Adaptive Fuzzy Output Tracking Control of MIMO Nonlinear Uncertain Systems

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***Abstract*—In this paper, the adaptive fuzzy tracking control problem is discussed for a class of uncertain multiple-input–multiple-output (MIMO) nonlinear systems with the block-triangular structure. The fuzzy logic systems are used to approximate the unknown nonlinear functions. By using the backstepping technique, the adaptive fuzzy tracking control design scheme is developed, which has minimal learning parameterizations. The adaptive fuzzy tracking controllers guarantee that the outputs of systems converge to a small neighborhood of the reference signals and all the signals in the closed-loop system are semiglobally uniformly ultimately bounded. Two examples are used to show the effectiveness of the approach.**

***Index Terms*—Adaptive fuzzy control, backstepping, multiple-input–multiple-output (MIMO) nonlinear systems, output tracking, uncertainty.**

I. INTRODUCTION

IN PRACTICE, most plants are nonlinear and contain uncertainties. During the past years, many people have devoted a lot of effort to both theoretical research and implementation techniques to handle nonlinear control problems. In [13] a genetic-algorithm-based fuzzy modelling approach was proposed to generate Takagi–Sugeno–Kang (TSK) models. A simple but effective fuzzy-rule-based models of complex systems from input–output data was developed in [12]. Fuzzy control methodology has emerged in recent years as a promising way to deal with the control problems of nonlinear systems containing highly uncertain nonlinear functions. It has been shown that fuzzy logic systems can be used to approximate any nonlinear function over a convex compact region [20] and [21]. Based on this observation, many systematic fuzzy controller design methods have been developed to solve output tracking control problems for single-input–single-output (SISO) systems with unknown nonlinearities. Stable direct and indirect adaptive fuzzy control schemes were first developed to control uncertain nonlinear systems by Lyapunov function method [20]. Afterwards, several stable adaptive fuzzy control schemes have been introduced, respectively, for SISO nonlinear systems [2], [3], [15], [16], [19]. In recent years, the corresponding research results have been extended to multiple-input–multiple-output (MIMO) nonlinear systems [18]. The basic idea of these works is to use the fuzzy logic systems to approximate the unknown nonlinear functions in systems and design adaptive fuzzy controllers by using Lyapunov stability theory. All the results mentioned previously are obtained with the restriction that the system is feedback linearizable. This means that the unknown nonlinear functions satisfy the matching conditions. However, in practice, a large class of physical systems may contain unknown nonlinear functions which do not satisfy the matching conditions. In this case, the adaptive fuzzy control approaches mentioned previously fail.

Backstepping, which is based on the nonlinear stabilization technique of "adding an integrator" introduced in [17] and [1], and was first used in nonlinear adaptive control in [6], leads to the discovery of a structural strict feedback condition under which the systematic construction of robust control Lyapunov function is always possible. Up to now, backstepping-based adaptive control technique, which is mainly used to deal with the robust control of nonlinear systems with parametric uncertainties, has become one of the most popular design methods for a large class of nonlinear systems [9], [10], [14], [23].

Recently, the adaptive neural control approach based on backstepping design has been developed for nonlinear uncertain systems without the requirement of matching conditions. In [7], [8], and [24], stable neural controller design schemes were proposed for unknown nonlinear SISO systems via backstepping design technique. With the backstepping design technique, neural networks were mostly applied to approximate the unmatched and unknown nonlinearities, and then implement adaptive control using the conventional control technology. In [5], an adaptive neural control approach was proposed for a class of MIMO nonlinear systems with triangular structure in control input. By using the triangular property, an integral-type Lyapunov functions are introduced to construct a Lyapunov-based controller. Particularly, the further results on backstepping-based neural control for more general uncertain MIMO nonlinear systems, in which the unknown system state interconnections appear in every equation of each subsystem, have been proposed in [4]. The advantage of adaptive neural control based on backstepping methodology includes that both the parameters and the nonlinear functions can be unknown and the uncertainties in systems need not satisfy the matching conditions. Similar to neural networks, the fuzzy logic systems can be also used to uniformly approximate unknown nonlinear functions. Compared with neural networks, fuzzy logic systems can achieve the faster

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convergence because the fuzzy models perform a fuzzy blending of local models, and fuzzy logic systems are capable of accommodating both numerical data and expert knowledge as they can be represented by linguistic IF–THEN rules. However, compared with neural control, there are only a few results available in the literature on adaptive fuzzy control via backstepping design approach. More recently, the pioneering works on adaptive fuzzy control for unknown nonlinear systems via backstepping method was proposed in [22], where the T–S type fuzzy logic systems are used to approximate the unknown nonlinear functions, and the backstepping-based adaptive fuzzy tracking control methodology has been developed for strict-feedback SISO nonlinear systems. An adaptive fuzzy tracking controller has been constructed by using backstepping design technique. The advantage of the scheme proposed in [22] is that it does not require matching conditions for the unknown nonlinear functions, and has less adaptive parameters. So far, there is no result on adaptive fuzzy output tracking control for MIMO nonlinear systems with unmatched nonlinear functions.

In this paper, we consider adaptive fuzzy control of a class of uncertain MIMO nonlinear systems with block-triangular forms. MIMO nonlinear systems with block-triangular forms was first proposed in [4]. The purpose of this paper is to develop an adaptive fuzzy control design method for output tracking control problems of MIMO nonlinear systems. By using the backstepping approach, a new systematic procedure is developed for the synthesis of the stable adaptive fuzzy output tracking controllers for MIMO nonlinear systems with block-triangular structure. The fuzzy logic systems are used to approximate the unknown nonlinear functions. The adaptive fuzzy controllers are constructed by using backstepping design technique. The proposed design scheme achieves semiglobal uniform ultimate boundedness of all the signals in the closed-loop systems. The tracking error is proven to converge to a small neighborhood of the origin. The adaptive laws in the paper are designed based on norms of unknown parameters in fuzzy logic approximators, so the number of the adaptive laws is reduced to the number of unknown nonlinear functions. As a result, adaptive controllers require less computation time.

II. Preliminaries and Problem Formulation

Consider a class of uncertain MIMO nonlinear systems described by the following differential equations:

$$\begin{aligned} \dot{x}_{i,i_j} &= g_{i,i_j}\left(\bar{x}_{1,(i_j-\rho_{1i})}, \bar{x}_{2,(i_j-\rho_{2i})}, \ldots, \bar{x}_{m,(i_j-\rho_{mi})}\right) x_{i,i_j+1} \\ &\quad + f_{i,i_j}\left(\bar{x}_{1,(i_j-\rho_{1i})}, \bar{x}_{2,(i_j-\rho_{2i})}, \ldots, \bar{x}_{m,(i_j-\rho_{mi})}\right) \\ \dot{x}_{i,\rho_i} &= g_{i,\rho_i}(\bar{x}_{1,(\rho_i-\rho_{1i})}, \bar{x}_{2,(\rho_i-\rho_{2i})}, \ldots, \bar{x}_{m,(\rho_i-\rho_{mi})})u_i \\ &\quad + f_{i,\rho_i}(\bar{x}, u_1, \ldots, u_{i-1}) \quad 1 \le i \le m; \quad 1 \le i_j < \rho_i \\ y_i &= x_{i,1} \end{aligned} \tag{1}$$

where $x_{i,i_j}, i_j = 1, \ldots, \rho_i$, are the state variables of the ith subsystem, $u_i \in R$ and $y_i \in R$ are the control input and output of the ith subsystem, respectively. $g_{i,i_j}(\cdot)$ and $f_{i,i_j}(\cdot)$ are unknown nonlinear functions, i, i_j and m are positive integers. For simplicity, throughout this paper, the following notations are used: $\bar{x}_{i_{ij}} = [x_{i,1} \ \ldots \ x_{i,i_j}]^T$, $\bar{x} = [\bar{x}_{1,\rho_1}^T \ \ldots \ \bar{x}_{m,\rho_m}^T]^T$ and $\rho_{ij} = \rho_i - \rho_j$ with ρ_i being the order of the ith subsystem.

Remark 1: If $i_j - \rho_{ki} \le 0$, then the corresponding variable $\bar{x}_{k,(i_j-\rho_{ki})}$ does not exist and does not appear in system (1). If $i_j - \rho_{ki} > 0$, $\bar{x}_{k,(i_j-\rho_{ki})}$ stands for the maximum number of state variables of the kth subsystem which are embedded in ith subsystem.

Remark 2: Note that the state variables of the kth subsystem appear in the i_jth equation of the ith subsystem in the form $\bar{x}_{k,q}$ with $q = (i_j - \rho_{ki})$, which implies that $q - \rho_{ik} < i_j + 1$, so the state variable x_{i,i_j+1} will not appear in the first q equations of the kth subsystem. Therefore, system (1) is called to be of the block-triangular form. MIMO nonlinear systems with the block-triangular form has been first proposed in [4]. It is clear that system (1) is equivalent to the system in [4]. See [4] for more details. Just by using the properties of block-triangular form, the output tracking control problem can be solved for uncertain MIMO nonlinear systems via backstepping approach.

The control objective is to design adaptive fuzzy controllers for system (1) such that: *i)* all the signals in the closed-loop system remain uniformly ultimately bounded, and *ii)* the output $y_i(t)$ follows a given reference signal $y_{id}(t)$.

From a mathematical point of view, fuzzy logic systems can be used as practical function approximators. Thus, the following fuzzy logic system is used to approximate a continuous function $f(x)$ defined on some compact set.

R_i: *IF* x_1 is F_1^i and,..., and x_n is F_n^i, *THEN* y is B^i $(i = 1, 2, \ldots, N)$ into fuzzy logic system

$$y(x) = \frac{\sum_{j=1}^{N} \Phi_j \prod_{i=1}^{n} \mu_{F_i^j}(x_i)}{\sum_{j=1}^{N} \left[\prod_{i=1}^{n} \mu_{F_i^j}(x_i)\right]}$$

where $x = [x_1, \ldots, x_n]^T \in R^n$, $j = 1, 2, \ldots, N$ and N is the total number of fuzzy rules. F_j^i and B^i are fuzzy sets. $\Phi_j = \max_{y \in R} \mu_{B^j(y)}$ and $\mu_{F_i^j}(x_i)$ is the membership of F_i^j. Let $p_j(x) = \left(\prod_{i=1}^{n} \mu_{F_i^j}(x_i) / \sum_{j=1}^{N} \left[\prod_{i=1}^{n} \mu_{F_i^j}(x_i)\right]\right)$, $P(x) = [p_1(x), p_2(x), \ldots, p_N(x)]^T$ and $\Phi = [\Phi_1, \ldots, \Phi_N]^T$. The fuzzy logic system above can be expressed as follows.

$$y(x) = \Phi^T P(x). \tag{2}$$

If all memberships are chosen as Gussian functions, then we have the following lemma [20].

Lemma 1: Let $f(x)$ is a continuous function defined on a compact set Ω. Then for any given constant $\varepsilon > 0$, there exists a fuzzy logic system (2) such that

$$\sup_{x \in \Omega} \left| f(x) - \Phi^T P(x) \right| \le \varepsilon.$$

Assumption 1: There exist constants a_{i,i_j}^m and a_{i,i_j}^M such that for $i = 1, \ldots, m$ and $j = 1, \ldots, \rho_i$

$$\begin{aligned} 0 < a_{i,i_j}^m &\le \left| g_{i,i_j}\left(\bar{x}_{1,(i_j-\rho_{1i})}, \bar{x}_{2,(i_j-\rho_2)}, \ldots, \bar{x}_{m,(i_j-\rho_{mi})}\right) \right| \\ &\le a_{i,i_j}^M. \end{aligned}$$

It is obvious that Assumption 1 requires the unknown functions g_{i,i_j} are not zero. Without loss of generality, it is assumed that $g_{i,i_j} > 0$.

Lemma 2: Let $M(x_1, x_2, \ldots, x_n)$ be a real-valued continuous function and satisfy $0 < a_m \le M(x_1, x_2, \ldots, x_n) \le a_M$ with a_m and a_M being two constants. Define function $V(t)$ as follows:

$$V(t) = \int_0^{z(t)} \sigma M(x_1, x_2, \ldots, x_{k-1}, \sigma + \beta(t), x_{k+1}, \ldots, x_n) d\sigma$$

where $z(t)$ and $\beta(t)$ are real-value functions with $t \in [0, \infty)$. Then the integral function $V(t)$ has the following properties.

1)

$$\frac{1}{2} a_m z^2(t) \le V(t) \le \frac{1}{2} a_M z^2(t).$$

2)

$$\begin{aligned} \frac{d}{dt} V(t) = {} & z(t) M(x_1, x_2, \ldots, x_{k-1}, z(t) + \beta(t), x_{k+1}, \ldots, x_n) \dot{z}(t) \\ & + \dot{\beta}(t) z(t) M(x_1, x_2, \ldots, x_{k-1}, z(t) + \beta(t), x_{k+1}, \ldots, x_n) \\ & + z^2(t) \dot{x}_i(t) \int_0^1 \theta \sum_{i=1, i \neq k}^{n-1} \frac{\partial}{\partial x_i} M (x_1, x_2, \ldots, x_{k-1}, z(t) + \beta(t), x_{k+1}, \ldots, x_n) d\theta \\ & - z(t) \dot{\beta}(t) \int_0^1 M(x_1, x_2, \ldots, x_{k-1}, \theta z(t) + \beta(t), x_{k+1}, \ldots, x_n) d\theta. \end{aligned}$$

Proof: Conclusion 1 is obtained immediately from the inequalities $0 < a_m \le M(x_1, x_2, \ldots, x_n) \le a_M$. To get the second conclusion, differentiating $V(t)$ gives

$$\begin{aligned} \frac{d}{dt} V(t) = {} & z(t) M(x_1, x_2, \ldots, x_{k-1}, z(t) + \beta(t), x_{k+1}, \ldots, x_n) \dot{z}(t) \\ & + \int_0^{z(t)} \sigma \dot{\beta}(t) \frac{\partial}{\partial \sigma} M(x_1, x_2, \ldots, x_{k-1}, \sigma + \beta(t), x_{k+1}, \ldots, x_n) d\sigma \\ & + \int_0^{z(t)} \sigma \dot{x}_i(t) \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} M(x_1, x_2, \ldots, x_{k-1}, \sigma + \beta(t), x_{k+1}, \ldots, x_n) d\sigma. \end{aligned} \tag{3}$$

A simple calculation gives (4), as shown at the bottom of the page. By using the transformation $\sigma = z(t)\theta$, the following result is obtained:

$$\begin{aligned} & \int_0^{z(t)} \sigma \dot{x}_i(t) \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} M(x_1, x_2, \ldots, x_{k-1}, \sigma + \beta(t), x_{k+1}, \ldots, x_n) d\sigma \\ & = z^2(t) \dot{x}_i(t) \int_0^1 \theta \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} M(x_1, x_2, \ldots, x_{k-1}, \theta z(t) + \beta(t), x_{k+1}, \ldots, x_n) d\theta. \end{aligned} \tag{5}$$

Similarly

$$\begin{aligned} & \dot{\beta}(t) \int_0^{z(t)} M(x_1, x_2, \ldots, x_{k-1}, \sigma + \beta(t), x_{k+1}, \ldots, x_n) d\sigma \\ & = z(t) \dot{\beta}(t) \int_0^1 M(x_1, x_2, \ldots, x_{k-1}, \theta z(t) + \beta(t), x_{k+1}, \ldots, x_n) d\theta. \end{aligned} \tag{6}$$

Therefore, Conclusion 2 follows immediately from substituting (4)–(6) into (3).

III. Adaptive Fuzzy Control Design

In this section, the backstepping design technique is used to design tracking controllers for all the subsystems of (1). Note that all the subsystems in (1) are interconnected, the stability analysis of whole closed-loop MIMO system becomes difficult. But with the block-triangular property, it is feasible to design

$$\begin{aligned} & \int_0^{z(t)} \sigma \dot{\beta}(t) \frac{\partial}{\partial \sigma} M(x_1, x_2, \ldots, x_{k-1}, \sigma + \beta(t), x_{k+1}, \ldots, x_n) d\sigma \\ & = \dot{\beta}(t) \int_0^{z(t)} \sigma \frac{\partial}{\partial \sigma} M(x_1, x_2, \ldots, x_{k-1}, \sigma + \beta(t), x_{k+1}, \ldots, x_n) d\sigma \\ & = \dot{\beta}(t) \sigma M(x_1, x_2, \ldots, x_{k-1}, \sigma + \beta(t), x_{k+1}, \ldots, x_n) \mid_0^{z(t)} \\ & \quad - \dot{\beta}(t) \int_0^{z(t)} M(x_1, x_2, \ldots, x_{k-1}, \sigma + \beta(t), x_{k+1}, \ldots, x_n) d\sigma \\ & = \dot{\beta}(t) z(t) M(x_1, x_2, \ldots, x_{k-1}, z(t) + \beta(t), x_{k+1}, \ldots, x_n) \\ & \quad - \dot{\beta}(t) \int_0^{z(t)} M(x_1, x_2, \ldots, x_{k-1}, \sigma + \beta(t), x_{k+1}, \ldots, x_n) d\sigma \end{aligned} \tag{4}$$

a full state feedback controller and prove the stability of the closed-loop system.

For the ith subsystem of (1), the backstepping design procedure contains ρ_i steps. At the Step i_j $(1 \le i_j \le \rho_i - 1)$, fuzzy logic systems will be employed to approximate the unknown nonlinear function, and then an intermediate feedback control α_{i,i_j} will be developed, which guarantees the stability of the i_jth subsystem in the ith subsystem with respect to a Lyapunov function V_{i,i_j}. The tracking controller u_i will be designed at the ρ_i step. In the following, we will give the procedure of backstepping design for the ith subsystem.

1) Step 1: Define the tracking error variable $z_{i,1} = x_{i,1} - y_{id}$. Then from the first differential equation in the ith subsystem, the following differential equation can be obtained:

$$\dot{z}_{i,1} = g_{i,1}x_{i,2} + f_{i,1} - \dot{y}_{id} \tag{7}$$

where $g_{i,1}$ and $f_{i,1}$ are the unknown functions of the variables $\bar{x}_{1,(1-\rho_{i1})}, \ldots, \bar{x}_{m,(1-\rho_{im})}$, and $\rho_{ik} = \rho_i - \rho_k, k = 1, 2, \ldots, m$. If $1 - \rho_{ik} \le 0$, the corresponding state variable $\bar{x}_{k,(1-\rho_{ik})}$ does not appear in (7). Consider a Lyapunov function candidate as

$$V_{i,1} = \int_0^{z_{i,1}} \sigma \bar{M}_{i,1}(\sigma + y_{id})d\sigma + \frac{1}{2r_{i,1}}\tilde{\theta}_{i,1}^2 \tag{8}$$

where

$$\begin{aligned}\bar{M}_{i,1}&(\sigma + y_{id}) \\ &= g_{i,1}^{-1}(\bar{x}_{1,(1-\rho_{i1})}, \ldots, \bar{x}_{i-1,(1-\rho_{ii-1})} \\ &\qquad (\sigma + y_{id}), \bar{x}_{i+1,(1-\rho_{ii+1})}, \ldots, \bar{x}_{m,(1-\rho_{im})})\end{aligned}$$

$\tilde{\theta}_{i,1} = \theta_{i,1} - \hat{\theta}_{i,1}, \theta_{i,1}$ is an unknown parameter, which will be specified later, and $\hat{\theta}_{i,1}$ is its estimation. Then, by Lemma 2 the time derivative of $V_{i,1}$ is given by

$$\begin{aligned}\dot{V}_{i,1} &= z_{i,1}g_{i,1}^{-1}\dot{z}_{i,1} + \dot{y}_{id}z_{i,1}g_{i,1}^{-1} \\ &\quad - z_{i,1}\dot{y}_{id}\int_0^1 \bar{M}_{i,1}(\theta z_{i,1} + y_{id})d\theta \\ &\quad + \sum_{j=1,j\ne i}^{m}\sum_{s=1}^{1-\rho_{ij}} z_{i,1}^2\dot{x}_{j,s}\int_0^1 \theta\frac{\partial \bar{M}_{i,1}(\theta z_{i,1} + y_{id})}{\partial x_{j,k}}d\theta \\ &\quad - \frac{1}{r_{i,1}}\tilde{\theta}_{i,1}\dot{\hat{\theta}}_{i,1} \\ &= z_{i,1}\left(x_{i,2} + g_{i,1}^{-1}f_{i,1}\right) \\ &\quad - z_{i,1}\dot{y}_{id}\int_0^1 \bar{M}_{i,1}(\theta z_{i,1} + y_{id})d\theta \\ &\quad + \sum_{j=1,j\ne i}^{m}\sum_{s=1}^{1-\rho_{ij}} z_{i,1}^2\dot{x}_{j,s}\int_0^1 \theta\frac{\partial \bar{M}_{i,1}(\theta z_{i,1} + y_{id})}{\partial x_{j,k}}d\theta \\ &\quad - \frac{1}{r_{i,1}}\tilde{\theta}_{i,1}\dot{\hat{\theta}}_{i,1} \\ &= z_{i,1}\left(x_{i,2} + \hat{f}_{i,1}\right) - \frac{1}{r_{i,1}}\tilde{\theta}_{i,1}\dot{\hat{\theta}}_{i,1}\end{aligned} \tag{9}$$

where

$$\begin{aligned}\hat{f}_{i,1} &= f_{i,1}g_{i,1}^{-1} - \dot{y}_{id}\int_0^1 \bar{M}_{i,1}(\theta z_{i,1} + y_{id})d\theta \\ &\quad + \sum_{j=1,j\ne i}^{m}\sum_{s=1}^{1-\rho_{ij}} z_{i,1}\dot{x}_{j,s}\int_0^1 \theta\frac{\partial \bar{M}_{i,1}(\theta z_{i,1} + y_{id})}{\partial x_{j,k}}d\theta.\end{aligned}$$

According to Lemma 1, for a given $\varepsilon_{i,1} > 0$, there exists a fuzzy logic system $\Phi_{i,1}^T P_{i,1}(X_{i,1})$, which can be employed to approximate $\hat{f}_{i,1}$, such that

$$\hat{f}_{i,1} = \Phi_{i,1}^T P_{i,1}(X_{i,1}) + \delta_{i,1}(X_{i,1}) \quad |\delta_{i,1}(X_{i,1})| \le \varepsilon_{i,1} \tag{10}$$

where

$$\begin{aligned}X_{i,1} = \Big[&\bar{x}_{1,(1-\rho_{i1})}^T, \ldots, \bar{x}_{i-1,(1-\rho_{ii-1})}^T, x_{i,1} \\ &\bar{x}_{i+1,(1-\rho_{ii-1})}^T, \ldots, \bar{x}_{m,(1-\rho_{im})}^T\Big]^T\end{aligned}$$

and $\delta_{i,1}(X_{i,1})$ is the approximation error. It is evident that (10) can be rewritten as

$$\hat{f}_{i,1} = \|\Phi_{i,1}\|\,\Phi_{i,1}^{*T}P_{i,1}(X_{i,1}) + \delta_{i,1}(X_{i,1}) \tag{11}$$

with $\Phi_{i,1}^* = (\Phi_{i,1}/\|\Phi_{i,1}\|)$. Substituting (11) into (9) produces

$$\begin{aligned}\dot{V}_{i,1} = z_{i,1}&\left(x_{i,2} + \|\Phi_{i,1}\|\,\Phi_{i,1}^{*T}P_{i,1}(X_{i,1}) + \delta_{i,1}(X_{i,1})\right) \\ &- \frac{1}{r_{i,1}}\tilde{\theta}_{i,1}\dot{\hat{\theta}}_{i,1}.\end{aligned} \tag{12}$$

Now, let $\theta_{i,1} = \|\Phi_{i,1}\|^2$. By using the well-known inequality $x^Ty \le (1/2)\varepsilon^{-2}x^Tx + (1/2)\varepsilon^2y^Ty$ with a constant $\varepsilon > 0$, the following inequality can be obtained:

$$\begin{aligned}z_{i,1}&\|\Phi_{i,1}\|\,\Phi_{i,1}^{*T}P_{i,1}(X_{i,1}) + z_{i,1}\delta_{i,1}(X_{i,1}) \\ &\le \frac{1}{2}a_{i,1}^{-2}z_{i,1}^2P_{i,1}^TP_{i,1}\theta_{i,1} + \frac{1}{2}a_{i,1}^2\Phi_{i,1}^{*T}\Phi_{i,1}^* \\ &\quad + \frac{1}{2}c_{i,1}^{-2}z_{i,1}^2 + \frac{1}{2}c_{i,1}^2\delta_{i,1}^2 \\ &\le \frac{1}{2}a_{i,1}^{-2}z_{i,1}^2P_{i,1}^TP_{i,1}\theta_{i,1} + \frac{1}{2}a_{i,1}^2 \\ &\quad + \frac{1}{2}c_{i,1}^{-2}z_{i,1}^2 + \frac{1}{2}c_{i,1}^2\varepsilon_{i,1}^2\end{aligned} \tag{13}$$

where $a_{i,1}$ and $c_{i,1}$ are positive constants. Note that $\theta_{i,1} = \tilde{\theta}_{i,1} + \hat{\theta}_{i,1}$, substituting (13) into (12) leads to

$$\begin{aligned}\dot{V}_{i,1} &\le z_{i,1}\left(x_{i,2} + \frac{1}{2}a_{i,1}^{-2}z_{i,1}P_{i,1}^TP_{i,1}\theta_{i,1} + \frac{1}{2}c_{i,1}^{-2}z_{i,1}\right) \\ &\quad + \frac{1}{2}a_{i,1}^2 + \frac{1}{2}c_{i,1}^2\varepsilon_{i,1}^2 - \frac{1}{r_{i,1}}\tilde{\theta}_{i,1}\dot{\hat{\theta}}_{i,1} \\ &= z_{i,1}\left(x_{i,2} + \frac{1}{2}a_{i,1}^{-2}z_{i,1}P_{i,1}^TP_{i,1}\hat{\theta}_{i,1} + \frac{1}{2}c_{i,1}^{-2}z_{i,1}\right) \\ &\quad + \frac{1}{r_{i,1}}\tilde{\theta}_{i,1}\left(\frac{r_{i,1}}{2a_{i,1}^2}z_{i,1}^2P_{i,1}^TP_{i,1} - \dot{\hat{\theta}}_{i,1}\right) \\ &\quad + \frac{1}{2}a_{i,1}^2 + \frac{1}{2}c_{i,1}^2\varepsilon_{i,1}^2.\end{aligned} \tag{14}$$

Now, choose the intermediate virtual control input $\alpha_{i,1}$ as

$$\alpha_{i,1} = -\left(\lambda_{i,1} + \frac{1}{2}c_{i,1}^{-2}\right) z_{i,1} - \frac{1}{2}a_{i,1}^{-2} z_{i,1} P_{i,1}^T P_{i,1}\hat{\theta}_{i,1} \quad (15)$$

where $\lambda_{i,1} > 0$ is the design constant. Then, by using (15), (14) can be expressed as

$$\begin{aligned}\dot{V}_{i,1} \le & -\lambda_{i,1} z_{i,1}^2 + z_{i,1}(x_{i,2} - \alpha_{i,1}) \\ & + \frac{1}{r_{i,1}}\tilde{\theta}_{i,1}\left(\frac{r_{i,1}}{2} a_{i,1}^{-2} z_{i,1}^2 P_{i,1}^T P_{i,1} - \dot{\hat{\theta}}_{i,1}\right) \\ & + \frac{1}{2}a_{i,1}^2 + \frac{1}{2}c_{i,1}^2 \varepsilon_{i,1}^2. \end{aligned} \quad (16)$$

2) Step 2: Define $z_{i,2} = x_{i,2} - \alpha_{i,1}$, then

$$\dot{z}_{i,2} = g_{i,2} x_{i,3} + f_{i,2} - \dot{\alpha}_{i,1} \quad (17)$$

where $g_{i,2}$ and $f_{i,2}$ are the functions of the variables $\bar{x}_{1,(2-\rho_{i1})}, \ldots \bar{x}_{m,(2-\rho_{im})}$, and $\dot{\alpha}_{i,1} = (\partial\alpha_{i,1}/\partial y_{id})\dot{y}_{id} + (\partial\alpha_{i,1}/\partial\hat{\theta}_{i,1})\dot{\hat{\theta}}_{i,1} + \sum_{j=1}^{m}\sum_{k=1}^{1-\rho_{ij}} (\partial\alpha_{i,1}/\partial x_{jk})\dot{x}_{jk}$.

Then, take a Lyapunov function candidate as

$$V_{i,2} = V_{i,1} + \int_0^{z_{i,2}} \sigma \bar{M}_{i,2}(x_{i,1}, \sigma + \alpha_{i,1}) d\sigma + \frac{1}{2r_{i,2}}\tilde{\theta}_{i,2}^2 \quad (18)$$

where $\bar{M}_{i,2} = g_{i,2}^{-1}\Big(\bar{x}_{1,(2-\rho_{i1})}, \ldots, \bar{x}_{i-1,(2-\rho_{ii-1})}, x_{i,1}, \sigma + \alpha_{i,1}, \bar{x}_{i+1,(2-\rho_{ii+1})}, \ldots, \bar{x}_{m,(2-\rho_{im})}\Big)$ and $\tilde{\theta}_{i,2} = \theta_{i,2} - \hat{\theta}_{i,2}$. According to Lemma 2 the derivative of $V_{i,2}$ is given by (19) and (20), as shown at the bottom of the page. Define

$$\begin{aligned}\hat{f}_{i,2} = & f_{i,2} g_{i,2}^{-1} - \dot{\alpha}_{i,1}\int_0^1 \bar{M}_{i,2}(\theta z_{i,2} + \alpha_{i,1}) d\theta \\ & + \sum_{j=1,j\neq i}^{m}\sum_{s=1}^{2-\rho_{ij}} z_{i,2}\dot{x}_{j,s} \\ & \times \int_0^1 \theta \frac{\partial}{\partial x_{j,k}} \bar{M}_{i,2}(\theta z_{i,2} + \alpha_{i,1}) d\theta \\ & + z_{i,2}\dot{x}_{i,1}\int_0^1 \theta \frac{\partial}{\partial x_{i,1}} \bar{M}_{i,2}(\theta z_{i,2} + \alpha_{i,1}) d\theta. \end{aligned}$$

$$\begin{aligned}\dot{V}_{i,2} = & \dot{V}_{i,1} + z_{i,2} g_{i,2}^{-1}\dot{z}_{i,2} + \dot{\alpha}_{i,1} z_{i,2} g_{i,2}^{-1} \\ & - z_{i,2}\dot{\alpha}_{i,1}\int_0^1 \bar{M}_{i,2}(\theta z_{i,2} + \alpha_{i,1}) d\theta \\ & + \sum_{j=1,j\neq i}^{m}\sum_{s=1}^{2-\rho_{ij}} z_{i,2}^2 \dot{x}_{j,s}\int_0^1 \theta \frac{\partial}{\partial x_{j,k}} \bar{M}_{i,2}(\theta z_{i,2} + \alpha_{i,1}) d\theta \\ & + z_{i,2}^2 \dot{x}_{i,1}\int_0^1 \theta \frac{\partial}{\partial x_{i,1}} \bar{M}_{i,2}(\theta z_{i,2} + \alpha_{i,1}) d\theta - \frac{1}{r_{i,2}}\tilde{\theta}_{i,2}\dot{\hat{\theta}}_{i,2} \end{aligned} \quad (19)$$

$$\begin{aligned} = & \dot{V}_{i,1} + z_{i,2}\Bigg[x_{i,3} + f_{i,2} g_{i,2}^{-1} - \dot{\alpha}_{i,1} g_{i,2}^{-1} \\ & \quad - \dot{\alpha}_{i,1}\int_0^1 \bar{M}_{i,2}(\theta z_{i,2} + \alpha_{i,1}) d\theta + \dot{\alpha}_{i,1} g_{i,2}^{-1} \\ & \quad + \sum_{j=1,j\neq i}^{m}\sum_{s=1}^{2-\rho_{ij}} z_{i,2}\dot{x}_{j,s}\int_0^1 \theta \frac{\partial}{\partial x_{j,k}} \bar{M}_{i,2}(\theta z_{i,2} + \alpha_{i,1}) d\theta \\ & \quad + z_{i,2}\dot{x}_{i,1}\int_0^1 \theta \frac{\partial}{\partial x_{i,1}} \bar{M}_{i,2}(\theta z_{i,2} + \alpha_{i,1}) d\theta\Bigg] \\ & - \frac{1}{r_{i,2}}\tilde{\theta}_{i,2}\dot{\hat{\theta}}_{i,2} \\ = & \dot{V}_{i,1} + z_{i,2}\Bigg[x_{i,3} + f_{i,2} g_{i,2}^{-1} \\ & \quad - \frac{1}{b_{i,2}}\dot{\alpha}_{i,1}\int_0^1 \bar{M}_{i,2}(\theta z_{i,2} + \alpha_{i,1}) d\theta \\ & \quad + \sum_{j=1,j\neq i}^{m}\sum_{s=1}^{2-\rho_{ij}} z_{i,2}\dot{x}_{j,s}\int_0^1 \theta \frac{\partial}{\partial x_{j,k}} \bar{M}_{i,2}(\theta z_{i,2} + \alpha_{i,1}) d\theta \\ & \quad + z_{i,2}^2 \dot{x}_{i,1}\int_0^1 \theta \frac{\partial}{\partial x_{i,1}} \bar{M}_{i,2}(\theta z_{i,2} + \alpha_{i,1}) d\theta\Bigg] \\ & - \frac{1}{r_{i,2}}\tilde{\theta}_{i,2}\dot{\hat{\theta}}_{i,2} \end{aligned} \quad (20)$$

By Lemma 1, for constant $\varepsilon_{i,2} > 0$, a fuzzy logic system $\Phi_{i,2}^T P_{i,2}(X_{i,2})$ can be used to approximate the unknown function $\hat{f}_{i,2}$ such that

$$\hat{f}_{i,2} = \Phi_{i,2}^T P_{i,2}(X_{i,2}) + \delta_{i,2}(X_{i,2}) \quad |\delta_{i,2}(X_{i,2})| \le \varepsilon_{i,2}$$

where

$$X_{i,2} = \left[\bar{x}_{1,(2-\rho_{i1})}^T, \ldots, \bar{x}_{i-1,(2-\rho_{ii-1})}^T, \bar{x}_{i,2}^T, \bar{x}_{i+1,(2-\rho_{ii+1})}^T \ldots, \bar{x}_{m,(2-\rho_{im})}^T\right]^T.$$

Similar to (11) in step 1), $\hat{f}_{i,2}$ can be rewritten as

$$\hat{f}_{i,2} = \left\|\Phi_{i,2}^T\right\| \Phi_{i,2}^{*T} P_{i,2}^T(X_{i,2}) + \delta_{i,2}(X_{i,2}) \tag{21}$$

with $\Phi_{i,2}^* = (\Phi_{i,2}/\|\Phi_{i,2}\|)$. Substituting (21) into (20) yields

$$\dot{V}_{i,2} = \dot{V}_{i,1} + z_{i,2}\big[x_{i,3} + \|\Phi_{i,2}\| \Phi_{i,2}^{*T} P_{i,2}^T(X_{i,2}) + \delta_{i,2}(X_{i,2})\big] - \frac{1}{r_{i,1}}\tilde{\theta}_{i,2}\dot{\hat{\theta}}_{i,2}. \tag{22}$$

Define $\theta_{i,2} = \|\Phi_{i,2}\|^2$. Then the following inequality can be obtained:

$$\begin{aligned} &z_{i,2}\,\|\Phi_{i,2}\|\,\Phi_{i,2}^{*T} P_{i,2}^T(X_{i,2}) + z_{i,2}\delta_{i,2}(X_{i,2}) \\ &\quad\le \frac{1}{2}a_{i,2}^{-2} z_{i,2}^2 P_{i,2}^T P_{i,2}\theta_{i,2} + \frac{1}{2}a_{i,2}^2 \\ &\qquad + \frac{1}{2}c_{i,2}^{-2} z_{i,2}^2 + \frac{1}{2}c_{i,2}^2\varepsilon_{i,2}^2. \end{aligned} \tag{23}$$

Furthermore, it follows from substituting (16) and (23) into (22) that

$$\dot{V}_{i,2} \le -\lambda_{i,1} z_{i,1}^2 + \frac{1}{r_{i,1}}\tilde{\theta}_{i,1}\left(\frac{r_{i,1}}{2}a_{i,1}^{-2} z_{i,1}^2 P_{i,1}^T P_{i,1} - \dot{\hat{\theta}}_{i,1}\right) \tag{24}$$

$$\begin{aligned} &+ \frac{1}{2}a_{i,1}^2 + \frac{1}{2}c_{i,1}^2\varepsilon_{i,1}^2 \\ &+ z_{i,2}\left[x_{i,3} + z_{i,1} + \frac{1}{2}a_{i,2}^{-2}\theta_{i,2} z_{i,2} P_{i,2}^T P_{i,2} + \frac{1}{2}c_{i,2}^{-2} z_{i,2}\right] \\ &+ \frac{1}{2}a_{i,2}^2 + \frac{1}{2}c_{i,2}^2\varepsilon_{i,2}^2 - \frac{1}{r_{i,2}}\tilde{\theta}_{i,2}\dot{\hat{\theta}}_{i,2}. \end{aligned} \tag{25}$$

Now, choose

$$\alpha_{i,2} = -\left(\lambda_{i,2} + \frac{1}{2}c_{i,2}^{-2}\right) z_{i,2} - \frac{1}{2}a_{i,2}^{-2}\hat{\theta}_{i,2} z_{i,2} P_{i,2}^T P_{i,2} - z_{i,1}.$$

Then, (25) becomes

$$\begin{aligned} \dot{V}_{i,2} \le \sum_{j=1}^{2}\Bigg(&-\lambda_{i,j} z_{i,j}^2 \\ &+ \frac{1}{r_{i,j}}\tilde{\theta}_{i,j}\left(\frac{r_{i,j}}{2}a_{i,1}^{-2} z_{i,j}^2 P_{i,j}^T P_{i,j} - \dot{\hat{\theta}}_{i,j}\right) \\ &+ \frac{1}{2}a_{i,j}^2 + \frac{1}{2}c_{i,j}^2\varepsilon_{i,2}^2\Bigg) + z_{i,2}\left(x_{i,3} - \alpha_{i,2}\right) \end{aligned}$$

where the equality $\theta_{i,2} = \tilde{\theta}_{i,2} + \hat{\theta}_{i,2}$ is used.

3) Step k: $(1 \le k \le \rho_i - 1)$ In general, let

$$X_{i,k} = \left[\bar{x}_{1,(k-\rho_{i1})}^T, \ldots, \bar{x}_{i-1,(k-\rho_{ii-1})}^T, \bar{x}_{i,k}^T, \bar{x}_{i+1,(k-\rho_{ii+1})}^T, \ldots, \bar{x}_{m,(k-\rho_{im})}^T\right]^T$$

and suppose that the fuzzy logic system $\Phi_{i,k}^T P_{i,k}(X_{i,k})$ is used to approximate the unknown function

$$\begin{aligned} \hat{f}_{i,k} = {}& f_{i,k} g_{i,k}^{-1} - \dot{\alpha}_{i,k-1}\int_0^1 \bar{M}_{i,k}(\theta z_{i,k} + \alpha_{i,k-1})d\theta \\ &+ \sum_{j=1, j\ne i}^{m}\sum_{s=1}^{k-\rho_{ij}} z_{i,k}\dot{x}_{j,s} \\ &\times \int_0^1 \theta\frac{\partial}{\partial x_{j,s}}\bar{M}_{i,k}(\theta z_{i,k} + \alpha_{i,k-1})d\theta \\ &+ \sum_{s=1}^{k-1} z_{i,k}\dot{x}_{i,s}\int_0^1 \theta\frac{\partial}{\partial x_{j,s}}\bar{M}_{i,k}(\theta z_{i,k} + \alpha_{i,k-1})d\theta \end{aligned}$$

such that

$$\hat{f}_{i,k} = \Phi_{i,k}^T P_{i,k}(X_{i,k}) + \delta_{i,k}(X_{i,k}) \quad |\delta_{i,k}(X_{i,k})| \le \varepsilon_{i,k}$$

where $\bar{M}_{i,k} = g_{i,k}^{-1}\Big(\bar{x}_{1,(k-\rho_{i1})}, \ldots, \bar{x}_{i-1,(k-\rho_{ii-1})}, \bar{x}_{i,k-1}, \theta z_{i,k} + \alpha_{i,k-1}, \ldots, \bar{x}_{m,(k-\rho_{im})}\Big)$ with $z_{i,k} = x_{i,k} - \alpha_{i,k-1}$, and $\varepsilon_{i,k}$ is a given positive constant. By choosing

$$\alpha_{i,k} = -\left(\lambda_{i,k} + \frac{1}{2}c_{i,k}^{-2}\right) z_{i,k} - \frac{1}{2}a_{i,k}^{-2}\hat{\theta}_{i,k} z_{i,k} P_{i,k}^T P_{i,k} - z_{i,k-1} \tag{26}$$

the Lyapunov function

$$V_{i,k} = V_{i,k-1} + \int_0^{z_{i,k}} \sigma\bar{M}_{i,k}(\sigma + \alpha_{i,k-1})d\sigma + \frac{1}{2r_{i,k}}\tilde{\theta}_{i,k}^2$$

satisfies the following inequality:

$$\begin{aligned} \dot{V}_{i,k} \le \sum_{j=1}^{k}\Bigg(&-\lambda_{i,j} z_{i,j}^2 + \frac{1}{r_{i,j}}\tilde{\theta}_{i,j}\left(\frac{r_{i,j}}{2a_{i,j}^2} z_{i,j}^2 P_{i,j}^T P_{i,j} - \dot{\hat{\theta}}_{i,j}\right) + \frac{1}{2}\left(a_{i,j}^2 + c_{i,j}^2\varepsilon_{i,j}^2\right)\Bigg) \\ &+ z_{i,k}\left(x_{i,k+1} - \alpha_{i,k}\right) \end{aligned} \tag{27}$$

where $\tilde{\theta}_{i,k} = \theta_{i,k} - \hat{\theta}_{i,k}$ with $\theta_{i,k} = \|\Phi_{i,k}\|^2$.

4) Step ρ_i: Define $z_{i,\rho_i} = x_{i,\rho_i} - \alpha_{i,\rho_i-1}$. Then, differentiating z_{i,ρ_i} gives

$$\dot{z}_{i,\rho_i} = g_{i,\rho_i} u_i + f_{i,\rho_i}(\bar{x}) - \dot{\alpha}_{i,\rho_i-1} \tag{28}$$

with $g_{i,\rho_i} = g_{i,\rho_i}\left(\bar{x}_{1,(\rho_i-\rho_{i1})} \quad \ldots \quad \bar{x}_{m,(\rho_i-\rho_{im})}\right)$. Consider the following Lyapunov function candidate:

$$V_{i,\rho_i} = V_{i,\rho_i-1} + \int_0^{z_{i,\rho_i}} \sigma\bar{M}_{i,\rho_i}(\sigma + \alpha_{i,\rho_i-1})d\sigma + \frac{1}{2r_{i,\rho_i}}\tilde{\theta}_{i,\rho_i}^2 \tag{29}$$

where $\bar{M}_{i,\rho_i} = g_{i,\rho_i}^{-1}\left(\bar{x}_{1,(\rho_i-\rho_{i1})}, \ldots, \bar{x}_{i-1,(\rho_i-\rho_{ii-1})}, \bar{x}_{i,\rho_i-1}, \sigma + \alpha_{i,\rho_i-1}, \ldots, \bar{x}_{m,(\rho_i-\rho_{im})}\right)$ and $\tilde{\theta}_{i,\rho_i} = \theta_{i,\rho_i} - \hat{\theta}_{i,\rho_i}$.

By Lemma 2, the derivative of V_{i,ρ_i} is given by

$$\begin{aligned}\dot{V}_{i,\rho_i} =& \dot{V}_{i,\rho_i-1} + z_{i,\rho_i} g_{i,\rho_i}^{-1} \dot{z}_{i,\rho_i} + \dot{\alpha}_{i,\rho_i-1} z_{i,\rho_i} g_{i,\rho_i}^{-1} \\ & - z_{i,\rho_i} \dot{\alpha}_{i,\rho_i-1} \int_0^1 \bar{M}(\theta z_{i,\rho_i} + \alpha_{i,\rho_i-1}) d\theta \\ & + z_{i,\rho_i}^2 \sum_{j=1,j\neq i}^{m} \sum_{s=1}^{\rho_i-\rho_{ij}} \dot{x}_{j,s} \int_0^1 \theta \frac{\partial}{\partial x_{i,j}} \bar{M} \\ & \times (\theta z_{i,\rho_i} + \alpha_{i,\rho_i-1}) d\theta \\ & + z_{i,\rho_i}^2 \sum_{s=1}^{\rho_i-1} \dot{x}_{i,s} \int_0^1 \theta \frac{\partial}{\partial x_{i,j}} \bar{M}(\theta z_{i,\rho_i} + \alpha_{i,\rho_i-1}) d\theta \\ & - \frac{1}{r_{i,\rho_i}} \tilde{\theta}_{i,\rho_i} \dot{\hat{\theta}}_{i,\rho_i} \end{aligned} \tag{30}$$

$$= \dot{V}_{i,\rho_i-1} + z_{i,\rho_i} \left[u_i + \hat{f}_{i,\rho_i} \right] - \frac{1}{r_{i,\rho_i}} \tilde{\theta}_{i,\rho_i} \dot{\hat{\theta}}_{i,\rho_i} \tag{31}$$

where

$$\begin{aligned}\hat{f}_{i,\rho_i} =& f_{i,\rho_i} g_{i,\rho_i}^{-1} - \dot{\alpha}_{i,\rho_i-1} \int_0^1 \bar{M}(\theta z_{i,\rho_i} + \alpha_{i,\rho_i-1}) d\theta \\ & + \sum_{j=1,j\neq i}^{m} \sum_{s=1}^{\rho_i-\rho_{ij}} z_{i,\rho_i} \dot{x}_{j,s} \int_0^1 \theta \frac{\partial}{\partial x_{i,j}} \bar{M} \\ & \times (\theta z_{i,\rho_i} + \alpha_{i,\rho_i-1}) d\theta \\ & + z_{i,\rho_i} \sum_{s=1}^{\rho_i-1} \dot{x}_{i,s} \int_0^1 \theta \frac{\partial}{\partial x_{i,j}} \bar{M}(\theta z_{i,\rho_i} + \alpha_{i,\rho_i-1}) d\theta. \end{aligned}$$

Since $\hat{f}_{i,\rho_i}$ is unknown, by Lemma 1, for $\varepsilon_{i,\rho_i} > 0$, a fuzzy logic system $\Phi_{i,\rho_i}^T P_{i,\rho_i}(X_{i,\rho_i})$ is used to approximate $\hat{f}_{i,\rho_i}$ such that

$$\hat{f}_{i,\rho_i} = \Phi_{i,\rho_i}^T P_{i,\rho_i}(\bar{x}) + \delta_{i,\rho_i}(\bar{x}) \quad |\delta_{i,\rho_i}(\bar{x})| \leq \varepsilon_{i,\rho_i}$$

which can be expressed as follows:

$$\hat{f}_{i,\rho_i} = \left\| \Phi_{i,\rho_i}^T \right\| \Phi_{i,\rho_i}^{*T} P_{i,\rho_i}(\bar{x}) + \delta_{i,\rho_i}(\bar{x}) \tag{32}$$

where $\Phi_{i,\rho_i}^* = (\Phi_{i,\rho_i}/\|\Phi_{i,\rho_i}\|)$. Then, by taking (27) into account with $k = \rho_i - 1$, and substituting (32) into (31), the following inequality can be obtained:

$$\begin{aligned}\dot{V}_{i,\rho_i} \leq& \sum_{j=1}^{\rho_i-1} \Bigg(-\lambda_{i,j} z_{i,j}^2 + \frac{1}{r_{i,j}} \tilde{\theta}_{i,j} \Bigg(\frac{r_{i,j}}{2a_{i,j}^2} z_{i,j}^2 P_{i,j}^T P_{i,j} \\ & - \dot{\hat{\theta}}_{i,j} + \frac{1}{2} \left(a_{i,j}^2 + c_{i,j}^2 \varepsilon_{i,j}^2 \right) \Bigg) \Bigg) \\ & + z_{i,\rho_i} \Big(u_i + z_{i,\rho_i-1} \\ & + \|\Phi_{i,\rho_i}\| \Phi_{i,\rho_i}^{*T} P_{i,\rho_i}(\bar{x}) + \delta_{i,\rho_i}(\bar{x}) \Big) \\ & - \frac{1}{r_{i,\rho_i}} \tilde{\theta}_{i,\rho_i} \dot{\hat{\theta}}_{i,\rho_i}. \end{aligned} \tag{33}$$

Define $\theta_{i,\rho_i} = \|\Phi_{i,\rho_i}\|^2$. Then, a simple calculation shows that

$$\begin{aligned} z_{i,\rho_i} & \|\Phi_{i,\rho_i}\| \Phi_{i,\rho_i}^{*T} P_{i,\rho_i}(\bar{x}) + z_{i,\rho_i} \delta_{i,\rho_i}(\bar{x}) \\ & \leq \frac{1}{2} a_{i,\rho_i}^{-2} \theta_{i,\rho_i} z_{i,\rho_i}^2 P_{i,\rho_i}^T P_{i,\rho_i} + \frac{1}{2} a_{i,\rho_i}^2 \\ & \quad + \frac{1}{2} c_{i,\rho_i}^{-2} z_{i,\rho_i}^2 + \frac{1}{2} c_{i,\rho_i}^2 \varepsilon_{i,\rho_i}^2. \end{aligned} \tag{34}$$

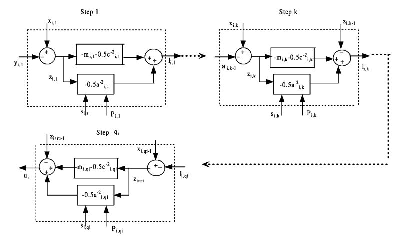

Fig. 1. Structure of the u_i.

Substituting (34) into (33) gives

$$\begin{aligned}\dot{V}_{i,\rho_i} \leq& \sum_{j=1}^{\rho_i-1} \Bigg(-\lambda_{i,j} z_{i,j}^2 + \frac{1}{r_{i,j}} \tilde{\theta}_{i,j} \Bigg(\frac{r_{i,j}}{2a_{i,j}^2} z_{i,j}^2 P_{i,j}^T P_{i,j} \\ & - \dot{\hat{\theta}}_{i,j} \Bigg) + \frac{1}{2} \left(a_{i,j}^2 + c_{i,j}^2 \varepsilon_{i,j}^2 \right) \Bigg) \\ & + z_{i,\rho_i} \Bigg[u_i + z_{i,\rho_i-1} + \frac{1}{2} a_{i,\rho_i}^{-2} \theta_{i,\rho_i} z_{i,\rho_i} P_{i,\rho_i}^T P_{i,\rho_i} \\ & + \frac{1}{2} c_{i,\rho_i}^{-2} z_{i,\rho_i} \Bigg] \\ & - \frac{1}{r_{i,\rho_i}} \tilde{\theta}_{i,\rho_i} \dot{\hat{\theta}}_{i,\rho_i} + \frac{1}{2} a_{i,\rho_i}^2 + \frac{1}{2} c_{i,\rho_i}^2 \varepsilon_{i,\rho_i}^2. \end{aligned} \tag{35}$$

By choosing

$$\begin{aligned} u_i =& - \left(\lambda_{i,\rho_i} + \frac{1}{2} c_{i,\rho_i}^{-2} \right) z_{i,\rho_i} - z_{i,\rho_i-1} \\ & - \frac{1}{2} a_{i,\rho_i}^{-2} \hat{\theta}_{i,\rho_i} z_{i,\rho_i} P_{i,\rho_i}^T P_{i,\rho_i} \end{aligned} \tag{36}$$

and using $\theta_{i,\rho_i} = \tilde{\theta}_{i,\rho_i} + \hat{\theta}_{i,\rho_i}$, (35) becomes

$$\begin{aligned}\dot{V}_{i,\rho_i} \leq& \sum_{j=1}^{\rho_i} \Bigg(-\lambda_{i,j} z_{i,j}^2 + \frac{1}{r_{i,j}} \tilde{\theta}_{i,jj} \Bigg(\frac{r_{i,j}}{2a_{i,j}^2} z_{i,j}^2 P_{i,j}^T P_{i,j} \\ & - \dot{\hat{\theta}}_{i,j} \Bigg) + \frac{1}{2} \left(a_{i,j}^2 + c_{i,j}^2 \varepsilon_{i,j}^2 \right) \Bigg). \end{aligned} \tag{37}$$

The design procedure of the controller can be visualized from the block diagram shown in Fig. 1.

Remark 3: Note that the parameters $\theta_{i,j}$ are introduced to estimate the norms of the unknown parameters $\Phi_{i,k}$ of the fuzzy logic approximators $\Phi_{i,k}^T P_{i,k}(X_{i,k})$, which reduces the number of adaptive laws to the number of unknown nonlinear functions in the original system. As a result, the computation burden is reduced considerably.

IV. Analysis of Stability

The backstepping-based fuzzy adaptive control design scheme has been developed in above section. In this section, we will propose the following adaptive laws

$$\begin{aligned} \dot{\hat{\theta}}_{i,j} =& \frac{r_{i,j}}{2a_{i,j}^2} z_{i,j}^2 P_{i,j}^T P_{i,j} - k_{i,j} \hat{\theta}_{i,j} \quad 1 \leq i \leq m; \\ & 1 \leq j \leq \rho_i \end{aligned} \tag{38}$$

and analyze the stability of the closed-loop system. The main result is summarized in the following theorem.

Theorem 1: Consider the system (1), together with Assumption 1. Suppose that for $1 \leq i \leq m$; $1 \leq j \leq \rho_i$, the packaged unknown functions $\hat{f}_{i,j}$ can be approximated by the fuzzy logic systems in the sense that the approximating error $\delta_{i,j}$ are bounded. Then the fuzzy adaptive controllers in (36), the intermediate virtual control $\alpha_{i,j}$ in (26) and the adaptive laws in (38) for $\theta_{i,j}$ can make all signals in the closed-loop system remain bounded. Furthermore, given any scalar $\varepsilon > 0$, the controller parameters can be tuned such that $\lim_{t\to\infty} \|z_{i,1}\|^2 \leq \varepsilon^2$.

Proof: For stability analysis, consider the Lyapunov candidate function

$$V = \sum_{i=1}^{m} V_{i,\rho_i}. \tag{39}$$

From (37) and (38), the time derivative of (39) can be expressed as

$$\dot{V} \leq \sum_{i=1}^{m}\sum_{j=1}^{\rho_i}\left(-\lambda_{i,j}z_{i,j}^2 + \frac{k_{i,j}}{r_{i,j}}\tilde{\theta}_{i,j}\hat{\theta}_{i,j} + \frac{1}{2}\left(a_{i,j}^2 + c_{i,j}^2\varepsilon_{i,j}^2\right)\right). \tag{40}$$

Note that

$$\begin{aligned}\tilde{\theta}_{i,j}\hat{\theta}_{i,j} &= -\tilde{\theta}_{i,j}\left(\theta_{i,j} - \hat{\theta}_{i,j} - \theta_{i,j}\right) = -\tilde{\theta}_{i,j}^2 + \tilde{\theta}_{i,j}\theta_{i,j}\\ &\leq -\frac{1}{2}\tilde{\theta}_{i,j}^2 + \frac{1}{2}\theta_{i,j}^2.\end{aligned}$$

Substituting this inequality into (40) produces

$$\dot{V} \leq \sum_{i=1}^{m}\sum_{j=1}^{\rho_i}\left(-\lambda_{i,j}z_{i,j}^2 - \frac{k_{i,j}}{2r_{i,j}}\tilde{\theta}_{i,j}^2 + \frac{k_{i,j}}{2r_{i,j}}\theta_{i,j}^2 + \frac{1}{2}\left(a_{i,j}^2 + c_{i,j}^2\varepsilon_{i,j}^2\right)\right). \tag{41}$$

It is clear that (41) can be rewritten as

$$\dot{V} \leq \sum_{i=1}^{m}\sum_{j=1}^{\rho_i}\left(-\frac{2\lambda_{i,j}a_{i,j}^m}{2a_{i,j}^m i,j}z_{i,j}^2 - \frac{k_{i,j}}{2r_{i,j}}\tilde{\theta}_{i,j}^2 + \frac{k_{i,j}}{2r_{i,j}}\theta_{i,j}^2 + \frac{1}{2}\left(a_{i,j}^2 + c_{i,j}^2\varepsilon_{i,j}^2\right)\right). \tag{42}$$

Since Assumption 2 implies that $\left(a_{i,j}^M\right)^{-1} \leq g_{i,j}^{-1} \leq \left(a_{i,j}^m\right)^{-1}$, it follows from Lemma 2 that

$$-\frac{1}{2a_{i,j}^m}z_{i,j}^2 \leq -\int_0^{z_{i,}} \sigma M_{i,j}(\bar{x}_{i,j-1}, \sigma + \alpha_{i,j-1})d\sigma. \tag{43}$$

Consequently, it follows from (42) and (43) that

$$\begin{aligned}\dot{V} \leq \sum_{i=1}^{m}\sum_{j=1}^{\rho_i}\Bigg(&-2\lambda_{i,j}a_{i,j}^m\int_0^{z_{i,}} \sigma M_{i,j}(\bar{x}_{i,j-1}, \sigma\\ &+ \alpha_{i,j-1})d\sigma - \frac{k_{i,j}}{2r_{i,j}}\tilde{\theta}_{i,j}^2 + \frac{k_{i,j}}{2r_{i,j}}\theta_{i,j}^2\\ &+ \frac{1}{2}\left(a_{i,j}^2 + c_{i,j}^2\varepsilon_{i,j}^2\right)\Bigg).\end{aligned} \tag{44}$$

Define $a_0 = \min\left\{2\lambda_{i,j}a_{i,j}^m, k_{i,j}\right\}$ and $b_0 = \sum_{j=1}^{\rho_i}\left((k_{i,j}/2r_{i,j})\theta_{i,j}^2 + (1/2)\left(a_{i,j}^2 + c_{i,j}^2\varepsilon_{i,j}^2\right)\right)$. Then, (44) becomes

$$\dot{V} \leq -a_0\sum_{i=1}^{m}\sum_{j=1}^{\rho_i} V_{i,j} + b_0 = -a_0V + b_0$$

which implies that for $t \geq 0$

$$V \leq \left(V(0) - \frac{b_0}{a_0}\right)e^{-a_0t} + \frac{b_0}{a_0}.$$

Furthermore, all $z_{i,j}$ and $\hat{\theta}_{i,j}$ belong to the compact set $\Omega = \left\{\left(z_{i,j}, \hat{\theta}_{i,j}\right) \mid V \leq (b_0/a_0)\right\}$. This proves that $z_{i,j}$ and $\hat{\theta}_{i,j}$ are bounded.

Note that $k_{i,j}$, $r_{i,j}$, $a_{i,j}$ and $c_{i,j}$ are design parameters, and $\theta_{i,j}$, $b_{i,j}$ and $a_{i,j}^M$ are constants. Thus for any $\varepsilon > 0$, the inequality $(b_0/a_0) \leq \left(\varepsilon^2/\max\left\{2a_{i,1}^M\right\}\right)$ can be obtained by appropriately choosing these design parameters. In addition, according to Assumption 2 and Lemma 2, for each i

$$\frac{1}{2a_{i,1}^M}\|z_{i,1}\|^2 \leq \int_0^{z_{i,1}} \sigma\bar{M}_{i,1}(\sigma + y_{id})\,d\sigma = V_{i,1}.$$

Furthermore, $\|z_{i,1}\|^2 \leq 2a_{i,1}^M V_{i,1} \leq 2a_{i,1}^M(V(0) - (b_0/a_0))e^{-a_0t} + 2a_{i,1}^M(b_0/a_0)$, which implies that

$$\lim_{t\to\infty}\|z_{i,1}\|^2 \leq \varepsilon^2.$$

For the proof of the boundedness for the original state variables $x_{i,j}$, see [4].

Remark 4: The previous analysis shows that tracking errors depends on $\theta_{i,j}$, $\varepsilon_{i,j}$, $b_{i,j}$, and $a_{i,1}^M$. Because $\theta_{i,j}$, $b_{i,j}$ and $a_{i,1}^M$ are unknown, an explicit estimation of the tracking errors is impossible. However, it is clear that reducing $k_{i,j}$, $a_{i,j}$ and $c_{i,j}$, meanwhile increasing $r_{i,j}$, will lead to smaller tracking errors.

V. Simulations

In this section, the adaptive fuzzy approach is applied to the following two examples to verify its effectiveness.

1) Example 1: Consider a two continuous stirred tank reactor process, which is described by the following differential equations [11]:

$$\begin{aligned}\dot{x}_{11} &= b_{11}x_{12} \quad y_1 = x_{11}\\ \dot{x}_{12} &= b_{12}u_1\\ \dot{x}_{21} &= b_{21}x_{22} + \phi_{21}(x_{11}, x_{21}) + \Phi x_{31}\\ \dot{x}_{22} &= b_{22}u_2 + \phi_{22}(x_{21}, x_{22}) \quad y_2 = x_{21}\\ \dot{x}_{31} &= b_{31}x_{32} + \phi_{31}(x_{11}, x_{12}, x_{21}, x_{31}) + \Psi w\\ \dot{x}_{32} &= b_{32}u_3 + \phi_{32}(x_{31}, x_{32}) \quad y_3 = x_{31}\end{aligned} \tag{45}$$

where

$$\begin{aligned}&b_{11} = 1 \quad b_{12} = 1 \quad b_{21} = \frac{UA}{\rho c_p V} \quad b_{22} = \frac{F_{j2}}{V_j}\\ &b_{31} = \frac{UA}{\rho c_p V} \quad b_{32} = \frac{F_{j1}}{V_j} \quad \Psi = \frac{F_0}{V} \quad \Phi = \frac{F + F_R}{V}\\ &\phi_{21} = \frac{F + F_R}{V}T_1^d - \frac{F + F_R}{V}\left(x_{21} + T_2^d\right)\\ &\qquad - \frac{\alpha\lambda}{\rho c_p}\left(x_{11} + C_{A2}^d\right)e^{-\left(E/R\left(x_{21} + T_2^d\right)\right)}\\ &\qquad - \frac{UA}{\rho c_p V}\left(x_{21} + T_2^d - T_{j2}^d\right),\end{aligned}$$

$$\begin{aligned}
\phi_{22} &= \frac{F_{j2}}{V_j}\left(T_{j20}^d - x_{22} - T_{j2}^d\right) \\
&\quad + \frac{UA}{\rho_j c_j V_j}\left(x_{21} + T_2^d - x_{22} - T_{j2}^d\right) \\
\phi_{31} &= \frac{F_0}{V}T_0^d - \frac{F+F_R}{V}\left(x_{31} + T_1^d\right) + \frac{F_R}{V}\left(x_{21} + T_2^d\right) \\
&\quad - \frac{\alpha\lambda}{\rho c_p} C_{A1} e^{-\left(E/R\left(x_{31}+T_1^d\right)\right)} \\
&\quad - \frac{UA}{\rho c_p V}\left(x_{31} + T_1^d - T_{j1}^d\right) \\
\phi_{32} &= \frac{F_{j1}}{V_j}\left(T_{j10}^d - x_{32} - T_{j1}^d\right) \\
&\quad + \frac{UA}{\rho_j c_j V_j}\left(x_{31} + T_1^d - x_{32} - T_{j1}^d\right) \\
C_{A1} &= \frac{V}{F+F_R}\Bigg(x_{12} + \frac{F+F_R}{V}\left(x_{11} + C_{A2}^d\right) \\
&\qquad + \alpha\left(x_{11} + C_{A2}^d\right) e^{-\left(E/R\left(x_{21}+T_2^d\right)\right)}\Bigg)
\end{aligned}$$

with $V_{j1} = V_{j2} = V_j$, $V_1 = V_2 = V$, $F_0 = F_2 = F$ and the values of the process parameters are provided as follows:

$$\begin{aligned}
\alpha &= 7.08 \times 10^{10}\ \mathrm{h}^{-1} \\
\rho &= 800.9189\ \frac{\mathrm{kg}}{\mathrm{m}^3} \\
\rho_j &= 997.9450\ \frac{\mathrm{kg}}{\mathrm{m}^3} \\
\lambda &= -3.1644 \times 10^7\ \frac{\mathrm{J}}{\mathrm{mol}} \\
R &= 1679.2\ \frac{\mathrm{J}}{\mathrm{mol}^\circ\mathrm{C}} \\
E &= 3.1644 \times 10^7\ \frac{\mathrm{J}}{\mathrm{mol}} \\
c_\rho &= 1395.3\ \frac{\mathrm{J}}{\mathrm{kg}^\circ\mathrm{C}} \\
c_j &= 1860.3\ \frac{\mathrm{J}}{\mathrm{kg}^\circ\mathrm{C}} \\
U &= 1.3625 \times 10^6\ \frac{\mathrm{J}}{\mathrm{hm}^{2\circ}\mathrm{C}} \\
F &= 2.8317\ \frac{\mathrm{m}^3}{\mathrm{h}} \\
F_R &= 1.4158\ \frac{\mathrm{m}^3}{\mathrm{h}} \\
F_{j1} &= 1.4130\ \frac{\mathrm{m}^3}{\mathrm{h}} \\
F_{j2} &= 1.4130\ \frac{\mathrm{m}^3}{\mathrm{h}} \\
T_0^d &= 703.7^\circ\mathrm{C} \\
T_1^d &= 750^\circ\mathrm{C} \\
T_2^d &= 737.5^\circ\mathrm{C} \\
T_{j1}^d &= 740.8^\circ\mathrm{C} \\
T_{j2}^d &= 727.6^\circ\mathrm{C} \\
T_{j10}^d &= 629.2^\circ\mathrm{C} \\
T_{j20}^d &= 608.2^\circ\mathrm{C}
\end{aligned}$$

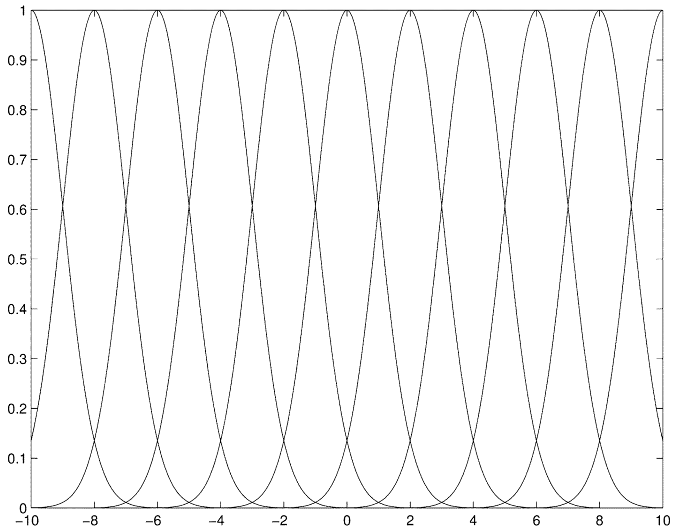

Fig. 2. Fuzzy rule base on $[-10, 10]$.

$$\begin{aligned}
C_{A0}^d &= 18.3728\ \frac{\mathrm{mol}}{\mathrm{m}^3} \\
C_{A1}^d &= 12.3061\ \frac{\mathrm{mol}}{\mathrm{m}^3} \\
C_{A2}^d &= 10.4178\ \frac{\mathrm{mol}}{\mathrm{m}^3} \\
V &= 1.3592\ \mathrm{m}^3 \\
V_j &= 0.1090\ \mathrm{m}^3 \\
A &= 23.2\ \mathrm{m}^3.
\end{aligned}$$

The reference signals are assumed to be

$$\begin{aligned}
y_{1d} &= 5\sin(t) + 2\sin(0.5t) \\
y_{2d} &= 3\cos(2t) + \cos(t) \\
y_{3d} &= 2\sin(t) + \cos(4t).
\end{aligned}$$

The control objective is to design adaptive fuzzy controllers such that the outputs y_i follows y_{id} for $i = 1, 2, 3$, under the condition that in the system (45) the parameters b_{ij} and the functions ϕ_{ij} ($i = 2, 3$; $j = 1, 2$) are completely unknown. In the simulation, eleven fuzzy sets are defined over interval $[-10, 10]$ for all $x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}$, and by choosing the partitioning points as $-10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10$ the fuzzy membership functions are given as follows:

$$\begin{aligned}
&\mu_{F_{ij}^1}(x_{ij}) = e^{-0.5(x_{ij}+10)^2} \quad && \mu_{F_{ij}^2}(x_{ij}) = e^{-0.5(x_{ij}+8)^2} \\
&\mu_{F_{ij}^3}(x_{ij}) = e^{-0.5(x_{ij}+6)^2} && \mu_{F_{ij}^4}(x_{ij}) = e^{-0.5(x_{ij}+4)^2} \\
&\mu_{F_{ij}^5}(x_{ij}) = e^{-0.5(x_{ij}+2)^2} && \mu_{F_{ij}^6}(x_{ij}) = e^{-0.5(x_{ij})^2} \\
&\mu_{F_{ij}^7}(x_{ij}) = e^{-0.5(x_{ij}-2)^2} && \mu_{F_{ij}^8}(x_{ij}) = e^{-0.5(x_{ij}-4)^2} \\
&\mu_{F_{ij}^9}(x_{ij}) = e^{-0.5(x_{ij}-6)^2} && \mu_{F_{ij}^{10}}(x_{ij}) = e^{-0.5(x_{ij}-8)^2} \\
&\mu_{F_{ij}^{11}}(x_{ij}) = e^{-0.5(x_{ij}-10)^2}.
\end{aligned}$$

The fuzzy rule base on the interval $[-10, 10]$ is shown in Fig. 2.

Let

$$P_{21}^k = \frac{\mu_{F_{11}^k}(x_{11})\mu_{F_{21}^k}(x_{21})\mu_{F_{31}^k}(x_{31})}{\sum_{k=1}^{11}\left(\mu_{F_{11}^k}(x_{11})\mu_{F_{21}^k}(x_{21})\mu_{F_{31}^k}(x_{31})\right)}$$
$$P_{22}^k = \frac{\Pi_{i=1}^2\Pi_{j=1}^2\mu_{F_{ij}^k}(x_{ij})\mu_{F_{31}^k}(x_{31})}{\sum_{k=1}^{11}\left(\Pi_{i=1}^2\Pi_{j=1}^2\mu_{F_{ij}^k}(x_{ij})\mu_{F_{31}^k}(x_{31})\right)}$$
$$P_{31}^k = \frac{\Pi_{j=1}^2\mu_{F_{1j}^k}(x_{1j})\mu_{F_{21}^k}(x_{21})\mu_{F_{31}^k}(x_{31})}{\sum_{k=1}^{11}\left(\Pi_{j=1}^2\mu_{F_{1j}^k}(x_{1j})\mu_{F_{21}^k}(x_{21})\mu_{F_{31}^k}(x_{31})\right)}$$
$$P_{32}^k = \frac{\Pi_{i=1}^3\Pi_{j=1}^2\mu_{F_{ij}^k}(x_{ij})}{\sum_{k=1}^{11}\left(\Pi_{i=1}^3\Pi_{j=1}^2\mu_{F_{ij}^k}(x_{ij})\right)}.$$

Then

$$\begin{aligned}
P_{2,1} &= \left[P_{21}^1, P_{21}^2, \ldots, P_{21}^{11}\right]^T\\
P_{2,2} &= \left[P_{22}^1, P_{22}^2, \ldots, P_{22}^{11}\right]^T\\
P_{3,1} &= \left[P_{31}^1, P_{31}^2, \ldots, P_{31}^{11}\right]^T\\
P_{3,2} &= \left[P_{32}^1, P_{32}^2, \ldots, P_{32}^{11}\right]^T.
\end{aligned}$$

Furthermore, the fuzzy controllers and adaptive laws are constructed as follows:

$$\begin{aligned}
u_1 &= -\lambda_{1,2}(x_{1,2}-\alpha_{1,1}) - (x_{1,1}-y_{1d}) + \dot{\alpha}_{1,1}\\
u_2 &= -\left(\lambda_{2,2}+0.5c_{2,2}^{-2}\right)(x_{2,2}-\alpha_{2,1}) - (x_{2,1}-y_{2d})\\
&\quad - \frac{1}{2a_{2,2}^2}(x_{2,2}-\alpha_{2,1})P_{2,2}^T P_{2,2}\hat{\theta}_{2,2}\\
u_3 &= -\left(\lambda_{3,2}+0.5c_{3,2}^{-2}\right)(x_{3,2}-\alpha_{3,1}) - (x_{3,2}-y_{3d})\\
&\quad - \frac{1}{2a_{3,2}^2}(x_{3,2}-\alpha_{3,1})P_{3,2}^T P_{3,2}\hat{\theta}_{3,2}\\
\dot{\hat{\theta}}_{2,1} &= \frac{r_{2,1}}{2a_{2,1}^2}(x_{2,1}-y_{2d})^2 P_{2,1}^T P_{2,1} - k_{2,1}\hat{\theta}_{2,1}\\
\dot{\hat{\theta}}_{2,2} &= \frac{r_{2,2}}{2a_{2,2}^2}(x_{2,2}-\alpha_{2,1})^2 P_{2,2}^T P_{2,2} - k_{2,2}\hat{\theta}_{2,2}\\
\dot{\hat{\theta}}_{3,1} &= \frac{r_{3,1}}{2a_{3,1}^2}(x_{3,1}-y_{3d})^2 P_{3,1}^T P_{3,1} - k_{3,1}\hat{\theta}_{3,1}\\
\dot{\hat{\theta}}_{3,2} &= \frac{r_{3,2}}{2a_{3,2}^2}(x_{3,2}-\alpha_{3,1})^2 P_{3,2}^T P_{3,2} - k_{3,2}\hat{\theta}_{3,2}
\end{aligned}$$

with

$$\begin{aligned}
\alpha_{1,1} &= -\lambda_{1,1}(x_{1,1}-y_{1d}) + \dot{y}_{1d}\\
\alpha_{2,1} &= -\left(\lambda_{2,1}+0.5c_{2,1}^{-2}\right)(x_{2,1}-y_{2d})\\
&\quad - \frac{1}{2a_{2,1}^2}(x_{2,1}-y_{2d})P_{2,1}^T P_{2,1}\hat{\theta}_{2,1}\\
\alpha_{3,1} &= -\left(\lambda_{3,1}+0.5c_{3,1}^{-2}\right)(x_{3,1}-y_{3d})\\
&\quad - \frac{1}{2a_{3,1}^2}(x_{3,1}-y_{3d})P_{3,1}^T P_{3,1}\hat{\theta}_{3,1}.
\end{aligned}$$

The simulation was carried out with the disturbance $w = 10e^{-0.15t}\sin(t)$ and the initial conditions

$$\begin{gathered}
x_{1,1}(0) = -2 \quad x_{12}(0) = -1.9966 \quad x_{21}(0) = 5\\
x_{22}(0) = -4 \quad x_{31}(0) = 2 \quad x_{32}(0) = -4\\
\hat{\theta}_{21} = \hat{\theta}_{22} = \hat{\theta}_{31} = \hat{\theta}_{32} = 0
\end{gathered}$$

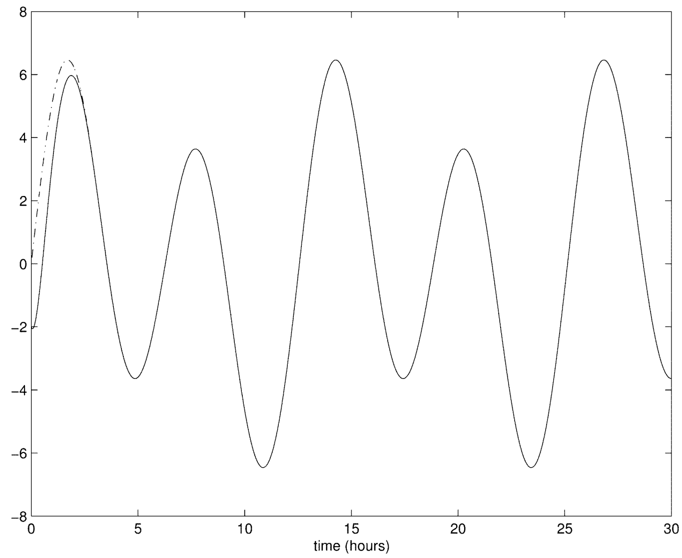

Fig. 3. Output y_1 (“_”) follows the reference y_{1d} (“_”).

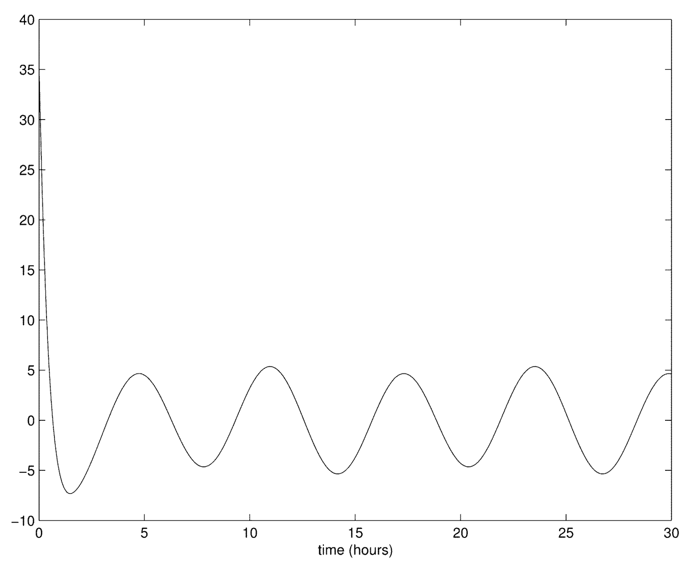

Fig. 4. Control input u_1.

The design parameters are chosen as

$$\begin{gathered}
\lambda_{1,1} = 1.5 \quad \lambda_{1,2} = 2\\
\lambda_{2,1} = \lambda_{2,2} = 30 \quad \lambda_{3,1} = \lambda_{3,2} = 40\\
r_{2,1} = r_{2,2} = r_{3,1} = r_{3,2} = 1\\
a_{2,1} = a_{2,2} = a_{3,1} = a_{3,2} = 1\\
c_{2,1} = c_{2,2} = c_{3,1} = c_{3,2} = 1\\
k_{2,1} = k_{2,2} = k_{3,1} = k_{3,2} = 0.5.
\end{gathered}$$

The simulation results are shown in Figs. 3–10. From the observation of the simulation results, it is clear that even though the exact information on the nonlinear functions in the system is not available, the adaptive fuzzy controllers guarantee the good tracking performance.

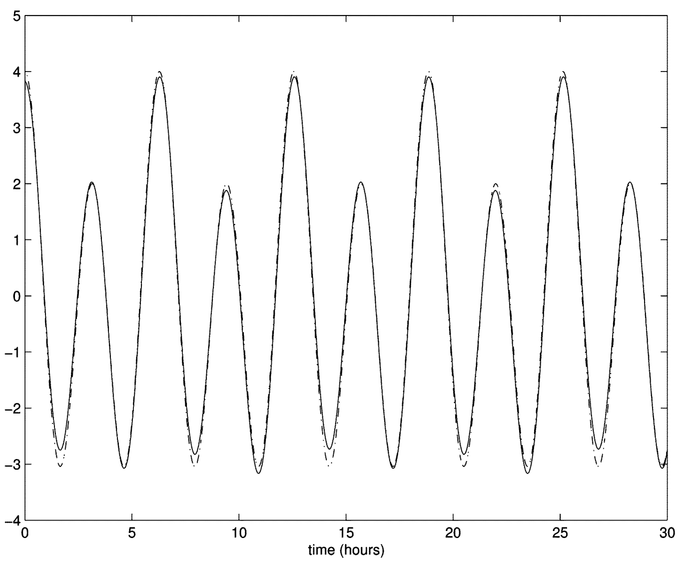

Fig. 5. Output y_2 ("_") follows the reference y_{2d} ("_").

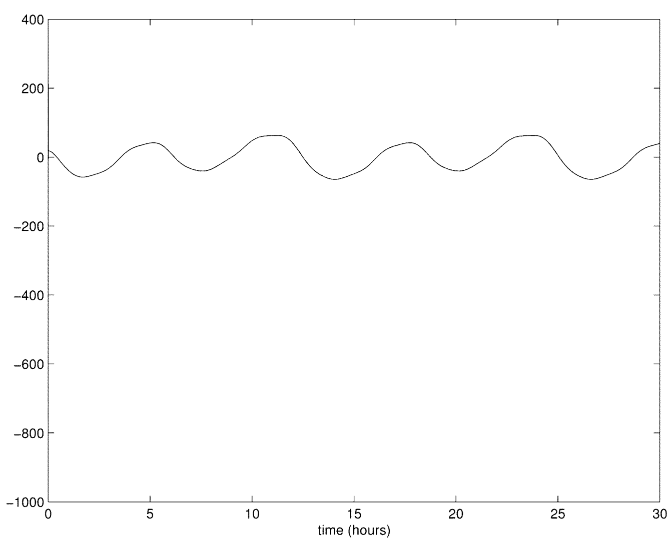

Fig. 6. Control input u_2.

2) Example 2 [4]: In the first example, $g_{i,j} = 1$ for $i = 1, 2, 3$ and $j = 1, 2$, that is, all $g_{i,j}$ are constant. Now, let us consider the following MIMO nonlinear system:

$$
\begin{aligned}
\dot{x}_{1,1} &= f_{1,1}(x_{1,1}x_{2,1}) + g_{1,1}(x_{1,1}x_{2,1})x_{1,2}\\
\dot{x}_{1,2} &= f_{1,2}(\bar{x}_1\bar{x}_2) + g_{1,2}(\bar{x}_1\bar{x}_2)u_1\\
\dot{x}_{1,1} &= f_{2,1}(x_{1,1}x_{2,1}) + g_{2,1}(x_{1,1}x_{2,1})x_{2,2}\\
\dot{x}_{2,2} &= f_{2,2}(\bar{x}_1\bar{x}_2) + g_{2,2}(\bar{x}_1\bar{x}_2)u_2\\
y_1 &= x_{1,1} \quad y_2 = x_{2,1}
\end{aligned}
\tag{46}
$$

where $g_{i,j}$ and $f_{i,j}$ are unknown nonlinear functions with $i = 1, 2$ and $j = 1, 2$. The control objective is to design fuzzy controllers such that the outputs of (46) follow the given signals y_{1d} and y_{2d}. To this end, the fuzzy membership functions are chosen as the same as in Example 1. Define

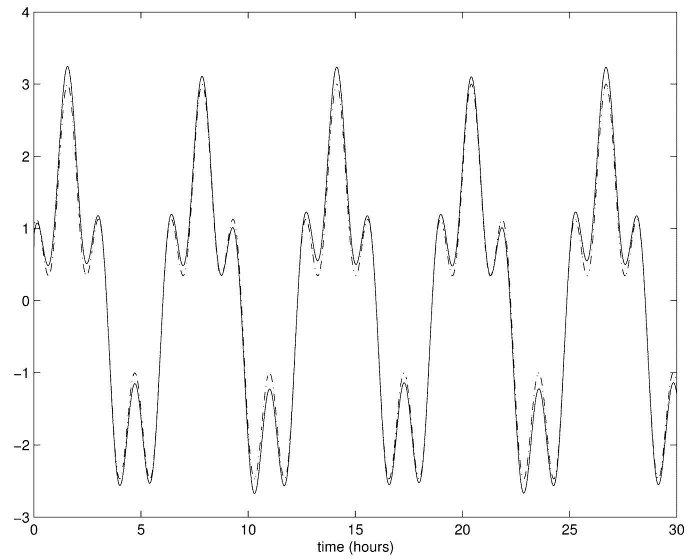

Fig. 7. Output y_3 ("_") follows the reference y_{3d} ("_").

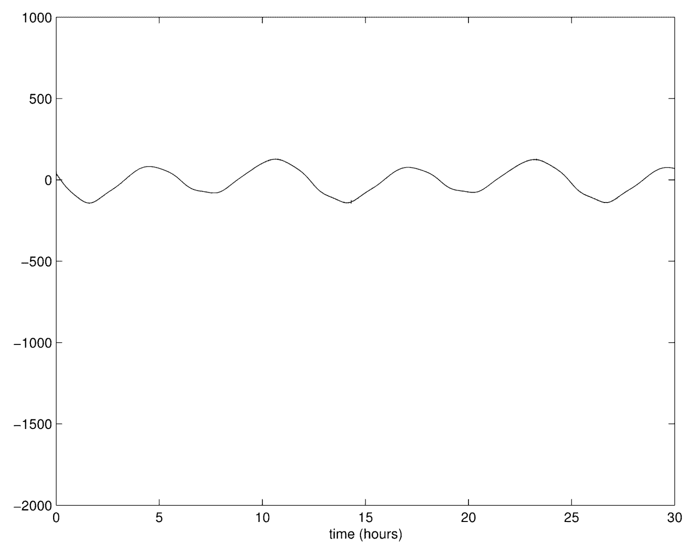

Fig. 8. Control input u_3.

$$
\begin{aligned}
P_{1,1}^k &= \frac{\mu_{F_{ij}^k}(x_{1,1})\mu_{F_{21}^k}(x_{2,1})}{\sum_{k=1}^{11}\left(\mu_{F_{ij}^k}(x_{1,1})\mu_{F_{21}^k}(x_{2,1})\right)}\\
P_{1,2}^k &= \frac{\mu_{F_{ij}^k}(x_{1,1})\mu_{F_{21}^k}(x_{21})}{\sum_{k=1}^{11}\left(\mu_{F_{ij}^k}(x_{1,1})\mu_{F_{21}^k}(x_{2,1})\right)}\\
P_{2,1}^k &= \frac{\Pi_{i=1}^2\Pi_{j=1}^2\mu_{F_{ij}^k}(x_{ij})}{\sum_{k=1}^{11}\left(\Pi_{i=1}^2\Pi_{j=1}^2\mu_{F_{ij}^k}(x_{ij})\right)},\\
P_{2,2}^k &= \frac{\Pi_{i=1}^2\Pi_{j=1}^2\mu_{F_{ij}^k}(x_{ij})}{\sum_{k=1}^{11}\left(\Pi_{i=1}^2\Pi_{j=1}^2\mu_{F_{ij}^k}(x_{ij})\right)}.
\end{aligned}
$$

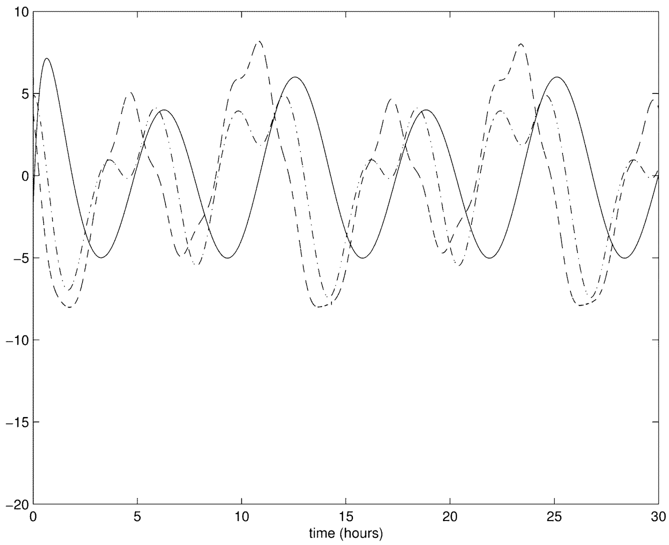

Fig. 9. State variables $x_{1,2}$ ("–"), $x_{2,2}$ ("–") and $x_{3,2}$ ("—").

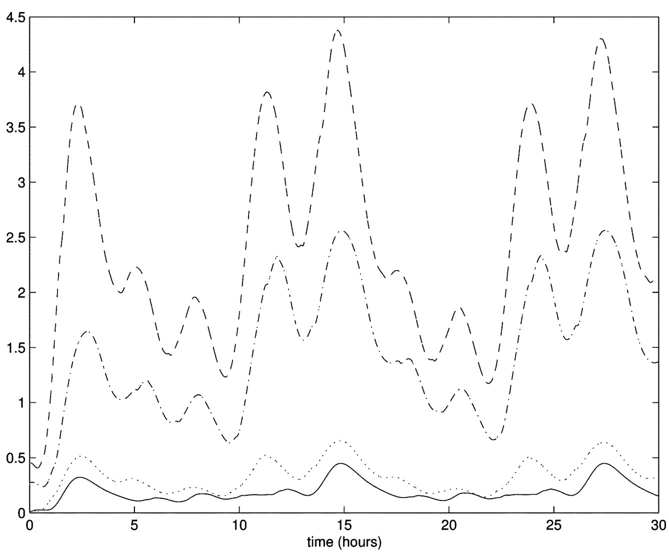

Fig. 10. Adaptive variables $10\hat{\theta}_{2,1}$ ("–"), $\hat{\theta}_{2,2}$ ("–"), $15\hat{\theta}_{3,1}$ (":"), and $\hat{\theta}_{3,2}$ ("—").

Then

$$P_{1,1} = P_{2,1} = \left[P_{1,1}^{1}, P_{1,1}^{2}, \ldots, P_{1,1}^{11}\right]^{T}$$
$$P_{1,2} = P_{2,2} = \left[P_{1,2}^{1}, P_{1,2}^{2}, \ldots, P_{1,2}^{11}\right]^{T}.$$

Furthermore, the adaptive fuzzy controllers are constructed as follows:

$$\begin{aligned} u_1 = &- \left(\lambda_{1,2} + \frac{1}{2c_{1,2}^2}\right)(x_{1,2} - \alpha_{1,1}) - (x_{1,1} - y_{1d}) \\ &- \frac{1}{2a_{1,2}^2}(x_{1,2} - \alpha_{1,1}) P_{1,2}^T P_{1,2} \hat{\theta}_{1,2} \\ u_2 = &- \left(\lambda_{2,2} + \frac{1}{2c_{2,2}^2}\right)(x_{2,2} - \alpha_{2,1}) - (x_{2,1} - y_{2d}) \\ &- \frac{1}{2a_{2,2}^2}(x_{2,2} - \alpha_{2,1}) P_{2,2}^T P_{2,2} \hat{\theta}_{2,2} \end{aligned}$$

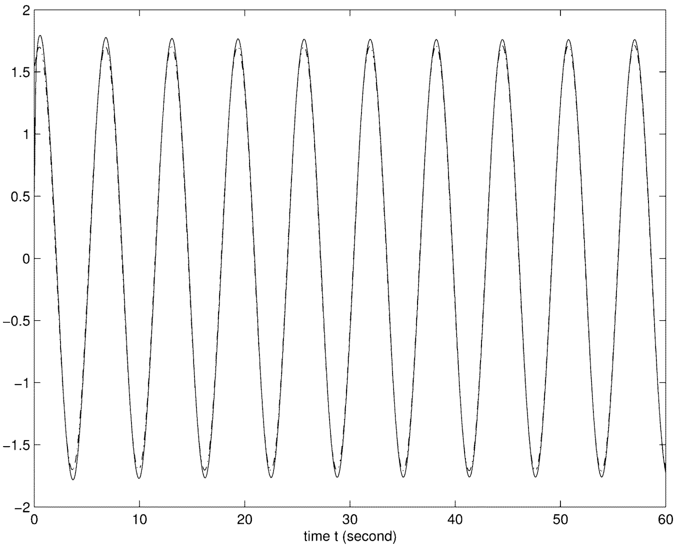

Fig. 11. Output $y_{1,1}$ ("–") follows the reference y_{1d} ("–").

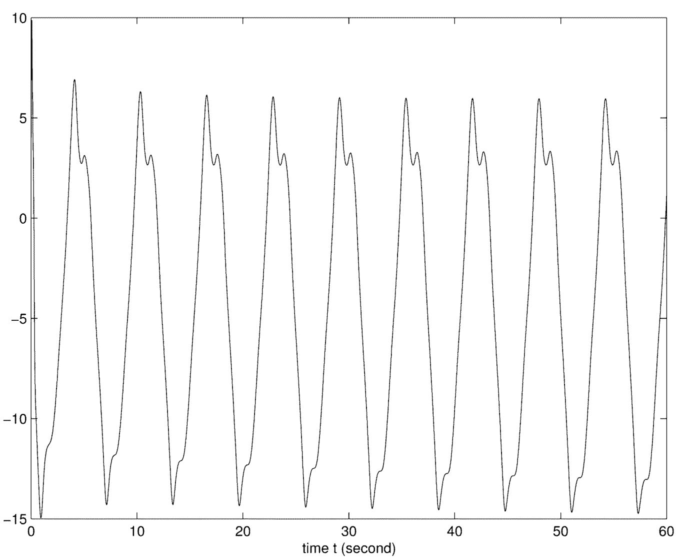

Fig. 12. Control input u_1.

with

$$\begin{aligned} \alpha_{1,1} = &- \left(\lambda_{11} + \frac{1}{2c_{1,1}^2}\right)(x_{1,1} - y_{1d}) \\ &- \frac{1}{2a_{1,1}^2}(x_{1,1} - y_{1d}) \hat{\theta}_{1,1} P_{1,1}^T P_{1,1} \\ \alpha_{2,1} = &- \left(\lambda_{21} + \frac{1}{2c_{2,1}^2}\right)(x_{2,1} - y_{2d}) \\ &- \frac{1}{2a_{2,1}^2}(x_{2,1} - y_{2d}) \hat{\theta}_{2,1} P_{2,1}^T P_{2,1} \end{aligned}$$

and the adaptive laws are given by

$$\begin{aligned} \dot{\hat{\theta}}_{1,1} &= \frac{r_{11}}{b_{11}^2}(x_{1,1} - y_{1d})^2 P_{1,1}^T P_{1,1} - k_{11}\hat{\theta}_{1,1} \\ \dot{\hat{\theta}}_{1,2} &= \frac{r_{12}}{b_{12}^2}(x_{1,2} - \alpha_{1,1})^2 P_{1,2}^T P_{1,2} - k_{12}\hat{\theta}_{1,2} \\ \dot{\hat{\theta}}_{2,1} &= \frac{r_{21}}{b_{21}^2}(x_{2,1} - y_{2d})^2 P_{2,1}^T P_{2,1} - k_{21}\hat{\theta}_{2,1} \\ \dot{\hat{\theta}}_{2,2} &= \frac{r_{22}}{b_{22}^2}(x_{2,2} - \alpha_{2,1})^2 P_{2,2}^T P_{2,2} - k_{22}\hat{\theta}_{2,2}. \end{aligned}$$

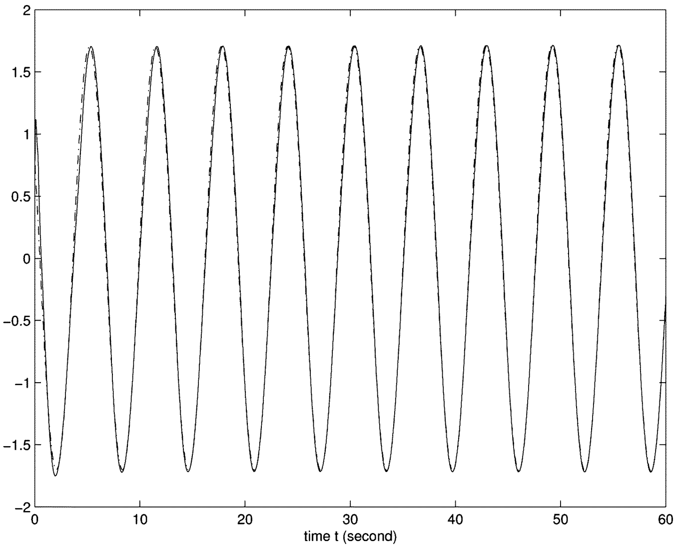

Fig. 13. Output $y_{2,1}$ (“_”) follows the reference y_{2d} (“_”).

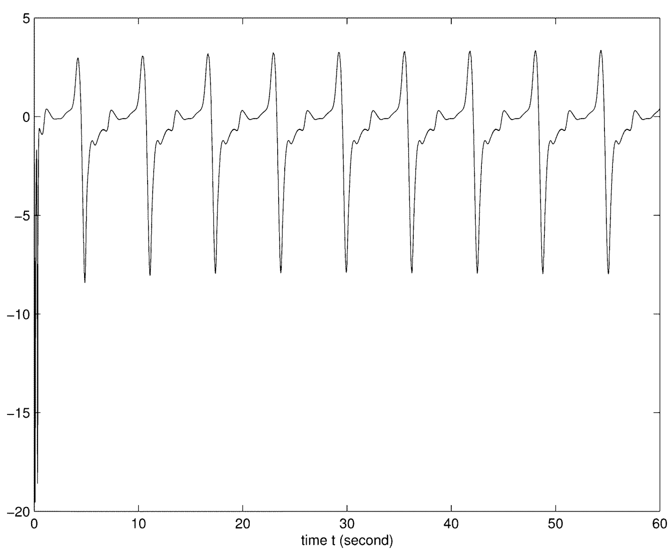

Fig. 14. Control input u_2.

The reference signals are generated by the following system:

$$\begin{aligned}\dot{x}_{d1} &= x_{d2}\\ \dot{x}_{d2} &= -x_{d1} + \beta\left(1 - x_{d1}^2\right)x_{d2}\\ y_{id} &= x_{di}, \qquad i = 1, 2.\end{aligned}$$

The design parameters are chosen to be

$$\begin{aligned}\lambda_{1,1} &= \lambda_{1,2} = \lambda_{2,1} = \lambda_{2,2} = 1.5\\ a_{1,1} &= a_{1,2} = a_{2,1} = a_{2,2} = 0.25\\ c_{1,1} &= c_{1,2} = c_{2,1} = c_{2,2} = 0.25\\ r_{1,1} &= r_{1,2} = r_{2,1} = r_{2,2} = 1.5;\\ k_{1,1} &= k_{1,2} = k_{2,1} = k_{2,2} = 0.025.\end{aligned}$$

As [4], the nonlinear functions in the system (46) are chosen as follows:

$$\begin{aligned}g_{i,1} &= 1 + 0.1x_{1,1}^2x_{2,1}^2 \quad g_{1,2} = 2 + \cos(x_{1,1}x_{2,1})\\ g_{2,1} &= 2 + 2\sin(x_{1,1}x_{2,1}) \quad g_{2,2} = e^{x_{1,1}} + e^{-x_{2,1}}\end{aligned}$$

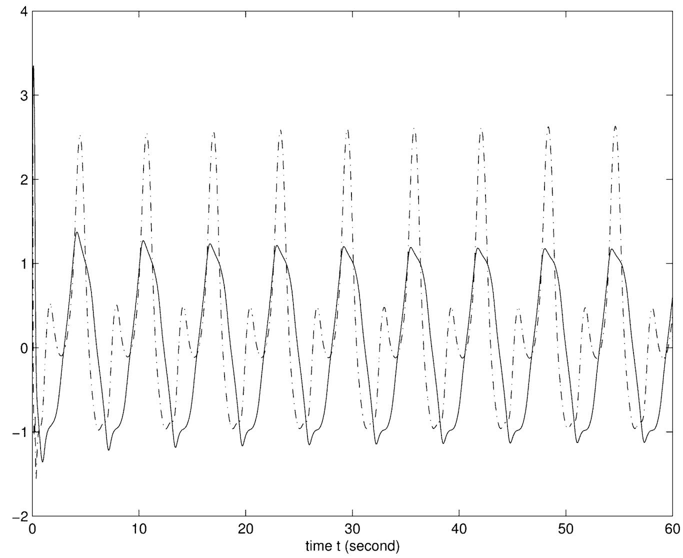

Fig. 15. State variables $x_{1,2}$ (“_”) and $x_{2,2}$ (“_”).

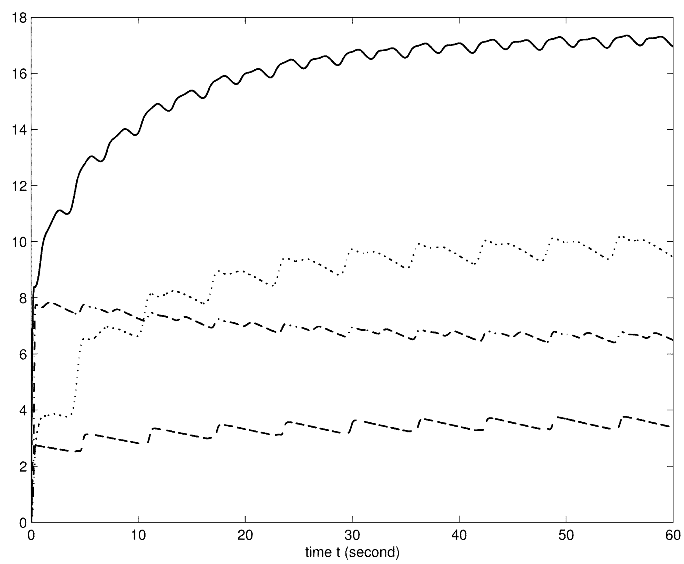

Fig. 16. Adaptive variables $10\hat{\theta}_{1,1}$ (“_”), $5\hat{\theta}_{1,2}$ (“_”), $5\hat{\theta}_{2,1}$ (“:”), and $\hat{\theta}_{2,2}$ (“__”).

$$\begin{aligned}f_{1,1} &= 0.5\left(x_{1,1} + x_{2,1}\right) \quad f_{1,2} = x_{1,1}x_{1,2} + x_{21}x_{2,2}\\ f_{2,1} &= x_{1,1}x_{2,1} \quad f_{2,2} = \left(x_{1,1}x_{1,2} + x_{2,1}x_{2,2} + u_1\right)^2.\end{aligned}$$

The simulation was performed under the initial conditions $x_{1,1}(0) = 0.5$, $x_{1,2}(0) = 2$, $x_{2,1}(0) = 0.7$, $x_{2,2}(0) = 1$, $\hat{\theta}_{1,1}(0) = \hat{\theta}_{1,2}(0) = \hat{\theta}_{2,1}(0) = \hat{\theta}_{2,2}(0) = 0$, $x_{d1}(0) = 1.5$, $x_{d2}(0) = 0.8$, and $\beta = 0.001$. The simulation results are shown in Figs. 11–16. By comparing with the simulation results in [4], it is seen that the adaptive fuzzy controllers in this paper achieve the better tracking performance than adaptive neural network controllers, but the control effort is larger. In addition, the number of the adaptive laws in [4] is equal to the dimension of the unknown vector in the nonlinear function approximator multiplied by the number of the unknown nonlinear functions, whereas the number of the adaptive laws in this paper is equal to the number of the unknown nonlinear functions. As a result, the adaptive controllers in this paper require less computation time than [4]. However, it should be pointed out that the design idea in this paper can be also used to reduce the computation time for neural networks.

VI. Conclusion

In this paper, the output tracking control problem has been considered for a class of uncertain nonlinear MIMO systems in block-triangular form. The unknown functions in systems are not linearly parameterized and have no *priori* knowledge of the bounded functions. Fuzzy logic systems are used to approximate these unknown nonlinear functions. By means of backstepping design technique, the adaptive fuzzy tracking control scheme has been developed for nonlinear MIMO systems. The proposed controllers guarantee that the outputs of the closed-loop system follow the reference signals, and achieve uniform ultimate boundedness of all the signals in the closed-loop system. The main feature of the control scheme proposed in this paper is that the number of the adaptive parameters is independent of the number of the state variables and the selection of fuzzy rules, which results in less computation burden.

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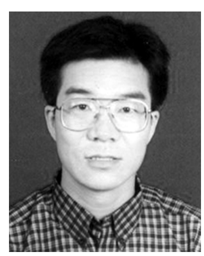

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