CPSC 121 - MATHEMATICAL PROOFS AND SEQUENTIAL CIRCUITS SOLUTIONS

Note the definition: $f(n) \in O(g(n)) \equiv \exists c \in \mathbb{R}_+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \rightarrow f(n) \leq cg(n)$

Problem 1. Given that $f(n) = n^3 + 2n^2 + 96$, prove that $f(n) \in O(n^3)$.

Solution:
Choose $c = 99$ and $n_0 = 1$.
Consider an arbitrary $n \in \mathbb{N}$ and assume that $n \geq n_0 = 1$.
We note that for our arbitrary $n \geq 1$:

- $n^3 \leq n^3$
- $2n^2 \leq 2n^3$
- $96 \leq 96n^3$

Then it follows that:

$$\begin{aligned} f(n) &= n^3 + 2n^2 + 96 \\ &\leq n^3 + 2n^3 + 96n^3 \\ &= 99n^3 \\ &= cn^3 \end{aligned}$$

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Problem 2. Given that $f(n) = n^3$ and $g(n) = 9988n^2$, prove that $f(n) \notin O(g(n))$.

Solution: We will prove the negation of the Big-O definition, that is:

$$\begin{aligned} f(n) \notin O(g(n)) &\equiv \sim \exists c \in \mathbb{R}_+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \rightarrow f(n) \leq cg(n) \\ &\equiv \forall c \in \mathbb{R}_+, \forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq n_0 \wedge f(n) > cg(n) \end{aligned}$$

Now, beginning the actual proof, consider an arbitrary $c \in \mathbb{R}_+$ and consider an arbitrary $n_0 \in \mathbb{N}$.
Choose $n = \max\{n_0, 9988c\} + 1$, (we note that the condition $n \geq n_0$ is clearly satisfied).
For the second condition that we need to show $f(n) > cg(n)$:

$$\begin{aligned} f(n) &= n^3 \\ &= n \cdot n^2 \\ &> 9988cn^2 \quad \text{by the choice of } n = \max\{n_0, 9988c\} + 1 \\ &= cg(n) \end{aligned}$$

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Problem 3. Prove that for every distinct pair of real numbers, there exists a third real number that is strictly between the first two numbers.

Solution: In order to help guide us through our prrof, we can translate the statement as follows:
$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x < y \rightarrow \exists z \in \mathbb{R}, x < z < y$
Now, to the actual proof, consider an arbitrary $x \in \mathbb{R}$, consider an aribtrary $y \in \mathbb{R}$, and assume that $x < y$
Next, choose $z = \frac{x+y}{2}$
Next, we verify that $x < \frac{x+y}{2} < y$:

$$\begin{aligned} x &= \tfrac{x+x}{2} \\ &< \tfrac{x+y}{2} && \text{by } x < y \\ &< \tfrac{y+y}{2} && \text{by } x < y \\ &= y \end{aligned}$$

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Problem 4. We define the set $[0,1) \subset \mathbb{R}$ as $[0,1) = \{x \in \mathbb{R} | 0 \leq x < 1\}$

(1) Prove that $[0,1)$ has no maximum (largest) element.

Solution: We will proceed by proof by contradiction.
Suppose $[0,1)$ has a maximum element, call this maximum element $\alpha = \max\{[0,1)\}$ where by definition $\alpha \in [0,1)$.

Consider $\gamma = \frac{\alpha+1}{2}$

Then:

$$\begin{aligned} \alpha &= \tfrac{\alpha+\alpha}{2} \\ &< \tfrac{\alpha+1}{2} && \text{By defintion of } \alpha \in [0,1) \text{ and hence } \alpha < 1 \\ &< \tfrac{1+1}{2} && \text{By defintion of } \alpha \in [0,1) \text{ and hence } \alpha < 1 \\ &= 1 \end{aligned}$$

Since $0 \leq \alpha < \gamma < 1$, then $\gamma \in [0,1)$ by definition of the set $[0,1)$.

However, since $\alpha < \gamma$ and γ is also an element of $[0,1)$, this contradicts α being the maximum (largest) element of $[0,1)$. Therefore the set $[0,1)$ has no maximum element.

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(2) Prove that 1 is the least upper bound of $[0,1)$. For a number $s \in \mathbb{R}$ to be the least upper bound of a set $A \subset \mathbb{R}$, it must:

(a) Be an upper bound of the set A, specifically, it must satisfy: $\forall x \in A, x \leq s$
(b) Be the smallest (minimum) upper bound. If S is the set of upper bounds of A, s is the least upper bound of A if $\forall s' \in S, s \leq s'$

Solution We will first prove condition (a), that 1 is an upper bound of $[0,1)$ using a direct proof.

Proof of $\forall x \in [0,1), x \leq 1$*:*

Consider an arbitrary $x \in [0,1)$

By definition of $[0,1)$, $0 \leq x < 1$ which implies that $x \leq 1$.

Therefore, 1 is an upper bound of $[0,1)$.

Proof by contradiction that 1 *is the least upper upper bound of* $[0,1)$*:*

Suppose that $[0, 1)$ has an upper bound that is smaller than 1, let us call this upper bound s. By this assumption $s < 1$.

Consider $\gamma = \frac{s+1}{2}$

$$\begin{array}{rcll} s & = & \frac{s+s}{2} & \\ & < & \frac{s+1}{2} & \text{By assumption of } s < 1 \\ & < & \frac{1+1}{2} & \text{By assumption of } s < 1 \\ & = & 1 & \end{array}$$

Thus, $0 \leq s < \gamma < 1$ ($0 \leq s$ must follow as $0 \in [0, 1)$ and s is an upper bound and otherwise we would obtain a contradiction at this point).

Therefore $0 \leq \gamma < 1$ and hence $\gamma \in [0, 1)$

However, since $s < \gamma$ and γ is an element of $[0, 1)$ this contradicts s being an upper bound of $[0, 1)$. Therefore, it must be the case that 1 is the least upper bound of $[0, 1)$.

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Problem 5. The following is a very incomplete sequential circuit whose output should be the average (more specifically, the floor of the average) of the 8-bit values that were present at the input every time the clock ticked. Complete the circuit so it behaves according to this description.

Hint: you will need two adders, two registers, a clock, and maybe some other simpler components (gates, constants, etc). The divider divides the top input by the bottom input and outputs the floor of the quotient. Do not worry about its behaviour when the bottom input is 0.

Solution: We use one register (the bottom one in the figure) to keep track of the number of values we have seen, and one register (the one on the top) to keep track of the sum of the value we have seen so far.