CPSC 121 TUTORIAL - PROPOSITIONAL LOGIC SOLUTIONS

Problem 1. Show the following logical equivalences using the known logical equivalences. Explicitly indicate the name of every known logical equivalence that you have used.

(1) $\sim(\sim(p \vee q \vee p) \wedge p) \equiv T$
(2) $p \vee q \equiv \sim(\sim p \wedge ((\sim q \wedge \sim p) \vee (\sim q \wedge p)))$
(3) $(\sim p \vee s) \wedge (p \rightarrow (s \rightarrow \sim p)) \equiv \sim p$
(4) $(p \vee q) \equiv (\sim q \rightarrow p) \vee \sim(\sim p \wedge \sim q) \vee ((p \wedge \sim p) \wedge (q \vee \sim q))$

Solution:

(1) Show $\sim(\sim(p \vee q \vee p) \wedge p) \equiv T$.

$\sim(\sim(p \vee q \vee p) \wedge p)$	
$\equiv \sim(\sim(p \vee q \vee p)) \vee \sim p$	by De Morgan's Law
$\equiv (p \vee q \vee p) \vee \sim p$	by Double Negative Law
$\equiv p \vee q \vee p \vee \sim p$	by Associative Law
$\equiv p \vee p \vee \sim p \vee q$	by Commutative Law
$\equiv p \vee \sim p \vee q$	by Idempotent Law
$\equiv T \vee q$	by Negation Law
$\equiv T$	by Universal Bound Law

(2) Show $p \vee q \equiv \sim(\sim p \wedge ((\sim q \wedge \sim p) \vee (\sim q \wedge p)))$.

$\sim(\sim p \wedge ((\sim q \wedge \sim p) \vee (\sim q \wedge p)))$	
$\equiv \sim\sim p \vee \sim((\sim q \wedge \sim p) \vee (\sim q \wedge p))$	by De Morgan's Law
$\equiv p \vee \sim((\sim q \wedge \sim p) \vee (\sim q \wedge p))$	by Double Negation Law
$\equiv p \vee (\sim(\sim q \wedge \sim p) \wedge \sim(\sim q \wedge p))$	by De Morgan's Law
$\equiv p \vee ((\sim\sim q \vee \sim\sim p) \wedge (\sim\sim q \vee \sim p))$	by De Morgan's Law
$\equiv p \vee ((q \vee p) \wedge (q \vee \sim p))$	by Double Negation Law
$\equiv p \vee (q \vee (p \wedge \sim p))$	by Distributive Law
$\equiv p \vee q \vee F$	by Negation Law
$\equiv p \vee q$	by Identity Law

(3) Show $(\sim p \vee s) \wedge (p \rightarrow (s \rightarrow \sim p)) \equiv \sim p$.

$(\sim p \vee s) \wedge (p \rightarrow (s \rightarrow \sim p))$	
$\equiv (\sim p \vee s) \wedge (\sim p \vee (\sim s \vee \sim p))$	by Definition of Implication
$\equiv (\sim p \vee s) \wedge (\sim p \vee (\sim p \vee \sim s))$	by Commutative Law
$\equiv (\sim p \vee s) \wedge ((\sim p \vee \sim p) \vee \sim s)$	by Associative Law
$\equiv (\sim p \vee s) \wedge (\sim p \vee \sim s)$	by Idempotent Law
$\equiv \sim p \vee (s \wedge \sim s)$	by Distributive Law
$\equiv \sim p \vee F$	by Negation Law
$\equiv \sim p$	by Identity Law

(4) Show $(p \vee q) \equiv (\sim q \rightarrow p) \vee \sim(\sim p \wedge \sim q) \vee ((p \wedge \sim p) \wedge (q \vee \sim q))$.

$(\sim q \rightarrow p) \vee \sim(\sim p \wedge \sim q) \vee ((p \wedge \sim p) \wedge (q \vee \sim q))$	
$\equiv (\sim q \rightarrow p) \vee \sim(\sim p \wedge \sim q) \vee (F \wedge T)$	by Negation Laws
$\equiv (\sim q \rightarrow p) \vee \sim(\sim p \wedge \sim q) \vee F$	by Universal Bound Law
$\equiv (\sim q \rightarrow p) \vee \sim(\sim p \wedge \sim q)$	by Identity Law
$\equiv (\sim q \rightarrow p) \vee (\sim\sim p \vee \sim\sim q)$	by De Morgan's Law
$\equiv (\sim q \rightarrow p) \vee (p \vee q)$	by Double Negation Law
$\equiv (\sim p \rightarrow q) \vee (p \vee q)$	by Property of Contrapositive Propositions
$\equiv (\sim\sim p \vee q) \vee (p \vee q)$	by Definition of Implication
$\equiv (p \vee q) \vee (p \vee q)$	by Double Negation Law
$\equiv p \vee q$	by Idempotent Law

Problem 2. For each proposition below, indicate whether it is (1) a tautology, (2) a contradiction, or (3) neither a tautology nor a contradiction.

(1) $p \rightarrow (q \rightarrow \sim p)$
(2) $(\sim(p \oplus q)) \leftrightarrow (\sim q \oplus p)$
(3) $((p \leftrightarrow q) \leftrightarrow r) \rightarrow (p \vee q \vee r)$
(4) $\sim(\sim(p \rightarrow q) \vee (p \wedge q)) \wedge p$

Solution:

(1) $p \rightarrow (q \rightarrow \sim p)$

	$p \rightarrow (q \rightarrow \sim p)$	
$\equiv$	$\sim p \vee (\sim q \vee \sim p)$	by Definition of Implication
$\equiv$	$(\sim p \vee \sim p) \vee \sim q$	by Commutative Law and Associative Law
$\equiv$	$\sim p \vee \sim q$	by Idempotent Law

The proposition is true when at least one of p and q is false. The proposition is false when both of p and q are true. Therefore, It is neither a tautology nor a contradiction.

(2) $(\sim(p \oplus q)) \leftrightarrow (\sim q \oplus p)$

We write its truth table.

p	q	$p \oplus q$	$\sim(p \oplus q)$	$\sim q \oplus p$	$(\sim(p \oplus q)) \leftrightarrow (\sim q \oplus p)$
1	1	0	1	1	1
1	0	1	0	0	1
0	1	1	0	0	1
0	0	0	1	1	1

It is a tautology.

(3) $((p \leftrightarrow q) \leftrightarrow r) \rightarrow (p \vee q \vee r)$

We write its truth table.

p	q	r	$p \leftrightarrow q$	$(p \leftrightarrow q) \leftrightarrow r$	$p \vee q \vee r$	$((p \leftrightarrow q) \leftrightarrow r) \rightarrow (p \vee q \vee r)$
1	1	1	1	1	1	1
1	1	0	1	0	1	1
1	0	1	0	0	1	1
1	0	0	0	1	1	1
0	1	1	0	0	1	1
0	1	0	0	1	1	1
0	0	1	1	1	1	1
0	0	0	1	0	0	1

It is a tautology.

(4) $\sim(\sim(p \to q) \vee (p \wedge q)) \wedge p$

$$
\begin{aligned}
& \sim(\sim(p \to q) \vee (p \wedge q)) \wedge p & \\
\equiv\ & \sim\sim(p \to q) \wedge \sim(p \wedge q) \wedge p & \text{by De Morgan's Law} \\
\equiv\ & (p \to q) \wedge \sim(p \wedge q) \wedge p & \text{by Double Negation Law} \\
\equiv\ & (\sim p \vee q) \wedge \sim(p \wedge q) \wedge p & \text{by Definition of Implication} \\
\equiv\ & (\sim p \vee q) \wedge (\sim p \vee \sim q) \wedge p & \text{by De Morgan's Law} \\
\equiv\ & (\sim p \vee (q \wedge \sim q)) \wedge p & \text{by Distributive Law} \\
\equiv\ & (\sim p \vee F) \wedge p & \text{by Negation Law} \\
\equiv\ & (\sim p) \wedge p & \text{by Identity Law} \\
\equiv\ & F & \text{by Negation Law}
\end{aligned}
$$

It is a contradiction.

Problem 3. Given the truth table below, using a Karnaugh-map, design a circuit to implement the function using as few gates as possible.

p	q	r	f
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Solution: Below we construct a Karnaugh-map for the above truth table, placing the variable p on the rows and the variables qr on the columns. The columns and rows are ordered in Gray code order rather than binary order, that is, between adjacent cells, including wrap-around, only a single variable (or bit) differs.

p \ qr	00	01	11	10
0	1	0	0	1
1	1	1	1	1

After constructing the Karnaugh-map, we group all of the cells which contain the value 1 into rectangles (or squares) which must be of an area that is a power of 2. Additionally, these rectangular groupings can wrap around the Karnaugh-map and multiple groupings can cover a single cell. The collection of groupings that we construct will allow us to construct a sum-of-products expression for our truth table, where each grouping will be a product, a conjunction of variables (and negated variables), and the the sum of these products will be a disjunction, a single OR gate, taking as input the conjunctions corresponding to each rectangular grouping.

We minimize the number of required gates by grouping the cells into rectangles of maximum area. We can, for instance, cover the cells $000, 100, 010$, and 110 with a 2×2 rectangle that wraps around the Karnaugh-map. We indicate this rectangle with a red border. Additionally, to cover the remaining 1-valued cells we can make a 1×4 rectangle covering the cells $100, 101, 111$, and 110. We indicate this rectangle with a blue border.

p \ qr	00	01	11	10
0	1	0	0	1
1	1	1	1	1

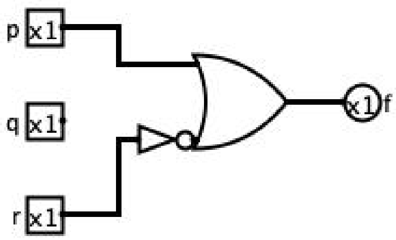
Now that each 1-valued cell is covered by a rectangular grouping, for each rectangular grouping we can construct a logical expression that represents the cells of that grouping. For each rectangular grouping, we obtain a expression by taking the conjunction of the variables that do not change within that grouping, and if a variable takes on the value 0 in that grouping we negate it.

For the red square, we notice that only the variable r remains constant with value 0, and we can account for cells covered by this rectangle with the logical expression: $\sim r$.

For the blue rectangle, we notice that only the variable p remains constant with value 1, and we can account for cells covered by this rectangle with the logical expression: p.

As there are only two rectangles covering the 1 entries of the Karnaugh-map, we can combine the above logical expressions with a disjunction in order to construct an expression that accounts for all 1-valued cells: $\sim r \vee p$.

Hence, the function described by the truth table can be implemented by the following circuit:



Problem 4. Given the truth table below, using a Karnaugh-map, design a circuit to implement the function using as few gates as possible.

p	q	r	s	f
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	1
1	1	1	0	0
1	1	1	1	1

We will follow the exact same procedure as in problem 3, however, we will omit most of our reasoning in our solution as it is identical to that of problem 3's solution.

We begin by first constructing the Karnaugh-map for the above truth table, placing the variables pq on the rows and the variables rs on the columns. Additionally, we order the rows and columns in Gray code order.

pq \ rs	00	01	11	10
00	1	1	0	1
01	1	1	1	1
11	0	1	1	0
10	0	1	0	0

We can construct the following rectangular grouping of adjacent 1-valued cells, while maximizing the area of each grouping:

- 0000, 0001, 0100, and 0101 with the colour red.
- 0000, 0010, 0100, and 0110 with the colour green.
- 0001, 0101, 1101, and 1001 with the colour blue.
- 0101, 0111, 1101, and 1111 with the colour orange.

pq \ rs	00	01	11	10
00	1	1	0	1
01	1	1	1	1
11	0	1	1	0
10	0	1	0	0

Now that each 1-valued cell is covered by a rectangular grouping, we can construct a logical expression, that is a conjunction of at most 4 variables, for each grouping:

- Red grouping: $\sim p \wedge \sim r$
- Green grouping: $\sim p \wedge \sim s$
- Blue grouping: $\sim r \wedge s$

- Orange grouping: $q \wedge s$

Consequently, taking the disjunction of the four expressions above, we obtain a logical expression for the function described by the truth table in this question: $(\sim p \wedge \sim r) \vee (\sim p \wedge \sim s) \vee (\sim r \wedge s) \vee (q \wedge s)$.

Finally, we implement the above expression as the following circuit: