HW 1 SOLUTIONS

Due 23:59 September 13, 2019

CS ID 1: __________

CS ID 2: __________

Instructions:

1. Do not change the problem statements we are giving you. Simply add your solutions by editing this latex document.
2. Take as much space as you need for each problem. You'll tell us where your solutions are when you submit your paper to gradescope.
3. Export the completed assignment as a PDF file for upload to gradescope.
4. On gradescope, upload only **one** copy per partnership. (Instructions for uploading to gradescope will be posted on the HW1 page of the course website.)

Academic Conduct: I certify that my assignment follows the academic conduct rules for of CPSC 221 as outlined on the course website. As part of those rules, when collaborating with anyone outside my group, (1) I and my collaborators took no record but names away, and (2) after a suitable break, my group created the assignment I am submitting without help from anyone other than the course staff.

1. (2 points) Using 140 characters or less, post a synopsis of your favorite movie to the course piazza space under the "HW1 tell me something!" notice, so that your post is visible to everyone in the class, and tagged by #HW1num1. Also, use Piazza's code-formatting tools to write a *private* post to course staff that includes at least 5 lines of code. It can be code of your own or from a favorite project—it doesn't even have to be syntactically correct—but it must be formatted as a code block in your post, and also include the tag #HW1num1. (Hint: Check http://support.piazza.com/customer/portal/articles/1774756-code-blocking). Finally, please write the two post numbers corresponding to your posts here:

Favorite Movie Post (Public) number:	
Formatted Code Post (Private) number:	

2. (16 points) In this problem, you will be a math detective! Some of the symbols and functions may not be familiar to you, but with a little digging, reading, and observing, you'll be able to figure them out. Your task is to simplify each of the following expressions as much as possible, **without using a calculator (either hardware or software)**. Do not approximate. Express all rational numbers as improper fractions. You may assume that n is an integer greater than 1. Show your work in the space provided, and write your final answer in the box.

(a) i. $3 + 6 + 9 + 12 + \cdots + 3n$

Answer for (a.i):	$3(n+1)n/2$

This is the sum $3\sum_{i=1}^{n} n = 3(n+1)n/2$.

ii. $\displaystyle\sum_{i=1}^{n}(3+4i)$

Answer for (a.ii):	$n(2n+5)$

The sum splits into $(\sum_{i=1}^{n} 3) + (4\sum_{i=1}^{n} i)$. The first part is $3n$ and the second is $4(n+1)n/2 = 2(n+1)n$. Together, that's $n(2n+5)$.

(b) i. $\displaystyle\sum_{r=2}^{\infty}\left(\frac{3}{5}\right)^r$

Answer for (b.i):	$9/10$

This is the sum of a geometric series with common ratio $3/5$. Since $3/5 < 1$, the sum converges. If S is the sum, then $\left(\frac{3}{5}\right) S$ is the sum of all the terms in S except for the first. So $\left(\frac{3}{5}\right) S = S - (3/5)^2$, which implies $S = (3/5)^2/(1-3/5) = 9/10$.

ii. $\displaystyle\sum_{r=(-2)}^{n} \left(\frac{1}{n}\right)^r$

Answer for (b.ii):	$\dfrac{n^3 - (1/n)^n}{n-1}$

This is the sum of $n+3$ terms in a geometric series with common ratio $1/n$. If S is the sum, then $S - S/n = (1/n)^{-2} - (1/n)^{n+1}$ (since all terms with corresponding powers of $1/n$ cancel), so $S = \frac{n^2-(1/n)^{n+1}}{1-1/n} = \frac{n^3-(1/n)^n}{n-1}$.

(c) i. $\displaystyle\sum_{k=1}^{n} k2^{n-k}$

Answer for (c.i):	$(n-1)2^{n+1} + 2$

If S is the sum, then $2S - S = 2^n + 2^{n-1} + 2^{n-2} + \cdots + 2^1 - n = 2^{n+1} - 2 - n$.

ii. $\displaystyle\prod_{k=0}^{n} (1 + a^{2^k})$ (Hint: Try multiplying by $(1-a)$.)

Answer for (c.ii):	$\dfrac{1 - a^{2^{n+1}}}{1-a}$

$$
\begin{aligned}
(1-a)\prod_{k=0}^{n}(1+a^{2^k}) &= (1-a)(1+a)(1+a^2)(1+a^4)(1+a^8)\cdots(1+a^{2^n}) \\
&= (1-a^2)(1+a^2)(1+a^4)(1+a^8)\cdots(1+a^{2^n}) \\
&= (1-a^4)(1+a^4)(1+a^8)\cdots(1+a^{2^n}) \\
&\vdots \\
&= (1-a^{2^{n+1}})
\end{aligned}
$$

Alternatively, you might also observe that the original product equals $1 + a + a^2 + \cdots + a^{2^{n+1}-1}$ by multiplying it all out.

(d) i. $7^{333} \bmod 10$

Answer for (d.i):	7

Notice that $7^4 = 1 \bmod 10$. This implies that $7^{4k} = 1 \bmod 10$ for positive integers k since if $x = a \bmod n$ and $y = b \bmod n$ then $xy \bmod n = ab \bmod n$. Since $333 = 4 \cdot 83 + 1$, $7^{333} = 7^{4 \cdot 83} 7^1 = 7 \bmod 10$.

ii. $16^{333} \bmod 14$

Answer for (d.ii):	8

First, $16^{333} = 2^{4 \cdot 333}$. Also, by the above observation, $2^{x+4} \bmod 14 = (2^x \bmod 14)(16 \bmod 14) \bmod 14 = 2^{x+1} \bmod 14$. By repeatedly (443 times) applying this equivalence, we obtain $2^{4 \cdot 333} \bmod 14 = 2^3 \bmod 14 = 8$.

(e) i. $32^{(\log_2 n)/4}$

Answer for (e.i):	$n^{5/4}$

$$32^{(\log_2 n)/4} = 2^{5(\log_2 n)/4} = n^{5/4}.$$

ii. $\dfrac{\log_{512} n}{\log_2 n}$

Answer for (e.ii):	1/9

$$\frac{\log_{512} n}{\log_2 n} = \frac{\log_2 n / \log_2 512}{\log_2 n} = 1/\log_2 512 = 1/9.$$

3. (9 points)

   (a) (2 points) Fill in the blanks:

   Theorem: For any $c \in \underline{\ \mathbb{Z}\ }$, and for any $n \in \underline{\ \mathbb{N}\ }$, there exists a $k \in \mathbb{Z}$ so that $c^n - 1 = k \cdot (c - 1)$.

   (b) (3 points) Prove the theorem from the previous part.

   This theorem is just a different way of expressing the sum of a geometric series: $\sum_{k=0}^{n-1} c^k = \frac{c^n - 1}{c - 1}$. Since c and n are integers, then the sum is an integer.

   (c) (4 points) Prove that $5^n \cdot 2^{2n} - 1$ is divisible by 19.

   The expression $5^n \cdot 2^{2n} - 1$ can be rewritten as $20^n - 1$. From the previous theorem, $\frac{20^n - 1}{19}$ is an integer. Thus, 19 divides $5^n \cdot 2^{2n} - 1$.

4. (12 points)

   (a) (3 points) Complete the 5-by-5 magic square below so that every row, every column, and every diagonal have the same sum, and so that each integer from 1 through 25 is used exactly once.

19	**25**	8	11	**2**
13	**1**	17	24	10
22	9	**15**	**3**	**16**
5	18	**21**	7	14
6	12	**4**	**20**	23

   (b) (3 points) Give an expression for the column sum of an n-by-n magic square.

Answer for (b):	$\frac{n(n^2+1)}{2}$

   The row sum is the sum of all the values in the table, divided by n, the number of rows. Since the numbers in the table are $1 \ldots n^2$, The total value of all cells is $\displaystyle\sum_{k=1}^{n^2} k = \frac{n^2(n^2+1)}{2}$.

   (c) (3 points) Consider an arrangement of the positive integers, grouped as shown, so that the kth group has k elements: $(1), (2, 3), (4, 5, 6), (7, 8, 9, 10), \ldots$. Find a formula for the sum of the k numbers in the kth group.

Answer for (c):	$\frac{k(k^2+1)}{2}$

   (d) (3 points) Prove that your formula from the previous part is correct.

The kth group of k elements ends at value $1+2+\ldots+k = \frac{k(k+1)}{2}$, and the previous group ends at value $1+2+\ldots+(k-1) = \frac{k(k-1)}{2}$, so the sum of interest is:

$$
\begin{aligned}
\sum_{i=1}^{\frac{k(k+1)}{2}} i - \sum_{i=1}^{\frac{k(k-1)}{2}} i &= \frac{1}{2}\left(\frac{k(k+1)}{2}\right)\cdot\left(\frac{k(k+1)}{2}+1\right) - \frac{1}{2}\left(\frac{k(k-1)}{2}\right)\cdot\left(\frac{k(k-1)}{2}+1\right) \\
&= \frac{1}{8}\left(k(k+1)\cdot(k(k+1)+2) - k(k-1)\cdot(k(k-1)+2)\right) \\
&= \frac{1}{8}\left[(k^2+k)\cdot(k^2+k+2) - (k^2-k)\cdot(k^2-k+2)\right] \\
&= \frac{1}{8}\left[(k^4+k^3+2k^2+k^3+k^2+2k) - (k^4-k^3+2k^2-k^3+k^2-2k)\right] \\
&= \frac{1}{8}(4k^3+4k) \\
&= \frac{k(k^2+1)}{2}
\end{aligned}
$$

5. (12 points) Indicate for each of the following pairs of expressions $(f(n), g(n))$, whether $f(n)$ is O, Ω, or Θ of $g(n)$. Prove your answers. Note, if you choose O or Ω, you must also show that the relationship is not Θ.

   (a) $f(n) = n \log n$, and $g(n) = 3n \log n - n \log(n^2 + 2)$.

Answer for (a):	$f(n) = \Theta(g(n))$

SOLUTION:

First, argue that $f(n) \in O(g(n))$: We will show that if $n \geq 9$, $n \log n \leq 6n \log n - 2n \log(n^2 + 2)$ (i.e. $n_0 = 9, c = 2$).

$$\begin{aligned}
& n \log n && \leq 6n \log n - 2n \log(n^2 + 2) && \\
\iff \quad & 5n \log n && \geq 2n \log(n^2 + 2) && \text{by algebra, and since } n \geq 1 \\
\iff \quad & 5 \log n && \geq 2 \log(n^2 + 2) && \text{since } n > 0 \\
\iff \quad & \log n^5 && \geq \log(n^2 + 2)^2 && \text{by defn of log} \\
\iff \quad & n^5 && \geq (n^2 + 2)^2 && \text{since } n \geq 1 \text{ and log is increasing} \\
\iff \quad & n^5 && \geq n^4 + 4n^2 + 4 && \text{algebra}
\end{aligned}$$

Since $n \geq 9$, $n^5 = n(n^4) \geq 9(n^4) \geq n^4 + 4n^2 + 4$ and we're done.

■

Next prove that $f(n) \in \Omega(g(n))$. We will show that if $n \geq 1$, then $n \log n \geq 3n \log n - n \log(n^2 + 2)$ (i.e. $n_0 = 1$, $c = 1$)

$$\begin{aligned}
& n \log n && \geq 3n \log n - n \log(n^2 + 2) && \\
\iff \quad & \log n && \geq 3 \log n - \log(n^2 + 2) && \text{since } n \geq 1.
\end{aligned}$$

Now, using log rules:

$$\begin{aligned}
3 \log n - \log(n^2 + 2) &= \log n^3 - \log(n^2 + 2) \\
&= \log \frac{n^3}{n^2 + 2} \\
&\leq \log n
\end{aligned}$$

■

(b) $f(n) = \frac{1}{2}n^3$, and $g(n) = n^2 + 4n + 37$.

Answer for (b):	$f(n) \in \Omega(g(n))$

SOLUTION:

Let $c = 2$, and $n_0 = 42$, and consider an arbitrary $n \geq 42$. Observe that $n^2 + 4n + 37 \leq 42n^2 \leq n^3 = 2 \cdot \frac{n^3}{2}$, since $n \geq 42$, so $f(n) \in \Omega(g(n))$.

■

Next we show that $f(n) \notin O(g(n))$.

To do this, we negate the definition of $O()$ and show that for any choice of c and n_0, we can find an n so $f(n) > cg(n)$

Consider arbitrary c and n_0, and let $n = \max\{n_0, 84c\} + 1$. Then,

$$\begin{aligned} & n && > 84c \\ \implies & n^3 && > 2cn^2 + 8cn^2 + 74cn^2 \\ \implies & n^3 && > 2cn^2 + 8cn + 74c \\ \implies & \tfrac{1}{2}n^3 && \geq c(n^2 + 4n + 37) \end{aligned}$$

■

6. (16 points) For each C++ function below, give the tightest asymptotic upper bound that you can determine for the function's **runtime**, in terms of the input parameter. Where prompted, also compute the return value of the function. Express both runtime and return value in simplest terms. For all function calls, assume that the input parameter $n \geq 1$.

(a)

```
int coffee(int n) {
   int s = n * n;
   for (int q = 0; q < n; q++)
      s = s - q;
   for (int q = n; q > 0; q--)
      s = s - q;
   return s + 2;
}
```

The first for-loop decrements s by $n(n-1)/2$. The second for-loop decrements s by $n(n+1)/2$. So s, which is initially n^2, has value 0 by the end and the function returns 2.

Return value for (a):	2

Both for-loops take time n.

Running time of (a):	$O(n)$

(b)

```
int tea(int n) {
   int r = 0;
   for (int i = 1; i < n*n*n; i = i * 2)
      r++;
   return r * r;
}
```

The for-loop executes $\lceil \lg(n^3) \rceil = \lceil 3 \lg n \rceil$ times, so the return value is $\lceil 3 \lg n \rceil^2$ and the running time is $O(\log n)$.

Return value for (b):	$\lceil 3 \lg n \rceil^2$

Running time of (b):	$O(\log n)$

(c)

```
int mocha(int n) {
   int r = 0;
   for (int i=0; i<=n; i = i+16)
      for (int j=0; j<i; j++)
         r++;
   return r;
}
```

The inner for-loop executes i times. The outer for-loop executes the inner for-loop with $i = 0, 16, 32, \ldots, 16\lfloor n/16 \rfloor$. So the total number of times r is incremented is $\sum_{k=0}^{\lfloor n/16 \rfloor} 16k = 16\lfloor n/16\rfloor(\lfloor n/16\rfloor + 1)/2 = 8\lfloor n/16\rfloor(\lfloor n/16\rfloor + 1)$.

Return value for (c):	$8\lfloor n/16\rfloor(\lfloor n/16\rfloor + 1)$

Running time of (c):	$O(n^2)$

(d)

```
int espresso(int n) {
    int j=0;
   for (int k = 16; coffee(k) * mocha(k) - k <= n; k+=16) {
      j++;
     cout << "I am having so much fun with asymptotics!" << endl;
   }
}
return j;
```

Since k is always divisible by 16, the value of `coffee(k) * mocha(k) - k` is $2 \times 8(k/16)(k/16+1) - k = k^2/16$. Thus the for-loop executes until k reaches (or exceeds) $4\sqrt{n}$, which happens after $\lfloor \sqrt{n}/4 \rfloor$ iterations.

Return value for (d):	$\lfloor \sqrt{n}/4 \rfloor$

Each iteration takes constant time plus $O(k)$ time to calculate `coffee(k)` and $O(k^2)$ time to calculate `mocha(k)`, which means each iteration takes $O(k^2)$ time. The total time is asymptotically

$$\sum_{i=1}^{\sqrt{n}/4} O((16i)^2) = O\left((16)^2 \sum_{i=1}^{\sqrt{n}/4} i^2\right) = O\left(\sum_{i=1}^{\sqrt{n}} i^2\right) = O(n^{3/2})$$

since each term in the sum is at most $\sqrt{n}$. This, perhaps surprisingly, is tight.

Running time of (d):	$O(n^{3/2})$

(e)

```
int latte(int n) {
    int j = 0;
    for (int k = 0; k < coffee(k) * n; k++)
        j = j + 2;
   return espresso(mocha(j));
}
```

The for-loop executes $2n$ times to produce $j = 4n$. The call to `coffee(k)` takes $O(k)$ time, so the total running time of the for-loop is $\sum_{k=0}^{2n} O(k) = O(n^2)$. The call to `mocha(j)` with $j = 4n$ takes $O(n^2)$ time to produce the value $x = 8\lfloor 4n/16 \rfloor(\lfloor 4n/16 \rfloor + 1)$. This value is then given as input to `espresso`, which takes $O(x^{3/2}) = O(n^3)$ time to complete. The total running time is $O(n^2) + O(n^2) + O(n^3) = O(n^3)$.

Running time of (e):	$O(n^3)$

Blank sheet for extra work.