CPSC 121 - MATHEMATICAL PROOFS

1. Direct Proof

Problem 1. Prove that the fourth power of a positive odd integer can be written in the form $8m + 1$, where m is a non-negative integer. Hint: a positive odd integer can be written as $2i + 1$, where i is a non-negative integer.

> **Definition.** The *floor function* assigns to the real number x the largest integer that is less than or equal to x. In other words, the floor function rounds a real number down to the nearest integer. The value of the floor function is denoted by $\lfloor x \rfloor$.
>
> **Definition.** The *ceiling function* assigns to the real number x the smallest integer that is greater than or equal to x. In other words, the ceiling function rounds a real number up to the nearest integer. The value of the ceiling function is denoted by $\lceil x \rceil$.
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> **Example.** $\lfloor \pi \rfloor = 3$. $\lceil \pi \rceil = 4$. $\lfloor -\frac{1}{2} \rfloor = -1$. $\lceil -\frac{1}{2} \rceil = 0$. $\lfloor \sqrt{2} \rfloor = 1$. $\lceil \sqrt{2} \rceil = 2$. $\lfloor -5 \rfloor = -5$. $\lceil -5 \rceil = -5$.

Problem 2. Prove that for any positive integer x, if x is one more than a multiple of 3, then the sum: $2 \cdot \lfloor \frac{x}{3} \rfloor + \lceil \frac{x}{3} \rceil = x$

Problem 3. Prove that for any integer m, if m is a perfect square, then $m + 2$ is not a perfect square.

2. Proof by Contraposition

Problem 4. Prove that for any integer n, if $n^2 + 8n - 1$ is even, then n is odd.

Problem 5. Prove that for any integers a, b and n, if $n \nmid (a \cdot b)$, then $n \nmid a$ and $n \nmid b$.

Problem 6. Prove that for any $x, y \in \mathbb{R}$, if $y^3 + yx^2 \leq x^3 + xy^2$, then $y \leq x$.

3. Proof by Contradiction

Problem 7. Prove that for any two real numbers a and b, if a is rational and ab is irrational, then b is irrational.

Problem 8. Prove that there do not exist integers a and b such that $7a + 21b = 1$.

Problem 9. Prove that for any integers a, b, and c, if $a^2 + b^2 = c^2$, then a is even or b is even.