F$ is equal to $3A$, or its supplement.*

Point order: $a, b, c, e, f, c_1, b_1, x$.
Hypotheses: coll$(a, b, f)$, coll$(a, c, e)$, perp$(a, b, c, f)$,
perp$(a, c, b, e)$, midpoint$(c_1, a, b)$, midpoint$(b_1, a, c)$, coll$(x, c_1, e)$, coll$(x, b_1, f)$.
Conclusion: $3[ba, ac] - [b_1x, xc_1]$.

The Machine Proof
$-[xb_1, xc_1] - 3[ca, ba]$
(Since $b_1$, $f$, $x$ are collinear; $c_1$, $e$, $x$ are collinear;
$[xb_1, xc_1] = [b_1f, c_1e]$.)
$= -[b_1f, c_1e] - 3[ca, ba]$
(Since circumcenter($b_1$,$f$, $a$, $c$ ); $[b_1f, c_1e] = [b_1f, fa] + [fa, c_1e] = -[c_1e, fa] + [fc, ca] + 1$.)
$= [c_1e, fa] - [fc, ca] - 3[ca, ba] - 1$
(Since circumcenter($c_1$,$e$, $a$, $b$ ); $[c_1e, fa] = [c_1e, ea] + [ea, fa] = -[fa, ea] + [eb, ba] + 1$.)
$= -[fc, ca] - [fa, ea] + [eb, ba] - 3[ca, ba]$
(Since $fc \perp ab$; $[fc, ca] = -[ca, ba] + 1$.)
$= -[fa, ea] + [eb, ba] - 2[ca, ba] - 1$
(Since $a$, $b$, $f$ are collinear; $[fa, ea] = -[ea, ba]$.)
$= [eb, ba] + [ea, ba] - 2[ca, ba] - 1$
(Since $eb \perp ac$; $[eb, ba] = [ca, ba] + 1$.)
$= [ea, ba] - [ca, ba]$
(Since $a$, $c$, $e$ are collinear; $[ea, ba] = [ca, ba]$.)
$= 0$

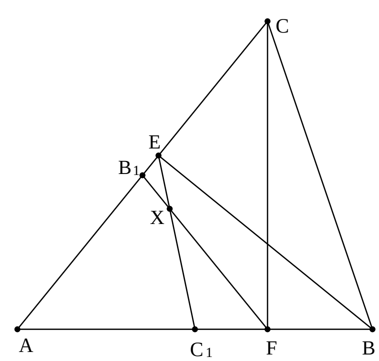

Figure 87

**Example 88** *Show that the symmetrics, $P$, $Q$ of a given point $L$ with respect to the sides $Ox$, $Oy$ of a given angle, and the points $P_1 = (LQ, Ox)$, $Q_1 = (LP, Oy)$ lie on a circle passing through $O$.*

Point order: $o, x, y, l, a, b, p, q, p_1, q_1, i$.
Hypotheses: perp$(l, a, o, x)$, coll$(a, o, x)$, perp$(l, b, o, y)$, coll$(o, b, y)$, midpoint$(a, l, p)$, midpoint$(b, l, q)$, coll$(p_1, l, q)$, coll$(o, x, p_1)$, coll$(q_1, l, p)$, coll$(q_1, o, y)$.
Conclusion: cyclic$(o, p, q, p_1)$.

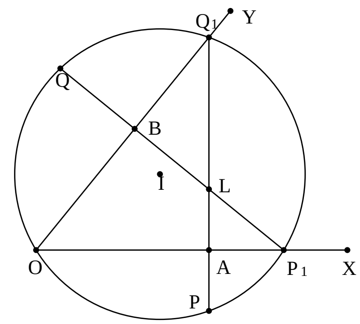

Figure 88

The Machine Proof

$[p_1p, p_1o] - [qp, qo]$

(Since $p_1o \perp la$; $[p_1p, p_1o] = [p_1p, al] + 1$.)

$= [p_1p, al] - [qp, qo] + 1$

(Since $al \perp ox$; $[p_1p, al] = [p_1p, xo] + 1$.)

$= [p_1p, xo] - [qp, qo]$

(Since $p_1p = p_1l$ $b$, $l$, $p_1$ are collinear; $[p_1p, xo] = [p_1p, pl] + [pl, xo] = [pl, bl] + [pl, xo]$.)

$= -[qp, qo] + [pl, bl] + [pl, xo]$

(Since $qp \parallel ab$; $[qp, qo] = -[qo, ba]$.)

$= [qo, ba] + [pl, bl] + [pl, xo]$

(Since $b$, $l$, $q$ are collinear; $oq = ol$ $[qo, ba] = [qo, ql] + [lb, ba] = -[ba, bl] + [bl, lo]$.)

$= [pl, bl] + [pl, xo] - [ba, bl] + [bl, lo]$

(Since $a$, $l$, $p$ are collinear; $[pl, bl] = -[bl, al]$.)

$= [pl, xo] - [ba, bl] - [bl, al] + [bl, lo]$

(Since $bl \parallel bl$; $-[ba, bl] - [bl, al] = -[ba, al]$.)

$= [pl, xo] - [ba, al] + [bl, lo]$

(Since $pl \perp xo$; $[pl, xo] = 1$.)

$= -[ba, al] + [bl, lo] + 1$

(Since $a$, $b$, $l$, $o$ are cyclic; $[ba, al] = [bo, lo]$.)

$= [bl, lo] - [bo, lo] + 1$

(Since $b$, $o$, $y$ are collinear; $[bo, lo] = -[lo, yo]$.)

$= [bl, lo] + [lo, yo] + 1$

(Since $bl \perp oy$; $[bl, lo] = -[lo, yo] + 1$.)

$= 0$

**Example 89** *Show that the nine-point center of the triangle $IBC$ lies on the internal bisector of the angle $A_1$ of the complementary triangle $A_1B_1C_1$ of the given triangle $ABC$.*

Point order: $b, c, a, i, a_1, b_1, c_1, p, q, n$.

Hypotheses: incenter$(i, a, b, c)$, midpoint$(p, b, i)$, midpoint$(q, c, i)$, pbisector$(n, a_1, p)$, pbisector$(n, a_1, q)$, midpoint$(a_1, c, b)$, midpoint$(b_1, c, a)$, midpoint$(c_1, a, b)$.

Conclusion: eqangle$(b_1, a_1, n, n, a_1, c_1)$.

The Machine Proof

$-[na_1, c_1a_1] - [na_1, b_1a_1]$

(Since $c_1a_1 \parallel ca$; $[na_1, c_1a_1] = [na_1, ac]$.)

$= -[na_1, b_1a_1] - [na_1, ac]$

(Since $b_1a_1 \parallel ba$; $[na_1, b_1a_1] = [na_1, ab]$.)

$= -[na_1, ac] - [na_1, ab]$

(Since circumcenter($n$,$a_1$, $q$, $p$ ); $[na_1, ac] = [na_1, a_1q] + [a_1q, ac] = -[qp, pa_1] + [qa_1, ac] + 1$.)

$= -[na_1, ab] + [qp, pa_1] - [qa_1, ac] - 1$

(Since circumcenter($n$,$a_1$, $q$, $p$ ); $[na_1, ab] = [na_1, a_1q] + [a_1q, ab] = -[qp, pa_1] + [qa_1, ab] + 1$.)

$= 2[qp, pa_1] - [qa_1, ac] - [qa_1, ab]$

(Since $qp \parallel bc$; $pa_1 \parallel ci$; $[qp, pa_1] = -[ic, cb]$.)

$= -[qa_1, ac] - [qa_1, ab] - 2[ic, cb]$

(Since $qa_1 \parallel bi$; $[qa_1, ac] = [ib, ac]$.)

$= -[qa_1, ab] - 2[ic, cb] - [ib, ac]$

(Since $qa_1 \parallel bi$; $[qa_1, ab] = [ib, ab]$.)

$= -2[ic, cb] - [ib, ac] - [ib, ab]$

(Since $i$ is the incenter of triangle $b$, $a$, $c$ $[ib, ac] = -2[ic, cb] + [ib, cb]$.)

$= -[ib, ab] - [ib, cb]$

(Since $i$ is the incenter of triangle $b$, $a$, $c$ $[ib, ab] = -[ib, cb]$.)

$= 0$

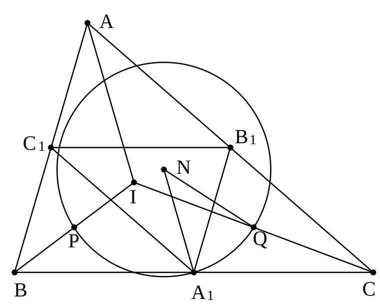


Figure 89

**Example 90** *If the perpendiculars dropped from a point $D$ of the circumcircle $(O)$ of the triangle $ABC$ upon the sides, meet $(O)$ in the points $A_1$, $B_1$, $C_1$, show that the two triangle $ABC$, $A_1B_1C_1$ are congruent and symmetrical with respect to an axis.*

Point order: $a, b, c, d, o, f, g, c_1, b_1$.

Hypotheses: pbisector$(o, a, b)$, pbisector$(o, a, c)$, pbisector$(o, a, d)$, coll$(a, b, g)$, coll$(a, c, f)$, perp$(a, b, d, g)$, perp$(a, c, d, f)$, coll$(d, g, c_1)$, pbisector$(o, a, c_1)$, coll$(d, f, b_1)$, pbisector$(o, a, b_1)$.

Conclusion: para$(c_1, c, b_1, b)$.

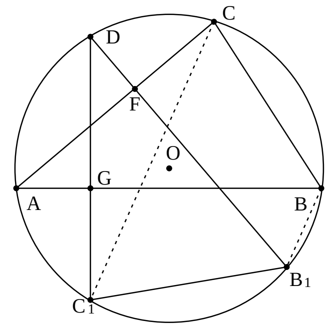

Figure 90

The Machine Proof

$-[b_1b, c_1c]$

(Since $b_1$, $d$, $f$ are collinear; $b_1$, $b$, $d$, $c_1$ are cyclic; $[b_1b, c_1c] = [b_1b, b_1d] + [df, c_1c] = -[c_1d, c_1b] - [c_1c, fd]$.)

$= [c_1d, c_1b] + [c_1c, fd]$

(Since $c_1d \perp ab$; $[c_1d, c_1b] = -[c_1b, ba] + 1$.)

$= [c_1c, fd] - [c_1b, ba] + 1$

(Since $fd \perp ac$; $[c_1c, fd] = [c_1c, ca] + 1$.)

$= [c_1c, ca] - [c_1b, ba]$

(Since $c$, $c_1$, $a$, $b$ are cyclic; $[c_1c, ca] = [c_1b, ba]$.)

$= 0$

**Example 91** *The perpendiculars through the points $P$, $P_1$ to the Simson lines $P_1(ABC)$, $P(ABC)$, respectively, intersect on the circumcircle of the triangle $ABC$.*

Point order: $a, b, c, o, p, p_1, g, g_1, f, f_1, k$.

Hypotheses: pbisector$(o, a, b)$, pbisector$(o, a, c)$, pbisector$(o, a, p)$, pbisector$(o, a, p_1)$, perp$(p, g, a, b)$, coll$(g, a, b)$, perp$(p, f, a, c)$, coll$(a, c, f)$, perp$(p_1, g_1, a, b)$, coll$(g_1, a, b)$, perp$(p_1, f_1, a, c)$, coll$(a, c, f_1)$, perp$(k, p, f_1, g_1)$, perp$(k, p_1, f, g)$.

Conclusion: cyclic$(p, p_1, a, k)$.

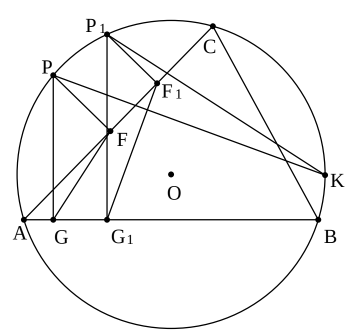

Figure 91

The Machine Proof

$[kp_1, kp] - [p_1a, pa]$

(Since $kp_1 \perp gf$; $kp \perp g_1f_1$; $[kp_1, kp] = -[f_1g_1, fg]$.)

$= -[f_1g_1, fg] - [p_1a, pa]$

(Since $a$, $c$, $f_1$ are collinear; $f_1$, $g_1$, $a$, $p_1$ are cyclic; $[f_1g_1, fg] = [f_1g_1, f_1a] + [ac, fg] = -[fg, ca] + [g_1p_1, p_1a]$.)

$= [fg, ca] - [g_1p_1, p_1a] - [p_1a, pa]$

(Since $p_1a \parallel p_1a$; $-[g_1p_1, p_1a] - [p_1a, pa] = -[g_1p_1, pa]$.)

$= [fg, ca] - [g_1p_1, pa]$

(Since $a$, $c$, $f$ are collinear; $f$, $g$, $a$, $p$ are cyclic; $[fg, ca] = [fg, fa] = [gp, pa]$.)

$= -[g_1p_1, pa] + [gp, pa]$

(Since $g_1p_1 \parallel pg$; $[g_1p_1, pa] = [gp, pa]$.)

$= 0$

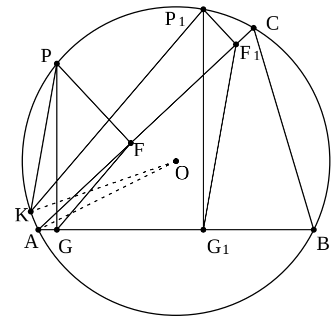


Figure 92

**Example 92** *The parallels through the points $P$, $P_1$ to the Simson lines $P_1(ABC)$, $P(ABC)$, respectively, intersect on the circumcircle of the triangle $ABC$.*

Point order: $a, b, c, o, p, p_1, g, g_1, f, f_1, k$.

Hypotheses: pbisector$(o, a, b)$, pbisector$(o, a, c)$, pbisector$(o, a, p)$, pbisector$(o, a, p_1)$, perp$(p, g, a, b)$, coll$(g, a, b)$, perp$(p, f, a, c)$, coll$(a, c, f)$, perp$(p_1, g_1, a, b)$, coll$(g_1, a, b)$, perp$(p_1, f_1, a, c)$, coll$(a, c, f_1)$, para$(k, p, f_1, g_1)$, para$(k, p_1, f, g)$.

Conclusion: cyclic$(p, p_1, a, k)$.

The Machine Proof

$[kp_1, kp] - [p_1a, pa]$

(Since $kp_1 \parallel gf$; $kp \parallel g_1f_1$; $[kp_1, kp] = -[f_1g_1, fg]$.)

$= -[f_1g_1, fg] - [p_1a, pa]$

(Since $a$, $c$, $f_1$ are collinear; $f_1$, $g_1$, $a$, $p_1$ are cyclic; $[f_1g_1, fg] = [f_1g_1, f_1a] + [ac, fg] = -[fg, ca] + [g_1p_1, p_1a]$.)

$= [fg, ca] - [g_1p_1, p_1a] - [p_1a, pa]$

(Since $p_1a \parallel p_1a$; $-[g_1p_1, p_1a] - [p_1a, pa] = -[g_1p_1, pa]$.)

$= [fg, ca] - [g_1p_1, pa]$

(Since $a$, $c$, $f$ are collinear; $f$, $g$, $a$, $p$ are cyclic; $[fg, ca] = [fg, fa] = [gp, pa]$.)

$= -[g_1p_1, pa] + [gp, pa]$

(Since $g_1p_1 \parallel pg$; $[g_1p_1, pa] = [gp, pa]$.)

$= 0$

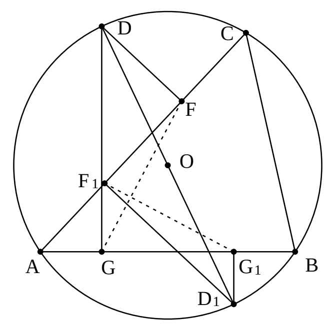


Figure 93

**Example 93** *The Simson lines of two diametrically opposite points are perpendicular.*

Point order: $a, b, c, d, o, d_1, f, g, f_1, g_1, k, a_1, b_1, c_1, n$.

Hypotheses: circumcenter$(o, a, b, c, d, d_1)$, coll$(a, b, g)$, coll$(a, c, f)$, perp$(a, b, d, g)$, perp$(a, c, d, f)$, midpoint$(o, d, d_1)$, coll$(a, b, g_1)$, coll$(a, c, f_1)$, perp$(a, b, d_1, g_1)$, perp$(a, c, d_1, f_1)$, coll$(g, f, k)$, coll$(g_1, f_1, k)$, midpoint$(a_1, c, b)$, midpoint$(b_1, a, c)$, midpoint$(c_1, a, b)$, pbisector$(n, c_1, a_1)$, pbisector$(n, c_1, b_1)$, pbisector$(n, c_1, k)$.

Conclusion: perp$(f, g, f_1, g_1)$.

The Machine Proof

$-[g_1f_1, gf] + 1$

(Since $a$, $b$, $g_1$ are collinear; $g_1$, $f_1$, $a$, $d_1$ are cyclic;
$[g_1f_1, gf] = [g_1f_1, g_1a] + [ab, gf] = [f_1d_1, d_1a] - [gf, ba]$.)
$= -[f_1d_1, d_1a] + [gf, ba] + 1$
(Since $f_1d_1 \parallel df$; $[f_1d_1, d_1a] = [fd, d_1a]$.)
$= [gf, ba] - [fd, d_1a] + 1$
(Since $a$, $b$, $g$ are collinear; $g$, $f$, $a$, $d$ are cyclic; $[gf, ba] = [gf, ga] = [fd, da]$.)
$= -[fd, d_1a] + [fd, da] + 1$
(Since $fd \parallel fd$; $-[fd, d_1a] + [fd, da] = [d_1a, da]$.)
$= [d_1a, da] + 1$
(Since $d_1a \perp da$; $[d_1a, da] = 1$.)
$= 0$

**Example 94** *The four Simson lines of a point of a circle for the four triangles determined by the vertices of a quadrilateral inscribed in that circle, admit the point considered for their Miquel point.*

Point order: $a, b, c, d, p, o, e, f, g, h, i$.
Hypotheses: cyclic$(a, b, c, d, p)$, perp$(p, g, a, b)$, perp$(p, h, b, c)$, perp$(p, e, c, d)$, perp$(p, f, d, a)$, coll$(g, a, b)$, coll$(h, b, c)$, coll$(e, c, d)$, coll$(f, d, a)$, coll$(i, g, f)$, coll$(i, e, h)$.
Conclusion: cyclic$(g, h, p, i)$.

The Machine Proof
$[ih, ig] - [hp, gp]$
(Since $e$, $h$, $i$ are collinear; $f$, $g$, $i$ are collinear; $[ih, ig] = [he, gf]$.)
$= [he, gf] - [hp, gp]$
(Since $b$, $c$, $h$ are collinear; $h$, $e$, $c$, $p$ are cyclic;
$[he, gf] = [he, hc] + [cb, gf] = -[gf, cb] + [ep, pc]$.)
$= -[hp, gp] - [gf, cb] + [ep, pc]$
(Since $hp \perp bc$; $gp \perp ab$; $[hp, gp] = [cb, ba]$.)
$= -[gf, cb] + [ep, pc] - [cb, ba]$
(Since $cb \parallel cb$; $-[gf, cb] - [cb, ba] = -[gf, ba]$.)
$= -[gf, ba] + [ep, pc]$
(Since $a$, $b$, $g$ are collinear; $g$, $f$, $a$, $p$ are cyclic;
$[gf, ba] = [gf, ga] = [fp, pa]$.)
$= -[fp, pa] + [ep, pc]$
(Since $fp \perp ad$; $[fp, pa] = -[pa, da] + 1$.)
$= [ep, pc] + [pa, da] - 1$
(Since $ep \perp cd$; $[ep, pc] = -[pc, dc] + 1$.)
$= -[pc, dc] + [pa, da]$
(Since $c$, $p$, $d$, $a$ are cyclic; $[pc, dc] = [pa, da]$.)

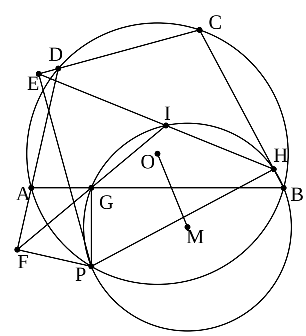

Figure 94

$= 0$

**Example 95** *The traces, on the circumcircle of a triangle, of a median and the corresponding symmedian determine a line parallel to the side of the triangle opposite the vertex considered.*

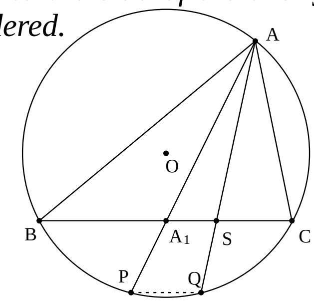

Figure 95

Point order: $a, b, c, o, a_1, s, q, p$.

Hypotheses: circumcenter$(o, a, b, c, p, q)$, midpoint$(a_1, b, c)$, coll$(a, a_1, p)$, eqangle$(b, a, s, a_1, a, c)$, coll$(b, s, c)$, coll$(a, s, q)$.

Conclusion: para$(p, q, b, c)$.

The Machine Proof

$[pq, cb]$

(Since $a$, $a_1$, $p$ are collinear; $p$, $q$, $a$, $b$ are cyclic; $[pq, cb] = [pq, pa] + [aa_1, cb] = [qb, ba] + [a_1a, cb]$.)

$= [qb, ba] + [a_1a, cb]$

(Since $a$, $q$, $s$ are collinear; $q$, $b$, $a$, $c$ are cyclic; $[qb, ba] = [qb, qa] + [as, ba] = [sa, ba] + [cb, ca]$.)

$= [sa, ba] + [a_1a, cb] + [cb, ca]$

(Since $cb \parallel cb$; $[a_1a, cb] + [cb, ca] = [a_1a, ca]$.)

$= [sa, ba] + [a_1a, ca]$

(Since $\angle[ab, as] = \angle[ac, aa_1]$ ; $[sa, ba] = -[a_1a, ca]$.)

$= 0$

**Example 96** *If from a point on the symmedian perpendiculars are drawn to the including sides of the triangle, the line joining the feet of these perpendiculars is perpendicular to the corresponding median of the triangle.*

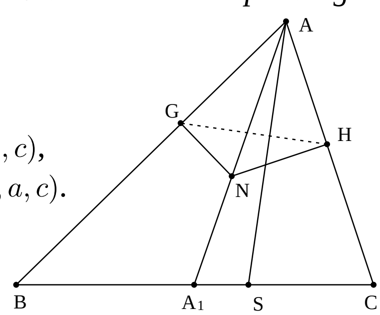

Figure 96

Point order: $b, c, a, a_1, s, n, g, h$.

Hypotheses: midpoint$(b, a_1, c)$, eqangle$(b, a, s, a_1, a, c)$, coll$(b, s, c)$, coll$(n, a, a_1)$, perp$(n, g, a, b)$, coll$(a, g, b)$, perp$(n, h, a, c)$, coll$(h, a, c)$.

Conclusion: perp$(g, h, a, s)$.

The Machine Proof

$[hg, sa] + 1$

(Since $a$, $c$, $h$ are collinear; $h$, $g$, $a$, $n$ are cyclic; $[hg, sa] = [hg, ha] + [ac, sa] = [gn, na] - [sa, ac]$.)

$= [gn, na] - [sa, ac] + 1$

(Since $gn \perp ba$; $[gn, na] = -[na, ab] + 1$.)

$= -[na, ab] - [sa, ac]$

(Since $a$, $a_1$, $n$ are collinear; $[na, ab] = [a_1a, ab]$.)

$= -[sa, ac] - [a_1a, ab]$

(Since $\angle[ba, as]$ = $\angle[aa_1, ca]$ ; $[sa, ac] = [sa, ba] + [ba, ac] = -[a_1a, ac] - [ac, ab]$.)
$= [a_1a, ac] - [a_1a, ab] + [ac, ab]$
(Since $a_1a \parallel a_1a$; $[a_1a, ac] - [a_1a, ab] = -[ac, ab]$.)
$= 0$

**Example 97** *The median and the symmedian of a triangle $ABC$ issued from $A$ meet the circumcircle in $P$, $Q$. Show that the Simson lines of $P$, $Q$ are respectively perpendicular to $AP$, $AQ$.*

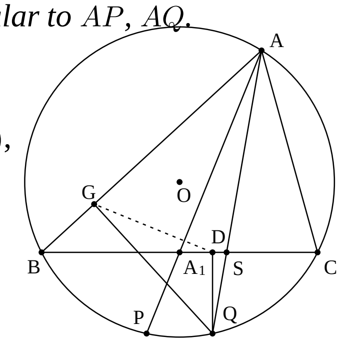

Figure 97

Point order: $a, b, c, o, a_1, s, q, p, d, g$.
Hypotheses: circumcenter$(o, a, b, c, p, q)$, midpoint$(a_1, b, c)$, coll$(a, a_1, p)$, eqangle$(b, a, s, a_1, a, c)$, coll$(b, s, c)$, coll$(a, s, q)$, coll$(d, b, c)$, perp$(q, d, b, c)$, coll$(a, b, g)$, perp$(q, g, a, b)$.
Conclusion: perp$(d, g, a, p)$.

The Machine Proof
$[gd, pa] + 1$
(Since $a$, $b$, $g$ are collinear; $g$, $d$, $b$, $q$ are cyclic; $[gd, pa] = [gd, gb] + [ba, pa] = [dq, qb] - [pa, ba]$.)
$= [dq, qb] - [pa, ba] + 1$
(Since $dq \parallel oa_1$; $[dq, qb] = -[qb, a_1o]$.)
$= -[pa, ba] - [qb, a_1o] + 1$
(Since $a$, $a_1$, $p$ are collinear; $[pa, ba] = [a_1a, ba]$.)
$= -[qb, a_1o] - [a_1a, ba] + 1$
(Since $a_1o \perp bc$; $[qb, a_1o] = [qb, cb] + 1$.)
$= -[qb, cb] - [a_1a, ba]$
(Since $b$, $q$, $c$, $a$ are cyclic; $[qb, cb] = [qa, ca]$.)
$= -[qa, ca] - [a_1a, ba]$
(Since $a$, $q$, $s$ are collinear; $[qa, ca] = [sa, ca]$.)
$= -[sa, ca] - [a_1a, ba]$
(Since $\angle[ab, as]$ = $\angle[aa_1, ac]$ ; $[sa, ca] = [sa, ab] + [ab, ca] = -[a_1a, ca] - [ca, ba]$.)
$= [a_1a, ca] - [a_1a, ba] + [ca, ba]$
(Since $a_1a \parallel a_1a$; $[a_1a, ca] - [a_1a, ba] = -[ca, ba]$.)
$= 0$

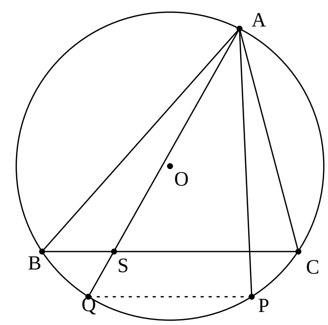

Figure 98

**Example 98** *The line joining the traces, on the circumcircle of a triangle, of two isogonal lines of an angle of the triangle is parallel to the side opposite the vertex considered.*

Point order: $a, b, c, o, p, s, q$.
Hypotheses: pbisector$(o, p, b)$, pbisector$(o, c, b)$, pbisector$(o, a, b)$,

eqangle$(b, a, s, p, a, c)$, coll$(b, s, c)$, coll$(a, s, q)$, pbisector$(o, q, b)$.
Conclusion: para$(p, q, b, c)$.

The Machine Proof

$[qp, cb]$

(Since $a$, $q$, $s$ are collinear; $q$, $p$, $a$, $c$ are cyclic; $[qp, cb] = [qp, qa] + [as, cb] = [sa, cb] + [pc, ca]$.)

$= [sa, cb] + [pc, ca]$

(Since $\angle[ab, as] = \angle[ap, ac]$ ; $[sa, cb] = [sa, ab] + [ab, cb] = -[pa, ca] - [cb, ba]$.)

$= [pc, ca] - [pa, ca] - [cb, ba]$

(Since $c$, $p$, $a$, $b$ are cyclic; $[pc, ca] = [pb, ba]$.)

$= [pb, ba] - [pa, ca] - [cb, ba]$

(Since circumcenter($o$,$p$, $b$, $a$ ); $[pb, ba] = [pb, oa] + [ba, ba] = [pa, oa] - [ba, ba] + 1$.)

$= [pa, oa] - [pa, ca] - [cb, ba] - [ba, ba] + 1$

(Since $pa \parallel pa$; $[pa, oa] - [pa, ca] = -[oa, ca]$.)

$= -[oa, ca] - [cb, ba] - [ba, ba] + 1$

(Since circumcenter($o$,$a$, $c$, $b$ ); $[oa, ca] = [oa, ac] = -[cb, ba] + 1$.)

$= -[ba, ba]$

(Since $[ba, ba] = 0$.)

$= 0$

**Example 99** *The line joining the two projections of a given point upon the sides of an angle is perpendicular to the isogonal conjugate of the line joining the given point to the vertex of the angle.*

Point order: $a, b, c, m, n, q, p$.
Hypotheses: eqangle$(b, a, m, n, a, c)$, perp$(m, q, a, b)$, coll$(q, a, b)$, perp$(m, p, a, c)$, coll$(p, a, c)$.
Conclusion: perp$(n, a, p, q)$.

The Machine Proof

$-[pq, na] + 1$

(Since $a$, $c$, $p$ are collinear; $p$, $q$, $a$, $m$ are cyclic; $[pq, na] = [pq, pa] + [ac, na] = [qm, ma] - [na, ca]$.)

$= -[qm, ma] + [na, ca] + 1$

(Since $qm \perp ab$; $[qm, ma] = -[ma, ba] + 1$.)

$= [na, ca] + [ma, ba]$

(Since $\angle[an, ac] = \angle[ab, am]$ ; $[na, ca] = -[ma, ba]$.)

$= 0$

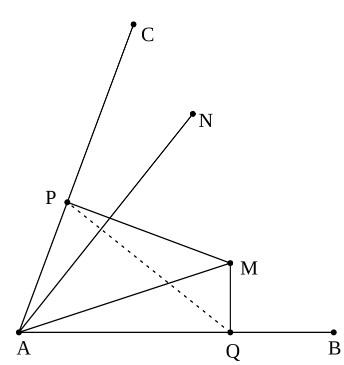


Figure 99

**Example 100** *$ABC$ is an equilateral triangle. Produce $AB$ to $D$ such that $BD = 2AB$. $F$ is the foot of the perpendicular line from $D$ to $BC$. Show that $AC \perp AF$.*

Point order: $a, b, c, m, d, f$.

Hypotheses: cong$(a, c, c, b)$, cong$(c, b, b, a)$, midpoint$(b, a, m)$, midpoint$(m, b, d)$, foot$(f, d, b, c)$.

Conclusion: perp$(c, a, a, f)$.

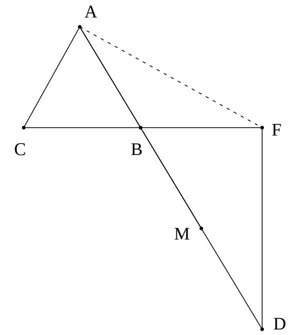


Figure 100

The Machine Proof

$-[fa, ca] + 1$

(Since $b$, $c$, $f$ are collinear; $f$, $a$, $c$, $m$ are cyclic; $[fa, ca] = [fa, fc] + [cb, ca] = -[mc, ma] + [cb, ca]$.)

$= [mc, ma] - [cb, ca] + 1$

(Since $mc \perp ac$; $[mc, ma] = -[ma, ca] + 1$.)

$= -[ma, ca] - [cb, ca]$

(Since $a$, $b$, $m$ are collinear; $[ma, ca] = -[ca, ba]$.)

$= -[cb, ca] + [ca, ba]$

(Since $\angle[ac, bc]$ = $\angle[ab, ac]$ ; $[cb, ca] = [cb, ac] + [ac, ca] = -[ca, ca] + [ca, ba]$.)

$= [ca, ca]$

(Since $[ca, ca] = 0$.)

$= 0$

**Example 101** *The two tangents to the circumcircle of $ABC$ at $A$ and $C$ meet at $E$. The mediator of $BC$ meet $AB$ at $D$. Show that $A$, $O$, $O$, and $D$ are cyclic.*

Point order: $a, b, c, o, h, d, e$.

Hypotheses: circumcenter$(o, a, b, c)$, midpoint$(h, c, b)$, coll$(d, a, b)$, coll$(d, o, h)$, perp$(e, a, a, o)$, perp$(e, c, c, o)$.

Conclusion: cyclic$(a, o, e, d)$.

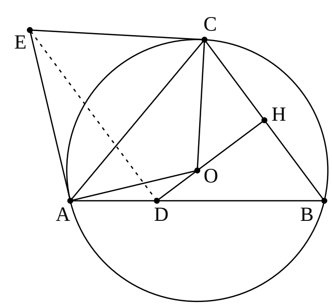


Figure 101

The Machine Proof

$-[eo, ea] + [do, da]$

(Since $ea \perp ao$; $[eo, ea] = [eo, oa] + 1$.)

$= -[eo, oa] + [do, da] - 1$

(Since $o$, $e$, $a$, $c$ are cyclic; $[eo, oa] = [ec, ca]$.)

$= -[ec, ca] + [do, da] - 1$

(Since $ec \perp co$; $[ec, ca] = [oc, ca] + 1$.)

$= [do, da] - [oc, ca]$

(Since $d$, $h$, $o$ are collinear; $a$, $b$, $d$ are collinear; $[do, da] = [ho, ba]$.)

$= [ho, ba] - [oc, ca]$

(Since $ho \perp bc$; $[ho, ba] = [cb, ba] + 1$.)

$= -[oc, ca] + [cb, ba] + 1$

(Since circumcenter($o$,$c$, $a$, $b$ ); $[oc, ca] = [oc, ca] = [cb, ba] + 1$.)

$= 0$

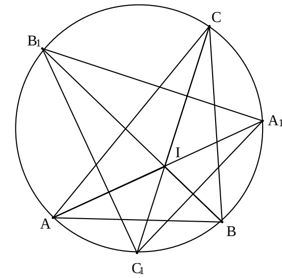


Figure 102

**Example 102** [2] *Let $ABC$ be inscribed in a circle and let $A_1$, $B_1$, and $C_1$ be the midpoints of the arc $BC$, $CA$, and $AB$ respectively. Show that the incenter of triangle $ABC$ is the orthocenter of triangle $A_1B_1C_1$.*

Point order: $a, b, c, a1, b1, c1$.

Hypotheses: cyclic$(a, b, c, a1, b1, c1)$, cong$(a1, b, a1, c)$, cong$(b1, a, b1, c)$, cong$(c1, a, c1, b)$.

Conclusion: perp$(a, a1, b1, c1)$.

The Machine Proof

$-[c1b1, a1a] + 1$

($[c1b1, a1a] = [c1b1, c1a] + [ac1, a1a] = [c1a, a1a] + [b1a1, a1a]$, because cyclic($c1$, $b1$, $a$, $a1$ ).)

$= -[c1a, a1a] - [b1a1, a1a] + 1$

($[c1a, a1a] = [c1a, ba] - [a1a, ba]$.)

$= -[c1a, ba] - [b1a1, a1a] + [a1a, ba] + 1$

($[c1a, ba] = [b1a, ca] + [a1a, ba] + 1$, because cyclic($a$, $b$, $c$, $a1$, $b1$, $c1$ ), $a1b = a1c$, $b1a = b1c$, $c1a = c1b$.)

$= -[b1a1, a1a] - [b1a, ca]$

($[b1a1, a1a] = -[b1a, ca]$, because cyclic($a1$, $b1$, $a$, $c$, $a$ ), $b1a = b1c$.)

$= 0$

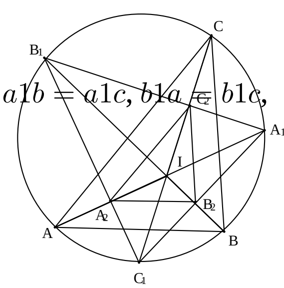


Figure 103

**Example 103** [3] *Continue from the preceding example. Show that the pedal triangle of triangle $A_1B_1C_1$ is homothetic to triangle $ABC$.*

Point order: $a, b, c, a1, b1, c1, a2, b2, c2$.

Hypotheses: cyclic$(a, b, c, a1, b1, c1)$, cong$(a1, b, a1, c)$, cong$(b1, a, b1, c)$, cong$(c1, a, c1, b)$, foot$(a2, a1, b1, c1)$, foot$(b2, b1, a1, c1)$, foot$(c2, c1, a1, b1)$.

Conclusion: para$(a, b, a2, b2)$.

[2]This example is a problem proposed in American Mathematics Monthly, 1993 (Problem 10317)
[3]This example is a problem proposed in American Mathematics Monthly, 1993 (Problem 10317)

The Machine Proof

$-[b2a2, ba]$

([$b2a2, ba] = [b2a2, b2a1] + [a1c1, ba] = [a2b1, b1a1] + [c1a1, ba]$, because collinear($a1$, $b2$, $c1$ ), cyclic($b2$, $a2$, $a1$, $b1$).)

$= -[a2b1, b1a1] - [c1a1, ba]$

($[a2b1, b1a1] = [c1b1, b1a1]$, because collinear($a2$, $b1$, $c1$ ).)

$= -[c1b1, b1a1] - [c1a1, ba]$

($[c1b1, b1a1] = [c1b1, b1b] - [b1a1, b1b]$.)

$= -[c1b1, b1b] - [c1a1, ba] + [b1a1, b1b]$

($[c1b1, b1b] = -[c1b, ba]$, because cyclic($b1$, $c1$, $b$, $a$, $b$ ), $c1b = c1a$.)

$= -[c1a1, ba] + [c1b, ba] + [b1a1, b1b]$

($[c1a1, ba] = [c1a1, c1b] + [bc1, ba] = [c1b, ba] + [b1a1, b1b]$, because cyclic($c1$, $a1$, $b$, $b1$ ).)

$= 0$

## 2.3 Examples Proved by Combining Forward Chaining with Backward Chaining

The examples in this subsection are the most difficult problems solved by our prover. Their proof need the database containing information about equal-angles. Generally speaking, the information in GIB is not obvious and need further explanation.

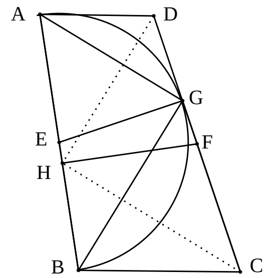


Figure 104

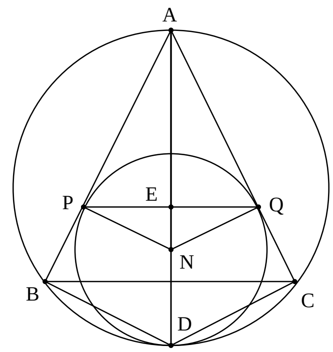


Figure 105

**Example 104** [4] *In quadrilateral $ABCD$, $BC \parallel AD$ and the circle with $AB$ as its diameter is tangent to $CD$. Show that the circle with $CD$ as its diameter is tangent to $AB$.*

Point order: $a, b, c, d, e, g, f, h$.

Hypotheses: para$(a, d, b, c)$, midpoint$(e, a, b)$, foot$(g, e, c, d)$, perp$(a, g, b, g)$, midpoint$(f, c, d)$, foot$(h, f, a, b)$.

Conclusion: perp$(c, h, h, d)$.

The Machine Proof

$-[hd, hc] + 1$

(Since $a$, $b$, $h$ are collinear; $h$, $d$, $a$, $g$ are cyclic; $h$, $c$, $b$, $g$ are cyclic; $[hd, hc] = [hd, ha] + [hb, hc] = [gd, ga] - [gc, gb]$.)

$= -[gd, ga] + [gc, gb] + 1$

(Since $c$, $d$, $g$ are collinear; $-[gd, ga] + [gc, gb] = -[gb, ga]$.)

$= -[gb, ga] + 1$

(Since $gb \perp ga$, $[gb, ga] = 1$.)

$= 0$

The following information are in the GIB.

Points $a$, $d$, $g$, $h$ are cyclic, because $\angle[ad, cd] = \angle[ab, gh]$.
This equation if further derived from the following two equations:
$\angle[ad, cd] = \angle[ef, cd]$, because $EF \parallel AD$;
$\angle[ab, gh] = \angle[ef, cd]$, because $g$, $e$, $f$, $h$ are cyclic.

Points $b$, $c$, $g$, $h$ are cyclic, because $\angle[bc, cd] = \angle[ab, gh]$.
This equation if further derived from the following two equations:
$\angle[bc, cd] = \angle[ef, cd]$, because $EF \parallel BC$;
$\angle[ab, gh] = \angle[ef, cd]$, because $g$, $e$, $f$, $h$ are cyclic.

**Example 105** [5] *In triangle $ABC$, $AB = AC$. A circle is tangent to the circumcircle of triangle $ABC$ and is tangent to $AB$, $AC$ at $P$ and $Q$. Show that the midpoint of the $PQ$ is the incenter of triangle $ABC$.*

[4]This example is from the 1984 International Mathematical Olympiad

Point order: $p, q, a, n, d, b, c, e$.

Hypotheses: cyclic$(a, b, c, d)$, cong$(a, p, a, q)$, cong$(a, b, a, c)$, cong$(d, b, d, c)$, coll$(n, a, d)$, foot$(p, n, a, b)$, foot$(q, n, a, c)$, circumcenter$(n, d, p, q)$, coll$(e, p, q)$, coll$(e, a, d)$.

Conclusion: eqangle$(a, b, e, e, b, c)$.

The Machine Proof

$-[eb, cb] - [eb, ba]$

(Since $cb \parallel pq$; $[eb, cb] = [eb, qp]$.)

$= -[eb, ba] - [eb, qp]$

(Since $ba \perp pn$; $[eb, ba] = [eb, np] + 1$.)

$= -[eb, np] - [eb, qp] - 1$

(Since $np \perp pa$; $[eb, np] = [eb, ap] + 1$.)

$= -[eb, ap] - [eb, qp]$

(Since $e$, $p$, $q$ are collinear; $e$, $b$, $p$, $d$ are cyclic;
$[eb, ap] = [eb, ep] + [pq, ap] = [bd, dp] - [ap, qp]$.)

$= -[eb, qp] - [bd, dp] + [ap, qp]$

(Since $e$, $p$, $q$ are collinear; $e$, $b$, $p$, $d$ are cyclic;
$[eb, qp] = [eb, ep] = [bd, dp]$.)

$= -2[bd, dp] + [ap, qp]$

(Since $bd \parallel pn$; $[bd, dp] = -[dp, np]$.)

$= 2[dp, np] + [ap, qp]$

(Since circumcenter($n$,$d$, $p$, $q$ ); $dn \perp pq$; $[dp, np] = [na, np]$.)

$= [na, np] + [ap, qp]$

(Since $na \perp pq$; $np \perp pa$; $[na, np] = -[ap, qp]$.)

$= 0$

The GIB contains the following circles.

$p$, $a$, $n$, $q$ are cyclic, because $pa \perp pn$; $qa \perp qn$.
$b$, $c$, $p$, $q$ are cyclic, because $\angle[pa, pq] = \angle[bc, qa]$.
$b$, $d$, $e$, $p$ are cyclic, because $\angle[pa, pq] = \angle[bc, ca] = \angle[db, ad]$.
$c$, $d$, $e$, $q$ are cyclic, because $\angle[ad, dc] = \angle[ab, bc] = \angle[pq, qa]$.

[5]This problem is from the 1978 International Mathematical Olympiad

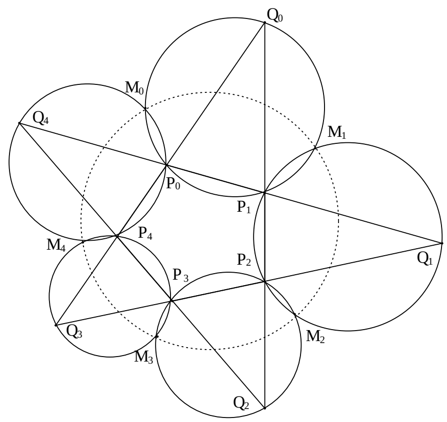


Figure 106

**Example 106 (The Five Circle Theorem)** *As in Figure 9, $P_0P_1P_2P_3P_4$ is a pentagon. $Q_i = P_{i-1}P_i \cap P_{i+1}P_{i+2}$, $M_i = circle(Q_{i-1}P_{i-1}P_i) \cap circle(Q_iP_iP_{i+1})$ (the subscripts are understood to be mod 4). Show that points $M_0$, $M_1$, $M_2$, $M_3$, $M_4$ are cyclic.*

Point order: $p_0, p_1, p_2, p_3, p_4, q_0, q_1, q_2, q_3, q_4, m_0, m_1, m_2, m_3, m_4$.

Hypotheses: coll$(q_0, p_1, p_2, q_2)$, coll$(q_1, p_2, p_3, q_3)$, coll$(q_2, p_3, p_4, q_4)$, coll$(q_3, p_4, p_0, q_0)$, coll$(q_4, p_0, p_1, q_1)$, cyclic$(m_0, m_1, q_0, p_0, p_1)$, cyclic$(m_0, m_4, q_4, p_0, p_4)$, cyclic$(m_3, m_4, q_3, p_3, p_4)$, cyclic$(m_3, m_2, q_2, p_2, p_3)$, cyclic$(m_1, m_2, q_1, p_1, p_2)$.

Conclusion: cyclic$(m_0, m_1, m_2, m_4)$.

The Machine Proof

$[m_4m_1, m_4m_0] - [m_2m_1, m_2m_0]$

(Since $m_1$, $m_4$, $p_0$, $q_1$ are cyclic; $m_0$, $m_4$, $p_0$, $p_4$ are cyclic;
$[m_4m_1, m_4m_0] = [m_4m_1, m_4p_0] + [m_4p_0, m_4m_0] = [m_1q_1, q_1p_0] - [m_0p_4, p_4p_0]$.)

$= -[m_2m_1, m_2m_0] + [m_1q_1, q_1p_0] - [m_0p_4, p_4p_0]$

(Since $m_1$, $m_2$, $p_1$, $p_2$ are cyclic; $m_0$, $m_2$, $p_1$, $q_4$ are cyclic;
$[m_2m_1, m_2m_0] = [m_2m_1, m_2p_1] + [m_2p_1, m_2m_0] = [m_1p_2, p_2p_1] - [m_0q_4, q_4p_1]$.)

$= [m_1q_1, q_1p_0] - [m_1p_2, p_2p_1] + [m_0q_4, q_4p_1] - [m_0p_4, p_4p_0]$

(Since $p_0$, $p_1$, $q_1$ are collinear; $q_1$, $m_1$, $p_1$, $p_2$ are cyclic;
$[m_1q_1, q_1p_0] = [q_1m_1, q_1p_1] = [m_1p_2, p_2p_1]$.)

$= [m_0q_4, q_4p_1] - [m_0p_4, p_4p_0]$

(Since $p_0$, $p_1$, $q_4$ are collinear; $q_4$, $m_0$, $p_0$, $p_4$ are cyclic;
$[m_0q_4, q_4p_1] = [q_4m_0, q_4p_0] = [m_0p_4, p_4p_0]$.)

$= 0$

The following information are in the GIB.

cyclic($m_3$, $m_0$, $p_4$, $q_0$, $q_2$)
cyclic($m_1$, $m_4$, $p_0$, $q_1$, $q_3$)
cyclic($m_1$, $m_3$, $p_2$, $q_0$, $q_3$)
cyclic($m_2$, $m_4$, $p_3$, $q_1$, $q_4$)

cyclic($m_2$, $m_0$, $p_1$, $q_2$, $q_4$)

The first of the five circles is the combination of the following two circles

cyclic($m_3$, $p_4$, $q_0$, $q_2$),

since $\angle[p_4m_3, q_0p_4] = \angle[q_2m_3, q_0q_2]$. Both the above two angles equal to $\angle[m_3p_3, p_2p_3]$, since cyclic$(p_3, p_4, q_3, m_3)$ and cyclic$(p_2, p_3, q_2, m_3)$.

cyclic($m_0$, $p_4$, $q_0$, $q_2$ ),

since $\angle[q_0m_0, q_0q_2] = \angle[p_4m_0, q_2p_4]$. Both the above two angles equal to $\angle[m_0p_0, p_0p_1]$, since cyclic$(p_0, p_4, q_4, m_0)$ and cyclic$(p_0, p_1, q_0, m_0)$.

**Example 107** *A line $AD$ through the vertex $A$ meets the circumcircle of the triangle $ABC$ in $D$. If $U$, $V$ are the orthocenters of the triangle $ABD$, $ACD$, respectively, prove that $UV$ is equal and parallel to $BC$.*

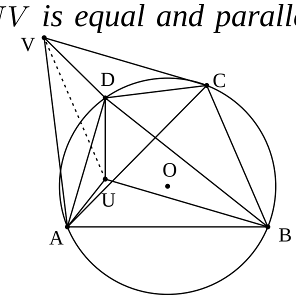

Figure 107

Point order: $a, b, c, d, u, v$.

Hypotheses: cyclic$(a, b, c, d)$, orthocenter$(u, a, b, d)$, orthocenter$(v, a, c, d)$.

Conclusion: para$(u, v, b, c)$.

The Machine Proof

$[vu, cb]$

(Since $v$, $u$, $a$, $d$ are cyclic; $va \perp cd$; $[vu, cb] = [vu, va] + [va, cb] = [ud, da] + [dc, cb] + 1$.)

$= [ud, da] + [dc, cb] + 1$

(Since $ud \perp ab$; $[ud, da] = -[da, ba] + 1$.)

$= [dc, cb] - [da, ba]$

(Since $c$, $d$, $b$, $a$ are cyclic; $[dc, cb] = [da, ba]$.)

$= 0$

**Example 108** *If $P$ is any point on a semicircle, diameter $AB$, and $BC$, $CD$ are two equal arcs, then if $E = CA \cap PB$, $F = AD \cap PC$, prove that $AD$ is perpendicular to $EF$.*

Point order: $a, b, c, p, o, d, e, f$.

Hypotheses: midpoint$(o, a, b)$, perp$(a, c, c, b)$, perp$(a, p, p, b)$, cong$(o, a, o, d)$, cong$(c, d, c, b)$, coll$(e, a, c)$, coll$(e, p, b)$, coll$(f, a, d)$, coll$(f, p, c)$.

Conclusion: perp$(e, f, a, d)$.

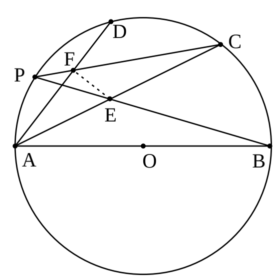

Figure 108

The Machine Proof

$[fe, da] + 1$

(Since $da \parallel co$; $[fe, da] = [fe, oc]$.)

$= [fe, oc] + 1$

(Since $a$, $d$, $f$ are collinear; $f$, $e$, $a$, $p$ are cyclic;
$[fe, oc] = [fe, fa] + [ad, oc] = [ep, pa] + [da, oc]$.)
$= [ep, pa] + [da, oc] + 1$
(Since $ep \perp pa$; $[ep, pa] = 1$.)
$= [da, oc]$
(Since $da \parallel oc$; $[da, oc] = 0$.)
$= 0$

**Example 109** *The centers of the four circles $(pqr)$, $(qrs)$, $(rsp)$, $(spq)$ and the Miquel point lie on the same circle.*

Point order: $a, b, d, c, e, f, m, o_1, o_2, o_3, o_4$.
Hypotheses: coll$(a, b, c)$, coll$(c, d, e)$, coll$(e, f, b)$, coll$(d, f, a)$, cyclic$(m, a, b, f)$, cyclic$(m, a, c, d)$, pbisector$(o_1, a, b)$, pbisector$(o_1, a, f)$, pbisector$(o_2, b, c)$, pbisector$(o_2, b, e)$, pbisector$(o_3, d, f)$, pbisector$(o_3, d, e)$, pbisector$(o_4, a, c)$, pbisector$(o_4, a, d)$.
Conclusion: cyclic$(o_1, o_2, o_3, o_4)$.

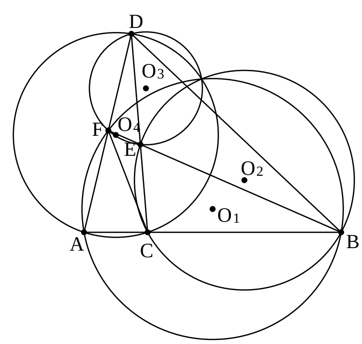

Figure 109

The Machine Proof
$[o_4o_2, o_4o_1] - [o_3o_2, o_3o_1]$
(Since $o_4o_2 \perp cm$; $[o_4o_2, o_4o_1] = -[o_4o_1, mc] + 1$.)
$= -[o_4o_1, mc] - [o_3o_2, o_3o_1] + 1$
(Since $o_4o_1 \perp am$; $[o_4o_1, mc] = -[mc, ma] + 1$.)
$= -[o_3o_2, o_3o_1] + [mc, ma]$
(Since $o_3o_2 \perp em$; $[o_3o_2, o_3o_1] = -[o_3o_1, me] + 1$.)
$= [o_3o_1, me] + [mc, ma] - 1$
(Since $o_3o_1 \perp fm$; $[o_3o_1, me] = [mf, me] + 1$.)
$= [mf, me] + [mc, ma]$
(Since $m$, $f$, $e$, $d$ are cyclic; $[mf, me] = [fd, ed]$.)
$= [mc, ma] + [fd, ed]$
(Since $m$, $c$, $a$, $d$ are cyclic; $[mc, ma] = [cd, da]$.)
$= [fd, ed] + [cd, da]$
(Since $ed \parallel cd$; $[fd, ed] + [cd, da] = [fd, da]$.)
$= [fd, da]$
(Since $a$, $d$, $f$ are collinear; $[fd, da] = 0$.)
$= 0$

**Example 110** *Continue from Example 109. The centers of the four circles $(pqr)$, $(qrs)$, $(rsp)$, $(spq)$ and the Miquel point lie on the same circle.*

Point order: $a, b, d, c, e, f, m, o_1, o_2, o_3, o_4$.

Hypotheses: coll$(a, b, c)$, coll$(c, d, e)$, coll$(e, f, b)$, coll$(d, f, a)$, cyclic$(m, a, b, f)$, cyclic$(m, a, c, d)$, pbisector$(o_1, a, b)$, pbisector$(o_1, a, f)$, pbisector$(o_2, b, c)$, pbisector$(o_2, b, e)$, pbisector$(o_3, d, f)$, pbisector$(o_3, d, e)$, pbisector$(o_4, a, c)$, pbisector$(o_4, a, d)$.

Conclusion: cyclic$(o_1, o_2, o_3, m)$.

The Machine Proof

$-[o_3o_2, o_3o_1] + [o_2m, o_1m]$

(Since $o_3o_2 \perp em$; $[o_3o_2, o_3o_1] = -[o_3o_1, me] + 1$.)

$= [o_3o_1, me] + [o_2m, o_1m] - 1$

(Since $o_3o_1 \perp fm$; $[o_3o_1, me] = [mf, me] + 1$.)

$= [o_2m, o_1m] + [mf, me]$

(Since circumcenter($o_2$,$m$, $e$, $b$ );
$[o_2m, o_1m] = [o_2m, me] + [me, o_1m] = -[o_1m, me] + [mb, eb] + 1$.)

$= -[o_1m, me] + [mf, me] + [mb, eb] + 1$

(Since circumcenter($o_1$,$m$, $a$, $b$ );
$[o_1m, me] = [o_1m, ma] + [ma, me] = -[me, ma] + [mb, ba] + 1$.)

$= [mf, me] + [me, ma] + [mb, eb] - [mb, ba]$

(Since $me \parallel me$; $[mf, me] + [me, ma] = [mf, ma]$.)

$= [mf, ma] + [mb, eb] - [mb, ba]$

(Since $mb \parallel mb$; $[mb, eb] - [mb, ba] = -[eb, ba]$.)

$= [mf, ma] - [eb, ba]$

(Since $m$, $f$, $a$, $b$ are cyclic; $[mf, ma] = [fb, ba]$.)

$= [fb, ba] - [eb, ba]$

(Since $b$, $e$, $f$ are collinear; $[fb, ba] = [eb, ba]$.)

$= 0$

## 3 Time and Proof Length Statistics

| Exs | St | Mt | Time | Exs | St | Mt | Time | Exs | St | Mt | Time |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 0.89 | 38 | 6 | 4 | 0.735 | 75 | 7 | 5 | 0.922 |
| 2 | 2 | 2 | 1.016 | 39 | 6 | 3 | 8.530 | 76 | 5 | 2 | 0.719 |
| 3 | 2 | 2 | 0.916 | 40 | 4 | 2 | 0.532 | 77 | 9 | 5 | 1.218 |
| 4 | 2 | 2 | 0.86 | 41 | 7 | 4 | 1.25 | 78 | 11 | 4 | 1.562 |
| 5 | 2 | 1 | 0.453 | 42 | 7 | 3 | 2.531 | 79 | 5 | 3 | 0.703 |
| 6 | 5 | 3 | 0.64 | 43 | 4 | 2 | 0.422 | 80 | 6 | 4 | 0.656 |
| 7 | 5 | 2 | 0.969 | 44 | 4 | 2 | 0.5 | 81 | 9 | 3 | 1.249 |
| 8 | 7 | 3 | 1.11 | 45 | 3 | 2 | 0.937 | 82 | 7 | 3 | 1.359 |
| 9 | 7 | 2 | 1.562 | 46 | 9 | 5 | 1.453 | 83 | 5 | 3 | 0.672 |
| 10 | 4 | 3 | 0.5 | 47 | 4 | 2 | 0.593 | 84 | 8 | 3 | 2.858 |
| 11 | 5 | 3 | 1.0 | 48 | 5 | 2 | 0.453 | 85 | 5 | 4 | 0.641 |
| 12 | 5 | 2 | 0.516 | 49 | 9 | 4 | 1.703 | 86 | 8 | 3 | 1.656 |
| 13 | 6 | 2 | 1.297 | 50 | 8 | 3 | 1.0 | 87 | 9 | 4 | 1.313 |
| 14 | 6 | 3 | 1.375 | 51 | 6 | 4 | 2.172 | 88 | 11 | 4 | 2.063 |
| 15 | 14 | 3 | 2.75 | 52 | 7 | 3 | 1.359 | 89 | 10 | 4 | 2.421 |
| 16 | 8 | 4 | 2.469 | 53 | 10 | 4 | 1.156 | 90 | 5 | 3 | 1.031 |
| 17 | 10 | 5 | 1.531 | 54 | 8 | 3 | 1.156 | 91 | 7 | 3 | 1.453 |
| 18 | 4 | 3 | 0.531 | 55 | 9 | 3 | 1.282 | 92 | 7 | 3 | 1.484 |
| 19 | 4 | 2 | 0.375 | 56 | 11 | 4 | 1.187 | 93 | 6 | 3 | 5.076 |
| 20 | 7 | 3 | 1.078 | 57 | 5 | 2 | 0.75 | 94 | 9 | 3 | 1.406 |
| 21 | 4 | 4 | 0.391 | 58 | 5 | 2 | 0.484 | 95 | 5 | 3 | 0.734 |
| 22 | 8 | 3 | 0.937 | 59 | 4 | 2 | 0.781 | 96 | 6 | 3 | 0.828 |
| 23 | 4 | 3 | 0.657 | 60 | 11 | 4 | 2.32 | 97 | 9 | 3 | 1.437 |
| 24 | 5 | 3 | 0.515 | 61 | 5 | 3 | 0.593 | 98 | 8 | 5 | 1.032 |
| 25 | 8 | 4 | 2.75 | 62 | 5 | 3 | 1.265 | 100 | 6 | 3 | 2.219 |
| 26 | 6 | 3 | 0.937 | 63 | 7 | 3 | 0.672 | 101 | 7 | 3 | 0.984 |
| 27 | 8 | 4 | 1.063 | 64 | 4 | 3 | 0.734 | 99 | 4 | 3 | 0.562 |
| 28 | 6 | 3 | 1.172 | 65 | 6 | 3 | 0.532 | 102 | 7 | 4 | 5.99 |
| 29 | 5 | 2 | 0.593 | 66 | 5 | 3 | 0.593 | 103 | 7 | 3 | 22.95 |
| 30 | 5 | 2 | 0.5 | 67 | 5 | 3 | 1.015 | 104 | 6 | 3 | 3.889 |
| 31 | 7 | 3 | 1.25 | 68 | 8 | 3 | 1.828 | 105 | 12 | 3 | 9.515 |
| 32 | 6 | 3 | 0.703 | 69 | 5 | 3 | 0.641 | 106 | 7 | 4 | 21.436 |
| 33 | 7 | 3 | 0.844 | 70 | 7 | 2 | 0.968 | 107 | 4 | 3 | 3.421 |
| 34 | 8 | 4 | 0.765 | 71 | 5 | 3 | 0.844 | 108 | 5 | 3 | 6.342 |
| 35 | 10 | 4 | 1.11 | 72 | 3 | 2 | 0.422 | 109 | 9 | 3 | 6.797 |
| 36 | 12 | 4 | 1.266 | 73 | 6 | 3 | 0.703 | 110 | 9 | 4 | 6.875 |
| 37 | 7 | 4 | 1.016 | 74 | 5 | 2 | 1.109 | | | | |

Table 1. Statistics for the Examples in This paper

We include four indexes for each theorem.

1. Exs indicates the example number.

2. St indicates the proof length.

3. Mt indicates the length of the maximal algebraic expressions in the proof.
4. Time indicates the cpu time used to obtain the first proof using the depth-first search strategy.