Linear Regression

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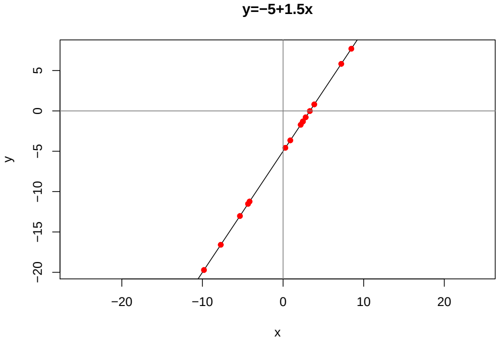
functional relationship



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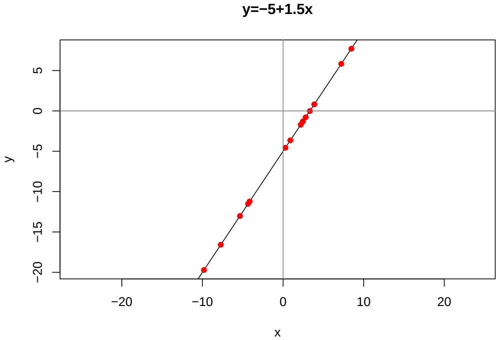

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What is the intercept?

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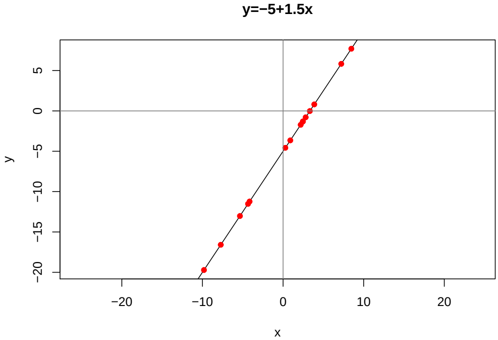

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What is the slope?

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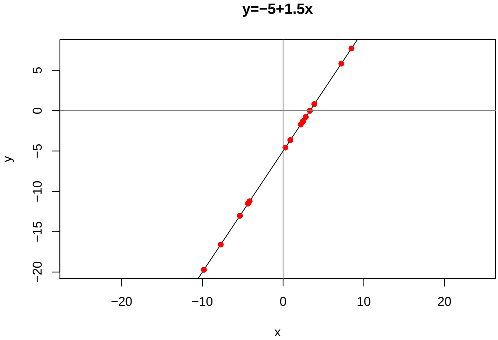

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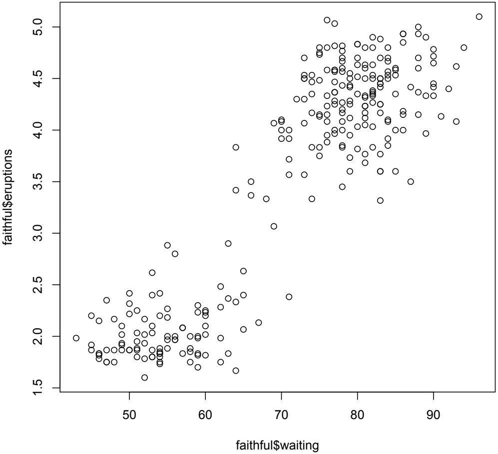
Real Example:
plot(faithful$waiting, faithful$eruptions)



Simple Linear Regression: The Model

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- ϵ_i is the error term.

Simple Linear Regression: Assumptions

- **Linearity**: The population regression line is straight; the relationship is linear.
- **Expected error is 0**: i.e. $E[\epsilon_i] = 0$ for all i. No observation is systematically too high or too low.
- **Constant Error**: i.e $Var[\epsilon_i] = \sigma^2$ for all i. The strength of the model is the same everywhere.
- **Uncorrelated errors**: Knowing the error of one observations gives no information about the size of any another error.

Note: Constant variance is called **homoscedasticity**.
Non-constant variance is called **heteroscedasticity**.

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- More about the error term ϵ_i
- We expect the error term to be symmetric about 0
- Have bell shaped distribution

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  In this example, $\hat{\beta}_0 = -1.87402$ and $\hat{\beta}_1 = 0.07563$

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- What's the predicted eruption time for a waiting time of 60min? $\hat{Y} = -1.87402 + 0.07563 * 60 = 2.66378$

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  plot(faithful$waiting, faithful$eruptions)
  fit = lm(eruptions ~ waiting, data = faithful)
  abline(fit)

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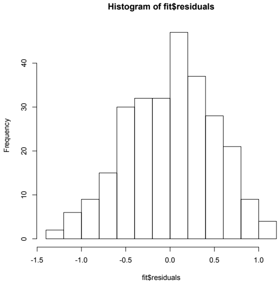
  hist(fit$residuals)



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  Associate with "t" comes a **p-value**. If the p-value is small, less than 0.05, we reject H_0

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  fit $=$ lm(eruptions $\sim$ waiting, data $=$ faithful)
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  fit = lm(eruptions $\sim$ waiting, data = faithful)
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```
> summary(fit)

Call:
lm(formula = eruptions ~ waiting, data = faithful)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.29917 -0.37689  0.03508  0.34909  1.19329 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) -1.874016   0.160143  -11.70   <2e-16 ***
waiting      0.075628   0.002219   34.09   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.4965 on 270 degrees of freedom
Multiple R-squared:  0.8115,	Adjusted R-squared:  0.8108 
F-statistic:  1162 on 1 and 270 DF,  p-value: < 2.2e-16
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Let's interpret this result

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- The t-values and p-values ($Pr(> |t|)$) are in the last columns
```

Simple Linear Regression: The Model

- Your turn: Run a Regression with the cars dataset in R
- Plot the observations and the line
- Run the hypothesis test

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fit = lm(eruptions ∼ waiting, data = faithful)
confint(fit)

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- Suppose we want to predict the eruption time for a waiting time of X = 73
- We can compute the predicted $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 = -1.87402 + 0.07563 * 73 = 3.64697$
- But R can give us a confidence interval for the prediction:
  new ¡- data.frame(waiting=73)
  predict(fit,new,interval="confidence")

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```
Analysis of Variance Table

Response: eruptions
           Df  Sum Sq Mean Sq F value    Pr(>F)
waiting     1 286.478 286.478  1162.1 < 2.2e-16 ***
Residuals 270  66.562   0.247
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
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Simple Linear Regression: The Model

- The ANOVA table

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- The ANOVA table
- df = degrees of freedom
- SS = Sum of Squares
- Note that SST = SSR + SSE

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- It can be shown that the slope $\beta_1 = \frac{sd(y)}{sd(x)} r$
- So we can estimate the correlation coefficient with the slope and vice versa.
- Exercise: Estimate the correlation coefficient of waiting and eruptions using the slope and the sd of each variable.

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$$R^2 = \frac{SSR}{SSTO} = 1 - \frac{SSE}{SSTO}$$

$$0 \leq R^2 \leq 1$$

- The closer to 1 R^2 is, the more variability is explained by your model (when linear)

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- R^2 in R:
  fit = lm(eruptions ~ waiting, data = faithful)
  summary(fit)
  "Multiple R-squared: 0.8115"

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  2. Regression models relate to association, not causality

  3. If the plot of the data does not look linear, then we need to find another (non-linear) model

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- Example

  Nonlinearity of the regression function