CPSC 121 - MATHEMATICAL PROOFS SOLUTIONS

1. Direct Proof

Problem 1. Prove that the fourth power of a positive odd integer can be written in the form $8m+1$, where m is a non-negative integer. Hint: a positive odd integer can be written as $2i+1$, where i is a non-negative integer.

Solution:

We can translate the theorem to predicate logic in order to guide us through our proof:

$\forall j \in \mathbb{Z}^+, (\exists k \in \mathbb{Z}, j = 2k+1) \rightarrow (\exists m \in \mathbb{Z}, j^4 = 8m+1)$

Proof:

Consider an unspecified positive odd integer j. Then we know that $j = 2k+1$, where k is a non-negative integer.

We need to choose a non-negative integer m such that $j^4 = 8m+1$.

Note that $j^4 = (2k+1)^4 = 16k^4 + 32k^3 + 16k^2 + 8k + 1 = 8(2k^4 + 4k^3 + 2k^2 + k) + 1, (1)$

Choose $m = 2k^4 + 4k^3 + 2k^2 + k$
By the closure of addition and multiplication of non-negative integers, $2k^4 + 4k^3 + 2k^2 + k$ is a non-negative integer and thus m is a non-negative integer.
By equation (1), $j^4 = 8m+1$.
QED

Definition. The *floor function* assigns to the real number x the largest integer that is less than or equal to x. In other words, the floor function rounds a real number down to the nearest integer. The value of the floor function is denoted by $\lfloor x \rfloor$.

Definition. The *ceiling function* assigns to the real number x the smallest integer that is greater than or equal to x. In other words, the ceiling function rounds a real number up to the nearest integer. The value of the ceiling function is denoted by $\lceil x \rceil$.

Example. $\lfloor \pi \rfloor = 3$. $\lceil \pi \rceil = 4$. $\lfloor -\frac{1}{2} \rfloor = -1$. $\lceil -\frac{1}{2} \rceil = 0$. $\lfloor \sqrt{2} \rfloor = 1$. $\lceil \sqrt{2} \rceil = 2$. $\lfloor -5 \rfloor = -5$. $\lceil -5 \rceil = -5$.

Problem 2. Prove that for any positive integer x, if x is one more than a multiple of 3, then the sum: $2 \cdot \lfloor \frac{x}{3} \rfloor + \lceil \frac{x}{3} \rceil = x$

Note: If x is one more than a multiple of three, then it can be written as $3k+1$ for some integer k.

Solution:

We can translate the theorem to predicate logic in order to guide us through our proof:

$\forall x \in \mathbb{Z}^+, (\exists k \in \mathbb{Z}, x = 3k+1) \rightarrow 2 \cdot \lfloor \frac{x}{3} \rfloor + \lceil \frac{x}{3} \rceil = x$

Proof:

Consider an unspecified integer x.
Assume that $x = 3k+1$ for some integer k.

It thus follows,

$$
\begin{aligned}
2 \cdot \lfloor \tfrac{x}{3} \rfloor + \lceil \tfrac{x}{3} \rceil &= 2 \cdot \lfloor \tfrac{3k+1}{3} \rfloor + \lceil \tfrac{3k+1}{3} \rceil \\
&= 2 \cdot \lfloor k + \tfrac{1}{3} \rfloor + \lceil k + \tfrac{1}{3} \rceil \\
&= 2k + (k+1) \\
&= 3k + 1 \\
&= x
\end{aligned}
$$

QED

Problem 3. Prove that for any integer m, if m is a perfect square, then $m+2$ is not a perfect square.

Solution:
We can translate the theorem to predicate logic in order to guide us through our proof:

$\forall m \in \mathbb{Z}, (\exists a \in \mathbb{Z}, m = a^2) \rightarrow \neg(\exists b \in \mathbb{Z}, m + 2 = b^2)$

Note: To prove that a number is not a perfect square, it is equivalent to proving that it is strictly in between two consecutive perfect squares p^2 and $(p+1)^2$.

Proof:

Consider an unspecified integer m and assume that m is a perfect square. Then $m = p^2$ where p is some integer.

Then $m + 2 = p^2 + 2$.

We need to show that $p^2 + 2$ is not a perfect square for any integer p.

Consider 2 cases.

- Case 1: $p = 0$

  Then $p^2 + 2 = 0^2 + 2 = 2$, which is not a perfect square.

- Case 2: $p > 0$

  Then $m = p^2$ is a perfect square and so the next larger perfect square is $(p+1)^2$. Consequently, any integer that is greater than p^2 and less than $(p+1)^2$ cannot be a perfect square.

  We will show that $p^2 + 2$ is greater than p^2 and less than $(p+1)^2$.

  $p^2 < p^2 + 2$ because 2 is strictly greater than 0.

  Since $p \in \mathbb{Z}$ and $p > 0$, it follows that $p \geq 1$. Hence we can argue:

$$
\begin{aligned}
m + 2 &= p^2 + 2 \\
&\leq p^2 + 2p \\
&< p^2 + 2p + 1 \\
&= (p+1)^2
\end{aligned}
$$

  Thus, $m + 2 = p^2 + 2 < (p+1)^2$

  In this case, $p^2 < p^2 + 2 < (p+1)^2$, showing that $m+2$ lies strictly between m and the next biggest perfect square, and therefore showing $m+2$ cannot be a perfect square.

QED

2. Proof by Contraposition

Problem 4. Prove that for any integer n, if $n^2 + 8n - 1$ is even, then n is odd.

Proof: We prove the contrapositive of the theorem. For any integer n, if n is even, then $n^2 + 8n - 1$ is odd.

Consider an unspecified integer n.
Assume that n is even.
Then $n = 2k$ for some integer k.
$n^2 + 8n - 1 = (2k)^2 + 8(2k) - 1 = 4k^2 + 16k - 1 = 2(2k^2 + 8k - 1) + 1$
Since $2k^2 + 8k - 1$ is an integer, $n^2 + 8n - 1$ is odd.

QED

Problem 5. Prove that for any integers a, b and n, if $n \nmid (a \cdot b)$, then $n \nmid a$ and $n \nmid b$.

Proof: We prove the contrapositive of the theorem. For any integers a, b and n, if $n \mid a$ or $n \mid b$, then $n \mid (a \cdot b)$.

Consider unspecified integers n, a, and b.
Without loss of generality, assume that $n \mid a$
Then $a = n \cdot p$ for some integer p.
$a \cdot b = (n \cdot p) \cdot b = n \cdot (p \cdot b)$.
Thus $n \mid (a \cdot b)$.

QED

Problem 6. Prove that for any $x, y \in \mathbb{R}$, if $y^3 + yx^2 \leq x^3 + xy^2$, then $y \leq x$.

Proof: We prove the contrapositive of the theorem. For any $x, y \in \mathbb{R}$, if $y > x$, then $y^3 + yx^2 > x^3 + xy^2$.

Consider unspecified real numbers x and y.
Assume that $y > x$
We know that $y^2 + x^2 \geq 0$ for any real numbers x and y.
Since x and y are different, $y^2 + x^2$ is positive.
Multiplying the inequality $y > x$ on both sides by the positive quantity $y^2 + x^2$ we obtain:

$$\begin{array}{rcl} (y^2 + x^2)y & > & (y^2 + x^2)x \\ y^3 + yx^2 & > & y^2x + x^3 \end{array}$$

QED

3. Proof by Contradiction

Problem 7. Prove that for any two real numbers a and b, if a is rational and ab is irrational, then b is irrational.

Proof: We prove this by contradiction. Assume that there are real numbers a and b such that a is rational, ab is irrational, and b is rational.

a is rational, so $a = \frac{x}{y}$ for some integers x and y, with $y \neq 0$.
b is rational, so $b = \frac{m}{n}$ for some integers m and n, with $n \neq 0$.
$ab = \frac{xm}{yn}$. xm and yn are both integers, and $yn \neq 0$, so ab is rational, and this contradicts our assumption that ab is irrational.

QED

Problem 8. Prove that there do not exist integers a and b such that $7a + 21b = 1$.

Proof: We prove this by contradiction. Assume that there exist integers a and b such that $7a + 21b = 1$.
$7a + 21b = 1$
$7(a + 3b) = 1$
$a + 3b$ is an integer, so the LHS of the equation is divisible by 7. This means that the RHS of the equation is also divisible by 7, but 1 is not divisible by 7. This is a contradiction.

QED

Problem 9. Prove that for any integers a, b, and c, if $a^2 + b^2 = c^2$, then a is even or b is even.

Proof: We prove this by contradiction. Assume that there are integers a, b, and c such that $a^2 + b^2 = c^2$ and a and b are both odd.

a is odd, so $a = 2m + 1$, for some integer m.
b is odd, so $b = 2n + 1$, for some integer n.

$$\begin{aligned} c^2 &= (2m+1)^2 + (2n+1)^2 \\ &= 4m^2 + 4m + 1 + 4n^2 + 4n + 1 \\ &= 4m^2 + 4m + 4n^2 + 4n + 2 \\ &= 2(2m^2 + 2m + 2n^2 + 2n + 1) \end{aligned}$$

Since $2m^2 + 2m + 2n^2 + 2n + 1$ is an integer. c^2 is even. So c is also even by the result of a proof covered in lecture..

Then $c = 2k$, for some integer k. $c^2 = 4k^2$. So c^2 is divisible by 4.
By our calculation above,

$$\begin{aligned} \tfrac{c^2}{4} &= \tfrac{4m^2+4m+4n^2+4n+2}{4} \\ &= m^2 + m + n^2 + n + \tfrac{1}{2} \end{aligned}$$

Since $m^2 + m + n^2 + n$ is an integer and $\frac{1}{2}$ is not, this contradicts our previous result that c^2 is divisible by 4.

QED