Traffic Flow on a Roundabout

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18.300 Principles of Continuum Dynamics

1 Introduction

Despite efforts to minimize carbon emissions around the world, the number of cars on the road continues to increase as more and more roads are built. Thus, it's imperative that, when we build new roads and intersections, we design them with the intention of maximizing the flow rate so that cars spend less time emitting fumes and get people to their destination faster.

In this paper, we'll consider the case of the roundabout (traffic circle), which has been shown experimentally to decrease the number of accidents and increase the flow of cars through a given intersection compared to the traditional lighted intersection.

To begin the investigation, we'll assume that density of cars is defined at all points in space and time and that they satisfy the continuity equation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0 \tag{1}$$

Where ρ is the density of cars at a point in space and time and J is the rate at which cars pass through at a given point in space and time. Because J represents the flux of cars through a point, it can be written as $J = u\rho$ where u is the velocity of the cars. We can then simply this equation into 1 equation of a single variable by substituting a constitutive law that relates the density of vehicles to their velocity. Although their are many different laws derived from experimental observations, in this paper we will use the Lighthill-Whitham-Richards model which states that

$$u(\rho) = u_{max}(1 - \frac{\rho}{\rho_{max}}) \tag{2}$$

where we define u_{max} and ρ_{max} on a given roadway. This makes sense intuitively, as the more cars in a given area then the slower they'll need to go to avoid an accident. Substituting this in for J, using the chain rule, and introducing the following dimensionless variables,

- $\hat{x} = \frac{x}{l}$
- $\hat{t} = \frac{u_{max}}{l} t$
- $\hat{\rho} = \frac{\rho}{\rho_{max}}$

where l is a length scale that need not be defined, we get the dimensionless equation

$$\frac{\partial \hat{\rho}}{\partial t} + (1 - 2\hat{\rho})\frac{\partial \hat{\rho}}{\partial x} = 0 \tag{3}$$

For the purpose of simplicity we'll drop the hats in further calculations.

2 Roundabout

To begin our investigation, we first consider an initially even distribution of vehicles entering the roundabout given by the function,

$$\rho(x, 0) = \rho_0(x) = \begin{cases} \rho_0, & x < 0 \\ 0, & x > 0 \end{cases} \tag{4}$$

Using the method of characteristics, we can represent the evolution of the traffic by following how lines of information called characteristics. We define the following paramaterization:

- $t(s, 0) = 0$
- $x(s, 0) = s$
- $\rho(s, 0) = \rho_0(s)$

By the parameterization we get that

- $t(s, \tau) = \tau$
- $x(s, \tau) = (1 - 2\rho_0(s))\tau + s$
- $\rho(s, \tau) = \rho$

Using our definition of $\rho_0(s)$, x can written in the form

$$x(s, \tau) = \begin{cases} (1 - 2\rho_0)\tau + s, & s < 0 \\ \tau + s, & s > 0 \end{cases} \tag{5}$$

$$s(x, t) = \begin{cases} x - (1 - 2\rho_0)t, & s < 0 \\ x - t, & s > 0 \end{cases} \tag{6}$$

$$\rho(x, t) = \begin{cases} \rho_0, & x < (1 - 2\rho_0)t \\ 0, & x > t \end{cases} \tag{7}$$

However, we note when we write $\rho(x, t)$, the boundaries of the piecewise function won't always line up. This indicates we need an expansion fan to define ρ at all points in time in space.

We write $\rho = F(\mu)$ where F is some function of $\mu = \frac{x}{t}$ for $1 - 2\rho_0 < \mu < 1$. Substituting this into equation (3) we get

$$-\frac{\mathrm{d}F}{\mathrm{d}\mu}\mu + (1 - 2F)\frac{\mathrm{d}F}{\mathrm{d}\mu} = 0 \tag{8}$$

Because we assume a non-constant solution, we can cancel out $\frac{\mathrm{d}F}{\mathrm{d}\mu}$ and solve for $F$to get,

$$F = \frac{1}{2}(1 - \mu) \tag{9}$$

$$\rho = \frac{1}{2}(1 - \frac{x}{t}) \tag{10}$$

Once we include the similarity solution, equation (7) becomes,

$$\rho(x,t) = \begin{cases} \rho_0, & x < (1 - 2\rho_0)t \\ \frac{1}{2}(1 - \frac{x}{t}), & (1 - 2\rho_0)t < x < t \\ 0, & x > t \end{cases} \tag{11}$$

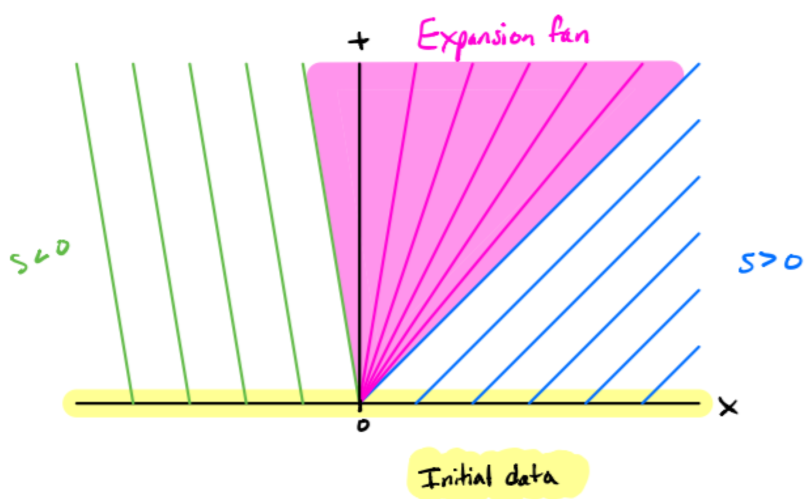

Figure 1: The characteristics lines for the system with $\rho_0 > 0.5$. The highlighted line represents the initial Cauchy data and the origin of all characteristics

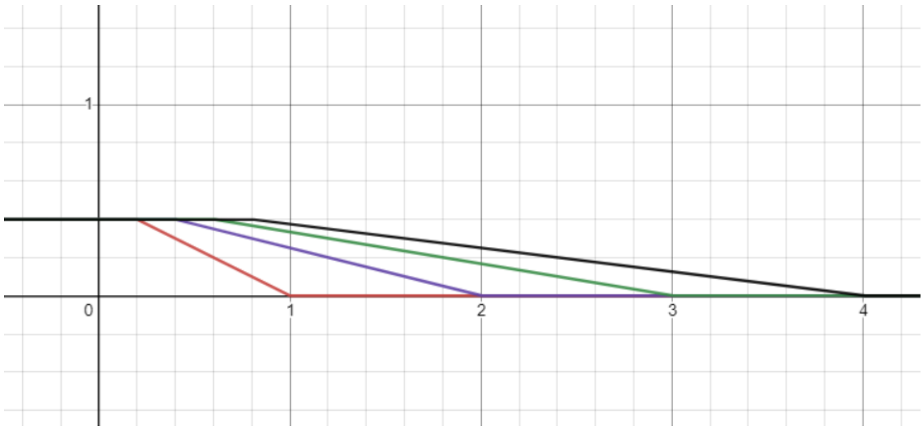

Figure 2: Graph of the evolution of ρ from time $1 < t < 4$

3 Shock propagation

As can be seen from the characteristic diagram, the solution, or the boundaries of the piecewise function directly, the front of the wave travels with velocity 1, thus traveling dimensionless distance t in time t. However, as the cars travel around the roundabout, because the domain is periodic, the wavefront eventually reaches the point at which the cars are entering the roundabout. This causes a shock in the flow of information.

There are 2 key cases to examine in the consideration of propagation of shocks, but for the sake of brevity, we'll only look at one case, where $\rho_0 < 0.5$. We define the length of the roundabout as L.

3.1 $\rho_0 < 0.5$

The first is the case where $\rho_0 < 0.5$. In this case, by the time the wavefront reaches the entrance, it collides with the portion of cars with density $\rho = \rho_0$ which moves forward in time. To see how the shock propagates, first we write equation (3) in conservation form,

$$\frac{\partial P}{\partial T} + \frac{\partial Q}{\partial x} = 0 \tag{12}$$

$$P = \rho \tag{13}$$

$$Q = \rho - \rho^2 \tag{14}$$

Then we apply the Rankine-Hugoniot condition to the characteristics, which states that the shock $x_s(t)$ propagates according to the equation

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \frac{[Q]_-^+}{[P]_-^+} \tag{15}$$

where $[Q]^+_-$ and $[P]^+_-$ are the differences of Q and P evaluated at u_+ and u_-. In the case of $\rho_0 < 0$, because the periodic domain, the characteristics entering the right collide with the characteristics on the left, so u_+ and u_- are defined as,

$$u_+ = \rho_0 \tag{16}$$

$$u_- = \frac{1}{2}(1 - \frac{x}{t}) \tag{17}$$

Simplifying equation (15) using Q and P we get

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = 1 - (u_+ + u_-) \tag{18}$$

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \frac{1}{2}(\frac{x}{t} - 1) - \rho_0 \tag{19}$$

Solving this differential equation using an integrating factor we get

$$x_s(t) = -(2\rho_0 + 1)t + C\sqrt{t} \tag{20}$$

Knowing that the shock begins at $x_s = t = L$, we can solve for the constant of integration to get the final equation

$$x_s(t) = -(2\rho_0 + 1)t + 2(\rho_0 + 1)\sqrt{Lt} \tag{21}$$

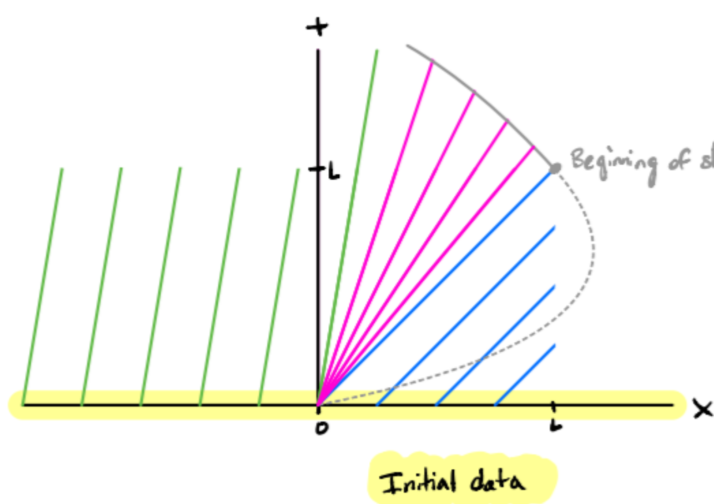

Figure 3: The characteristics of the system, where the solid gray line is the shock at which information does not pass through. Note how characteristics stop at $x < L$ to show one period of the domain.

As shown in the diagram, the shock propagates backwards along the roundabout for $\rho_0 < 0.5$

4 Simulation

To continue our investigation, we we'll now consider the case when, at the time the wavefront loops around the roundabout to reach the entrance again, the flow of cars into the roundabout stops, and the cars are free to evolve in a system where the number of cars is conserved throughout the domain.

4.1 Naive Approach

To do this, we'll write a discrete numerical approximation to equation (3) then write code using the numerical approximation to simulate the evolution of the closed roundabout system. We attempt this by creating a 2D mesh where the value of the system at each point is denoted u_j^n, $n < \frac{T}{\Delta t}$, $0 < j < N$, N is the number of spatial points. Equation (3) is rewritten with an $O(\Delta x^2, \Delta t)$ forward approximation of the time derivative and a central approximation of the spatial derivative.

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + (1 - 2u_j^n)\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0 \tag{22}$$

However, as seen in the figure 4, our numerical approximation introduces instability in our solution, This holds regardless of how we vary the mesh and parameters. To assess the stability of the method, we assume a period initial condition such that $u_j^n = U_n e^{ij\alpha k}$ where $\alpha = \frac{2\pi}{N}$ and k =1, 2, ..., N. When we substitute this into equation (22) and rearrange we get, for $a = (2u_j^n - 1)$,

$$U_{n+1}e^{ij\alpha k} = \frac{a\Delta t}{2\Delta x}(U_n e^{ij\alpha k}e^{i\alpha k} - U_n e^{ij\alpha k}e^{-i\alpha k}) + U_n e^{ij\alpha k} \tag{23}$$

$$U_{n+1} = \mu_k U_n \tag{24}$$

$$\mu_k = \frac{a\Delta t}{\Delta x} i \sin{(\alpha k)} + 1 \tag{25}$$

For the numerical method to be stable, $|\mu_k|^2 < 1$

$$|\mu_k|^2 = \frac{a^2 \Delta t^2}{\Delta x^2} \sin{(\alpha k)}^2 + 1 > 1 \tag{26}$$

Therefore, this approximation is unconditionally unstable.

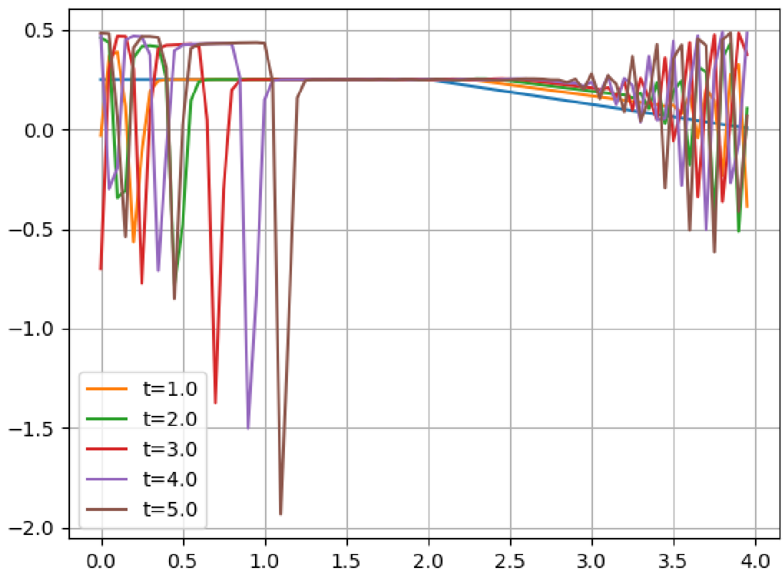

Figure 4: The result of using the naive approximation to evolve the system. Although it roughly includes propagation of the wave, it also introduces very large instability which increase as time goes on.

4.2 Nonlinear Lax-Wendroff

We can improve upon this approximation by using the nonlinear Lax-Wendroff method, which first calculates values at half-values in x and t. Because the method is intended for equations written in conservation form, $f = Q = \rho - \rho^2$.

$$u_{j+1/2}^{n+1/2} = \frac{1}{2}(u_{j+1}^n + u_j^n) - \frac{\Delta t}{2\Delta x}(f(u_{j+1}) - f(u_j^n)) \tag{27}$$

$$u_{j-1/2}^{n+1/2} = \frac{1}{2}(u_j^n + u_{j-1}^n) - \frac{\Delta t}{2\Delta x}(f(u_j) - f(u_{j-1}^n)) \tag{28}$$

Then, we compute the whole step,

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(f(u_{j+1/2}^{n+1/2}) - f(u_{j-1/2}^{n+1/2})) \tag{29}$$

In figures 4 and 5 we can see how the closed system evolves in time. We see that for the case where $\rho_0 < 0.5$, the density of cars throughout the roundabout slowly converges to ρ_0. For the case where ρ_0, however, the density converges to $\frac{\rho_0}{2}$.

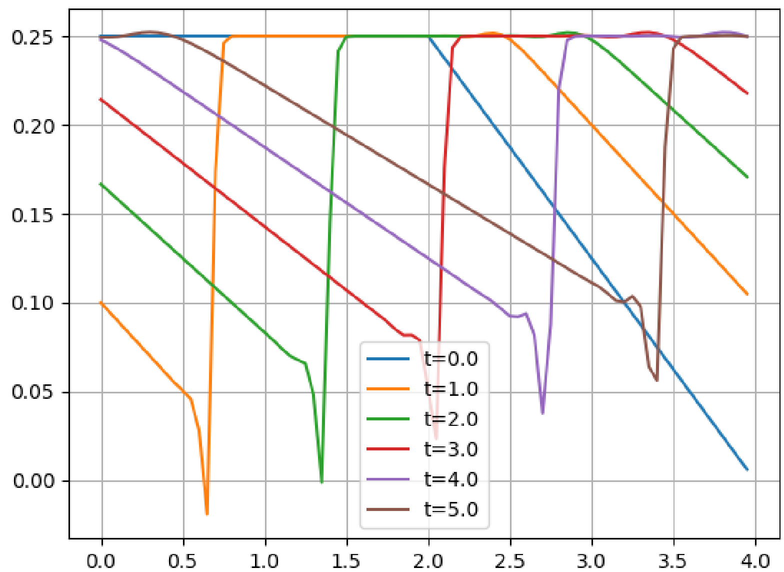

Figure 5: The simulation of $\rho(x,t)$ for $\rho_0 < 0.5$

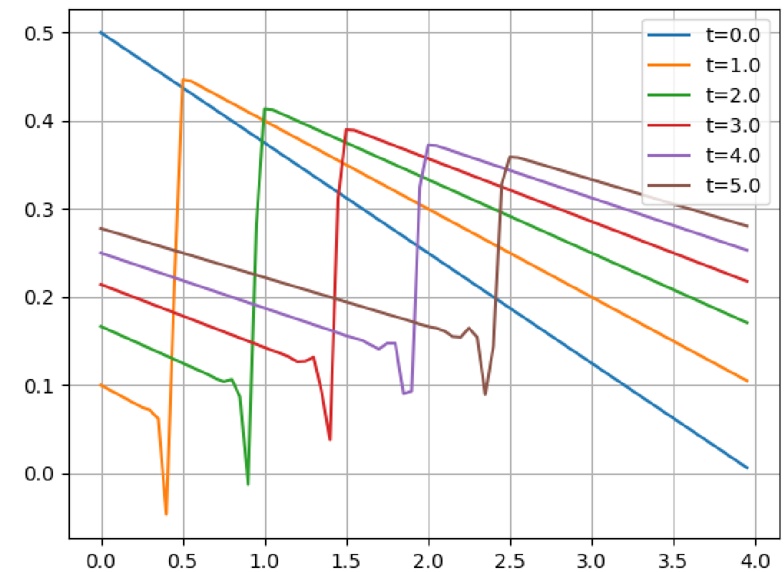

Figure 6: The simulation of $\rho(x,t)$ for $\rho_0 > 0.5$

5 Summary

In this paper we investigated traffic flow on a roundabout by devising an evolution equation for the density and an initial distribution of cars on the road. We then solved the system using the method of characteristics and identified the shock that formed when $\rho_0 < 0.5$.

We then sought to analyze the system once the influx of cars into the roundabout had stopped by using numerical simulation. We began with a naive approach which was then proven to be unconditionally unstable and redid the simulation using the nonlinear Lax-Wendroff method in cases where $\rho_0 < 0.5$ and $\rho_0 > 0.5$.

In potential future papers, how the system evolves when a sink term is added to equation (3) to represent cars entering and exiting the roundabout should be studied in more detail.

6 Works Cited

1. "Lax-Wendroff method." Wikipedia, Wikimedia Foundation, 20 May. 2021.