In[∘]:= (*Below is the self-adjoint version of the Wheeler-DeWitt equation for the first ordering:*)

```
(*-Y''[Q]-(3/(16Q^2)+((8πρQ)/(3)))Y[Q]==-k Y[Q]*)
```

(* For the case of particles with highest velocity and in the Eucledian metric, Q < 0, the density equals zero. The WdW equation the, for k not equal zero is solved by:*)

```
DSolve[-Y''[Q] - (3 / (16 Q^2)) Y[Q] == -k Y[Q], Y[Q], Q]
```

Out[∘]=

$$\left\{\left\{Y[Q] \to \sqrt{Q}\ \text{BesselJ}\left[\frac{1}{4}, -i\sqrt{k}\,Q\right] c_1 + \sqrt{Q}\ \text{BesselY}\left[\frac{1}{4}, -i\sqrt{k}\,Q\right] c_2\right\}\right\}$$

In[∘]:= (*This time $Q^{1/4}$ gives the solution to the WdW equation, which is equal to:*)

SolutionNegQ1st[Q_, k_, α_, β_] :=

$$Q^{3/4}\left(\text{BesselJ}\left[\frac{1}{4}, -i\sqrt{k}\,Q\right]\alpha + \text{BesselY}\left[\frac{1}{4}, -i\sqrt{k}\,Q\right]\beta\right)$$

(*The asymptotic limit for Q → -Infinity can be investigated by writing a first order Taylor series:*)

```
Normal[Series[SolutionNegQ1st[Q, k, α, β], {Q, -Infinity, 0}]]
```

Out[∘]=

$$-\frac{(-1)^{1/4}\left(\sqrt{k}\,Q - \sqrt{k\,Q^2}\right)\beta\,\text{Cos}\left[\frac{\pi}{8} - i\sqrt{k}\,Q\right]}{\sqrt{k}\ \sqrt{\pi}\ Q^{1/4}\ \sqrt{-i\sqrt{k}\,Q}} - \frac{\left(\sqrt{k}\,Q + \sqrt{k\,Q^2}\right)\beta\,\text{Cos}\left[\frac{\pi}{8} + i\sqrt{k}\,Q\right]}{\sqrt{k}\ \sqrt{\pi}\ Q^{1/4}\ \sqrt{-i\sqrt{k}\,Q}} +$$

$$\frac{\left(\sqrt{k}\,Q + \sqrt{k\,Q^2}\right)\alpha\,\text{Sin}\left[\frac{\pi}{8} - i\sqrt{k}\,Q\right]}{\sqrt{k}\ \sqrt{2\pi}\ Q^{1/4}\ \sqrt{-i\sqrt{k}\,Q}} + \frac{\left(\sqrt{k}\,Q + \sqrt{k\,Q^2}\right)\beta\,\text{Sin}\left[\frac{\pi}{8} - i\sqrt{k}\,Q\right]}{\sqrt{k}\ \sqrt{2\pi}\ Q^{1/4}\ \sqrt{-i\sqrt{k}\,Q}} +$$

$$\frac{(-1)^{3/4}\left(\sqrt{k}\,Q - \sqrt{k\,Q^2}\right)\alpha\,\text{Sin}\left[\frac{\pi}{8} + i\sqrt{k}\,Q\right]}{\sqrt{k}\ \sqrt{2\pi}\ Q^{1/4}\ \sqrt{-i\sqrt{k}\,Q}} + \frac{(-1)^{3/4}\left(\sqrt{k}\,Q - \sqrt{k\,Q^2}\right)\beta\,\text{Sin}\left[\frac{\pi}{8} + i\sqrt{k}\,Q\right]}{\sqrt{k}\ \sqrt{2\pi}\ Q^{1/4}\ \sqrt{-i\sqrt{k}\,Q}}$$

In[∘]:= (*In order to investigate whether this asymptotically goes to zero, it is best to write this in its exponential for:*)

$$\text{Expand}\Big[\text{Simplify}\Big[\text{TrigToExp}\Big[-\frac{(-1)^{1/4}\left(\sqrt{k}\,Q-\sqrt{k\,Q^2}\right)\beta\,\text{Cos}\left[\frac{\pi}{8}-\mathrm{i}\sqrt{k}\,Q\right]}{\sqrt{k}\,\sqrt{\pi}\,Q^{1/4}\sqrt{-\mathrm{i}\sqrt{k}\,Q}}-\frac{\left(\sqrt{k}\,Q+\sqrt{k\,Q^2}\right)\beta\,\text{Cos}\left[\frac{\pi}{8}+\mathrm{i}\sqrt{k}\,Q\right]}{\sqrt{k}\,\sqrt{\pi}\,Q^{1/4}\sqrt{-\mathrm{i}\sqrt{k}\,Q}}+\frac{\left(\sqrt{k}\,Q+\sqrt{k\,Q^2}\right)\alpha\,\text{Sin}\left[\frac{\pi}{8}-\mathrm{i}\sqrt{k}\,Q\right]}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}\sqrt{-\mathrm{i}\sqrt{k}\,Q}}+\frac{\left(\sqrt{k}\,Q+\sqrt{k\,Q^2}\right)\beta\,\text{Sin}\left[\frac{\pi}{8}-\mathrm{i}\sqrt{k}\,Q\right]}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}\sqrt{-\mathrm{i}\sqrt{k}\,Q}}+\frac{(-1)^{3/4}\left(\sqrt{k}\,Q-\sqrt{k\,Q^2}\right)\alpha\,\text{Sin}\left[\frac{\pi}{8}+\mathrm{i}\sqrt{k}\,Q\right]}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}\sqrt{-\mathrm{i}\sqrt{k}\,Q}}+\frac{(-1)^{3/4}\left(\sqrt{k}\,Q-\sqrt{k\,Q^2}\right)\beta\,\text{Sin}\left[\frac{\pi}{8}+\mathrm{i}\sqrt{k}\,Q\right]}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}\sqrt{-\mathrm{i}\sqrt{k}\,Q}}\Big]\Big]\Big]$$

Out[∘]=

$$-\frac{(-1)^{7/8}e^{\sqrt{k}\,Q}\sqrt{-\mathrm{i}\sqrt{k}\,Q}\,\alpha}{2\sqrt{k}\,\sqrt{\pi}\,Q^{1/4}}+\frac{(-1)^{7/8}e^{-\sqrt{k}\,Q}\sqrt{-\mathrm{i}\sqrt{k}\,Q}\,\alpha}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}}+\frac{(-1)^{3/8}e^{\sqrt{k}\,Q}\sqrt{-\mathrm{i}\sqrt{k}\,Q}\,\sqrt{k\,Q^2}\,\alpha}{2k\sqrt{\pi}\,Q^{5/4}}-\frac{\left(1-\frac{\mathrm{i}}{2}\right)(-1)^{7/8}e^{\sqrt{k}\,Q}\sqrt{-\mathrm{i}\sqrt{k}\,Q}\,\beta}{\sqrt{k}\,\sqrt{\pi}\,Q^{1/4}}-\frac{(-1)^{3/8}e^{-\sqrt{k}\,Q}\sqrt{-\mathrm{i}\sqrt{k}\,Q}\,\beta}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}}+\frac{(-1)^{7/8}e^{\sqrt{k}\,Q}\sqrt{-\mathrm{i}\sqrt{k}\,Q}\,\sqrt{k\,Q^2}\,\beta}{2k\sqrt{\pi}\,Q^{5/4}}$$

In[∘]:= (*For k=-1, exponential become complex, which will not converge to zero, therefore k has to equal 1 in order for the exponential to be real. Furthermore, exponentials with a negative exponent will go to Infinity for Q → -Infinity, while positive exponents will go to zero, therefore k needs to be set to 1:*)

AsymLimNegQ1st[Q_, α_, β_] :=

$$-\frac{(-1)^{7/8}e^{\sqrt{k}\,Q}\sqrt{-\mathrm{i}\sqrt{k}\,Q}\,\alpha}{2\sqrt{k}\,\sqrt{\pi}\,Q^{1/4}}+\frac{(-1)^{7/8}e^{-\sqrt{k}\,Q}\sqrt{-\mathrm{i}\sqrt{k}\,Q}\,\alpha}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}}+\frac{(-1)^{3/8}e^{\sqrt{k}\,Q}\sqrt{-\mathrm{i}\sqrt{k}\,Q}\,\sqrt{k\,Q^2}\,\alpha}{2k\sqrt{\pi}\,Q^{5/4}}-\frac{\left(1-\frac{\mathrm{i}}{2}\right)(-1)^{7/8}e^{\sqrt{k}\,Q}\sqrt{-\mathrm{i}\sqrt{k}\,Q}\,\beta}{\sqrt{k}\,\sqrt{\pi}\,Q^{1/4}}-\frac{(-1)^{3/8}e^{-\sqrt{k}\,Q}\sqrt{-\mathrm{i}\sqrt{k}\,Q}\,\beta}{\sqrt{k}\,\sqrt{2\pi}\,Q^{1/4}}+\frac{(-1)^{7/8}e^{\sqrt{k}\,Q}\sqrt{-\mathrm{i}\sqrt{k}\,Q}\,\sqrt{k\,Q^2}\,\beta}{2k\sqrt{\pi}\,Q^{5/4}}\ /.\ k\to 1$$

AsymLimNegQ1st[Q, α, β]

Out[∘]=

$$-\frac{(-1)^{7/8}e^{Q}\sqrt{-\mathrm{i}\,Q}\,\alpha}{2\sqrt{\pi}\,Q^{1/4}}+\frac{(-1)^{7/8}e^{-Q}\sqrt{-\mathrm{i}\,Q}\,\alpha}{\sqrt{2\pi}\,Q^{1/4}}+\frac{(-1)^{3/8}e^{Q}\sqrt{-\mathrm{i}\,Q}\,\sqrt{Q^2}\,\alpha}{2\sqrt{\pi}\,Q^{5/4}}-\frac{\left(1-\frac{\mathrm{i}}{2}\right)(-1)^{7/8}e^{Q}\sqrt{-\mathrm{i}\,Q}\,\beta}{\sqrt{\pi}\,Q^{1/4}}-\frac{(-1)^{3/8}e^{-Q}\sqrt{-\mathrm{i}\,Q}\,\beta}{\sqrt{2\pi}\,Q^{1/4}}+\frac{(-1)^{7/8}e^{Q}\sqrt{-\mathrm{i}\,Q}\,\sqrt{Q^2}\,\beta}{2\sqrt{\pi}\,Q^{5/4}}$$

In[*]:=
```
(*The next step is to eliminate exponentials with negative exponents by solving
 for one of the integration constants such that they cancel each other:*)
```
$$\text{Solve}\left[\frac{(-1)^{7/8}\, e^{-Q}\sqrt{-i\, Q}\;\alpha}{\sqrt{2\pi}\;Q^{1/4}}-\frac{(-1)^{3/8}\, e^{-Q}\sqrt{-i\, Q}\;\beta}{\sqrt{2\pi}\;Q^{1/4}}=0,\ \beta\right]$$

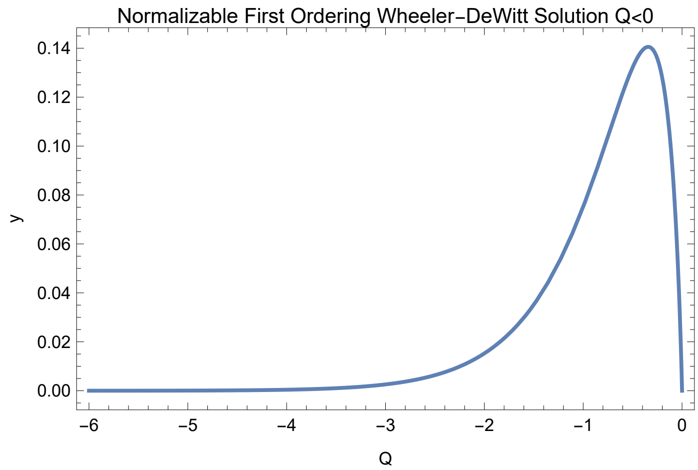
Out[*]=

$$\{\{\beta \to i\,\alpha\}\}$$

In[*]:=
```
(*Thus in order for the function to asymptotically go to zero at -Infinity,
we need the condition β=I α*)
(*This can be visually demonstrated using a plot and taking α=1 as an example:*)
```

In[*]:=
```
Plot[Abs[SolutionNegQ1st[Q, 1, 1, I]]^2,
 {Q, -6, 0}, Frame → True, FrameLabel → {"Q", "y"},
 PlotLabel → "Normalizable First Ordering Wheeler-DeWitt Solution Q<0"]
```

Out[*]=



In[*]:=
```
(*Also, to show that it is normalisable:*)
NIntegrate[Abs[SolutionNegQ1st[Q, 1, 1, I]]^2, {Q, -Infinity, 0}]
```

Out[*]=

0.156963

In[*]:=
```
(*Next, the same WdW will be solved for the positive Q, which corresponds to a non-
 zero energy density. For radiation-dominated universe, ρ=A/Q^2*)
(*The self adjoint version is:*)
DSolve[-Y''[Q] - (3 / (16 Q^2) + (8 π * A) / (3 Q)) Y[Q] == -k Y[Q], Y[Q], Q]
```

Out[*]=

$$\left\{\left\{Y[Q]\to c_1\,\text{WhittakerM}\left[\frac{4A\pi}{3\sqrt{k}},\ \frac{1}{4},\ 2\sqrt{k}\,Q\right]+c_2\,\text{WhittakerW}\left[\frac{4A\pi}{3\sqrt{k}},\ \frac{1}{4},\ 2\sqrt{k}\,Q\right]\right\}\right\}$$

In[ ]:=
```
(*This times Q^(1/4) gives the solution to the WdW:*)
SolutionPosQ1st[Q_, k_, α_, β_] :=
 Q^(1/4) (α WhittakerM[4 A π/(3 Sqrt[k]), 1/4, 2 Sqrt[k] Q] + β WhittakerW[4 A π/(3 Sqrt[k]), 1/4, 2 Sqrt[k] Q])
(*The asymptotic limit for Q →
 Infinity can be investigated by writing a first order Taylor series again:*)
Normal[Series[Q^(1/4) (α WhittakerM[4 A π/(3 Sqrt[k]), 1/4, 2 Sqrt[k] Q] + β WhittakerW[4 A π/(3 Sqrt[k]), 1/4, 2 Sqrt[k] Q]),
  {Q, Infinity, 0}]]
```

Out[ ]=

$$2^{\frac{4 A \pi}{3 \sqrt{k}}} e^{-\sqrt{k} Q} Q^{1/4} \left(\sqrt{k} Q\right)^{\frac{4 A \pi}{3 \sqrt{k}}} \beta + \frac{2^{-1-\frac{4 A \pi}{3 \sqrt{k}}} e^{\sqrt{k} Q} \sqrt{\pi} Q^{1/4} \left(\sqrt{k} Q\right)^{-\frac{4 A \pi}{3 \sqrt{k}}} \alpha}{\text{Gamma}\left[\frac{3}{4} - \frac{4 A \pi}{3 \sqrt{k}}\right]} - \frac{2^{-1+\frac{4 A \pi}{3 \sqrt{k}}} e^{-\sqrt{k} Q} \sqrt{\pi} \left(-\sqrt{k} Q\right)^{\frac{1}{4}+\frac{4 A \pi}{3 \sqrt{k}}} \left(\sqrt{k} Q\right)^{3/4} \alpha}{\sqrt{k} Q^{3/4} \text{Gamma}\left[\frac{3}{4} + \frac{4 A \pi}{3 \sqrt{k}}\right]}$$

In[∘]:=

$$2^{\frac{4 A \pi}{3 \sqrt{k}}} e^{-\sqrt{k} Q} Q^{1/4} \left(\sqrt{k} Q\right)^{\frac{4 A \pi}{3 \sqrt{k}}} \beta + \frac{2^{-1-\frac{4 A \pi}{3 \sqrt{k}}} e^{\sqrt{k} Q} \sqrt{\pi} Q^{1/4} \left(\sqrt{k} Q\right)^{-\frac{4 A \pi}{3 \sqrt{k}}} \alpha}{\text{Gamma}\left[\frac{3}{4} - \frac{4 A \pi}{3 \sqrt{k}}\right]} - \frac{2^{-1+\frac{4 A \pi}{3 \sqrt{k}}} e^{-\sqrt{k} Q} \sqrt{\pi} \left(-\sqrt{k} Q\right)^{\frac{1}{4}+\frac{4 A \pi}{3 \sqrt{k}}} \left(\sqrt{k} Q\right)^{3/4} \alpha}{\sqrt{k} Q^{3/4} \text{Gamma}\left[\frac{3}{4} + \frac{4 A \pi}{3 \sqrt{k}}\right]}$$

AsymLimPosQ1st[Q_, α_, β_] :=

$$2^{\frac{4 A \pi}{3 \sqrt{k}}} e^{-\sqrt{k} Q} Q^{1/4} \left(\sqrt{k} Q\right)^{\frac{4 A \pi}{3 \sqrt{k}}} \beta + \frac{2^{-1-\frac{4 A \pi}{3 \sqrt{k}}} e^{\sqrt{k} Q} \sqrt{\pi} Q^{1/4} \left(\sqrt{k} Q\right)^{-\frac{4 A \pi}{3 \sqrt{k}}} \alpha}{\text{Gamma}\left[\frac{3}{4} - \frac{4 A \pi}{3 \sqrt{k}}\right]} - \frac{2^{-1+\frac{4 A \pi}{3 \sqrt{k}}} e^{-\sqrt{k} Q} \sqrt{\pi} \left(-\sqrt{k} Q\right)^{\frac{1}{4}+\frac{4 A \pi}{3 \sqrt{k}}} \left(\sqrt{k} Q\right)^{3/4} \alpha}{\sqrt{k} Q^{3/4} \text{Gamma}\left[\frac{3}{4} + \frac{4 A \pi}{3 \sqrt{k}}\right]} \text{ /. } k \to 1$$

AsymLimPosQ1st[Q, α, β]

Out[∘]=

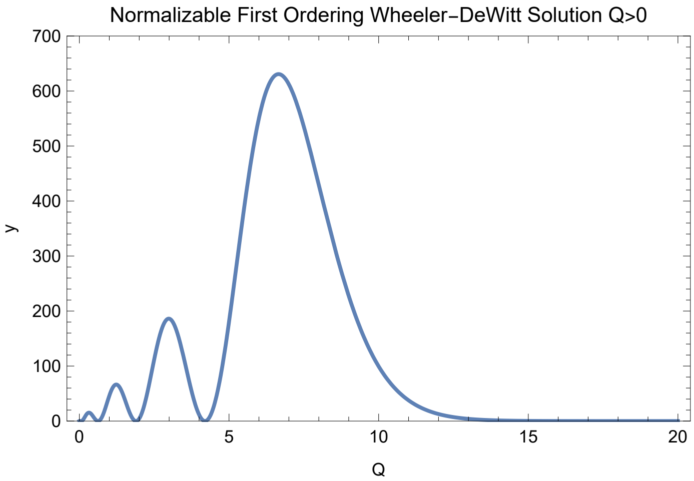
$$2^{\frac{4 A \pi}{3 \sqrt{k}}} e^{-\sqrt{k} Q} Q^{1/4} \left(\sqrt{k} Q\right)^{\frac{4 A \pi}{3 \sqrt{k}}} \beta + \frac{2^{-1-\frac{4 A \pi}{3 \sqrt{k}}} e^{\sqrt{k} Q} \sqrt{\pi} Q^{1/4} \left(\sqrt{k} Q\right)^{-\frac{4 A \pi}{3 \sqrt{k}}} \alpha}{\text{Gamma}\left[\frac{3}{4} - \frac{4 A \pi}{3 \sqrt{k}}\right]} - \frac{2^{-1+\frac{4 A \pi}{3 \sqrt{k}}} e^{-\sqrt{k} Q} \sqrt{\pi} \left(-\sqrt{k} Q\right)^{\frac{1}{4}+\frac{4 A \pi}{3 \sqrt{k}}} \left(\sqrt{k} Q\right)^{3/4} \alpha}{\sqrt{k} Q^{3/4} \text{Gamma}\left[\frac{3}{4} + \frac{4 A \pi}{3 \sqrt{k}}\right]}$$

Out[∘]=

$$2^{\frac{4 A \pi}{3}} e^{-Q} Q^{\frac{1}{4}+\frac{4 A \pi}{3}} \beta + \frac{2^{-1-\frac{4 A \pi}{3}} e^{Q} \sqrt{\pi} Q^{\frac{1}{4}-\frac{4 A \pi}{3}} \alpha}{\text{Gamma}\left[\frac{3}{4} - \frac{4 A \pi}{3}\right]} - \frac{2^{-1+\frac{4 A \pi}{3}} e^{-Q} \sqrt{\pi} (-Q)^{\frac{1}{4}+\frac{4 A \pi}{3}} \alpha}{\text{Gamma}\left[\frac{3}{4} + \frac{4 A \pi}{3}\right]}$$

In[ ]:=
```
(*Once again, for positive k limits can converge to zero at Infinity,
while for negative k they are sinusoidal. Furthermore,
positive exponents will go to infinity,
while negative exponents will go to zero, therefore α has to be set to zero:*)
(*This can be visually shown by plotting,
with β=1 and A=1 for plotting purposes:*)
Plot[Abs[SolutionPosQ1st[Q, 1, 0, 1]]^2 /. {A → 1}, {Q, 0, 20},
 PlotRange → {0, 700}, Frame → True, FrameLabel → {"Q", "y"},
 PlotLabel → "Normalizable First Ordering Wheeler-DeWitt Solution Q>0"]
```

Out[ ]=



In[ ]:=

In[ ]:=
```
(*To check that this can be normalised, a numerical integration can be done:*)
NIntegrate[Abs[SolutionPosQ1st[Q, 1, 0, 1]]^2 /. {A → 1}, {Q, 0, 20}]
```

Out[ ]=

2410.67

In[ ]:=
```
(* Important
 note: the Whittaker function does not have any poles (except for at infinity),
see https://dlmf.nist.gov/13.14,
so integration over all (pos) Q cannot blow up due to
 poles. (the integration might, of course, still be divergent due to infinite area).
  However, plotting a Whittaker in Mathematica does state infinities: *)
```

In[ ]:=

In[ ]:=
```
(*Next, The behaviour of the two solutions will be investigated at Q=0:*)
(*To do so, for each a Taylor series around Q=
 0 will be taken to see whether both solutions go to the same value as Q→0:*)
PowerExpand[Normal[Series[SolutionNegQ1st[Q, 1, 1, I], {Q, 0, 1}]]]
```

Out[ ]=

$$-\frac{(-1)^{5/8}\, 2^{1/4} \sqrt{Q}\ \text{Gamma}\left[\frac{1}{4}\right]}{\pi} - \frac{(-1)^{7/8}\, Q \left(2^{3/4} \pi - i\, 2^{1/4}\, \text{Gamma}\left[-\frac{1}{4}\right] \times \text{Gamma}\left[\frac{5}{4}\right]\right)}{2 \pi\, \text{Gamma}\left[\frac{5}{4}\right]}$$

In[ ]:=
```
% /. Q → 0
```

Out[ ]=

0

In[∘]:=
```
PowerExpand[Normal[Series[SolutionPosQ1st[Q, 1, 0, 1], {Q, 0, 1}]]]
```

Out[∘]=

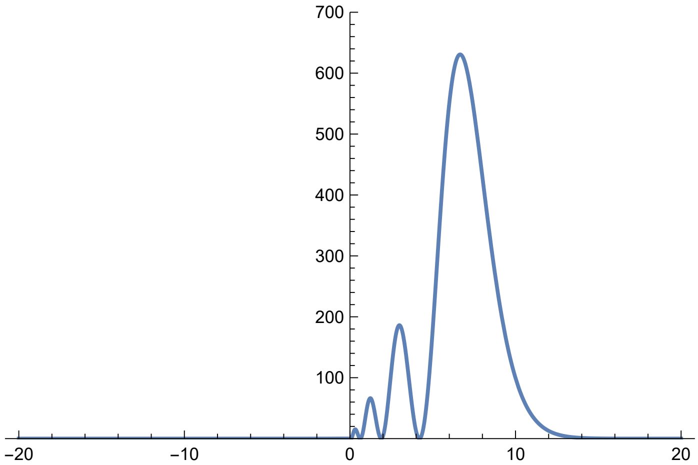
$$-\frac{2\times 2^{3/4}\sqrt{\pi}\,Q}{\text{Gamma}\left[\frac{1}{4}-\frac{4A\pi}{3}\right]}+\frac{2^{1/4}\sqrt{\pi}\,\sqrt{Q}}{\text{Gamma}\left[\frac{3}{4}-\frac{4A\pi}{3}\right]}$$

In[∘]:=

In[∘]:=
```
% /. Q → 0
```

In[∘]:=
```
(*Both go to zero as Q→0, this can be plotted for illustration:*)
Plot[{Abs[SolutionNegQ1st[Q, 1, 1, I]]^2 × UnitStep[-Q] +
    Abs[SolutionPosQ1st[Q, 1, 0, 1]]^2 × UnitStep[Q] /. A → 1},
 {Q, -20, 20}, PlotRange → {0, 700}]
```

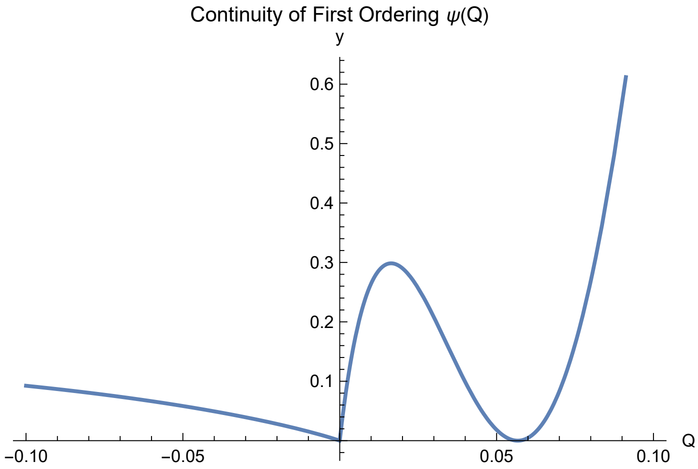
Out[∘]=



In[∘]:=

```
(*Zooming in around Q=0:*)
Plot[{Abs[SolutionNegQ1st[Q, 1, 1, I]]^2 × UnitStep[-Q] +
    Abs[SolutionPosQ1st[Q, 1, 0, 1]]^2 × UnitStep[Q] /. A → 1},
 {Q, -0.1, 0.1}, AxesLabel → {"Q", "Probability Density"},
 PlotLabel → "Continuity of First Ordering ψ(Q)"]
```

Out[∘]=



In[∘]:=
```
(*Next the derivatives will be checked for continuity at Q=0,
this will be done by Tayloring both of the derivatives of the solutions around 0:*)
Simplify[D[SolutionNegQ1st[Q, 1, α, I α], Q]]
```

Out[∘]=

$$\frac{1}{4\,Q^{1/4}}\alpha\left(-2\,i\,Q\,\text{BesselJ}\left[-\frac{3}{4},-i\,Q\right]+3\,\text{BesselJ}\left[\frac{1}{4},-i\,Q\right]+2\,i\,Q\,\text{BesselJ}\left[\frac{5}{4},-i\,Q\right]+2\,Q\,\text{BesselY}\left[-\frac{3}{4},-i\,Q\right]+3\,i\,\text{BesselY}\left[\frac{1}{4},-i\,Q\right]-2\,Q\,\text{BesselY}\left[\frac{5}{4},-i\,Q\right]\right)$$

In[∘]:=

$$\text{Normal}\left[\text{Series}\left[\frac{1}{4\,Q^{1/4}}\alpha\left(-2\,i\,Q\,\text{BesselJ}\left[-\frac{3}{4},-i\,Q\right]+3\,\text{BesselJ}\left[\frac{1}{4},-i\,Q\right]+2\,i\,Q\,\text{BesselJ}\left[\frac{5}{4},-i\,Q\right]+2\,Q\,\text{BesselY}\left[-\frac{3}{4},-i\,Q\right]+3\,i\,\text{BesselY}\left[\frac{1}{4},-i\,Q\right]-2\,Q\,\text{BesselY}\left[\frac{5}{4},-i\,Q\right]\right),\{Q,0,1\}\right]\right]$$

Out[∘]=

$$-\frac{i\,\alpha\left(3\,\text{Gamma}\left[\frac{1}{4}\right]-4\,\text{Gamma}\left[\frac{5}{4}\right]\right)}{2\times2^{3/4}\,\pi\,(-i\,Q)^{1/4}\,Q^{1/4}}+\left(Q^{3/4}\,\alpha\left(-3\,i\,2^{3/4}\,\pi\,\text{Gamma}\left[\frac{1}{4}\right]-4\,i\,2^{3/4}\,\pi\,\text{Gamma}\left[\frac{5}{4}\right]-3\times2^{1/4}\,\text{Gamma}\left[-\frac{1}{4}\right]\times\text{Gamma}\left[\frac{1}{4}\right]\times\text{Gamma}\left[\frac{5}{4}\right]+4\times2^{1/4}\,\text{Gamma}\left[\frac{1}{4}\right]\times\text{Gamma}\left[\frac{3}{4}\right]\times\text{Gamma}\left[\frac{5}{4}\right]\right)\right)\Big/\left(8\,\pi\,(-i\,Q)^{3/4}\,\text{Gamma}\left[\frac{1}{4}\right]\times\text{Gamma}\left[\frac{5}{4}\right]\right)$$

In[∘]:=

$$\text{NearZeroNegQ1st}[Q\_,\alpha\_]:=-\frac{i\,\alpha\left(3\,\text{Gamma}\left[\frac{1}{4}\right]-4\,\text{Gamma}\left[\frac{5}{4}\right]\right)}{2\times2^{3/4}\,\pi\,(-i\,Q)^{1/4}\,Q^{1/4}}+\left(Q^{3/4}\,\alpha\left(-3\,i\,2^{3/4}\,\pi\,\text{Gamma}\left[\frac{1}{4}\right]-4\,i\,2^{3/4}\,\pi\,\text{Gamma}\left[\frac{5}{4}\right]-3\times2^{1/4}\,\text{Gamma}\left[-\frac{1}{4}\right]\times\text{Gamma}\left[\frac{1}{4}\right]\times\text{Gamma}\left[\frac{5}{4}\right]+4\times2^{1/4}\,\text{Gamma}\left[\frac{1}{4}\right]\times\text{Gamma}\left[\frac{3}{4}\right]\times\text{Gamma}\left[\frac{5}{4}\right]\right)\right)\Big/\left(8\,\pi\,(-i\,Q)^{3/4}\,\text{Gamma}\left[\frac{1}{4}\right]\times\text{Gamma}\left[\frac{5}{4}\right]\right)$$

```
Limit[NearZeroNegQ1st[Q, α], Q → 0, Direction → +1]
```

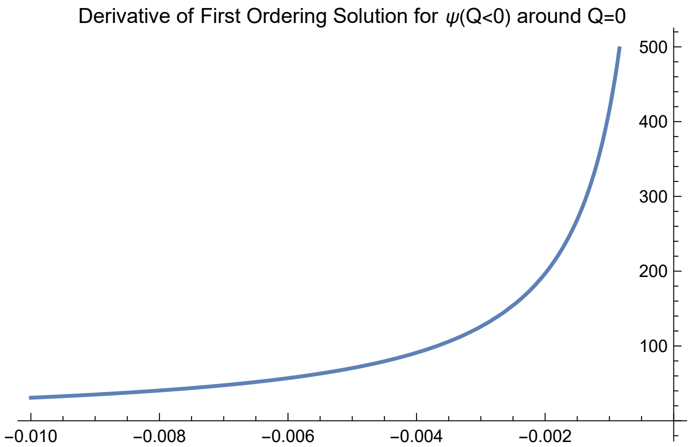
Out[∘]=

$$\alpha\left(\left(-(-1)^{1/8}\right)\infty\right)$$

In[ ]:=

```
(*And in a plot:*)
Plot[Abs[NearZeroNegQ1st[Q, 1]]^2, {Q, -0.01, 0}, FrameLabel → {"Q", "y"},
 PlotLabel → "Derivative of First Ordering Solution for ψ(Q<0) around Q=0"]
```

Out[ ]=



In[ ]:=

```
(*Next, the same for the positive Q solution:*)
Simplify[D[SolutionPosQ1st[Q, 1, 0, β], Q]]
```

Out[ ]=

$$\frac{\beta\left((3-16A\pi+12Q)\,\text{WhittakerW}\left[\frac{4A\pi}{3},\frac{1}{4},2Q\right]-12\,\text{WhittakerW}\left[1+\frac{4A\pi}{3},\frac{1}{4},2Q\right]\right)}{12\,Q^{3/4}}$$

In[ ]:=

$$\text{Normal}\left[\text{Series}\left[\frac{1}{12\,Q^{3/4}}\beta\left((3-16A\pi+12Q)\,\text{WhittakerW}\left[\frac{4A\pi}{3},\frac{1}{4},2Q\right]-12\,\text{WhittakerW}\left[1+\frac{4A\pi}{3},\frac{1}{4},2Q\right]\right),\{Q,0,1\}\right]\right]$$

Out[ ]=

$$\begin{aligned}&\frac{1}{12}Q\beta\left(-12\times2^{3/4}\left(\frac{2\sqrt{\pi}}{\text{Gamma}\left[-\frac{3}{4}-\frac{4A\pi}{3}\right]}-\frac{2\sqrt{\pi}\,(-3-16A\pi)}{9\,\text{Gamma}\left[-\frac{3}{4}-\frac{4A\pi}{3}\right]}\right)+\right.\\
&\left.\quad 2^{3/4}(3-16A\pi)\left(\frac{2\sqrt{\pi}}{\text{Gamma}\left[\frac{1}{4}-\frac{4A\pi}{3}\right]}-\frac{2\sqrt{\pi}\,(9-16A\pi)}{9\,\text{Gamma}\left[\frac{1}{4}-\frac{4A\pi}{3}\right]}\right)-\frac{24\times2^{3/4}\sqrt{\pi}}{\text{Gamma}\left[\frac{1}{4}-\frac{4A\pi}{3}\right]}\right)+\\
&\frac{1}{12}\beta\left(\frac{24\times2^{3/4}\sqrt{\pi}}{\text{Gamma}\left[-\frac{3}{4}-\frac{4A\pi}{3}\right]}-\frac{2\times2^{3/4}\sqrt{\pi}\,(3-16A\pi)}{\text{Gamma}\left[\frac{1}{4}-\frac{4A\pi}{3}\right]}\right)+\\
&\frac{1}{12}\sqrt{Q}\,\beta\left(-12\times2^{3/4}\left(-\frac{\sqrt{\frac{\pi}{2}}}{\text{Gamma}\left[-\frac{1}{4}-\frac{4A\pi}{3}\right]}+\frac{\sqrt{\frac{\pi}{2}}\,(-9-16A\pi)}{3\,\text{Gamma}\left[-\frac{1}{4}-\frac{4A\pi}{3}\right]}\right)+\right.\\
&\left.\quad 2^{3/4}(3-16A\pi)\left(-\frac{\sqrt{\frac{\pi}{2}}}{\text{Gamma}\left[\frac{3}{4}-\frac{4A\pi}{3}\right]}+\frac{\sqrt{\frac{\pi}{2}}\,(3-16A\pi)}{3\,\text{Gamma}\left[\frac{3}{4}-\frac{4A\pi}{3}\right]}\right)+\frac{12\times2^{1/4}\sqrt{\pi}}{\text{Gamma}\left[\frac{3}{4}-\frac{4A\pi}{3}\right]}\right)+\\
&\frac{\beta\left(-\frac{12\times2^{1/4}\sqrt{\pi}}{\text{Gamma}\left[-\frac{1}{4}-\frac{4A\pi}{3}\right]}+\frac{2^{1/4}\sqrt{\pi}\,(3-16A\pi)}{\text{Gamma}\left[\frac{3}{4}-\frac{4A\pi}{3}\right]}\right)}{12\sqrt{Q}}\end{aligned}$$

In[∘]:= **NearZeroPosQ1st[Q_, β_, A_] :=**

$$
\begin{aligned}
&\frac{1}{12} Q \beta \left(-12 \times 2^{3/4} \left(\frac{2 \sqrt{\pi}}{\text{Gamma}\left[-\frac{3}{4}-\frac{4 A \pi}{3}\right]}-\frac{2 \sqrt{\pi} (-3-16 A \pi)}{9 \,\text{Gamma}\left[-\frac{3}{4}-\frac{4 A \pi}{3}\right]}\right)+\right.\\
&\qquad \left. 2^{3/4} (3-16 A \pi) \left(\frac{2 \sqrt{\pi}}{\text{Gamma}\left[\frac{1}{4}-\frac{4 A \pi}{3}\right]}-\frac{2 \sqrt{\pi} (9-16 A \pi)}{9 \,\text{Gamma}\left[\frac{1}{4}-\frac{4 A \pi}{3}\right]}\right)-\frac{24 \times 2^{3/4} \sqrt{\pi}}{\text{Gamma}\left[\frac{1}{4}-\frac{4 A \pi}{3}\right]}\right)+\\
&\frac{1}{12} \beta \left(\frac{24 \times 2^{3/4} \sqrt{\pi}}{\text{Gamma}\left[-\frac{3}{4}-\frac{4 A \pi}{3}\right]}-\frac{2 \times 2^{3/4} \sqrt{\pi} (3-16 A \pi)}{\text{Gamma}\left[\frac{1}{4}-\frac{4 A \pi}{3}\right]}\right)+\\
&\frac{1}{12} \sqrt{Q} \beta \left(-12 \times 2^{3/4} \left(-\frac{\sqrt{\frac{\pi}{2}}}{\text{Gamma}\left[-\frac{1}{4}-\frac{4 A \pi}{3}\right]}+\frac{\sqrt{\frac{\pi}{2}} (-9-16 A \pi)}{3 \,\text{Gamma}\left[-\frac{1}{4}-\frac{4 A \pi}{3}\right]}\right)+\right.\\
&\qquad 2^{3/4} (3-16 A \pi) \left(-\frac{\sqrt{\frac{\pi}{2}}}{\text{Gamma}\left[\frac{3}{4}-\frac{4 A \pi}{3}\right]}+\frac{\sqrt{\frac{\pi}{2}} (3-16 A \pi)}{3 \,\text{Gamma}\left[\frac{3}{4}-\frac{4 A \pi}{3}\right]}\right)+\\
&\qquad \left.\frac{12 \times 2^{1/4} \sqrt{\pi}}{\text{Gamma}\left[\frac{3}{4}-\frac{4 A \pi}{3}\right]}\right)+\frac{\beta \left(-\frac{12 \times 2^{1/4} \sqrt{\pi}}{\text{Gamma}\left[-\frac{1}{4}-\frac{4 A \pi}{3}\right]}+\frac{2^{1/4} \sqrt{\pi} (3-16 A \pi)}{\text{Gamma}\left[\frac{3}{4}-\frac{4 A \pi}{3}\right]}\right)}{12 \sqrt{Q}}
\end{aligned}
$$

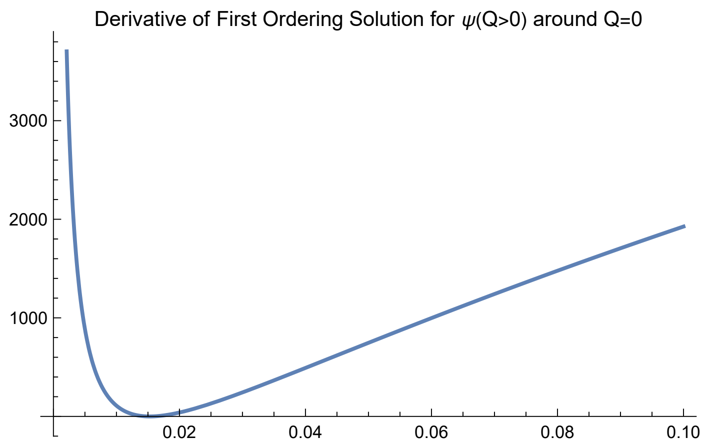
In[∘]:= **Limit[NearZeroPosQ1st[Q, β, A], Q → 0, Direction → -1]**

Out[∘]=

$$
\beta \infty \left(-\frac{12}{\text{Gamma}\left[-\frac{1}{4}-\frac{4 A \pi}{3}\right]}+\frac{3-16 A \pi}{\text{Gamma}\left[\frac{3}{4}-\frac{4 A \pi}{3}\right]}\right)
$$

In[∘]:=
```
(*And in a plot:*)
Plot[Abs[NearZeroPosQ1st[Q, 1, 1]]^2, {Q, 0, 0.1}, FrameLabel → {"Q", "y"},
 PlotLabel → "Derivative of First Ordering Solution for ψ(Q>0) around Q=0"]
```

Out[∘]=



In[∘]:= **(* So, what needs to be done next is to 'glue' the two infinities to each other, by choosing α and β appropriately. This will be done by making the difference of the Taylor series of the derivatives vanish: *)**

In[∘]:= **PowerExpand[Simplify[Solve[NearZeroNegQ1st[Q, α] − NearZeroPosQ1st[Q, β, A] == 0, α]]]**

Out[∘]=

$$\Bigg\{\Bigg\{\alpha \to -\Bigg(\Bigg(2\,(-1)^{1/8}\,\pi^{3/2}\,\sqrt{Q}\,\beta\,\text{Gamma}\left[\frac{1}{4}\right]\times\text{Gamma}\left[\frac{5}{4}\right]\Bigg(24\,\sqrt{2}\,\sqrt{Q}\,(-9+4\,(3+4\,A\,\pi)\,Q)\,\text{Gamma}\left[-\frac{1}{4}-\frac{4\,A\,\pi}{3}\right]\times\text{Gamma}\left[\frac{1}{4}-\frac{4\,A\,\pi}{3}\right]\times\text{Gamma}\left[\frac{3}{4}-\frac{4\,A\,\pi}{3}\right]+\text{Gamma}\left[-\frac{3}{4}-\frac{4\,A\,\pi}{3}\right]\Bigg(-36\,(-3+4\,(3+4\,A\,\pi)\,Q)\,\text{Gamma}\left[\frac{1}{4}-\frac{4\,A\,\pi}{3}\right]\times\text{Gamma}\left[\frac{3}{4}-\frac{4\,A\,\pi}{3}\right]+\text{Gamma}\left[-\frac{1}{4}-\frac{4\,A\,\pi}{3}\right]\Bigg(-3\left(9+36\,Q+256\,A^2\,\pi^2\,Q-48\,A\,\pi\,(1+Q)\right)\text{Gamma}\left[\frac{1}{4}-\frac{4\,A\,\pi}{3}\right]+2\,\sqrt{2}\,\sqrt{Q}\left(256\,A^2\,\pi^2\,Q-48\,A\,\pi\,(3+Q)+27\,(1+4\,Q)\right)\text{Gamma}\left[\frac{3}{4}-\frac{4\,A\,\pi}{3}\right]\Bigg)\Bigg)\Bigg)\Bigg)\Bigg/\Bigg(27\,\Bigg(\text{Gamma}\left[\frac{1}{4}\right]\Bigg(-i\,Q\left(3\,\text{Gamma}\left[-\frac{1}{4}\right]-4\,\text{Gamma}\left[\frac{3}{4}\right]\right)-2\,(-1)^{3/4}\,\sqrt{Q}\left(3\,\text{Gamma}\left[\frac{1}{4}\right]-4\,\text{Gamma}\left[\frac{5}{4}\right]\right)\Bigg)\text{Gamma}\left[\frac{5}{4}\right]+\sqrt{2}\,\pi\,Q\left(3\,\text{Gamma}\left[\frac{1}{4}\right]+4\,\text{Gamma}\left[\frac{5}{4}\right]\right)\Bigg)\text{Gamma}\left[-\frac{3}{4}-\frac{4\,A\,\pi}{3}\right]\times\text{Gamma}\left[-\frac{1}{4}-\frac{4\,A\,\pi}{3}\right]\times\text{Gamma}\left[\frac{1}{4}-\frac{4\,A\,\pi}{3}\right]\times\text{Gamma}\left[\frac{3}{4}-\frac{4\,A\,\pi}{3}\right]\Bigg)\Bigg)\Bigg\}\Bigg\}$$

In[•]:= (*Taking the limit Q→0 will lead to a relationship between α and β:*)

$$\text{Limit}\Big[-\Big(\Big(2(-1)^{1/8}\pi^{3/2}\sqrt{Q}\,\beta\,\text{Gamma}\big[\tfrac{1}{4}\big]\times\text{Gamma}\big[\tfrac{5}{4}\big]\Big(24\sqrt{2}\sqrt{Q}\,(-9+4(3+4A\pi)Q)\,\text{Gamma}\big[-\tfrac{1}{4}-\tfrac{4A\pi}{3}\big]\times\text{Gamma}\big[\tfrac{1}{4}-\tfrac{4A\pi}{3}\big]\times\text{Gamma}\big[\tfrac{3}{4}-\tfrac{4A\pi}{3}\big]+\text{Gamma}\big[-\tfrac{3}{4}-\tfrac{4A\pi}{3}\big]\Big(-36(-3+4(3+4A\pi)Q)\,\text{Gamma}\big[\tfrac{1}{4}-\tfrac{4A\pi}{3}\big]\times\text{Gamma}\big[\tfrac{3}{4}-\tfrac{4A\pi}{3}\big]+\text{Gamma}\big[-\tfrac{1}{4}-\tfrac{4A\pi}{3}\big]\Big(-3\big(9+36Q+256A^2\pi^2Q-48A\pi(1+Q)\big)\,\text{Gamma}\big[\tfrac{1}{4}-\tfrac{4A\pi}{3}\big]+2\sqrt{2}\sqrt{Q}\,\big(256A^2\pi^2Q-48A\pi(3+Q)+27(1+4Q)\big)\,\text{Gamma}\big[\tfrac{3}{4}-\tfrac{4A\pi}{3}\big]\Big)\Big)\Big)\Big)\Big/\Big(27\Big(\text{Gamma}\big[\tfrac{1}{4}\big]\Big(-iQ\Big(3\,\text{Gamma}\big[-\tfrac{1}{4}\big]-4\,\text{Gamma}\big[\tfrac{3}{4}\big]\Big)-2(-1)^{3/4}\sqrt{Q}\Big(3\,\text{Gamma}\big[\tfrac{1}{4}\big]-4\,\text{Gamma}\big[\tfrac{5}{4}\big]\Big)\Big)\text{Gamma}\big[\tfrac{5}{4}\big]+\sqrt{2}\,\pi Q\Big(3\,\text{Gamma}\big[\tfrac{1}{4}\big]+4\,\text{Gamma}\big[\tfrac{5}{4}\big]\Big)\Big)\,\text{Gamma}\big[-\tfrac{3}{4}-\tfrac{4A\pi}{3}\big]\times\text{Gamma}\big[-\tfrac{1}{4}-\tfrac{4A\pi}{3}\big]\times\text{Gamma}\big[\tfrac{1}{4}-\tfrac{4A\pi}{3}\big]\times\text{Gamma}\big[\tfrac{3}{4}-\tfrac{4A\pi}{3}\big]\Big)\Big),\ Q\to 0\Big]$$

Out[•]=

$$-\frac{\left(\frac{1}{3}+\frac{i}{3}\right)(-1)^{1/8}\,2^{2+\frac{8A\pi}{3}}\,\pi^2\,(9+16A\pi)\,\beta\,\text{Gamma}\big[\tfrac{5}{4}\big]\times\text{Gamma}\big[-\tfrac{3}{2}-\tfrac{8A\pi}{3}\big]}{\text{Gamma}\big[\tfrac{1}{4}\big]^2\,\text{Gamma}\big[-\tfrac{1}{4}-\tfrac{4A\pi}{3}\big]\times\text{Gamma}\big[\tfrac{1}{4}-\tfrac{4A\pi}{3}\big]\times\text{Gamma}\big[\tfrac{3}{4}-\tfrac{4A\pi}{3}\big]}\quad \text{if condition}$$

$$-\Big(\Big(\Big(\tfrac{1}{3}+\tfrac{i}{3}\Big)(-1)^{1/8}\,2^{2+\frac{8A\pi}{3}}\,\pi^2\,(9+16A\pi)\,\beta\,\text{Gamma}\big[\tfrac{5}{4}\big]\times\text{Gamma}\big[-\tfrac{3}{2}-\tfrac{8A\pi}{3}\big]\Big)\Big/\Big(\text{Gamma}\big[\tfrac{1}{4}\big]^2\,\text{Gamma}\big[-\tfrac{1}{4}-\tfrac{4A\pi}{3}\big]\times\text{Gamma}\big[\tfrac{1}{4}-\tfrac{4A\pi}{3}\big]\times\text{Gamma}\big[\tfrac{3}{4}-\tfrac{4A\pi}{3}\big]\Big)\Big)$$

if condition
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In[•]:= (*In numerical values:*)

$$\text{Simplify}\Big[\text{Abs}\Big[N\Big[-\Big(\Big(\Big(\tfrac{1}{3}+\tfrac{i}{3}\Big)(-1)^{1/8}\,2^{2+\frac{8A\pi}{3}}\,\pi^2\,(9+16A\pi)\,\beta\,\text{Gamma}\big[\tfrac{5}{4}\big]\times\text{Gamma}\big[-\tfrac{3}{2}-\tfrac{8A\pi}{3}\big]\Big)\Big/\Big(\text{Gamma}\big[\tfrac{1}{4}\big]^2\,\text{Gamma}\big[-\tfrac{1}{4}-\tfrac{4A\pi}{3}\big]\times\text{Gamma}\big[\tfrac{1}{4}-\tfrac{4A\pi}{3}\big]\times\text{Gamma}\big[\tfrac{3}{4}-\tfrac{4A\pi}{3}\big]\Big)\Big)\ /.\ A\to 1\Big]\Big]\Big]$$

Out[•]=

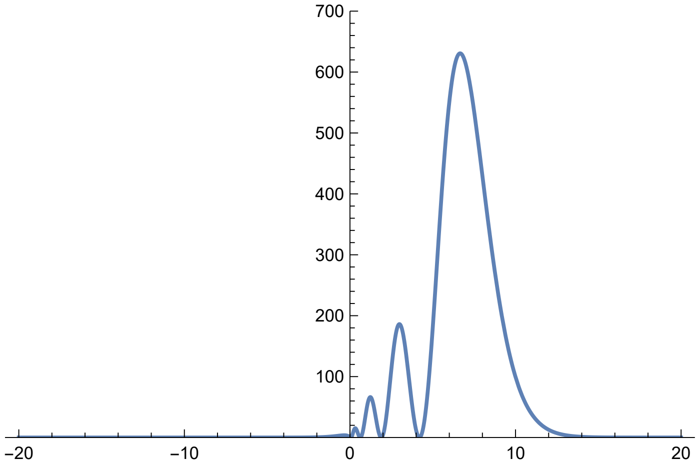
5.1291 Abs[β]

In[•]:=

(* Collecting all together, leads to a WdW solution for all Q,
that goes to zero at both infinities, is normalisable on both sides,
continuous at Q = 0, and whose derivative is continous at Q = 0 as well. *)

In[ ]:=
```
WdWsolutionAllQ1st[Q_, Aa_, β_] :=
 SolutionNegQ1st[Q, 1,
    -((((1/3 + i/3) (-1)^(1/8) 2^(2+8Aπ/3) π^2 (9 + 16 A π) β Gamma[5/4] × Gamma[-3/2 - 8Aπ/3]) /
       (Gamma[1/4]^2 Gamma[-1/4 - 4Aπ/3] × Gamma[1/4 - 4Aπ/3] × Gamma[3/4 - 4Aπ/3]))),
    I (-((((1/3 + i/3) (-1)^(1/8) 2^(2+8Aπ/3) π^2 (9 + 16 A π) β Gamma[5/4] × Gamma[-3/2 - 8Aπ/3]) /
        (Gamma[1/4]^2 Gamma[-1/4 - 4Aπ/3] × Gamma[1/4 - 4Aπ/3] × Gamma[3/4 - 4Aπ/3]))))] ×
    UnitStep[-Q] + SolutionPosQ1st[Q, 1, 0, β] × UnitStep[Q] /. A → Aa
(*Plotting this function, setting A=1 for plotting purposes:*)
Plot[Abs[WdWsolutionAllQ1st[Q, 1, 1]]^2, {Q, -20, 20}, PlotRange → {0, 700}]
```

Out[ ]=



In[ ]:=
```
(* Now to fix the remaining overall constant β by forcing
 the absolute square of the solution to be normalised to unity. *)
```

In[ ]:=
```
(* Of course, this depends on the value of A. For now, it will be set to unity. *)
```

In[ ]:=
```
(* Here is an example of what the integration will result in: *)
```

In[ ]:=
```
NIntegrate[Abs[WdWsolutionAllQ1st[Q, 1, 0.02035]]^2, {Q, -Infinity, Infinity}]
```

Out[ ]=
```
1.00002
```

In[ ]:=
```
(* Now follows a module that will calculate the constant β
 required to normalise the solution. One gets to put in the value of A,
the integration bounds (technically these are minus and plus infinity,
  but as explained above the integrals will converge very quickly to a finite value,
  so there is no need to have the integration range -∞,+∞),
and the desired accuracy of the numerical outcome for β. The higher DesiredAccuracy,
the more numerically accurate β will be returned. The
 output is a list of two elements, the first is the value of β,
the seond is the corresponing value of the numerical integration.*)
```

In[]:=
```
FindβforNormalisation1st[LowerIntegrationBound_,
  UpperIntegrationBound_, A_, DesiredAccuracy_] := Delete[{β = 10^-8;
   While[NIntegrate[Abs[WdWsolutionAllQ1st[Q, A, β / DesiredAccuracy]]^2,
      {Q, LowerIntegrationBound, UpperIntegrationBound}] < 1, β++];
   ToExpression[N[(β - 1) / DesiredAccuracy]], ToExpression[
    N[NIntegrate[Abs[WdWsolutionAllQ1st[Q, A, (β - 1) / DesiredAccuracy]]^2,
      {Q, LowerIntegrationBound, UpperIntegrationBound}]]], Clear[β];}, 3]
```

In[]:=
```
(* As an example, here is the outcome for β when A is 1 and the integration
 bounds increased. As expected and explained before, one needs only
 limited integration range to already end up with the converging limit. *)
```

In[]:=
```
FindβforNormalisation1st[-10, 10, 1, 1000]
```

Out[]=
```
{0.02, 0.926654}
```

In[]:=
```
FindβforNormalisation1st[-20, 20, 1, 1000]
```

Out[]=
```
{0.02, 0.96592}
```

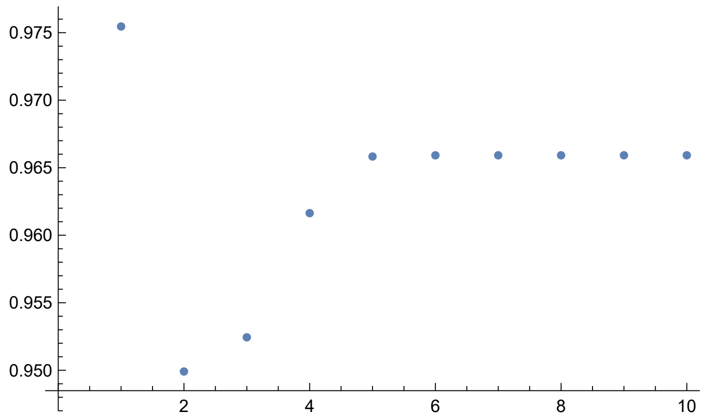
In[]:=
```
FindβforNormalisation1st[-30, 30, 1, 1000]
```

Out[]=
```
{0.02, 0.96592}
```

In[]:=
```
(* Or in a plot, to see the convergence of the numerical
 integration for increasing values of integration bounds: *)
```

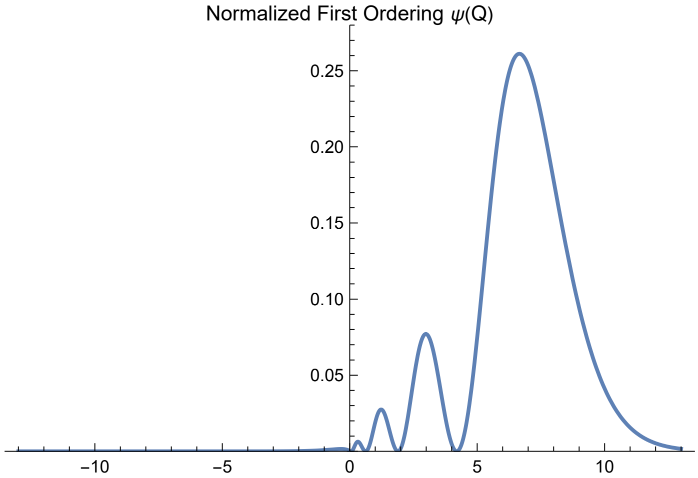
In[]:=
```
ListPlot[Table[Part[FindβforNormalisation1st[-3 k, 3 k, 1, 1000], 2], {k, 1, 10}]]
```

Out[]=



In[∘]:=
```
(* So, for this particular example, the value of β should be 0.02035. Then,
using this value, the final plot can be made: *)
Plot[Abs[WdWsolutionAllQ1st[Q, 1, 0.02035]]^2, {Q, -13, 13}, PlotRange → {0, 0.28},
 FrameLabel → {"Q", "y"}, PlotLabel → "Normalized First Ordering ψ(Q)"]
```

Out[∘]=



In[∘]:=
```
(*This can also be done for different A values*)
```

In[∘]:=
```
NIntegrate[Abs[(0.02035) * (5.1291) * SolutionNegQ1st[Q, 1, 1, I]]^2, {Q, -Infinity, 0}]
```

Out[∘]=

```
0.00171004
```

In[∘]:=
```
NIntegrate[Abs[(0.02035) * SolutionPosQ1st[Q, 1, 0, 1]]^2 /. {A → 1}, {Q, 0, Infinity}]
```

Out[∘]=

```
0.998313
```

In[∘]:=
```
(*For A>41*)
(*Gluing coefficient is:*)
```

$$\text{Simplify}\left[\text{Abs}\left[\text{N}\left[-\left(\left(\left(\frac{1}{3}+\frac{i}{3}\right)(-1)^{1/8}\,2^{2+\frac{8A\pi}{3}}\,\pi^2\,(9+16A\pi)\,\beta\,\text{Gamma}\left[\frac{5}{4}\right]\times\text{Gamma}\left[-\frac{3}{2}-\frac{8A\pi}{3}\right]\right)\Big/\left(\text{Gamma}\left[\frac{1}{4}\right]^2\text{Gamma}\left[-\frac{1}{4}-\frac{4A\pi}{3}\right]\times\text{Gamma}\left[\frac{1}{4}-\frac{4A\pi}{3}\right]\times\text{Gamma}\left[\frac{3}{4}-\frac{4A\pi}{3}\right]\right)\right)\ /.\ A\to 42\right]\right]\right]$$

Power: Infinite expression $\frac{1}{0.}$ encountered.

Power: Infinite expression $\frac{1}{0.^{1.}}$ encountered.

Power: Infinite expression $\frac{1}{0.}$ encountered.

General: Further output of Power::infy will be suppressed during this calculation.

Infinity: Indeterminate expression (0. + 0. i) β ComplexInfinity ComplexInfinity ComplexInfinity encountered.

Out[∘]=

```
Indeterminate
```

```
(*This means that for A>41 there cannot be probabilities*)
```

In[•]:=
```
(*For A≤41&≥21:*)
(*Gluing coefficient is:*)
```
$$\text{Simplify}\left[\text{Abs}\left[\text{N}\left[-\left(\left(\left(\frac{1}{3}+\frac{i}{3}\right)(-1)^{1/8}\,2^{2+\frac{8A\pi}{3}}\,\pi^2\,(9+16A\pi)\,\beta\,\text{Gamma}\left[\frac{5}{4}\right]\times\text{Gamma}\left[-\frac{3}{2}-\frac{8A\pi}{3}\right]\right)\Big/\left(\text{Gamma}\left[\frac{1}{4}\right]^2\text{Gamma}\left[-\frac{1}{4}-\frac{4A\pi}{3}\right]\times\text{Gamma}\left[\frac{1}{4}-\frac{4A\pi}{3}\right]\times\text{Gamma}\left[\frac{3}{4}-\frac{4A\pi}{3}\right]\right)\right)\ /.\ A\to 21\right]\right]\right]$$

```
0.`
(*This means that for this range of values the probability for a
 Euclidean universe is 0%, and for a Minkowskian the probability 10 100%*)
```

In[•]:=
```
(*For A=10:*)
(*Gluing coefficient is:*)
```
$$\text{Simplify}\left[\text{Abs}\left[\text{N}\left[-\left(\left(\left(\frac{1}{3}+\frac{i}{3}\right)(-1)^{1/8}\,2^{2+\frac{8A\pi}{3}}\,\pi^2\,(9+16A\pi)\,\beta\,\text{Gamma}\left[\frac{5}{4}\right]\times\text{Gamma}\left[-\frac{3}{2}-\frac{8A\pi}{3}\right]\right)\Big/\left(\text{Gamma}\left[\frac{1}{4}\right]^2\text{Gamma}\left[-\frac{1}{4}-\frac{4A\pi}{3}\right]\times\text{Gamma}\left[\frac{1}{4}-\frac{4A\pi}{3}\right]\times\text{Gamma}\left[\frac{3}{4}-\frac{4A\pi}{3}\right]\right)\right)\ /.\ A\to 10\right]\right]\right]$$

Out[•]=

$1.1479\times 10^{49}\ \text{Abs}[\beta]$

```
(*The normalization constant is:*)
```

In[•]:=
```
NIntegrate[Abs[WdWsolutionAllQ1st[Q, 10, 0.00000038957*^-45]]^2,
 {Q, -Infinity, Infinity}]
```

••• NIntegrate: NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in Q near {Q} = {56.9136}. NIntegrate obtained 1.0000345513774413` and 0.000016391934679637 63` for the integral and error estimates.

Out[•]=
```
1.00003
```

In[•]:=
```
(*The probability for a Euclidean universe is:*)
NIntegrate[
 Abs[(0.00000038957*^-45) * (1.1479021810202812`*^49) * SolutionNegQ1st[Q, 1, 1, I]]^2,
 {Q, -Infinity, 0}]
```

Out[•]=

3.1389×10^{-6}

In[•]:=
```
(*The probability for a Minkowskian universe is:*)
NIntegrate[Abs[(0.00000038957*^-45) * SolutionPosQ1st[Q, 1, 0, 1]]^2 /. {A → 10},
 {Q, 0, Infinity}]
```

••• NIntegrate: NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in Q near {Q} = {56.9136}. NIntegrate obtained 1.0000314126910084` and 0.000016370371402695823` for the integral and error estimates.

Out[•]=
```
1.00003
```

In[∘]:=
```
(*The probability for a Euclidean universe is
 0.000314% and for a Minkowskian universe it is 99.999686%*)

(*For A=5:*)
(*Gluing coefficient is:*)
Simplify[Abs[N[-((((1/3 + I/3) (-1)^(1/8) 2^(2+(8 A π)/3) π^2 (9 + 16 A π) β Gamma[5/4] × Gamma[-3/2 - (8 A π)/3])/
   (Gamma[1/4]^2 Gamma[-1/4 - (4 A π)/3] × Gamma[1/4 - (4 A π)/3] × Gamma[3/4 - (4 A π)/3]))) /. A → 5]]]
```
Out[∘]=
```
1.22396 × 10^18 Abs[β]
```
In[∘]:=
```
(*The normalization constant is:*)
NIntegrate[Abs[WdWsolutionAllQ1st[Q, 5, 0.00996113*^-18]]^2,
 {Q, -Infinity, Infinity}]
```
Out[∘]=
```
1.
```
In[∘]:=
```
(*The probability for a Euclidean universe is:*)
NIntegrate[
 Abs[(0.00996113*^-18) * (1.2239628302614828`*^18) * SolutionNegQ1st[Q, 1, 1, I]]^2,
 {Q, -Infinity, 0}]
```
Out[∘]=
```
0.0000233319
```
In[∘]:=
```
(*The probability for a Minkowskian universe is:*)
NIntegrate[
 Abs[(0.00996113*^-18) * SolutionPosQ1st[Q, 1, 0, 1]]^2 /. {A → 5}, {Q, 0, Infinity}]
```
Out[∘]=
```
0.99998
```
In[∘]:=
```
(*The probability for a Euclidean universe is
 0.002% and for a Minkowskian universe it is 99.998%*)

(*For A=0.5:*)
(*Gluing coefficient is:*)
Simplify[Abs[N[-((((1/3 + I/3) (-1)^(1/8) 2^(2+(8 A π)/3) π^2 (9 + 16 A π) β Gamma[5/4] × Gamma[-3/2 - (8 A π)/3])/
   (Gamma[1/4]^2 Gamma[-1/4 - (4 A π)/3] × Gamma[1/4 - (4 A π)/3] × Gamma[3/4 - (4 A π)/3]))) /. A → 0.5]]]
```
Out[∘]=
```
0.517435 Abs[β]
```
In[∘]:=
```
(*The normalization constant is:*)
NIntegrate[Abs[WdWsolutionAllQ1st[Q, 0.5, 0.361232]]^2, {Q, -Infinity, Infinity}]
```
Out[∘]=
```
1.
```

In[∘]:=
```
(*The probability for a Euclidean universe is:*)
NIntegrate[Abs[(0.361232) * (0.5174353615186497) * SolutionNegQ1st[Q, 1, 1, I]]^2,
 {Q, -Infinity, 0}]
```
Out[∘]=
```
0.00548379
```
In[∘]:=
```
(*The probability for a Minkowskian universe is:*)
NIntegrate[
 Abs[(0.361232) * SolutionPosQ1st[Q, 1, 0, 1]]^2 /. {A → 0.5}, {Q, 0, Infinity}]
```
Out[∘]=
```
0.994519
```
In[∘]:=
```
(*The probability for a Euclidean universe is
 0.548% and for a Minkowskian universe it is 99.452%*)


(*For A=0.01:*)
(*Gluing coefficient is:*)
```

$$\text{Simplify}\Big[\text{Abs}\Big[\text{N}\Big[-\Big(\Big(\Big(\frac{1}{3}+\frac{i}{3}\Big)(-1)^{1/8}\,2^{2+\frac{8A\pi}{3}}\,\pi^2\,(9+16A\pi)\,\beta\,\text{Gamma}\Big[\frac{5}{4}\Big]\times\text{Gamma}\Big[-\frac{3}{2}-\frac{8A\pi}{3}\Big]\Big)\Big/\Big(\text{Gamma}\Big[\frac{1}{4}\Big]^2\,\text{Gamma}\Big[-\frac{1}{4}-\frac{4A\pi}{3}\Big]\times\text{Gamma}\Big[\frac{1}{4}-\frac{4A\pi}{3}\Big]\times\text{Gamma}\Big[\frac{3}{4}-\frac{4A\pi}{3}\Big]\Big)\Big)\;/.\;A\to 0.01\Big]\Big]\Big]$$

Out[∘]=
```
1.19484 Abs[β]
```
In[∘]:=
```
(*The normalization constant is:*)
NIntegrate[Abs[WdWsolutionAllQ1st[Q, 0.01, 1.4339]]^2, {Q, -Infinity, Infinity}]
```
Out[∘]=
```
1.
```
In[∘]:=
```
(*The probability for a Euclidean universe is:*)
NIntegrate[Abs[(1.4339) * (1.1948364655512516) * SolutionNegQ1st[Q, 1, 1, I]]^2,
 {Q, -Infinity, 0}]
```
Out[∘]=
```
0.460735
```
In[∘]:=
```
(*The probability for a Minkowskian universe is:*)
NIntegrate[
 Abs[(1.4339) * SolutionPosQ1st[Q, 1, 0, 1]]^2 /. {A → 0.01}, {Q, 0, Infinity}]
```
Out[∘]=
```
0.539267
```

In[ ]:= (*The probability for a Euclidean universe is 46% and for a Minkowskian universe it is 54%*)

(*For A≤0.000001:*)
(*Gluing coefficient is:*)

$$\text{Simplify}\Big[\text{Abs}\Big[\text{N}\Big[-\Big(\Big(\Big(\frac{1}{3}+\frac{i}{3}\Big)(-1)^{1/8}\,2^{2+\frac{8A\pi}{3}}\,\pi^2\,(9+16A\pi)\,\beta\,\text{Gamma}\Big[\frac{5}{4}\Big]\times\text{Gamma}\Big[-\frac{3}{2}-\frac{8A\pi}{3}\Big]\Big)\Big/\Big(\text{Gamma}\Big[\frac{1}{4}\Big]^2\,\text{Gamma}\Big[-\frac{1}{4}-\frac{4A\pi}{3}\Big]\times\text{Gamma}\Big[\frac{1}{4}-\frac{4A\pi}{3}\Big]\times\text{Gamma}\Big[\frac{3}{4}-\frac{4A\pi}{3}\Big]\Big)\Big)\;/.\;A\to 0.000001\Big]\Big]\Big]$$

Out[ ]= 1.25331 Abs[β]

In[ ]:= (*The normalization constant is:*)

```
NIntegrate[Abs[WdWsolutionAllQ1st[Q, 0.000001, 1.424058]]^2,
 {Q, -Infinity, Infinity}]
```

Out[ ]= 1.

In[ ]:= (*The probability for a Euclidean universe is:*)

```
NIntegrate[Abs[(1.424058) * (1.2533084366720788) * SolutionNegQ1st[Q, 1, 1, I]]^2,
 {Q, -Infinity, 0}]
```

Out[ ]= 0.499997

In[ ]:= (*The probability for a Minkowskian universe is:*)

```
NIntegrate[
 Abs[(1.424058) * SolutionPosQ1st[Q, 1, 0, 1]]^2 /. {A → 0.000001}, {Q, 0, Infinity}]
```

Out[ ]= 0.500005

(*For values of A≤0.000001,
the probabilities for both Euclidean and Minkowskian universes are 50%*)