CPSC 121 SEQUENTIAL CIRCUITS AND PROOF BY INDUCTION SOLUTIONS

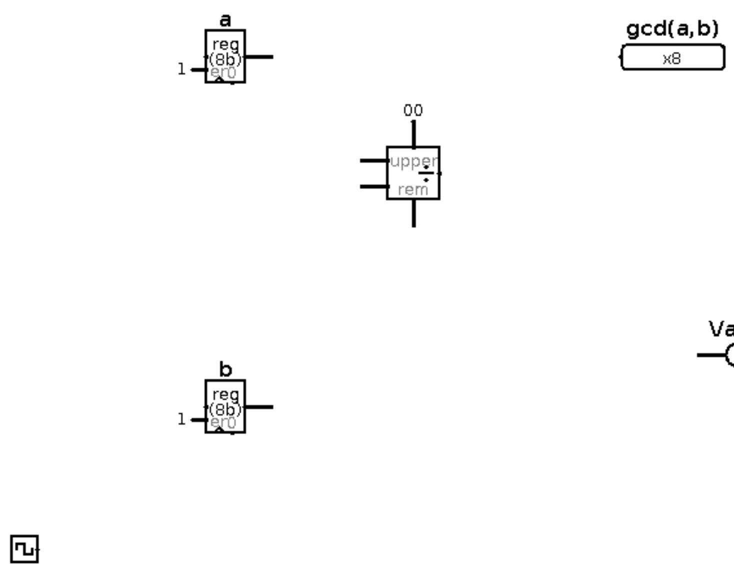
Problem 1. The Greatest Common Divisor (GCD) of two positive integers a, b is the largest positive integer that is a factor of both a and b. Being able to compute GCDs is useful because we can simplify a fraction a/b by dividing both a and b by GCD(a, b). Euclid (born around 325 BC) came up with the following algorithm to compute the GCD of two positive integers. This algorithm relies on the fact that GCD(a, b) = GCD (b, a mod b) as long as $b > 0$. Translated into Racket, since most of us can't read antique Greek, the algorithm becomes:

```
(define (gcd a b)
    (if (= b 0)
        a
        (gcd b (remainder a b))))
```

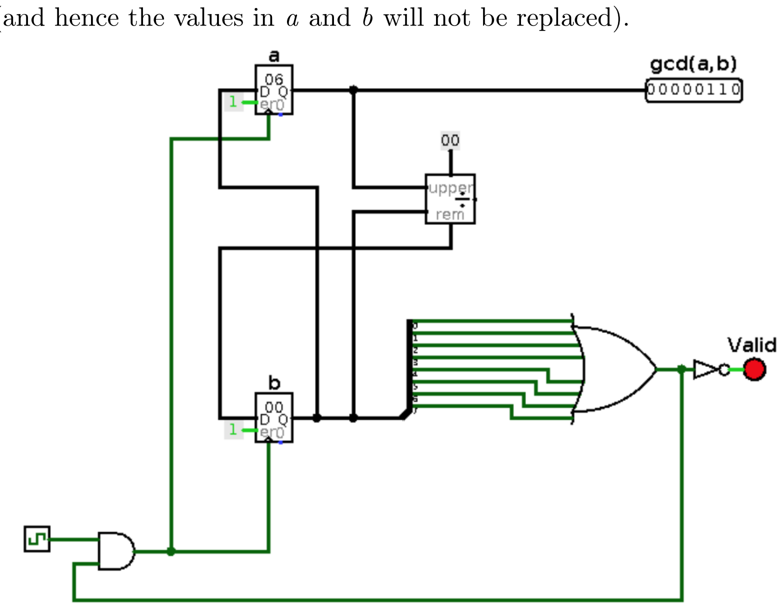
The following is an incomplete sequential circuit to compute the GCD of a pair of positive integers. The computation should start once the initial values of a and b have been entered into the two 8-bit registers `a` and `b`. A number of clock cycles later, the LED labeled `Valid` will be ON, and the output labeled `gcd(a,b)` will contain the GCD of a and b.

Complete the implementation of this circuit. The component labeled "upper" and "rem" is a Divider subcircuit: it takes one 16-bit input x (the 8 bits from the top input, and the 8 bits from the upper-left input) and one 8-bit input y (the 8 bits from the bottom-left input). It produces the quotient x/y (the value obtained by writing `(quotient x y)` in Dr. Racket) on the right output, and the remainder (`(remainder x y)`) on the bottom output.

Hint: consider the correspondence between the parameters in one recursive call and the parameters of the next one.



Solution: Here is the complete circuit. Note that the **AND** gate near the clock ensures that once $b = 0$, the clock will remain low (and hence the values in a and b will not be replaced).



Problem 2. Prove the following statement using induction, $\forall n \in \mathbb{Z}^+, \sum_{i=1}^{n} \frac{2}{3^i} = 1 - \left(\frac{1}{3}\right)^n$

Proof:

Base case:

For $n = 1$, the statement is true because $\sum_{i=1}^{1} \frac{2}{3^i} = \frac{2}{3} = 1 - \left(\frac{1}{3}\right)^1$

Induction Step:

We need to prove $\forall n \in \mathbb{Z}^+, \left(\sum_{i=1}^{n} \frac{2}{3^i} = 1 - \left(\frac{1}{3}\right)^n\right) \to \left(\sum_{i=1}^{n+1} \frac{2}{3^i} = 1 - \left(\frac{1}{3}\right)^{n+1}\right)$

Consider an unspecified positive integer n

Assume that $\sum_{i=1}^{n} \frac{2}{3^i} = 1 - \left(\frac{1}{3}\right)^n$ (*Induction Hypothesis*)

We need to show that $\sum_{i=1}^{n+1} \frac{2}{3^i} = 1 - \left(\frac{1}{3}\right)^{n+1}$

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{2}{3^i} &= \sum_{i=1}^{n} \frac{2}{3^i} + \left(\frac{2}{3^{n+1}}\right) \\
&= 1 - \left(\frac{1}{3}\right)^n + \left(\frac{2}{3^{n+1}}\right) && \text{by the inductive hypothesis} \\
&= 1 - 3\left(\frac{1}{3}\right)^{n+1} + 2\left(\frac{1}{3}\right)^{n+1} \\
&= 1 - (3-2)\left(\frac{1}{3}\right)^{n+1} \\
&= 1 - \left(\frac{1}{3}\right)^{n+1}
\end{aligned}$$

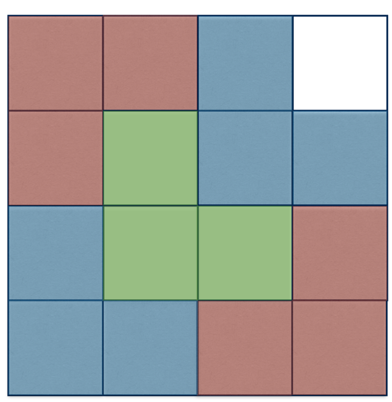
QED

Problem 3. You have a sheet of graph paper, containing squares. The squares are all equal size, and there are an equal number of squares in each row to the number of squares in each column. You have an infinite number of "L"-shaped blocks, called trominos, that can cover three of the squares on your paper in an "L" shape.

Prove that if your graph paper has 2^n by 2^n squares on it, where $n \in \mathbb{Z}^+$, you can cover it such that *all but one square* on your graph paper is covered by a single block and with none of the blocks you used hanging off the edge of the graph paper.

Hint: Prove the stronger claim that you can cover any $2^n \times 2^n$ graph paper with tromino blocks such that all but one corner square is uncovered.

Below is an example of a 4×4 grid covered with trominos such that all but one square (in this example a corner square) is uncovered.



Solution: We will prove a stronger claim, that for any 2^n by 2^n graph paper, we can cover all of its squares but for a single corner square, using tromino blocks. We can leave the corner square unspecified as any $2^n \times 2^n$ grid covered with trominos all but for a corner square can be rotated such that the uncovered corner square can be any of the four corners. Proving this stronger claim will also prove the original claim since the original claim does not specify which square is uncovered, instead, only requiring that exactly one square is uncovered.

Proof: We will prove our 'stronger' claim by induction on n, the size of the $2^n \times 2^n$ grid.

Base Case: For $n = 1$ we have a 2×2 grid. We can thus use a single tromino by placing it in the lower left corner of the grid such that all squares on the grid are covered except for the top right corner square.

Induction Step: Assume that for a $2^n \times 2^n$ grid we can cover all but one corner square using tromino blocks.

Consider a $2^{n+1} \times 2^{n+1}$ grid. We can divide this grid into four $2^n \times 2^n$ subgrids. Next, we can place a single tromino in the middle of the $2^{n+1} \times 2^{n+1}$ grid such that the tromino covers a square in three of the four subgrids.

For each of the three subgrids that have a square covered by the tromino we placed in the middle, each of them has one of their corner squares covered by the tromino. Therefore, we can cover all of the $2^n \times 2^n - 1$ remaining squares by the induction hypothesis, thus completely covering these subgrids.

For the $2^n \times 2^n$ subgrid that does not have a square covered by the middle tromino we had initially placed, we can cover all but a corner square of the $2^n \times 2^n$ squares using the induction hypothesis. In particular, we can choose to leave the corner square that corresponds to a corner of the larger $2^{n+1} \times 2^{n+1}$ grid.

Consequently we have now covered all of the squares of the $2^{n+1} \times 2^{n+1}$ graph paper except for one of the corner squares, completing the induction step.

QED