Hindmarsh-Rose Model

David Ulloa

18.353 Nonlinear Dynamics: Chaos

1 Introduction

The Hindmarsh-Rose model is a system of differential equations devised to study the spiking-bursting behavior of the membrane potential neuron. The exact equations are such:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y - x^3 + 3x^2 - z + I \tag{1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - 5x^2 - y \tag{2}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = r(4(x + \frac{8}{5}) - z) \tag{3}$$

The most relevant variable, $x(t)$, is a dimensionless quantity representing the electrical potential created by the sodium and potassium ions across the membrane of the neuron.

As is generally observed in real neurons, a spike consists of a small positive jerk in the potential. If the potential passes a certain threshold, it spikes the potential by briefly opening "fast" ion channels, then the channels close and potential dips back down below and overshoots the equilibrium potential before finally being brought back to equilibrium by the "slow" ion channels. The $y(t)$ variable represents flow of ions through the fast ion channels, and the $z(t)$ is the adaptation current, roughly corresponding to flow through the slow ion channels.

Although the original system had 8 different variables, physical constraints dictate a canonical set of values for those, leaving only 2 constants to vary: I and r. In typical studies of this model, r is the time scale of the neural adaptation and is on the order of 10^{-3} and I is the input current and ranges between -10 and 10. This study will sustain those values.

2 Cursory Analysis

2.1 Fixed Points

To find the fixed points of the system, where the neuron no change in membrane potential, we first find the nullclines of system by setting $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} = 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0 \rightarrow y = 1 - 5x^2 \tag{4}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = 0 \rightarrow z = 4(x + \frac{8}{5}) \tag{5}$$

Substituting those two equations into (1) gives the following cubic polynomial

$$x^3 + 2x^2 + 4x + (\frac{27}{5} - I) = 0 \tag{6}$$

Through graphical analysis, the above equation always has 2 complex roots and 1 real root and, for $|I| \leq 10$, the real root x satisfies $-2.67 \leq x \leq 0.756$. Notably about this polynomial, the value of the root does not depend on the value of r.

2.2 Linear Stability Analysis

Using linear stability analysis, we then proceed to calculate the stability of the fixed point by first finding the eigenvalues of the corresponding Jacobian matrix $J(x, y, z)$.

$$J(x, y, z) = \begin{bmatrix} -3x^2 + 6x & 1 & -1 \\ -10x & -1 & 0 \\ 4r & 0 & -r \end{bmatrix} \tag{7}$$

$$det(J - \lambda I) = (-3x^2 + 6x - \lambda)(1 + \lambda)(r + \lambda) - 10x(r + \lambda) - 4r(1 + \lambda) \tag{8}$$

$$= \lambda^3 + (1 + r - 6x + 3x^2)\lambda^2 + (3(r+1)x^2 + (4 - 6r)x + 5r)\lambda + r(3x^2 + 4x + 4) = 0 \tag{9}$$

From this equation, through calculator analysis we have that the fixed point always has 2 complex roots and 1 real root. Thus, the single fixed point has 3 possible classifications, depending on the values of r and I: an unstable node, saddle-focus node, or a stable node. To determine when the fixed point is a purely stable node, we invoke the Routh-Hurwitz criterion which states the following:

A fixed point is stable if and only if

$$a_0 > 0 \tag{10}$$

$$a_2 > 0 \tag{11}$$

$$a_1 a_2 > a_0 \tag{12}$$

For this particular system, the values for a_0, a_1, a_2 are the following:

$$a_0 = r(3x_*^2 + 4x_* + 4) \tag{13}$$

$$a_1 = 3(r+1)x_*^2 + (4-6r)x_* + 5r \tag{14}$$

$$a_2 = 3x_*^2 - 6x_* + r + 1 \tag{15}$$

Where x_* is the value of x at the fixed point which, as stated previously, depends only on I. The results of this analysis are summarized in the graphic below; the part of the graph colored green are the stable fixed points of the system in the $I - r$ plane.

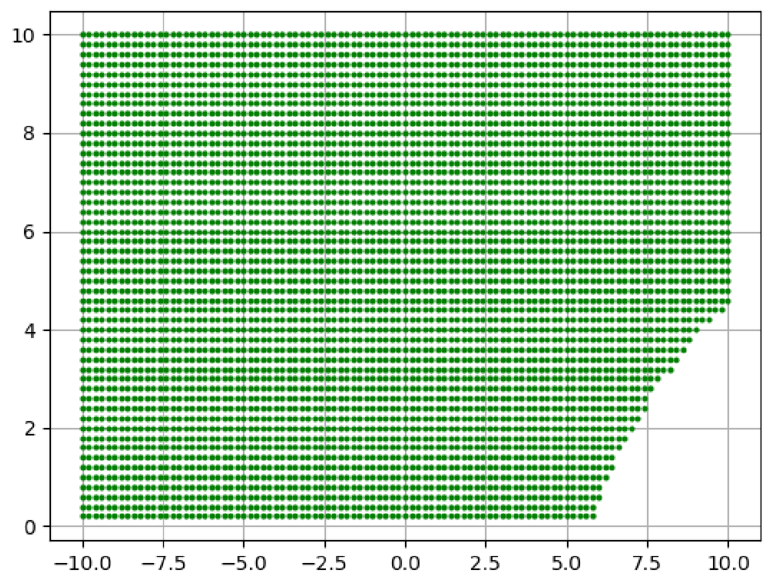

Figure 1: Stability diagram in the $I - r$ plane as given by the Routh-Hurwitz criterion

2.3 Hopf Bifurcation

As has been observed by other studies of Hindmarsh-Rose, the system has a Hopf bifurcation, where a limit cycle generates 2 identical limit cycles, along some boundary in the $I - r$ plane. In this study we will not analyze the stability of the limit cycles.

To find the location of the Hopf bifurcation, we'll substitute $\lambda = iw$ for some real constant w indicating a purely imaginary root to find the barrier analytically.

$$\lambda^3+(1+r-6x+3x^2)\lambda^2+(3(r+1)x^2+(4-6r)x+5r)\lambda+r(3x^2+4x+4)=0 \tag{16}$$

$$\lambda = iw \tag{17}$$

$$-iw^3-(1+r-6x+3x^2)w^2+(3(r+1)x^2+(4-6r)x+5r)iw+r(3x^2+4x+4)=0 \tag{18}$$

By equating the imaginary part with 0 and assuming $w \neq 0$ we have the following condition:

$$(3(r+1)x^2+(4-6r)x+5r)w-w^3=0 \tag{19}$$

$$w^2=3(r+1)x^2+(4-6r)x+5r \tag{20}$$

Substituting this into equation found by equating the real part with 0 we have the analytical equation of the boundary:

$$r(3x_*^2+4x_*+4)=(1+r-6x_*+3x_*^2)(3(r+1)x_*^2+(4-6r)x_*+5r) \tag{21}$$

Unfortunately, by the nature of equation (6), while an exact equation for the boundary is possible, it is intractable. So, we instead approximate equation (6) using a second-order implicit Taylor expansion around the point $(5.4, 0)$ to find an approximate equation for x_* in terms of I which we can then substitute into equation (21) and graph.

$$x^3+2x^2+4x+(\frac{27}{5}-I)=0 \tag{22}$$

$$\frac{\mathrm{d}x}{\mathrm{d}I}*(3x^2+4x+4)=1 \tag{23}$$

$$\frac{\mathrm{d}x}{\mathrm{d}I}=\frac{1}{4} \tag{24}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}I^2}*(3x^2+4x+4)+(\frac{\mathrm{d}x}{\mathrm{d}I})^2(6x+4)=0 \tag{25}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}I^2}=-\frac{1}{16} \tag{26}$$

Thus the Taylor approximation is,

$$x_*(I)=\frac{1}{4}(I-\frac{27}{5})-\frac{1}{32}(I-\frac{27}{5})^2 \tag{27}$$

We substitute this into (21) and plot in the $I-r$ plane.

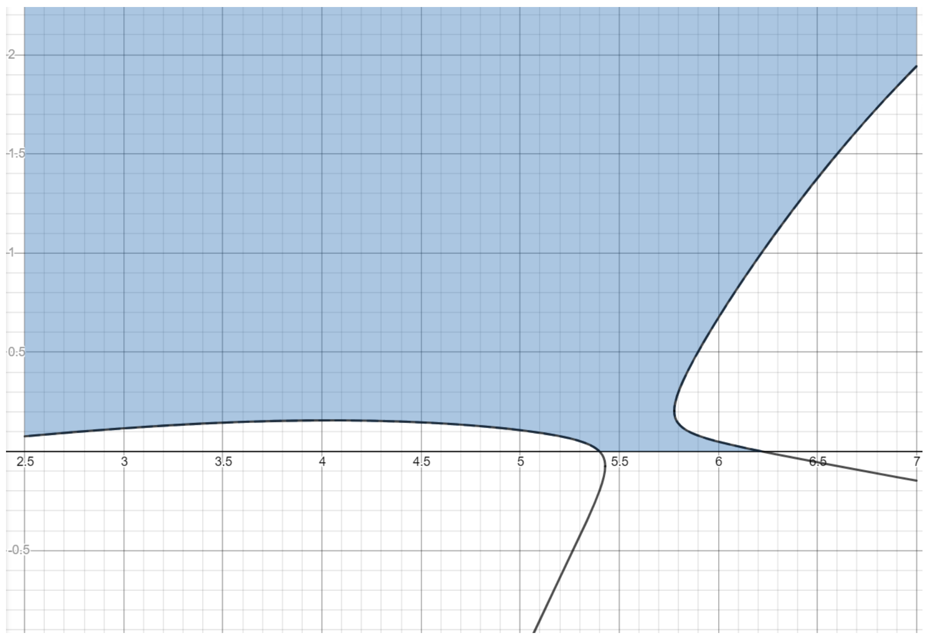

Figure 2: Hopf bifurcation boundary and stable region

The black line marks the Hopf bifurcation boundary. The part of the graph shown is where the Taylor expansion is a good approximation, x_* from $2.5 \leq I \leq 7$. Notably, the stable region marked in blue agrees with the stable region found by the Routh-Hurwitz criterion in Figure 1.

3 Chaos

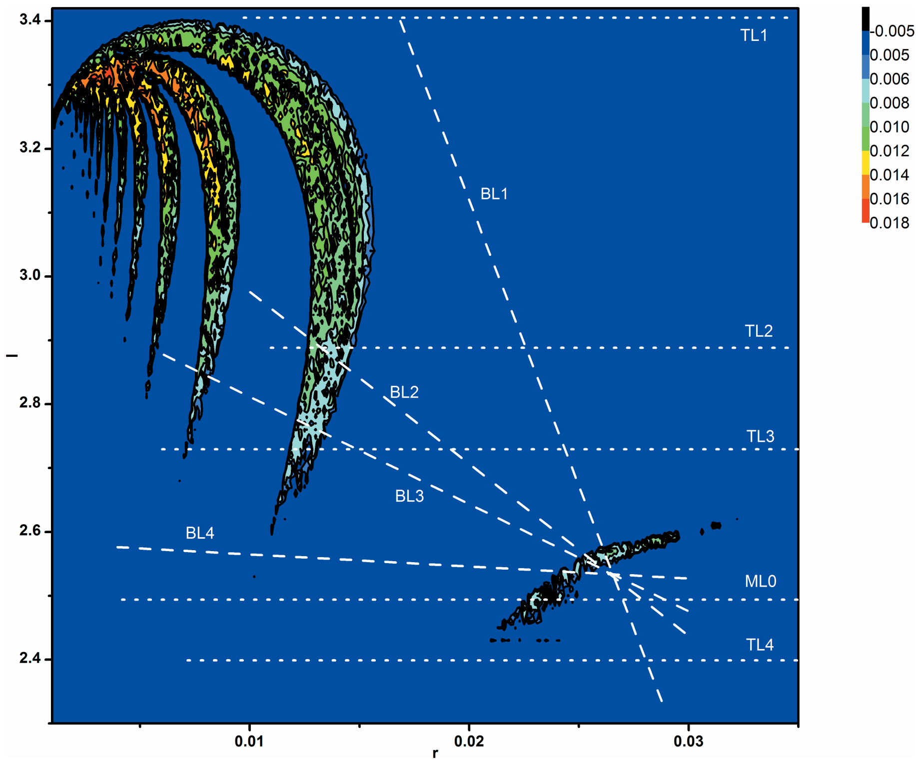

Figure 3: Detailed view of a portion of the $r - I$ plane[1]. Solid blue denotes periodic behavior of the system and colors noted on the right correspond to the maximal Lyapunov exponent (positive value implies chaotic behavior)

Now that we have a good preliminary understanding of the system and its various characteristic parameter regimes, we can begin to explore these regimes in greater detail: particularly the chaotic regimes. The study of these chaotic dynamics are especially relevant in neuroscience because these dynamics are well-documented but difficult to understand phenomena found in real neurons.

Because the wing-shaped region has been well-studied in prior research, for this paper we focus primarily on small amorphous region on the bottom right of the plot. Specifically, we'll look at the evolution of the system under the parameter values $r = 0.025$ and $I = 2.55$. These fall into saddle-focus area given by Figure 2 as expected.

3.1 The Attractor

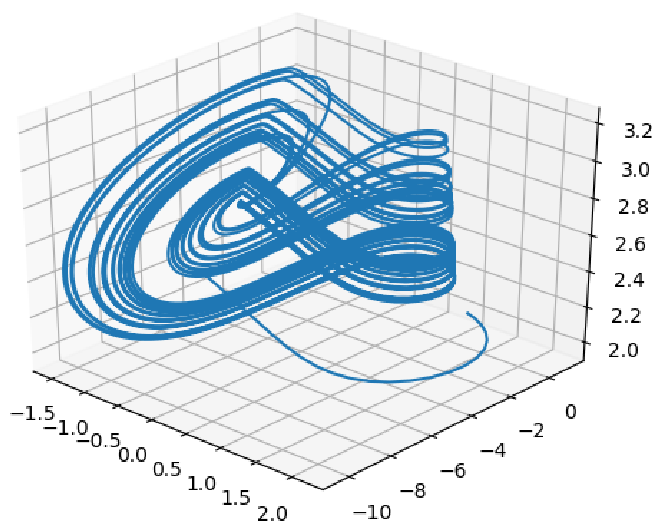

Figure 4: Evolution of the system for 2000 seconds starting from (1, 1, 2)

As supported graphically by the plot above, these parameter values give chaotic behavior. The trajectory oscillates between large, flat, region (corresponding to the recover phase of the spikes) and briefly jumps between the inner and outer ring through a small curl (corresponding to the spike). Moreover, the volume of the attractor appears to decay to 0, but proving so analytically is quite difficult.

3.2 Membrane Potential

However, because $x(t)$ is the most relevant variable in the system, it's best to isolate and plot it as a function of time, as in Figure 5. From this plot, the expected spiking period becomes clear, and although it may seem periodic, upon further inspection, there are slight differences between each spike.

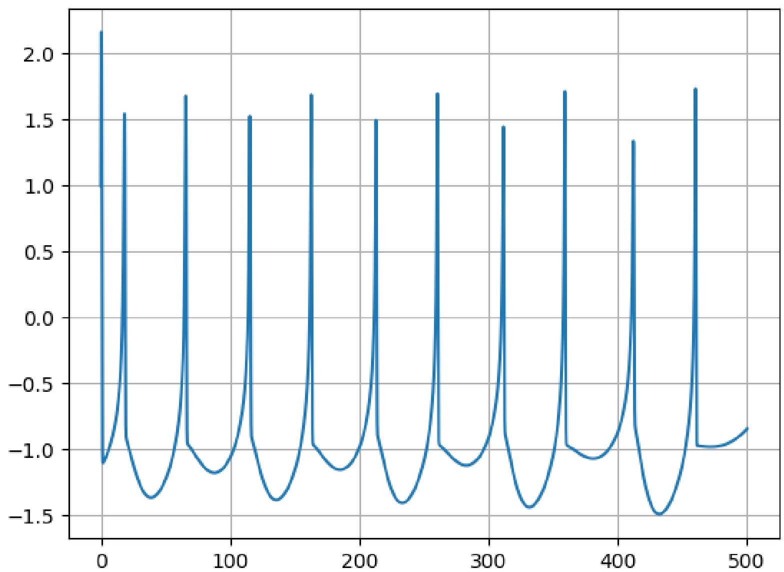

Figure 5: Evolution of the membrane potential for 500 seconds starting from (1, 1, 2)

3.3 Chaotic Map

To both make those differences more apparent and give a rough proof of the chaos of the membrane potential, we'll create a one dimensional map of the form

$$x_{n+1} = f(x_n) \tag{28}$$

where x_n is the value of the n^{th} peak of the potential. The points are (x_n, x_{n+1}) and the plot is given in Figure 6. From this plot, we can devise a rough approximation of f:

$$f(x) = 17e^{-x} \tag{29}$$

From the unimodality of the function and plotting the first 20 iterations of the map (Figure 7), we can say with high certainty that the membrane potential exhibits chaotic behavior for these parameter values.

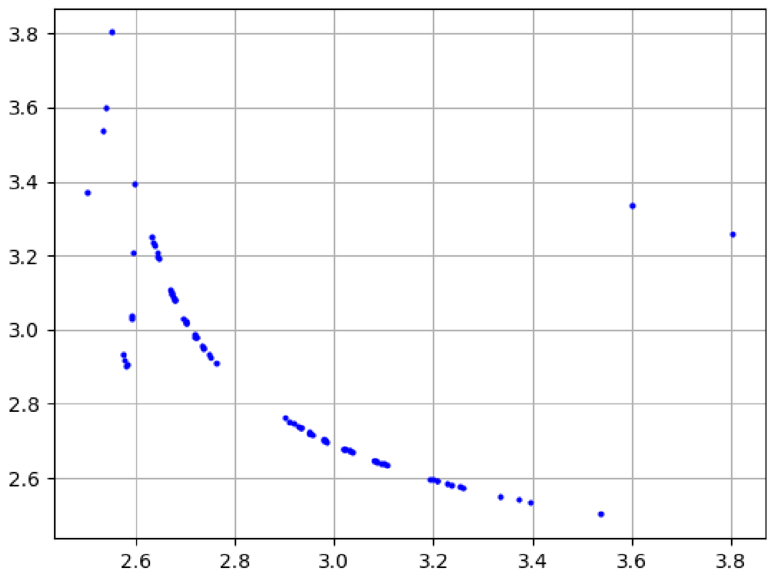

Figure 6: 1-dimensional map of the system. The horizontal axis represents the value of a peak of x and the vertical axis is the value of the subsequent peak

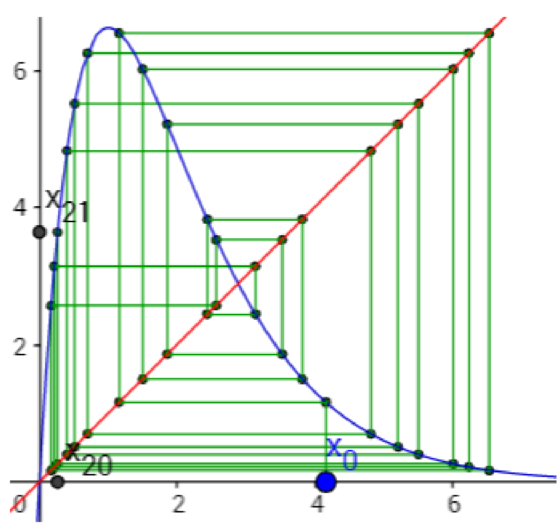

Figure 7: Cobwebbing of 20 iterations[3] of the map $x_{n+1} = 17x_ne^{-x_n}$

4 Summary

In this paper we studied the Hindmarsh-Rose model of a neuron, approximated the fixed point by do an implicit Taylor expansion to find the x value as function of I. Then, we did linear stability analysis alongside the Routh-Hurwitz criterion to find where the fixed point is stable. We then found an approximation of the Hopf bifurcation boundary for $2.5 \leq I \leq 7$ to confirm those results.

Next, we studied the chaotic regimes of the system, specifically the parameter values $r = 0.025$ and $I = 2.55$. To prove that those parameter values lead to chaos, we plotted $x(t)$ and created an approximate map to show its unimodality.

In potential future papers, the regimes with limit cycles and the wing-shaped chaotic regime should be studied in more detail.

5 Works Cited

1. Gu, Huaguang. "Biological Experimental Observations of an Unnoticed Chaos as Simulated by the Hindmarsh-Rose Model." PLOS ONE, Public Library of Science, 10 Dec. 2013.

2. "Hindmarsh–Rose Model." Wikipedia, Wikimedia Foundation, 1 Dec. 2020.

3. "Math Insight." Applet: Visualizing Function Iteration via Cobwebbing, Combined with Plot of Solution - Math Insight.