**EXERCISE 02: Ranking binary numbers with a perceptron**

Consider a perceptron which takes as input two binary numbers of 10 digits each and restitutes as an output either $+1$ or $-1$. The input vector $\vec{S}$ has 20 elements $S_i = \{\pm 1\}$. The first 10 elements represent the first number $n_T$ and the second 10 elements represent the second number $n_B$. The perceptron has $N = 20$ synaptic couplings $J_i$ and must be able to tell which number is larger by giving as an output $\sigma(\{S_i^\mu\}, \{J_i\}) = \mathrm{sign}(\sum_{i=1}^{N} S_i J_i)$ such that $\sigma = 1$ if $n_T > n_B$ and $\sigma = -1$ if $n_T \leq n_B$.

(1) Set up the perfect perceptron assigning to the synaptic weights the values $J_i^* = 2^{10-i} \quad 1 \leq i \leq 10$ and $J_i^* = -J_{i-10}^* \quad 11 \leq i \leq 20$.

(2) Define and evaluate a test error $\epsilon$ (counting the frequency of errors) of the perfect perceptron over a large number of trials where two random numbers between 0 and $2^{10}-1$ are ranked. The test error should be obtained comparing the results of the perfect perceptron with an independent benchmark method.

(3) Assign now to the synaptic weights random values extracted from a Gaussian distribution $\mathcal{N}(0, 1)$.

(4) Evaluate a test error $\epsilon$ (counting the frequency of errors) of the random perceptron over a large number of trials where two random numbers between 0 and $2^{10} - 1$ are ranked. The test error should be obtained comparing the results of the random perceptron with those of the perfect perceptron.

(5) Produce two sets of $P$ examples $\vec{\xi}^\mu$ with correct classifications $\sigma_T^\mu$, the first with $P = 500$ the second with $P = 2000$.

(6) For each training set train the perceptron using the following perceptron update rule: only if $\sigma(\{\xi_i^\mu\}, \{J_i\}) \neq \sigma_T^\mu$ update the synaptic couplings $J_i(t+1) = J_i(t) + \frac{1}{\sqrt{N}} \sigma_T^\mu \xi_i^\mu$. The rule should be applied across an entire cycle of the inputs in the training set, a training error $\epsilon_t$ is evaluated at the end of the cycle and the process repeated until $\epsilon_t = 0$.

(7) To study the evolution of the $J_i$ plot $\mathrm{sign}(J_i^* J_i) \log_2(|J_i|)$ versus $i$, the index of the synaptic weight, for the weights obtained with $P = 500$ and $P = 2000$ instances in the training set and comment on the result.

(8) Evaluate a test error $\epsilon$ (counting the frequency of errors) of the two trained perceptrons over a large number of trials where two random numbers between 0 and $2^{10}-1$ are ranked. The test error should be obtained comparing the results of the trained perceptrons with those of the perfect perceptron.