HW 3

Due: 23:00, Wednesday May 29, 2019

CS ID 1: d6a2b

CS ID 2: b9i2b

Instructions:

1. Do not change the problem statements we are giving you. Simply add your solutions by editing this latex document.

2. If you need more space, add a page between the existing pages using the `\newpage` command.

3. Include formatting to clearly distinguish your solutions from the given problem text (e.g. use a different font colour for your solutions). Improperly or insufficiently typeset submissions will receive a penalty.

4. Export the completed assignment as a PDF file for upload to Gradescope.

5. On Gradescope, upload only **one** copy per partnership. (Instructions for uploading to Gradescope are posted on the assignments page of the course website.)

6. During submission, for each question, please link ALL pages on which your solution appears. Submissions with several linking errors will incur a small penalty.

7. Late submissions will be accepted up to 24 hours past the deadline with a penalty of 20% of the assignment's maximum value

Academic Conduct: I certify that my assignment follows the academic conduct rules for of CPSC 121 as outlined on the course website. As part of those rules, when collaborating with anyone outside my group, (1) I and my collaborators took no record but names away, and (2) after a suitable break, my group created the assignment I am submitting without help from anyone other than the course staff.

Version history:

- 2019-05-22 02:52 – Initial version for release

1. **[9 marks]** Consider the following sets and predicates:

- D: video game developers
- G: video game titles
- P: hardware platforms
- $M(d, g)$: developer d made game g.
- $R(g, p)$: game g released on platform p.
- $B(d, g)$: developer d bribed a journalist to write a favourable review for game g.

Rewrite each of the following statements using **only** the quantifiers $\forall$ and $\exists$, the predicates M, R and B, the domains D, G, P, and $\mathbb{R}$ (the set of real numbers), logical connectives, and the operators $=$, $\neq$. $<$, $\leq$, $\geq$ and $>$.

If you feel a sentence is ambiguous, then state your assumed interpretation.

a. **[3 marks]** Every developer made a game that released on every platform.

b. **[3 marks]** Some developer bribed a journalist to write a favourable review for a game made by a different developer.

c. **[3 marks]** Some developer bribed journalists to write favourable reviews for every game that the developer made.

1(a).
$\forall x \in D, \exists y \in G, M(x, y) \rightarrow \forall z \in P, R(y, z)$

1(b).
$\exists x \in D, \exists y \in D, \exists z \in G, B(x, z) \land x \neq y \land M(y, z)$

1(c).
$\exists x \in D, \forall y \in G, M(x, y) \rightarrow B(x, y)$

2. **[9 marks]**

Using the definitions given in question 1, translate each of the following predicate logic statements into English. Try to make your English translations as natural sounding as possible.

a. **[3 marks]** $\exists d \in D, \forall x, y \in G, \exists p_1, p_2 \in P, M(d,x) \wedge M(d,y) \wedge R(x,p_1) \wedge R(y,p_2) \rightarrow p_1 = p_2 \rightarrow \sim(\exists g \in G, M(d,g) \wedge B(d,g))$.

b. **[3 marks]** $\exists d \in D, \forall g \in G, \sim M(d,g) \wedge (R(g, \text{Nintendo Switch}) \rightarrow B(d,g))$.

c. **[3 marks]** $\exists d \in D, \forall p \in P, \exists g_1, g_2 \in G, M(d,g_1) \wedge M(d,g_2) \wedge R(g_1,p) \wedge \sim R(g_2,p)$.

2(a).
If a developer only develop all games for a single platform, then it can't be that the developer made a game that they bribed a journalist for a positive review.

2(b).
A developer bribed a journalist for positive reviews on all the games released on Nintendo Switch that they didn't make.

2(c).
Some developer(s) release some games on exclusive (one platform only), and other games on all platforms.

3. **[9 marks]** Write the negation of each statement below. Then, write "Original" if the original statement is true, write "Negation" if the negated statement is true, or write "Neither" if the truth value cannot be determined.

   (a) **[3 marks]** $\forall x \in \mathbb{R}, x > 0 \rightarrow \forall y \in \mathbb{R}(y < x) \rightarrow (x + y < 2x)$

   (b) **[3 marks]** $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \frac{x}{y} \leq 1$

   (c) **[3 marks]** $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \forall z \in \mathbb{Z}, x^3 + y^3 + z^3 \geq 0$

3(a).
$\exists x \in \mathbb{R}, x > 0 \wedge (\exists y \in \mathbb{R}, y < x \wedge x + y \geq 2x)$
Original (2x = x+x > x+y if y<x)

3(b).
$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x/y > 1$
Neither (div 0 = null)

3(c).
$\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \exists z \in \mathbb{Z}, x^3 + y^3 + z^3 < 0$
Negation (+x, +y, -z)

4. **[9 marks]** Rewrite each of the following theorems using quantifiers and predicates. Note that the theorems are not *precisely* stated. It is up to you to choose reasonable sets from which the variables should be drawn. Further, notice that the definitions we have given you should be stated as predicates, but we would like *you* to formalize their specification. Make sure that your theorems have no unbound variables.

   (a) **[3 marks]** **Theorem**: For any non-zero real number x there is exactly one y such that $\frac{x}{y} = -1$.

   (b) **[3 marks]** **Theorem**: For any positive integer $x \geq 2$, x is prime if and only if x is evenly divisible by 1 and itself, and no other values in between.

   (c) **[3 marks]** We define "big-O" as follows:

   $$f(n) \in O(g(n)) : \quad \exists c \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \longrightarrow f(n) \leq cg(n)$$

   Theorem: Let k be a constant. If $f(n) \in O(h(n))$ then $k \cdot f(n) \in O(h(n))$.

4(a).
$\forall x \in \mathbb{R}, x \neq 0 \rightarrow (\exists y \in \mathbb{R}, \forall z \in \mathbb{R}, (x/y = -1) \iff (z = y))$

4(b).
$\forall x \in \mathbb{Z}, x \geq 2 \rightarrow (Prime(x) \iff (\exists y \in \mathbb{Z}, (x/1 = y) \leftrightarrow (x = y)))$

4(c).
$\exists k \in \mathbb{R}, f(n) \in O(h(n)) \rightarrow k \cdot f(n) \in O(h(n))$
Assuming f(n), h(n), and O(n) is previously defined in definition of O(n)

5. **[9 marks]** *Order Notation.* In this question we consider algorithms A, B with runtime functions $a(n), b(n)$.

   For each of the following three scenarios, indicate whether the claim is true or false, and give an informal argument to support your answer.

   (a) **[3 marks]** If $a(n) = 0.5n^3 + 4n^2$ and $b(n) = 256n^2$, then $a(n) \in O(b(n))$.

   (b) **[3 marks]** If $a(n) = 2^{3\log_2 n}$ and $b(n) = n^3 \cdot \log_2 n$, then $a(n) \in O(b(n))$.

   (c) **[3 marks]** If $a(n) = 4n^2, b(n) = 3n^4$ and $c(n) = n^4 - 10n^2$, then $a(n) + b(n) \in O(c(n))$.

5(a).
False, because the highest order term in a(n) is a polynomial of the third order, while in b(n) it is a polynomial of the second order. Over n, third order polynomial will outpace second order polynomial term, therefore a(n) cannot be within O(b(n)).

5(b).
True, because witness a(n) $\leq$ c*b(n) + n0, where c = 3/4 and n0 = 2.

5(c).
True, because the terms in a(n)+b(n) are the same order as the terms in c(n) (4 and 2, respectively). Therefore, there must exist some c and n0 such that a(n)+b(n) can be scaled to satisfy c*C(n)+n0 > a(n)+b(n).

6. **[6 marks]** Given a finite set S of n integers, $S = \{s_1, s_2, \ldots, s_n\}$, consider the problem of finding a subset of those integers, such that the sum of the items in the subset is equal to 0.

   One possible solution is to generate all possible subsets of S, and then calculate the sum of the values in each subset to see if it is equal to zero.

   a. **[4 marks]** The number of possible subsets for a set of n integers satisfies the following equation:

   $$B(n) = \begin{cases} 2B(n-1) & \text{if } n > 1 \\ 1 & \text{if } n = 0 \end{cases}$$

   For instance, when $n = 0$ there is only one possible subset: the empty set. When $n = 2$ there are four possible subsets:

   $$\begin{array}{ll} \{\} & \{s_1\} \\ \{s_2\} & \{s_1, s_2\} \end{array}$$

   Give a (very simple) mathematical expression for $B(n)$. Use the formula from the first part to justify your answer.

   b. **[2 marks]** Suppose you run an algorithm which takes a number n and a set S of that size and then outputs every subset on a machine where your algorithm takes .0001 seconds for each subset it outputs.

   Fill in the following table with the largest value of n for which your algorithm terminates for each duration indicated.

Duration	Largest value of n	Duration	Largest value of n
1 second		1 minute	
1 hour		1 day	
1 month		1 year	
10 years		100 years	

6(a).
We can observe a pattern:

n	$B(n)$	$Note$	2^n
0	1	Empty subset	1
1	2	Empty subset and itself	2
2	4	As shown	4
3	8	Previous subsets times 2	8
4	16		16
5	32		32

We can observe that B(n) = 2^n, as the values are equal. This makes sense since B(n) = 2B(n-1) = 2*$2^{(n-1)}$ = 2^n.

6(b).
t = duration in seconds = 0.0001 * B(n), therefore:
n = log(base2)(t/0.0001)
We can use this formula to calculate largest n (rounded down) for all t

Note that to account for leap years, we'll assume a year is 365.25 days long, and a month is 30.4375 days long.

Duration	t/0.0001	Largest n	Duration	t/0.0001	Largest n
1 second	10000	13	1 minute	600000	19
1 hour	36000000	25	1 day	864000000	29
1 month	26298000000	34	1 year	315576000000	38
10 years	3.15576e+12	41	100 years	3.15576e+13	44

Figure 1. This is a rare Gregor that only appears once every 1000 homework submissions. If you give extra points for Latex on this hw, you'll be blessed by rare Gregor with students with nice handwriting when grading exams for the next 2 yrs.