The Heat Equation and The Maximum Principle

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Outline

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2. Motivate the equation itself
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4. Define the Maximum Principle
5. Mention generalizations of the principle

The Heat Equation

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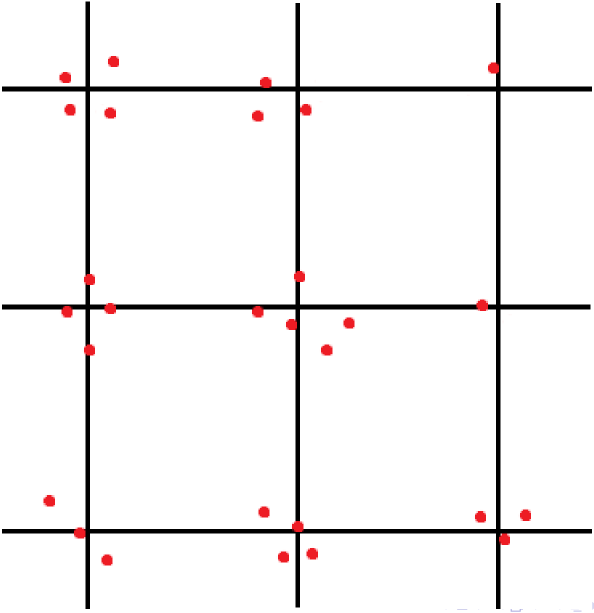
- $u(x, t)$ is a scalar field
- $x \in \mathbb{R}^n$

Motivating the Equation



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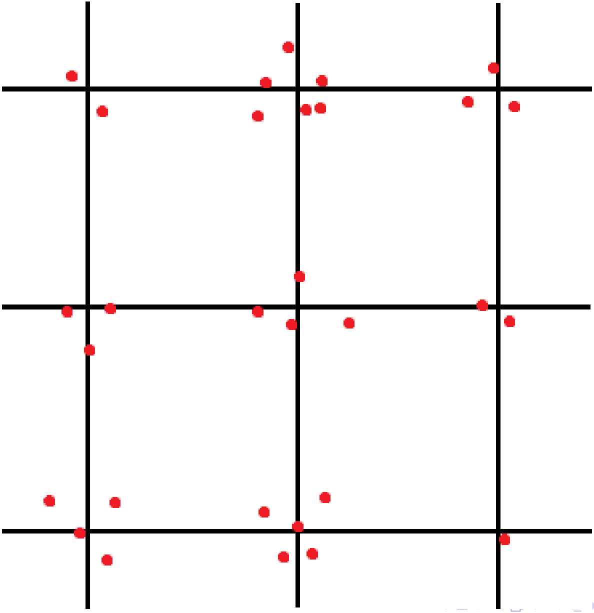

Motivating the Equation

$$p_{k+1}(x) = \frac{1}{2n} \sum_{y \sim x} p_k(y) \tag{2}$$

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$$\partial_t p_k(x) = \Delta p_k(x) \tag{4}$$

Motivating the Equation

$$\partial_t u = \Delta u \tag{5}$$

$$(\partial_t - \Delta) u = 0 \tag{6}$$

Constant Solution

Definition

Harmonic functions are functions that satisfy $\Delta u = 0$

Fundamental Solution

$$h(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \tag{7}$$

Heat Kernel

$$H(x, y, t) = h(x - y, t) \tag{8}$$

Heat Kernel

$$\lim_{t \to 0^+} \int_{\mathbb{R}^n} h(x, t) u(x) dx = u(0, 0) \tag{9}$$

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Heat Kernel

$$u(x,t) \equiv \int_{\mathbb{R}^n} H(x,y,t)v(y)dy \tag{11}$$

$$\lim_{t\to 0^+} u(x,t) = v(x) \tag{12}$$

The Maximum Principle

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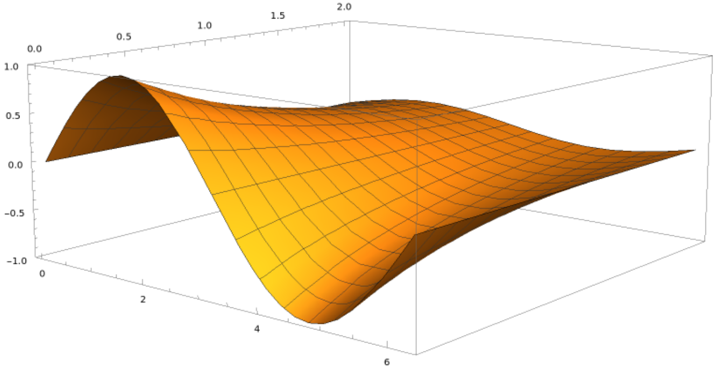
- The temperature at a, b is held at 0 for all time
- What can we say about the maximum and minimum temperature on the rod over a period of time $[0, T]$?

The Maximum Principle- Example

- $e^{-t} sinx : [0, 2\pi] \times [0, T] \to \mathbb{R}$

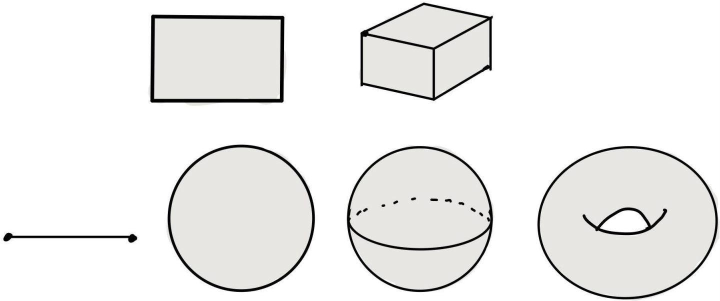


The Maximum Principle

- https://mathlets.org/mathlets/heat-equation/

The Maximum Principle

- Consider solutions $u : \Omega \times [0, T] \to \mathbb{R}$ where Ω is a compact domain.

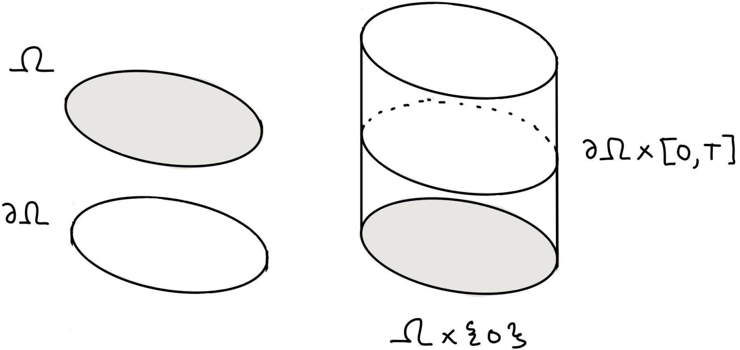


The Maximum Principle

Theorem

Let $u : \Omega \times [0, T] \to \mathbb{R}$ be a solution of the Heat Equation. The max. and min. of u are achieved on

$$(\Omega \times \{0\}) \cup (\partial\Omega \times [0, T])$$



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- Heuristically, $\Delta u(x, t)$ measures the difference between the space average of u in a small ball around (x, t) and $u(x, t)$.

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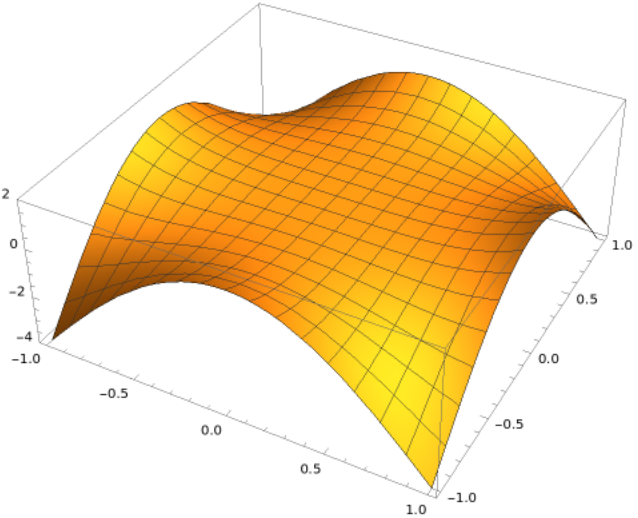
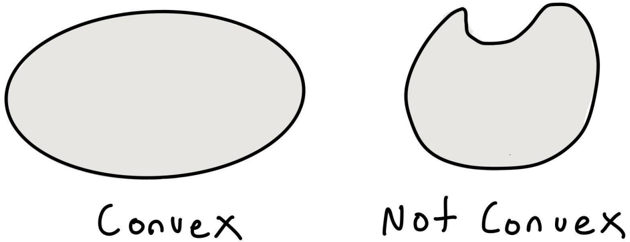
- Heuristically, $\Delta u(x,t)$ measures the difference between the space average of u in a small ball around (x,t) and $u(x,t)$.
- Since $\partial_t u = \Delta u$, we thus expect u to decrease at maximums and increase at minimums.

- Maximum Principle for solutions $\Delta u = 0$



Generalizations

- Convex sets



Generalizations

- By min-max principles, u stays inside the convex subset $[\min_{\overline{\Omega}} u, \max_{\overline{\Omega}} u]$

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Theorem

Let $u : \mathbb{R}^m \times [0, T] \to \mathbb{R}^k$ solve the Heat Equation. If $u(x, 0) \in C$ for some convex subset $C \subset \mathbb{R}^k$, then $u(x, t) \in C$ for all $t \in [0, T]$.

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- $\vec{u}(x,t) : \mathbb{R} \times [0, T] \to \mathbb{R}^2$ defined by

$$\begin{bmatrix} e^{-t} sin(x) \\ e^{-t} cos(x) \end{bmatrix}$$

Stays inside $[-1, 1] \times [-1, 1]$

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- Thank you very much for your attention!