CPSC 121 PROOF BY INDUCTION II

Problem 1. The Fibonacci sequence is defined by the equations:

$$\begin{aligned} F(0) &= 0 \\ F(1) &= 1 \\ F(n) &= F(n-1)+F(n-2) \quad \text{for every } n \geq 2 \end{aligned}$$

Using mathematical induction, prove that $F(n) \geq 1.6^{n-2}$ for every integer $n \geq 1$.

Problem 2. Randomized-quick select algorithm:

Find the i^{th} smallest element in an unsorted list as follows.

Choose a random element x of the list.
Divide the list into three sublists:

- list-smaller: elements smaller than x
- list-equal: elements equal to x
- list-larger: elements larger than x

Then:

- Search list-smaller if $i \leq$ length of list-smaller
- Search list-larger if $i >$ length of list-larger
- Otherwise return x

A student proved that the expected number of steps $S(n)$ of the algorithm on a list of n elements is:

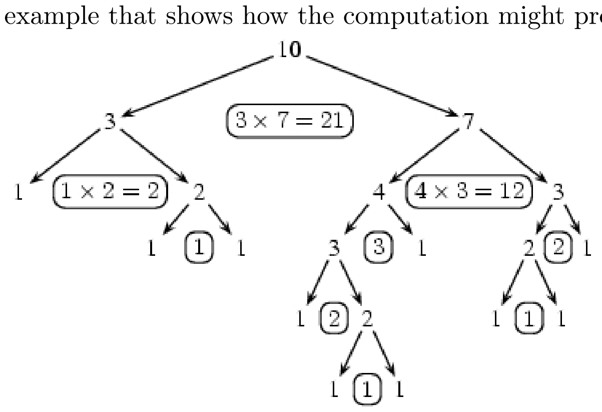
$S(1) = 4c$
$S(2) = 12c$
$S(3) = 20c$
$S(n) \leq 2cn + S(\lfloor \frac{3n}{4} \rfloor)$ for any integer$n \geq 4$

Prove the following theorem:

Theorem: For every integer $n \geq 1, S(n) \leq 8cn$. Where $S(n)$ is the expected number of steps of the randomized quick select algorithm for a list of n elements.

Problem 3. Suppose that you can buy chicken nuggets only in packs of 4, 6, and 7. What is the largest number of chicken nuggets you cannot buy using the available packs of 4, 6 and 7?

Problem 4. Suppose you begin with a pile of n cards, and split this pile into n piles of one card each by successively splitting a pile of cards into two smaller piles. Each time you split a pile, you multiply the number of cards in each of the two smaller piles you form, so that if these piles have r and s cards in them, respectively, you compute rs. Show that no matter how you split the piles, the sum of the products computed at each step equals $\frac{n(n-1)}{2}$. Here is an example that shows how the computation might proceed:



The sum is $21+2+12+1+3+2+2+1+1 = 45$, which is indeed $(10 \times 9)/2$.