CPSC 121 PROOF BY INDUCTION II SOLUTIONS

Problem 1. The Fibonacci sequence is defined by the equations:

$$\begin{array}{rcl} F(0) & = & 0 \\ F(1) & = & 1 \\ F(n) & = & F(n-1)+F(n-2) \quad \text{for every } n \geq 2 \end{array}$$

Using mathematical induction, prove that $F(n) \geq 1.6^{n-2}$ for every integer $n \geq 1$.

Proof

Base Cases: Because the value $F(n)$ depends on $F(n-1)$ and $F(n-2)$, we will need two base cases:

- For $n = 1$, $F(1) = 1 \geq 0.625 = 1.6^{-1} = 1.6^{1-2}$
- For $n = 2$, $F(2) = F(1) + F(0) = 1 \geq 1 = 1.6^{0} = 1.6^{2-2}$

Induction Step: Consider an unspecified positive integer $n \geq 3$. Assume that $F(n-1) \geq 1.6^{(n-1)-2}$ and $F(n-2) \geq 1.6^{(n-2)-2}$

$$\begin{array}{rcll} F(n) & = & F(n-1)+F(n-2) & \text{by defintion of the Fibonacci sequence} \\ & \geq & 1.6^{(n-1)-2}+1.6^{(n-2)-2} & \text{by the induction hypothesis} \\ & = & 1.6^{n-3}+1.6^{n-4} & \\ & = & (1.6+1)1.6^{n-4} & \\ & = & 2.6 \cdot 1.6^{n-4} & \\ & \geq & 2.56 \cdot 1.6^{n-4} & \\ & = & 1.6^{2} \cdot 1.6^{n-4} & \\ & = & 1.6^{n-2} & \end{array}$$

This completes the induction step and thus proving that $\forall n \in \mathbb{Z}^{+}, F(n) \geq 1.6^{n-2}$, QED

Problem 2. Randomized-quick select algorithm:

Find the i^{th} smallest element in an unsorted list as follows.

Choose a random element x of the list.
Divide the list into three sublists:

- list-smaller: elements smaller than x
- list-equal: elements equal to x
- list-larger: elements larger than x

Then:

- Search list-smaller if $i \leq$ length of list-smaller
- Search list-larger if $i >$ length of list-larger
- Otherwise return x

A student proved that the expected number of steps $S(n)$ of the algorithm on a list of n elements is:

$S(1) = 4c$
$S(2) = 12c$
$S(3) = 20c$
$S(n) \leq 2cn + S(\lfloor \frac{3n}{4} \rfloor)$ for any integer$n \geq 4$

Prove the following theorem:

Theorem: For every integer $n \geq 1, S(n) \leq 8cn$. Where $S(n)$ is the expected number of steps of the randomized quick select algorithm for a list of n elements.

Proof: We prove the theorem by induction.

Base cases:

We need to prove that $S(k) \leq 8ck$ for $1 \leq k \leq 3$

$$\begin{array}{llll} n=1, & S(1)=4c, & 8c \cdot 1 = 8c, & S(1) = 4c \leq 8c \\ n=2, & S(2)=12c, & 8c \cdot 2 = 16c, & S(2) = 12c \leq 16c \\ n=3, & S(3)=20c, & 8c \cdot 3 = 24c, & S(3) = 20c \leq 24c \end{array}$$

Induction step:

Consider an unspecified integer $n \geq 4$

Induction Hypothesis: assume that $S(k) \leq 8ck$ for any $1 \leq k \leq n-1$.

We need to prove that $S(n) \leq 8cn$

$$\begin{array}{rcl} S(n) & \leq & 2cn + S(\lfloor \frac{3n}{4} \rfloor) \\ & \leq & 2cn + 8c\lfloor \frac{3n}{4} \rfloor \text{ by the induction hypothesis since } \lfloor \frac{3n}{4} \rfloor \leq \frac{3n}{4} < n \\ & \leq & 2cn + 8c(\frac{3n}{4}) \text{ by the definition of floor} \\ & = & 2cn + 6cn \\ & = & 8cn \end{array}$$

QED

Problem 3. Suppose that you can buy chicken nuggets only in packs of 4, 6, and 7. What is the largest number of chicken nuggets you cannot buy using the available packs of 4, 6 and 7?

Solution

A set of 9 chicken nuggets cannot be made by any combination of packs of 4, 6, or 7, therefore 9 chicken nuggets cannot be purchased. Hence, if it can be shown that all quantities of chicken nuggets 10 or greater can be purchased using only packs of 4, 6, and 7, then 9 must be the largest number of chicken nuggets that cannot be bought.

Therefore, let us try to prove the theorem for any integer $n \geq 10$, we can buy n chicken nuggets using packs of 4, 6, and 7 only.

Proof: We prove the theorem by induction.

Base Cases:

Since we are trying to prove our theorem for $n \geq 10$, and our induction step begins at $n \geq 14$ we therefore need to prove that we can buy 10, 11, 12, and 13 chicken nuggets using only packs of 4, 6, and 7.

(1) We can buy 10 chicken nuggets by buying a pack of 4 nuggets and a pack of 6 nuggets.
(2) We can buy 11 chicken nuggets by buying a pack of 4 nuggets and a pack of 7 nuggets.
(3) We can buy 12 chicken nuggets by buying three packs of 4 nuggets.
(4) We can buy 13 chicken nuggets by buying a pack of 6 nuggets and a pack of 7 nuggets.

Induction Step:

Consider an unspecified integer $n \geq 14$

(*Note:* the choice to start the induction step from $n \geq 14$ is due to the subsequent steps in the induction proof. Since our proof utilizes the inductive hypothesis by utilizing our assumption that we can purchase $n - 4$ nuggets, and since we are trying to prove the theorem for $n \geq 10$, we must then choose to start our induction step at some integer j, such that $n \geq j > j - 4 \geq 10$. Therefore, the minimum choice of j in our case is $j = 10 + 4 = 14$. If we are proving the theorem for $n \geq 10$, but our induction step only proves the theorem for $n \geq 14$, then we now know that we need to explicitly prove the base cases for $n = 10, 11, 12$, and 13.

This explanation is not part of the formal proof for this problem, however, it is placed here to provide a sufficient explanation for the reader.)

Inductive hypothesis: assume that we can buy $k \in \mathbb{Z}^+$ chicken nuggets for any $10 \leq k \leq n - 1$

We need to show that we can buy n chicken nuggets.

By the induction hypothesis , suppose that we can buy $(n - 4)$ chicken nuggets using x 4 packs, y 6 packs, and z 7 packs. Where $x, y, z \in \mathbb{N}$.

Thus we can purchase: $n - 4 = 4x + 6y + 7z$

Next, we can purchase an additional pack of four chicken nuggets such that:

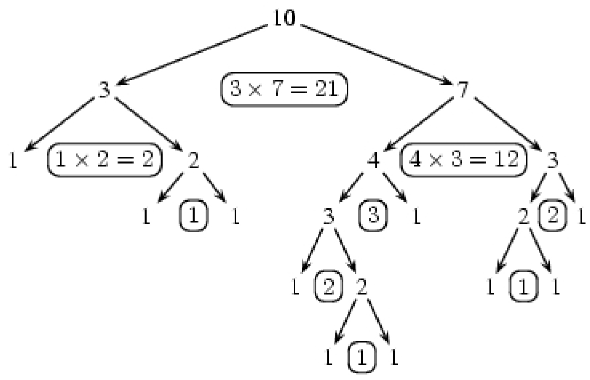
$n = 4x + 6y + 7z + 4 = 4(x + 1) + 6y + 7z$

Since $x + 1$ along with y and z are non-negative integers, it is possible for us to purchase n chicken nuggets using only packs of 4, 6, and 7. This completes the induction step.

QED

As it is shown that we can buy $n \geq 10$ chicken nuggets using only packs of 4, 6, and 7. Since 9 chicken nuggets cannot be bought from packs of 4, 6, and 7, it follows that 9 must be the largest number of chicken nuggets we cannot purchase.

Problem 4. Suppose you begin with a pile of n cards, and split this pile into n piles of one card each by successively splitting a pile of cards into two smaller piles. Each time you split a pile, you multiply the number of cards in each of the two smaller piles you form, so that if these piles have r and s cards in them, respectively, you compute rs. Show that no matter how you split the piles, the sum of the products computed at each step equals $\frac{n(n-1)}{2}$. Here is an example that shows how the computation might proceed:



The sum is $21 + 2 + 12 + 1 + 3 + 2 + 2 + 1 + 1 = 45$, which is indeed $(10 \times 9)/2$.

Solution:

We prove the claim by induction on n, the number of cards.

Base Case:

Our base case is $n = 1$. We have a single pile which cannot be split further, thus, we have no products of sub-piles to sum.

Consequently, the sum of products $0 = \frac{1(1-1)}{2}$, which proves the theorem for $n = 1$, completing the base case.

Induction Step:

Consider an arbitrary $n \in \mathbb{Z}^+$.

Assume $\forall i \in \mathbb{Z}^+, 1 \leq i \leq n$, no matter how a deck of i cards is split into i piles of one card each, the sum of the products computed at each splitting step equals $\frac{i(i-1)}{2}$.

Given a deck of $n + 1$ cards, consider an arbitrary splitting of this deck into k and $n + 1 - k$ sub-decks, where $k \in \mathbb{Z}^+$, and $k \leq n$.

Since $1 \leq k \leq n$, and $1 \leq n + 1 - k \leq n$, by the inductive hypothesis the decks of k and $n + 1 - k$ cards, no matter how they are split into k and $n + 1 - k$ piles of one card each, respectively, the sum of the products computed at each splitting step is equivalent to $\frac{k(k-1)}{2}$ and $\frac{(n+1-k)(n-k)}{2}$ respectively.

Now, the sum of products computed at each splitting step for the $n+1$ deck is equivalent to the sum of the product of the first splitting into two piles of k and $n + 1 - k$ cards plus the sum of the products of each of those two piles' sub-piles:

$$
\begin{aligned}
k(n+1-k) + \frac{k(k-1)}{2} + \frac{(n+1-k)(n-k)}{2} &= \frac{(k^2-k)+(n^2-2kn+n+k^2-k)+2k(n+1-k)}{2} \\
&= \frac{n^2+n}{2} \\
&= \frac{(n+1)n}{2}
\end{aligned}
$$

This completes the induction step.

QED