HW 4

Due: 23:00, Wednesday June 12, 2019

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Instructions:

1. Do not change the problem statements we are giving you. Simply add your solutions by editing this latex document. To make it easier for the TAs to find your solutions, please use the **soln** environment we provided as follows:

```
\begin{soln}
My solution is here.
\end{soln}
```

Your solution will then appear in blue, and be easier to differentiate from the questions.

2. Include formatting to clearly distinguish your solutions from the given problem text. Improperly or insufficiently typeset submissions will receive a penalty.

3. If you need more space, add a page between the existing pages using the `\newpage` command.

4. Export the completed assignment as a PDF file for upload to Gradescope.

5. On Gradescope, upload only **one** copy per partnership. (Instructions for uploading to Gradescope are posted on the assignments page of the course website.)

6. During submission, for each question, please link ALL pages on which your solution appears. Submissions with several linking errors will incur a small penalty.

7. Late submissions will be accepted up to 24 hours past the deadline with a penalty of 20% of the assignment's maximum value

Academic Conduct: I certify that my assignment follows the academic conduct rules for of CPSC 121 as outlined on the course website. As part of those rules, when collaborating with anyone outside my group, (1) I and my collaborators took no record but names away, and (2) after a suitable break, my group created the assignment I am submitting without help from anyone other than the course staff.

Version history:

- 2019-05-29 13:22 – Due date updated
- 2019-05-29 02:47 – Initial version for release

1. **[9 marks]** Each of the following theorems is either valid or invalid, but the proof given is incorrect even if the theorem is valid. Explain briefly what mistake was made in each case.

   a. **[3 marks]** **Theorem**: Let f, g and h be three functions from $\mathbf{N}$ into $\mathbf{R}^+$. If $f \in O(g)$ and $g \in O(h)$, then $f \in O(h)$.
   Proof: Consider three unspecified functions f, g and h from $\mathbf{N}$ into $\mathbf{R}^+$. Assume that $f \in O(g)$ and $g \in O(h)$. Since $f \in O(g)$, there is a real number c and a positive integer n_0 such that for every $n \geq n_0$, $f(n) \leq cg(n)$. Similarly, for every $n \geq n_0$, $g(n) \leq ch(n)$. Therefore, for every $n \geq n_0$, $f(n) \leq cg(n) \leq c(ch(n)) = c^2 h(n)$. Hence $f \in O(h)$ using the constants c^2 and n_0.

   The proof is wrong because the variable c in $f(n) \leq cg(n)$ and $g(n) \leq ch(n)$ are not the same variable, but rather a made up generalized variable for defining big-O formula. It would not end up as c$^2, butmorelike$f(n) $\leq c_1 g(n)$, $g(n) \leq c_2 h(n)$, and the final c being $c_1 * c_2$.

   b. **[3 marks]** **Theorem**: If $n^2 + n - 6 \geq 0$, then $n \geq 2$.
   Proof: When $n \geq 2$, we know that $n^2 \geq 4$, so $n^2 + n \geq 6$, and therefore $n^2 + n - 6 \geq 0$.

   The proof is wrong because the structure of the proof is wrong. The theorem is in the form of P->Q, where P=$n^2 + n - 6 \geq 0$, and Q=$n \geq 2$. The proof should use the structure of Q'->P' or simply P->Q, as opposed to currently using Q->P structure.

   This results in overlooking cases where n is less than or equal to -2. When $n^2 \geq 4$, $n \geq 2$ is not necessarily true, because n could have larger magnitude than -2 for $n^2 \geq 4$ to be true. Therefore, for cases of n is less than or equal to -2, $n^2 + n \geq 6$ can be false (ex: n = -2, $n^2 + n = (-2)^2 + (-2) = 2 \leq 6$).

   c. **[3 marks]** **Theorem**: No matter how we choose an integer n, the value $n^3 + n$ will be even.
   Proof: We use a proof by contradiction. Assume that the theorem is false. That is, $n^3 + n$ is odd. This is not true, as we can see by choosing $n = 2$ ($2^3 + 2 = 10$, which is even). So we have found a contradiction, which means the theorem is true.

   The proof is wrong, since for proof by contradiction, we have to use the contradicted statement to make assumptions that would contradict for all n, not find a single n that would proof the contradicted statement false.

   A correct version of this proof would've been more like: Suppose $n^3 + n = x$, where x is odd. We choose a general n to have properties of an even number. n is manipulated using $n^3 + n = x$ to get some odd x, where it is manipulated again to get n, where n now has properties of an odd number. Therefore, theorem is true because when the theorem is false, a contradiction happens (n can't be even and odd at the same time).

Each of the next four questions asks you to prove a theorem. When the theorem is stated in English, it would be an excellent idea to first rewrite it in predicate logic, and then think about the proof techniques we discussed in class, and the general "structure" of direct or indirect proofs for certain types of statements.

2. **[6 marks]** Prove that if n is a positive integer, then $n^3 + 4n + 2$ is not divisible by 4. Hint: divide the proof into two cases.

$\forall n \in \mathbb{Z}^+, n > 0 \to 0 \not\equiv n^3 + 4n + 2(MOD4)$

Proof:
Suppose n is made up of a and b, where a is all positive even integers (a = 2k for some positive integer k) and b is all positive odd integers (b = 2k + 1 for some positive integer k).

Case 1:
LHS $\equiv n^3 + 4n + 2(mod4)$
$\equiv a^3 + 4a + 2(mod4)$
$\equiv (2k)^3 + 4(2k) + 2(mod4)$
$\equiv 8k^3 + 8k + 2(mod4)$
$\equiv 4(2k^3 + 2k) + 2(mod4)$
$\equiv 2(mod4)$

Case 2:
LHS $\equiv n^3 + 4n + 2(mod4)$
$\equiv b^3 + 4b + 2(mod4)$
$\equiv (2k+1)^3 + 4(2k+1) + 2(mod4)$
$\equiv 8k^3 + 12k^2 + 14k + 7(mod4)$
$\equiv 8k^3 + 12k^2 + 12k + 4 + 2k + 3(mod4)$
$\equiv 4(2k^3 + 3k^2 + 3k + 1) + 2k + 3(mod4)$
$\equiv 2k + 3(mod4)$

Case 2.1: k is positive even integer (k = 2j for some positive integer j)
$\equiv 2(2j) + 3(mod4)$
$\equiv 4j + 3(mod4)$
$\equiv 3(mod4)$

Case 2.2: k is positive odd integer (k = 2j+1 for some positive integer j)
$\equiv 2(2j+1) + 3(mod4)$
$\equiv 4j + 5(mod4)$
$\equiv 4(j+1) + 1(mod4)$
$\equiv 1(mod4)$

We can se that for all cases, n is not divisible by 4, as the result of the MOD operation does not yield zero.

$\square$

3. **[6 marks]** Prove that for all integers a, b, and c, if a does not divide $(b-c)$ then a does not divide b or a does not divide c.

$\forall a, b, c \in \mathbb{Z}, \sim(a|(b-c)) \rightarrow \sim(a|b) \vee \sim(a|c)$

Proof:
We can do a proof by contradiction. Suppose the negation of $\forall a, b, c \in \mathbb{Z}, \sim(a|(b-c)) \rightarrow \sim(a|b) \vee \sim(a|c)$, or rewritten as $\exists a, b, c \in \mathbb{Z}, \sim(a|(b-c)) \wedge (a|b) \wedge (a|c)$, is true. For $\sim(a|(b-c)) \wedge (a|b) \wedge (a|c)$ to be true, all there terms has to be true. Therefore, $\sim(a|(b-c))$ is true and there does not exist some integer a,b,c,i where b-c=ia. We'll call this statement 1. Additionally, (a|b) and (a|c) are true and there exists some integer a,b,c,j,k where b=ja and c=ka. We'll call this statement 2 and 3, respectively.

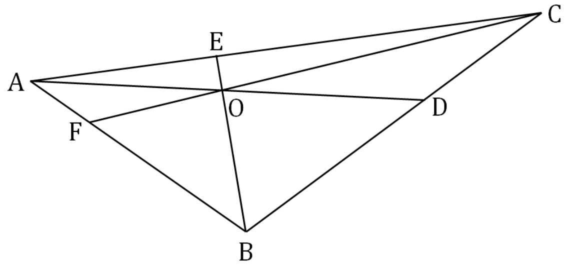
Suppose i=j-k. We can subtract statement 2 by statement 3 together to get:

b-c=ja-ka
b-c=(j-k)a
b-c=ia

We have deduced that statement 1 exists from statement 2 and 3, where both statement 2 and 3 can exist. However, we have stated that statement 1 can not exist for the negated theorem to be true. Therefore, the negated theorem is false, we have a contradiction, and the original theorem is true.

□

4. **[6 marks]** Consider a triangle ABC. Each angle is split a line segment, that all intersect at a common point O somewhere within the circle, but not exactly on any of the sides AB, BC, or AC. An example is shown in the figure below.



Is it possible to draw a triangle, with a common point O somewhere inside the triangle (but not exactly on the sides) such that $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} \neq 1$? In this equation above, XY means the length of the line segment created by the points X and Y. Explain how to draw such a triangle and the common point O, or prove that the product of the specified ratios is always equal to 1. Hint: try to relate side lengths to area, and try to find relations between ratios/sums/differences of side lengths and areas of different triangles.

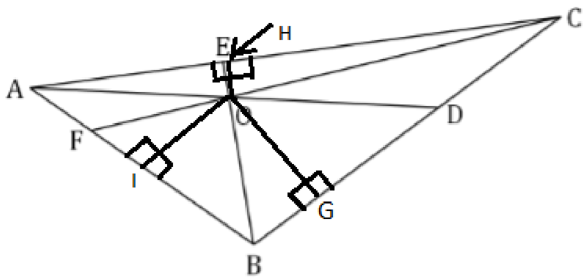
We can define x, y as any x and y coordinate of O, and WithinABC(x, y) as a function that produces true if the coordinates land inside triangle ABC defined in the illustration above, and false otherwise.
$\forall x, y \in \mathbb{R}, WithinABC(x, y) \rightarrow \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1$

Proof:
It is not possible since all positions of O will yield make $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1$ true.

There are two equations for an area of a triangle, where a, b, c are the sides of the triangle, and A, B, C are the angles of the triangle, corresponding respectively:

Area = (1/2)ab(sinC) = (1/2)ch
Where h is a line that is perpendicular to c and intersects the intersection between a and b.

To apply these formulas to this question, we have to label the triangle with more points, so we can draw "h" to satisfy "Area = (1/2)ch" for all triangles:



We can use these new points to write out the area formulas for the given triangles. I will be using $\triangle$ as shorthand for "Area of", $\angle$ for "Angle of", a line above characters for "Segment of", and [n] besides the question to number the corresponding equation. Note that the ordering of the letters does not matter (for example, $\overline{AO} = \overline{OA}$):

[1]$\triangle AOF = (1/2) * \overline{AO} * \overline{FO} * sin(\angle AOF)$
[2]$\triangle AOF = (1/2) * \overline{AF} * \overline{IO}$
[3]$\triangle FOB = (1/2) * \overline{FO} * \overline{BO} * sin(\angle FOB)$
[4]$\triangle FOB = (1/2) * \overline{FB} * \overline{IO}$
[5]$\triangle BOD = (1/2) * \overline{BO} * \overline{DO} * sin(\angle BOD)$
[6]$\triangle BOD = (1/2) * \overline{BD} * \overline{GO}$

$[7] \triangle DOC = (1/2) * \overline{DO} * \overline{CO} * sin(\angle DOC)$
$[8] \triangle DOC = (1/2) * \overline{DC} * \overline{GO}$
$[9] \triangle COE = (1/2) * \overline{CO} * \overline{EO} * sin(\angle COE)$
$[10] \triangle COE = (1/2) * \overline{CE} * \overline{HO}$
$[11] \triangle AOE = (1/2) * \overline{AO} * \overline{EO} * sin(\angle AOE)$
$[12] \triangle AOE = (1/2) * \overline{AE} * \overline{HO}$

Note that because $\overline{AD}$ and $\overline{FC}$ are straight and intersect to create $\triangle AOF$ and $\triangle COD$, $\triangle AOF$ and $\triangle COD$ are equal due to properties of vertically opposite angles. This also applies to $\triangle FOB$ and $\triangle EOC$, as well as $\triangle BOD$ and $\triangle AOE$. This gives use three more equations:

$[13] \triangle AOF = \triangle COD$
$[14] \triangle FOB = \triangle COE$
$[15] \triangle BOD = \triangle AOE$

To prove that $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1$ is true, it is useful to prove that $\frac{\triangle AOF}{\triangle FOB} \cdot \frac{\triangle BOD}{\triangle DOC} \cdot \frac{\triangle COE}{\triangle EOA} = 1$, or in other words $\triangle AOF * \triangle BOD * \triangle COE = \triangle FOB * \triangle DOC * \triangle EOA$ is true. We get this as a result:

$\triangle AOF * \triangle BOD * \triangle COE = \triangle FOB * \triangle DOC * \triangle EOA$ [Start]
$(1/2) * \overline{AO} * \overline{FO} * sin(\angle AOF) * (1/2) * \overline{BO} * \overline{DO} * sin(\angle BOD) * (1/2) * \overline{CO} * \overline{EO} * sin(\angle COE) = (1/2) * \overline{FO} * \overline{BO} * sin(\angle FOB) * (1/2) * \overline{DO} * \overline{CO} * sin(\angle DOC)(1/2) * \overline{AO} * \overline{EO} * sin(\angle AOE)$ [Substitution from equations 1,3,5,7,9,11]
$(1/2) * \overline{AO} * \overline{FO} * sin(\angle DOC) * (1/2) * \overline{BO} * \overline{DO} * sin(\angle AOE) * (1/2) * \overline{CO} * \overline{EO} * sin(\angle FOB) = (1/2) * \overline{FO} * \overline{BO} * sin(\angle FOB) * (1/2) * \overline{DO} * \overline{CO} * sin(\angle DOC)(1/2) * \overline{AO} * \overline{EO} * sin(\angle AOE)$ [Substitution from equations 13,14,15]
$(1/2) * (1/2) * (1/2)\overline{AO} * \overline{BO} * \overline{CO} * \overline{DO} * \overline{EO} * \overline{FO} * sin(\angle FOB) * sin(\angle DOC) * sin(\angle AOE) = (1/2) * (1/2) * (1/2)\overline{AO} * \overline{BO} * \overline{CO} * \overline{DO} * \overline{EO} * \overline{FO} * sin(\angle FOB) * sin(\angle DOC) * sin(\angle AOE)$ [Reordering of terms to make equating more obvious]

Thus, we have proved that $\frac{\triangle AOF}{\triangle FOB} \cdot \frac{\triangle BOD}{\triangle DOC} \cdot \frac{\triangle COE}{\triangle EOA} = 1$ is true. To prove that $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1$ is true, we have to substitute equations 2,4,6,8,10,12 into the proven area equating equation:

$\frac{\triangle AOF}{\triangle FOB} \cdot \frac{\triangle BOD}{\triangle DOC} \cdot \frac{\triangle COE}{\triangle EOA} = 1$ [Start]
$\frac{(1/2)*\overline{AF}*\overline{IO}}{(1/2)*\overline{FB}*\overline{IO}} \cdot \frac{(1/2)*\overline{BD}*\overline{GO}}{(1/2)*\overline{DC}*\overline{GO}} \cdot \frac{(1/2)*\overline{CE}*\overline{HO}}{(1/2)*\overline{AE}*\overline{HO}} = 1$ [Substitution from equations 2,4,6,8,10,12]
$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1$ [Cancellation of all $(1/2), \overline{IO}, \overline{GO}, \overline{HO}$ terms]

Therefore, we prove that it is not possible to draw a triangle, with a common point O somewhere inside the triangle (but not exactly on the sides), such that $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \neq 1$, since $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1$ is true for any arbitrary point.

$\square$

5. **[6 marks]** Suppose we have the following five propositions:

$X_1 \vee X_2 \vee Y_1$
$\sim Y_1 \vee X_3 \vee Y_2$
$\sim Y_2 \vee X_4 \vee Y_3$
$\sim Y_3 \vee X_5 \vee Y_4$
$\sim Y_4 \vee X_6 \vee X_7$

Using an *indirect* proof, show that every assignment of truth values to the eleven variables X_1, X_2, X_3, X_4, X_5, X_6, X_7, Y_1, Y_2, Y_3, Y_4 that makes every one of the five propositions True assigns the value True to *at least* one of X_1, X_2, X_3, X_4, X_5, X_6, X_7.

We define each statement above as a, b, c, d, e respectively.
$\forall X_1, X_2, X_3, X_4, X_5, X_6, X_7, Y_1, Y_2, Y_3, Y_4 \in Boolean, a \wedge b \wedge c \wedge d \wedge e \rightarrow X_1 \vee X_2 \vee X_3 \vee X_4 \vee X_5 \vee X_6 \vee X_7$

Proof:
We can use proof by contradiction. Suppose $a \wedge b \wedge c \wedge d \wedge e$ is true, but $\sim(X_1 \vee X_2 \vee X_3 \vee X_4 \vee X_5 \vee X_6 \vee X_7)$. We can use DeMorgan's theorem to change $\sim(X_1 \vee X_2 \vee X_3 \vee X_4 \vee X_5 \vee X_6 \vee X_7)$ to $\sim X_1 \wedge \sim X_2 \wedge \sim X_3 \wedge \sim X_4 \wedge \sim X_5 \wedge \sim X_6 \wedge \sim X_7$. This means that all X values has to be false.

Statement a is defined as $X_1 \vee X_2 \vee Y_1$, and since we know X_1 and X_2 is false, Y_1 has to be true for the statement to be true (property of Or gates).

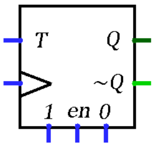
Using the same logic, we can say that for true b, Y_2 is true; for true c, Y_3 is true; for true d, Y_4 is true. Since Y_4 is true, and X_6 and X_7 are false, statement e will evaluate to false. However, e is true for $a \wedge b \wedge c \wedge d \wedge e$ to be true.

Therefore, we have arrived to a contradiction, and we can conclude that the original statement of $\forall X_1, X_2, X_3, X_4, X_5, X_6, X_7, Y_1, Y_2, Y_3, Y_4 \in Boolean, a \wedge b \wedge c \wedge d \wedge e \rightarrow X_1 \vee X_2 \vee X_3 \vee X_4 \vee X_5 \vee X_6 \vee X_7$ is true.

$\square$

6. **[17 marks]**

A *T flip-flop* ("toggle" flip-flop) is a synchronous device with the following specification:



T	function
0	$Q^+ = Q$
1	$Q^+ = \sim Q$

In other words, the state of the flip-flop remains unchanged if $T = 0$ during a rising clock edge, and the state of the flip-flop is negated (toggled) if $T = 1$ during a rising clock edge. The inputs 1, *en*, and 0 are asynchronous set/enable/reset – for the most part, *en* can be connected to constant-1, and set/reset can be connected to constant-0 during ordinary usage.

a. **[3 marks]** Logisim provides a T flip-flop in the "Memory" library. For this question, you will use a single J-K flip-flop and logic gates to construct your own T flip-flop.

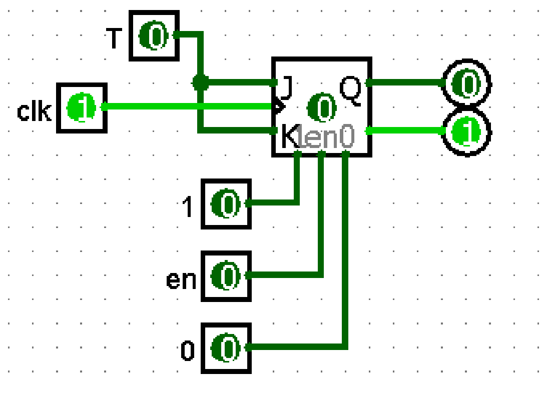
Complete the characteristic/excitation table below to determine the required inputs to the J-K flip-flop, and write your input equations for J and K.

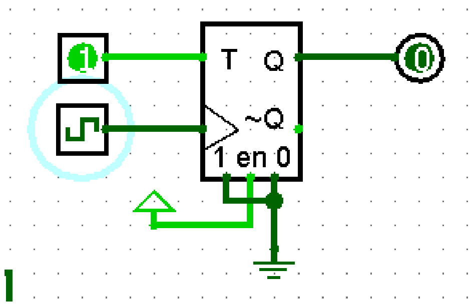
Q	T	Q^+	J	K
0	0	0	0	0
0	1	1	1	1
1	0	1	0	0
1	1	0	1	1

$J = T$

$K = T$

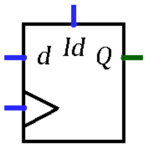
b. **[2 marks]** Based on your input equations determined earlier, in Logisim use a J-K flip-flop and logic gates to construct a circuit that has the required behaviour of a T flip-flop. Show your circuit diagram here. Adjust your device appearance so it matches the illustration given at the top of this page.





The Program Counter (PC) is a critical component of many past and modern CPUs, whose responsibility is to hold the address of the next program instruction to be retrieved from memory. In a typical program, instructions are executed sequentially, and so subsequent instructions are stored in consecutive addresses in memory, and the PC simply needs to increment its stored value to correctly reference the next instruction. However, many programs include branching conditionals and loops, where the next instruction to execute is found at a memory address which is not immediately following the current address stored in PC. Thus, the PC must have the ability to be loaded with an arbitrary address so that the instruction beginning the conditional branch or loop can be executed next. In summary, the PC can be described as an incrementing counter, with parallel load and clear. The full functionality will be described in a later part of this assignment, but we will achieve all the required functionality by first building simpler components which satisfy partial functionality.

We will begin by designing components to achieve the parallel load functionality. A 1-bit load register has the following specification:



ld	function
0	$Q^+ = Q$ (no change)
1	$Q^+ = d$

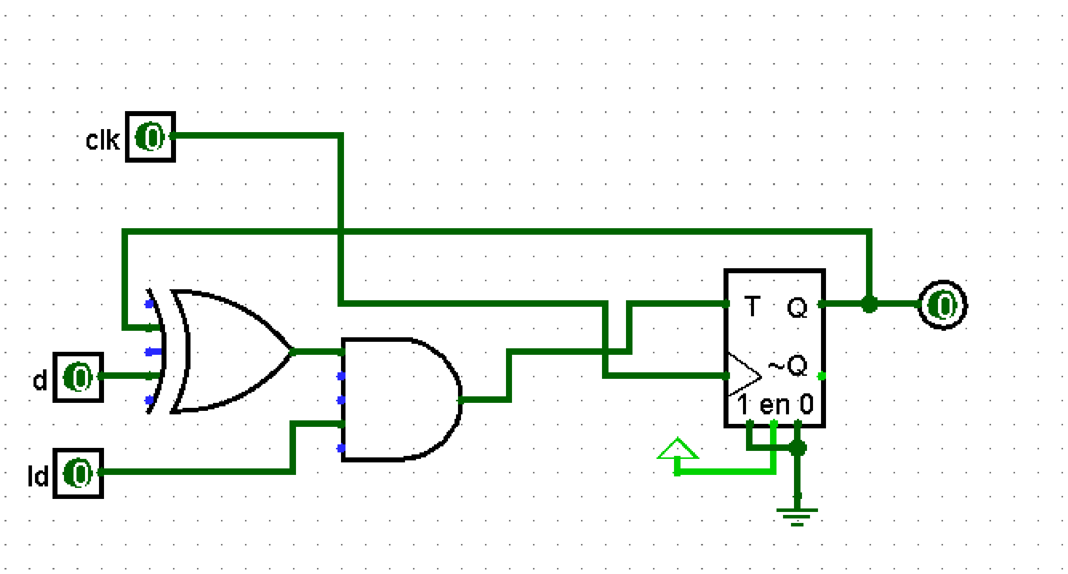
Specifically, when $ld = 0$, the device remains unchanged, and when $ld = 1$, the value observed on the input d is loaded into the device.

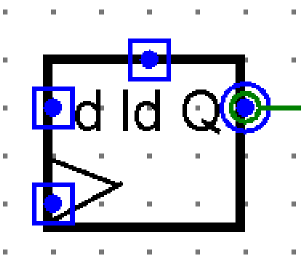
c. **[2 marks]** You will use the T flip-flop constructed in part 6b to construct the 1-bit load register. Complete the characteristic/excitation table below to determine the required inputs to the T flip-flop, and write your input equations for T.

Q	ld	d	Q^+	T
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	0
1	1	0	0	1
1	1	1	1	0

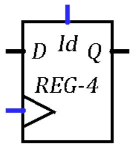
$T = ld \wedge (Q(+)d)$

d. **[2 marks]** Based on your input equation determined above, in Logisim use your T flip-flop and logic gates to construct a circuit that has the required behaviour of the 1-bit load register. It is recommended to connect power and ground devices to the asynchronous set/enable/reset inputs appropriately. Show your circuit diagram here. Adjust your device appearance so it matches the illustration given above.





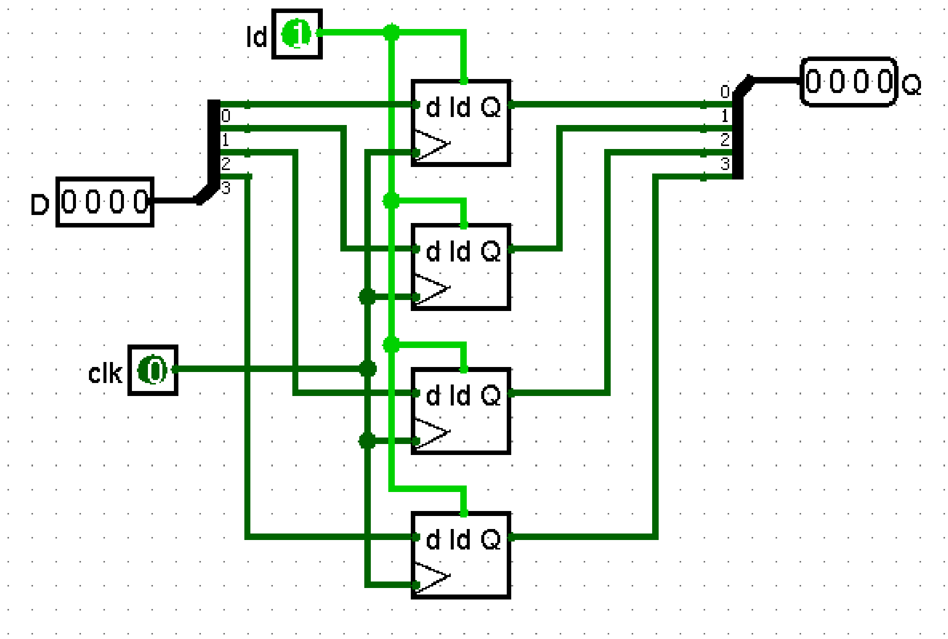
The 1-bit load register can be connected in parallel to construct larger parallel load registers. A 4-bit parallel load register is described below:

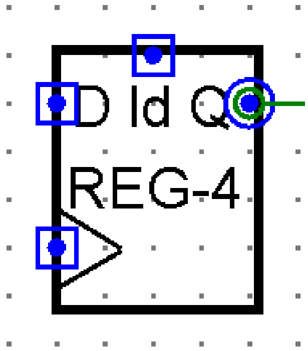


ld	function
0	No change
1	$Q \leftarrow D$

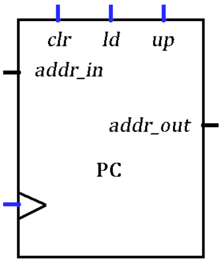
The behaviour is the same as the 1-bit load register, with the understanding that Q and D are 4-bit values.

e. **[2 marks]** Using four (4) of your 1-bit load registers, connect them in parallel to construct a 4-bit parallel load register. Use "splitter" devices to join your data inputs and outputs into single 4-bit buses. Show your circuit diagram here. Adjust your device appearance so it matches the illustration given above.





We are now ready to construct the Program Counter. It has the following specification:

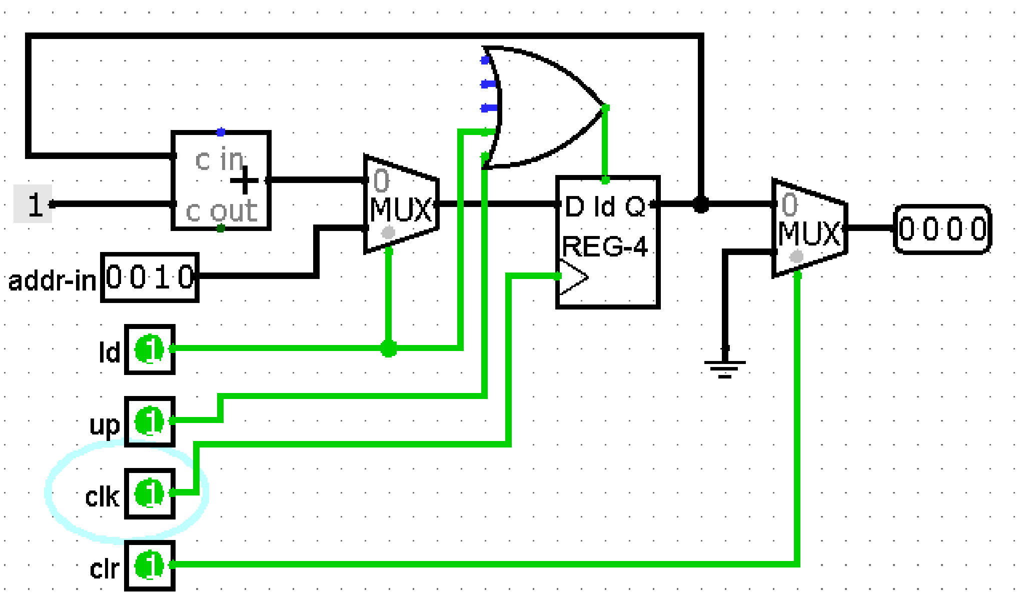


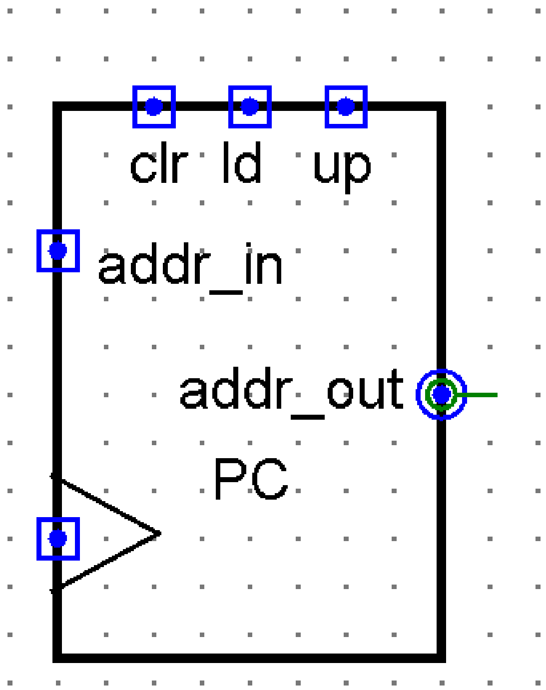
clr	*ld*	*up*	function
0	0	0	No change
0	0	1	$PC \leftarrow PC + 1$
0	1	X	$PC \leftarrow addr\_in$
1	X	X	$PC \leftarrow 0$

In the table above, PC is understood to mean the state/contents of the PC register.

Suppose $addr\_in$, and the register contents PC each have 4 bits. Then the characteristic table for this device would have $2^{11} = 2048$ rows! Fortunately instead of creating a purpose-built device using the characteristic table, we can observe the overall functionality from the functional description given above instead of analyzing the behaviour of individual bits.

f. **[6 marks]** In Logisim, using one (1) of your 4-bit parallel load registers you constructed, two (2) 4-bit 2×1 multiplexers, one (1) 4-bit full adder, and any logic gates and power/ground devices that you need, construct a circuit that satisfies the required behaviour of the program counter. Draw your circuit diagram here. To produce the necessary multiplexers in Logisim, set "Select Bits" to 1, "Data Bits" to 4, "Disabled Output" to Zero, "Include Enable?" to No.




clr
ld
up
addr_in
addr_out
PC