In[∘]:= (*Below is the self-adjoint version of the Wheeler-DeWitt equation for the third ordering:*)

```
(*-Y''[Q] + ((8πρQ)/3 - 3/(16Q^2))Y[Q]==k Y[Q],Y[Q],Q*)
```

In[∘]:= (* For the case of particles with highest velocity and in the Euclidian metric, Q < 0, the density equals zero. The WdW equation then, for k not equal to zero, is solved by: *)

```
DSolve[-Y''[Q] + (-3 / (16 Q^2)) Y[Q] == k Y[Q], Y[Q], Q]
```

Out[∘]=

$$\left\{\left\{Y[Q] \to \sqrt{Q}\ \text{BesselJ}\left[\frac{1}{4},\ \sqrt{k}\ Q\right]\ \mathbb{c}_1 + \sqrt{Q}\ \text{BesselY}\left[\frac{1}{4},\ \sqrt{k}\ Q\right]\ \mathbb{c}_2\right\}\right\}$$

In[∘]:= (* This solution times $Q^{1/4}$ gives the solution to the WdW equation, and hence is equal to: *)

```
SolutionNegQ3rd[Q_, k_, α_, β_] :=
```

$$Q^{3/4}\left(\text{BesselJ}\left[\frac{1}{4},\ \sqrt{k}\ Q\right]\alpha + \text{BesselY}\left[\frac{1}{4},\ \sqrt{k}\ Q\right]\beta\right)$$

In[∘]:= (* The asymptotic limit for Q → - Infinity can be investigated by writing a first order Taylor series: *)

```
Normal[Series[SolutionNegQ3rd[Q, k, α, β], {Q, -Infinity, 0}]]
```

Out[∘]=

$$-\frac{\left(\sqrt{k}\ Q - \mathbb{i}\ \sqrt{-k\,Q^2}\right)\beta\,\text{Cos}\left[\frac{\pi}{8} - \sqrt{k}\ Q\right]}{\sqrt{k}\ \sqrt{\pi}\ Q^{1/4}\ \sqrt{\sqrt{k}\ Q}} - \frac{(-1)^{1/4}\left(\sqrt{k}\ Q + \mathbb{i}\ \sqrt{-k\,Q^2}\right)\beta\,\text{Cos}\left[\frac{\pi}{8} + \sqrt{k}\ Q\right]}{\sqrt{k}\ \sqrt{\pi}\ Q^{1/4}\ \sqrt{\sqrt{k}\ Q}} +$$

$$\frac{(-1)^{3/4}\left(\sqrt{k}\ Q + \mathbb{i}\ \sqrt{-k\,Q^2}\right)\alpha\,\text{Sin}\left[\frac{\pi}{8} - \sqrt{k}\ Q\right]}{\sqrt{k}\ \sqrt{2\,\pi}\ Q^{1/4}\ \sqrt{\sqrt{k}\ Q}} + \frac{(-1)^{3/4}\left(\sqrt{k}\ Q + \mathbb{i}\ \sqrt{-k\,Q^2}\right)\beta\,\text{Sin}\left[\frac{\pi}{8} - \sqrt{k}\ Q\right]}{\sqrt{k}\ \sqrt{2\,\pi}\ Q^{1/4}\ \sqrt{\sqrt{k}\ Q}} +$$

$$\frac{\left(\sqrt{k}\ Q - \mathbb{i}\ \sqrt{-k\,Q^2}\right)\alpha\,\text{Sin}\left[\frac{\pi}{8} + \sqrt{k}\ Q\right]}{\sqrt{k}\ \sqrt{2\,\pi}\ Q^{1/4}\ \sqrt{\sqrt{k}\ Q}} + \frac{\left(\sqrt{k}\ Q - \mathbb{i}\ \sqrt{-k\,Q^2}\right)\beta\,\text{Sin}\left[\frac{\pi}{8} + \sqrt{k}\ Q\right]}{\sqrt{k}\ \sqrt{2\,\pi}\ Q^{1/4}\ \sqrt{\sqrt{k}\ Q}}$$

In[∘]:=

```
(* In order to investigate whether this asymptotically goes to zero,
it is best to write this in its exponential form: *)
Expand[Simplify[TrigToExp[
```

$$-\frac{\left(\sqrt{k}\,Q-i\sqrt{-k\,Q^2}\right)\beta\,\mathrm{Cos}\left[\frac{\pi}{8}-\sqrt{k}\,Q\right]}{\sqrt{k}\,\sqrt{\pi}\,Q^{1/4}\sqrt{\sqrt{k}\,Q}}-\frac{(-1)^{1/4}\left(\sqrt{k}\,Q+i\sqrt{-k\,Q^2}\right)\beta\,\mathrm{Cos}\left[\frac{\pi}{8}+\sqrt{k}\,Q\right]}{\sqrt{k}\,\sqrt{\pi}\,Q^{1/4}\sqrt{\sqrt{k}\,Q}}+$$

$$\frac{(-1)^{3/4}\left(\sqrt{k}\,Q+i\sqrt{-k\,Q^2}\right)\alpha\,\mathrm{Sin}\left[\frac{\pi}{8}-\sqrt{k}\,Q\right]}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}\sqrt{\sqrt{k}\,Q}}+$$

$$\frac{(-1)^{3/4}\left(\sqrt{k}\,Q+i\sqrt{-k\,Q^2}\right)\beta\,\mathrm{Sin}\left[\frac{\pi}{8}-\sqrt{k}\,Q\right]}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}\sqrt{\sqrt{k}\,Q}}+$$

$$\frac{\left(\sqrt{k}\,Q-i\sqrt{-k\,Q^2}\right)\alpha\,\mathrm{Sin}\left[\frac{\pi}{8}+\sqrt{k}\,Q\right]}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}\sqrt{\sqrt{k}\,Q}}+\frac{\left(\sqrt{k}\,Q-i\sqrt{-k\,Q^2}\right)\beta\,\mathrm{Sin}\left[\frac{\pi}{8}+\sqrt{k}\,Q\right]}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}\sqrt{\sqrt{k}\,Q}}]]]$$

Out[∘]=

$$-\frac{(-1)^{3/8}e^{i\sqrt{k}\,Q}\sqrt{\sqrt{k}\,Q}\,\alpha}{2\sqrt{k}\,\sqrt{\pi}\,Q^{1/4}}+\frac{(-1)^{3/8}e^{-i\sqrt{k}\,Q}\sqrt{\sqrt{k}\,Q}\,\alpha}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}}-$$

$$\frac{(-1)^{3/8}e^{i\sqrt{k}\,Q}\sqrt{\sqrt{k}\,Q}\,\sqrt{-k\,Q^2}\,\alpha}{2\,k\,\sqrt{\pi}\,Q^{5/4}}-\frac{\left(1-\frac{i}{2}\right)(-1)^{3/8}e^{i\sqrt{k}\,Q}\sqrt{\sqrt{k}\,Q}\,\beta}{\sqrt{k}\,\sqrt{\pi}\,Q^{1/4}}+$$

$$\frac{(-1)^{7/8}e^{-i\sqrt{k}\,Q}\sqrt{\sqrt{k}\,Q}\,\beta}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}}-\frac{(-1)^{7/8}e^{i\sqrt{k}\,Q}\sqrt{\sqrt{k}\,Q}\,\sqrt{-k\,Q^2}\,\beta}{2\,k\,\sqrt{\pi}\,Q^{5/4}}$$

In[∘]:=
```
(* Now, all these terms go to Infinity when Q does,
as long as the exponentials are imaginary. However, if the exponentials are real,
some of them go to zero whereas the others go to Infinity. To have this happen,
the exponentials must be made real,
which can only be done if k < 0. Doing so results in: *)
AsymLimNegQ3rd[Q_, α_, β_] :=
```

$$-\frac{(-1)^{3/8}\,e^{i\sqrt{k}\,Q}\sqrt{\sqrt{k}\,Q}\,\alpha}{2\sqrt{k}\,\sqrt{\pi}\,Q^{1/4}}+\frac{(-1)^{3/8}\,e^{-i\sqrt{k}\,Q}\sqrt{\sqrt{k}\,Q}\,\alpha}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}}-\frac{(-1)^{3/8}\,e^{i\sqrt{k}\,Q}\sqrt{\sqrt{k}\,Q}\,\sqrt{-k\,Q^2}\,\alpha}{2\,k\,\sqrt{\pi}\,Q^{5/4}}-\frac{\left(1-\frac{i}{2}\right)(-1)^{3/8}\,e^{i\sqrt{k}\,Q}\sqrt{\sqrt{k}\,Q}\,\beta}{\sqrt{k}\,\sqrt{\pi}\,Q^{1/4}}+\frac{(-1)^{7/8}\,e^{-i\sqrt{k}\,Q}\sqrt{\sqrt{k}\,Q}\,\beta}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}}-\frac{(-1)^{7/8}\,e^{i\sqrt{k}\,Q}\sqrt{\sqrt{k}\,Q}\,\sqrt{-k\,Q^2}\,\beta}{2\,k\,\sqrt{\pi}\,Q^{5/4}}\ /.\ k\to-1$$

```
AsymLimNegQ3rd[Q, α, β]
```

Out[∘]=

$$\frac{(-1)^{7/8}\,e^{-Q}\sqrt{i\,Q}\,\alpha}{2\sqrt{\pi}\,Q^{1/4}}-\frac{(-1)^{7/8}\,e^{Q}\sqrt{i\,Q}\,\alpha}{\sqrt{2\pi}\,Q^{1/4}}+\frac{(-1)^{3/8}\,e^{-Q}\sqrt{i\,Q}\,\sqrt{Q^2}\,\alpha}{2\sqrt{\pi}\,Q^{5/4}}+\frac{\left(\frac{1}{2}+i\right)(-1)^{3/8}\,e^{-Q}\sqrt{i\,Q}\,\beta}{\sqrt{\pi}\,Q^{1/4}}+\frac{(-1)^{3/8}\,e^{Q}\sqrt{i\,Q}\,\beta}{\sqrt{2\pi}\,Q^{1/4}}+\frac{(-1)^{7/8}\,e^{-Q}\sqrt{i\,Q}\,\sqrt{Q^2}\,\beta}{2\sqrt{\pi}\,Q^{5/4}}$$

In[∘]:=
```
(* The exponentials with -Q in the exponents are
  problematic: they go to Infinity as Q goes to minus infinity,
which is the domain of our interest. So, to avoid this from happening,
α and β have to be chosen such that these terms cancel each other.  *)
```

$$\mathrm{Solve}\left[\frac{(-1)^{7/8}\,e^{-Q}\sqrt{i\,Q}\,\alpha}{2\sqrt{\pi}\,Q^{1/4}}+\frac{(-1)^{3/8}\,e^{-Q}\sqrt{i\,Q}\,\sqrt{Q^2}\,\alpha}{2\sqrt{\pi}\,Q^{5/4}}+\frac{\left(\frac{1}{2}+i\right)(-1)^{3/8}\,e^{-Q}\sqrt{i\,Q}\,\beta}{\sqrt{\pi}\,Q^{1/4}}+\frac{(-1)^{7/8}\,e^{-Q}\sqrt{i\,Q}\,\sqrt{Q^2}\,\beta}{2\sqrt{\pi}\,Q^{5/4}}==0,\ \beta\right]$$

Out[∘]=

$$\left\{\left\{\beta\to\frac{-2\,Q^2\,\alpha-(2-2\,i)\,Q\,\sqrt{Q^2}\,\alpha}{Q^2+\left(2\,Q+\sqrt{Q^2}\right)^2}\right\}\right\}$$

In[∘]:=
```
(* Now, this makes it look as if β cannot be a constant (as it depends on Q),
but remember that Q is taken to the limit Q → -Infinity. So,
this limit applies to the expression for β as well. This then
 results in a perfectly constant expression for the required β: *)
```

$$\mathrm{Limit}\left[\frac{-2\,Q^2\,\alpha-(2-2\,i)\,Q\,\sqrt{Q^2}\,\alpha}{Q^2+\left(2\,Q+\sqrt{Q^2}\right)^2},\ Q\to-\mathrm{Infinity}\right]$$

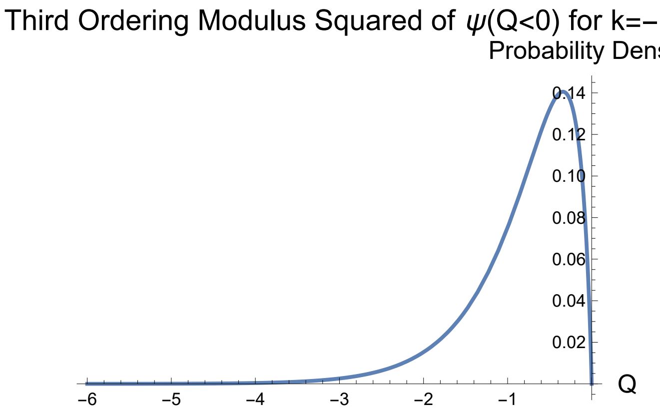
Out[∘]=

$$-i\,\alpha$$

In[*]:=
```
(* Hence, one should choose β to be minus I times α to make
  the solution found asymptotically go to zero as Q → -Infinity. *)
```

In[*]:=
```
(* It is easy to check that this is the case now
   (taking α = 1, because leaving α unspecified cannot be made into a plot): *)
Plot[Abs[SolutionNegQ3rd[Q, -1, 1, -I]]^2, {Q, -6, 0},
 AxesLabel → {Style["Q", Black, 13], Style["Probability Density", Black, 13]},
 PlotLabel → Style["Third Ordering Modulus Squared of ψ(Q<0) for k=-1", Black, 15],
 AxesStyle → Directive[Black, Thickness[0.0009]]]
```

Out[*]=



In[*]:=
```
(* Also, this is perfectly normalisable: *)
NIntegrate[Abs[SolutionNegQ3rd[Q, -1, 1, -I]]^2, {Q, -Infinity, 0}]
```

Out[*]=
```
0.156963
```

In[*]:=
```
(* Next, the same WdW will be solved for positive Q,
which corresponds to a non-zero energy density - ρ = A/Q^2*)
DSolve[-Y''[Q] + ((8 π × A) / (3 Q) - 3 / (16 Q^2)) Y[Q] == k Y[Q], Y[Q], Q]
```

Out[*]=

$$\left\{\left\{Y[Q] \to c_1\,\text{WhittakerM}\left[\frac{4\,i\,A\,\pi}{3\,\sqrt{k}}, \frac{1}{4}, 2\,i\,\sqrt{k}\,Q\right] + c_2\,\text{WhittakerW}\left[\frac{4\,i\,A\,\pi}{3\,\sqrt{k}}, \frac{1}{4}, 2\,i\,\sqrt{k}\,Q\right]\right\}\right\}$$

In[•]:=
```
(* This times Q^{1/4} gives the solution to the WdW: *)
SolutionPosQ3rd[Q_, k_, α_, β_] :=
```
$$Q^{1/4}\left(\alpha\,\text{WhittakerM}\left[\frac{4\,i\,A\,\pi}{3\sqrt{k}},\frac{1}{4},2\,i\sqrt{k}\,Q\right]+\beta\,\text{WhittakerW}\left[\frac{4\,i\,A\,\pi}{3\sqrt{k}},\frac{1}{4},2\,i\sqrt{k}\,Q\right]\right)$$
```
(* To fulfill one necessary condition for normalisation,
it will be checked what α and β should be to make
  the solution asymptotically go to zero as Q → Infinity. *)
Expand[Simplify[Normal[Series[SolutionPosQ3rd[Q, k, α, β], {Q, Infinity, 0}]]]]
```

Out[•]=

$$2^{\frac{4iA\pi}{3\sqrt{k}}}\,e^{-i\sqrt{k}\,Q}\,Q^{1/4}\left(i\sqrt{k}\,Q\right)^{\frac{4iA\pi}{3\sqrt{k}}}\beta+\frac{2^{-1-\frac{4iA\pi}{3\sqrt{k}}}\,e^{i\sqrt{k}\,Q}\sqrt{\pi}\,Q^{1/4}\left(i\sqrt{k}\,Q\right)^{-\frac{4iA\pi}{3\sqrt{k}}}\alpha}{\text{Gamma}\left[\frac{3}{4}-\frac{4iA\pi}{3\sqrt{k}}\right]}+\frac{i\,2^{-1+\frac{4iA\pi}{3\sqrt{k}}}\,e^{-i\sqrt{k}\,Q}\sqrt{\pi}\left(-i\sqrt{k}\,Q\right)^{\frac{1}{4}+\frac{4iA\pi}{3\sqrt{k}}}\left(i\sqrt{k}\,Q\right)^{3/4}\alpha}{\sqrt{k}\,Q^{3/4}\,\text{Gamma}\left[\frac{3}{4}+\frac{4iA\pi}{3\sqrt{k}}\right]}$$

In[•]:=
```
(*In order for exponential to be real, k has to be set to -1:*)
AsymLimPosQ3rd[Q_, α_, β_] :=
```
$$2^{\frac{4iA\pi}{3\sqrt{k}}}\,e^{-i\sqrt{k}\,Q}\,Q^{1/4}\left(i\sqrt{k}\,Q\right)^{\frac{4iA\pi}{3\sqrt{k}}}\beta+\frac{2^{-1-\frac{4iA\pi}{3\sqrt{k}}}\,e^{i\sqrt{k}\,Q}\sqrt{\pi}\,Q^{1/4}\left(i\sqrt{k}\,Q\right)^{-\frac{4iA\pi}{3\sqrt{k}}}\alpha}{\text{Gamma}\left[\frac{3}{4}-\frac{4iA\pi}{3\sqrt{k}}\right]}+\frac{i\,2^{-1+\frac{4iA\pi}{3\sqrt{k}}}\,e^{-i\sqrt{k}\,Q}\sqrt{\pi}\left(-i\sqrt{k}\,Q\right)^{\frac{1}{4}+\frac{4iA\pi}{3\sqrt{k}}}\left(i\sqrt{k}\,Q\right)^{3/4}\alpha}{\sqrt{k}\,Q^{3/4}\,\text{Gamma}\left[\frac{3}{4}+\frac{4iA\pi}{3\sqrt{k}}\right]}\;/.\;k\to -1$$
```
AsymLimPosQ3rd[Q, α, β]
```

Out[•]=

$$2^{\frac{4A\pi}{3}}\,e^{Q}\,(-Q)^{\frac{4A\pi}{3}}\,Q^{1/4}\beta+\frac{2^{-1-\frac{4A\pi}{3}}\,e^{-Q}\sqrt{\pi}\,(-Q)^{-\frac{4A\pi}{3}}\,Q^{1/4}\alpha}{\text{Gamma}\left[\frac{3}{4}-\frac{4A\pi}{3}\right]}+\frac{2^{-1+\frac{4A\pi}{3}}\,e^{Q}\sqrt{\pi}\,(-Q)^{3/4}\,Q^{-\frac{1}{2}+\frac{4A\pi}{3}}\alpha}{\text{Gamma}\left[\frac{3}{4}+\frac{4A\pi}{3}\right]}$$

In[•]:=
```
(*Now, as the limit goes to infinity,
positive exponential will go to infinity while negative exponentials
 will go to zero, so positive exponential need to vanish:*)
```
$$\text{Solve}\left[2^{\frac{4A\pi}{3}}\,e^{Q}\,(-Q)^{\frac{4A\pi}{3}}\,Q^{1/4}\beta+\left(2^{-1+\frac{4A\pi}{3}}\,e^{Q}\sqrt{\pi}\,(-Q)^{3/4}\,Q^{-\frac{1}{2}+\frac{4A\pi}{3}}\alpha\right)\Big/\text{Gamma}\left[\frac{3}{4}+\frac{4A\pi}{3}\right]==0,\alpha\right]$$

Out[•]=

$$\left\{\left\{\alpha\to-\frac{1}{\sqrt{\pi}}\,2\,(-Q)^{-\frac{3}{4}+\frac{4A\pi}{3}}\,Q^{\frac{3}{4}-\frac{4A\pi}{3}}\,\beta\,\text{Gamma}\left[\frac{3}{4}+\frac{4A\pi}{3}\right]\right\}\right\}$$

In[∘]:= (* Looks like a Q-dependent relation,
but the limit Q → Infinity need still be taken, resulting in a constant: *)

$$\text{Limit}\left[-\frac{1}{\sqrt{\pi}}\,2\,(-Q)^{-\frac{3}{4}+\frac{4A\pi}{3}}\,Q^{\frac{3}{4}-\frac{4A\pi}{3}}\,\beta\,\text{Gamma}\left[\frac{3}{4}+\frac{4A\pi}{3}\right],\,Q\to\text{Infinity}\right]$$

Out[∘]=

$$\frac{1}{\sqrt{\pi}}\,2\,(-1)^{\frac{1}{4}+\frac{4A\pi}{3}}\,\beta\,\text{Gamma}\left[\frac{3}{4}+\frac{4A\pi}{3}\right]$$

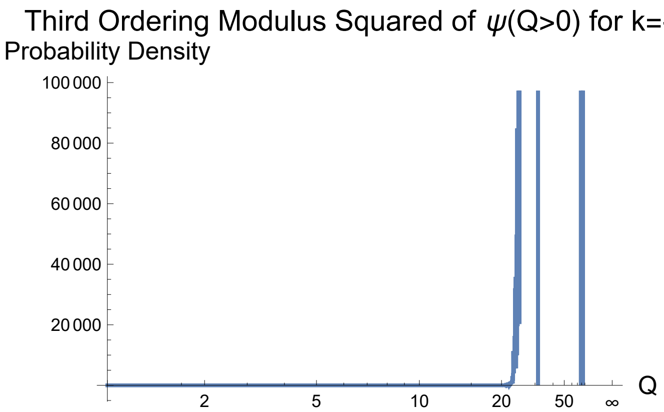
In[∘]:= (*So the value of α has to be $\frac{1}{\sqrt{\pi}}2\,(-1)^{\frac{1}{4}+\frac{4A\pi}{3}}\,\beta\,\text{Gamma}\left[\frac{3}{4}+\frac{4A\pi}{3}\right]$*)

(*Can be seen using a plot:*)

$$\text{Plot}\left[\text{Abs}\left[\text{SolutionPosQ3rd}\left[Q,\,-1,\,\frac{1}{\sqrt{\pi}}\,2\,(-1)^{\frac{1}{4}+\frac{4A\pi}{3}}\,\text{Gamma}\left[\frac{3}{4}+\frac{4A\pi}{3}\right],\,1\right]\right]\text{^}2\ /.\ \{A\to 1\},\right.$$

```
 {Q, 0, Infinity},
 AxesLabel → {Style["Q", Black, 13], Style["Probability Density", Black, 13]},
 PlotLabel → Style["Third Ordering Modulus Squared of ψ(Q>0) for k=-1", Black, 15],
 AxesStyle → Directive[Black, Thickness[0.0009]]]
```

Out[∘]=



(* As the plot abbove perfextly illustrates,
the positive Q solution does not converge to zero,
and instead blows up to infinity. That is due to the WhittakerM function,
it contains Kummer's HyperGeometricM function in the form $\text{WhittakerM}_{k,\mu}(Q) = e^{-z/2}z^{1/2+\mu}M(1+\mu-k,1+2\mu,z)$, where z=Q, which cannot be normalized *)