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On Correlated Queues

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Abstract We consider a type of correlated queue in which the service time of a customer depends on the inter-arrival time between him and his previous customer. We first derive an infinite system of linear equations for the moments of the system time, based on which we then develop several methods to calculate the moments of the system time by using MacLaurin series expansion and Padé approximation. In addition, we show how the moments and covariances of the inter-departure times of the correlated queue can be calculated based on the moments of the system time. Finally, numerical examples are provided to validate our methods.

Keywords Correlated queues · Waiting time · Departure process · MacLaurin series expansion · Padé approximation

Mathematics Subject Classification (2010) 60K25 · 68M20 · 90B22

1 Introduction

In this paper, we study a type of correlated G/G/1 queue with first-come first-served service discipline in which inter-arrival times and service times are correlated. Specifically, let A_n be the inter-arrival time between customers n and $n+1$ and S_n be the service time of customer n. We assume $S_{n+1} = p_n A_n + B_n$, where $\{A_n\}$, $\{p_n\}$ and $\{B_n\}$ are three sequences of i.i.d. non-negative random variables and are independent of each other. Such dependencies between the inter-arrival times and service times have been found in communications systems, transportation systems, and production processes, e.g., see [18, 12, 21, 36].

This type of correlated queue was first considered in [8] and then in [10, 11] in which it was assumed that p_n is a constant, B_n is zero, and A_n is exponentially distributed. [5] studied a more general case in which p_n is a binary random variable taking values 0 and 1 and A_n is exponentially distributed. Furthermore, [6] allowed p_n to be a general random variable and assumed that A_n and B_n are exponentially distributed.

Other related works include:

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1. Some extensions and modifications on the queue: [28] analyzed this type of queue with infinitely many servers; [3] for customer collection; [1] for Markov dependent inter-arrival and service times and [27] for discrete Markov arrival processes and phase-type service times.
2. The study on the effect of correlation on performance measures: The effect of correlation on the waiting time was first studied by [31, 30] via simulation and was then proved by [33, 4, 32] via analytical methods. [22] compared the correlated queue and the traditional G/G/1 queue. Recently, [7] used a simulation model to study the impact of correlation.
3. Extensions to more general correlation structures: Since p_n is necessarily non-negative in the model considered in this paper, only positive correlation can be taken into consideration. [23] developed a model in which negative correlation can be incorporated. The extension in [8] further allowed S_{n+1} and A_n to have bivariate exponential distribution (also see [9, 16, 17]). [24, 34] studied a queue where the joint density of the inter-arrival time and the service time is given by a mixture of joint density. More recently, [26] investigated a queue where the inter-arrival time and service time are correlated with Downton's bivariate exponential distribution.

In almost all the previous works, it has been assumed that A_n is exponential, and the performance measure considered is either a customer's waiting time or a busy period [29]. In this paper, we allow A_n to have more general types of distributions (i.e., phase-type distribution), and in addition to the waiting time and system time, we also analyze the departure process (i.e., the inter-departure times). We develop algorithms to calculate the moments of the system time as well as the moments and covariances of the inter-departure times. Our approach is based on the MacLaurin expansion method which was first developed by [15] for traditional G/G/1 queues with i.i.d. inter-arrival and service times and later applied to G/G/1 queues with Markov-modulated inter-arrival and service times [37]. Based on the MacLaurin expansion method, several approximation schemes, such as MacLaurin series approximation and Padé approximation, can be developed.

The remainder of this paper is organized as follows. In Section 2, we introduce some preliminaries of the correlated queue, derive the recursive formulas for the moments of the system time and waiting time based on which we propose four methods to calculate the moments. In Section 3, we first establish the relationship between the moments and covariances of the inter-departure times and the moments of the system time, and then show how the moments and covariances of the inter-departure times can be obtained. In Section 4, we present some numerical experiments to validate our method. Section 5 contains a conclusion and some discussions.

2 The system time

Let W_n and T_n be the waiting time and system time of customer n, respectively. We have

$$W_{n+1} = \max(T_n - A_n, 0), \tag{1}$$
$$\begin{aligned} T_{n+1} &= W_{n+1} + S_{n+1} \\ &= W_{n+1} + p_n A_n + B_n. \end{aligned} \tag{2}$$

We assume that the queue is stable and has steady-state waiting time W and system time T, i.e., $\lim_{n\to\infty} W_n \overset{d}{=} W$ and $\lim_{n\to\infty} T_n \overset{d}{=} T$ ($\overset{d}{=}$ denotes equal in distribution). Let A be a generic inter-arrival time with probability density function $f(x)$, S a generic service time, p a generic p_n, and B a generic B_n, then $S = pA + B$, where p, A, B are independent of each other. Based on (1) and (2), we have

$$W \overset{d}{=} \max(T - A, 0), \tag{3}$$
$$T \overset{d}{=} W + pA + B, \tag{4}$$

where T and A in the right-hand side of (3) are independent of each other, and (W, A), B, and p in the right-hand side of (4) are independent of each other (note that W and A are *not* independent of each other). We assume $f(x)$ can be expressed as its MacLaurin expansion:

$$f(x) = \sum_{i=0}^{\infty} \frac{\alpha_i}{i!} x^i,$$

for $x \geq 0$, where $\alpha_i = f^{(i)}(0)$ is the i-th derivative of $f(x)$ at $x = 0$. We note that all phase-type density distributions can be represented by their MacLaurin expansions.

To illustrate how our method works, we start with the special case in which p is a constant ($p < 1$ to ensure stability) and $B \equiv 0$. Then based on (4) we have

$$\frac{E[T^k]}{k!} = \sum_{j=0}^{k} p^{k-j} \frac{E[W^j A^{k-j}]}{j!(k-j)!}, \tag{5}$$

and based on (3)

$$\begin{aligned} E[W^k A^j] &= E\left[\int_0^T (T-x)^k x^j f(x) dx\right] \\ &= E\left[\int_0^T (T-x)^k x^j \sum_{i=0}^{\infty} \frac{\alpha_i}{i!} x^i dx\right] \\ &= \sum_{i=0}^{\infty} \frac{\alpha_i}{i!} E\left[\int_0^T (T-x)^k x^{i+j} dx\right] \\ &= \sum_{i=0}^{\infty} \frac{\alpha_i}{i!} \frac{k!(i+j)! E[T^{i+j+k+1}]}{(i+j+k+1)!}, \end{aligned} \tag{6}$$

for $k \geq 1$. In deriving (6), we have assumed that summation, integration, and expectation can be exchanged freely. These exchanges can be made more rigorous by using similar methods as in [15, 19, 20]. Combining (5) and (6), we have

$$\frac{E[T^k]}{k!} = \beta_k p^k + \sum_{i=0}^{\infty} \alpha_i \left(\sum_{j=0}^{k-1} C_{i+j}^i p^j\right) \frac{E[T^{k+i+1}]}{(k+i+1)!}, \tag{7}$$

where $\beta_k = E[A^k]/k!$ and $C_{i+j}^i = (i+j)!/(i!j!)$. (7) gives an infinite system of linear equations with respect to the moments of the system time. In what follows, we show the infinite system of linear equations defined by (7) has one and only one bounded solution under appropriate conditions. Suppose
A1. All the moments of A are finite, and $E[A^k] \leq k!(c_A)^k$ for $k = 1, 2, \ldots$, where $c_A > 0$ is a constant.
A2. $|\alpha_i| \leq (c_f)^{i+1}$ for $i = 1, 2, \ldots$, where $c_f > 0$ is a constant.

Without loss of generality, we assume c_f in (A2) is small enough such that $c_f < (1-p)/2$. (Recall $p < 1$.) Otherwise, we consider a correlated queue with inter-arrival time cA, where $c_f/c < (1-p)/2$. The waiting time and system time of this queue are cW and cT, respectively. The probability density function of cA is $\tilde{f}(x) = f(x/c)/c$. It is clear that $\tilde{f}(x)$ can also be represented by its MacLaurin series expansion; furthermore,

$$\left|\frac{d^i \tilde{f}(0)}{dx^i}\right| = \left|\frac{\alpha_i}{c^{i+1}}\right| \leq \left(\frac{c_f}{c}\right)^{i+1}.$$

We have

$$\begin{aligned}
\sum_{i=0}^{\infty}\left|\frac{\alpha_i}{i!}\left(\sum_{j=0}^{k-1}\frac{(i+j)!}{j!}p^j\right)\right| &\le \sum_{j=0}^{k-1}p^j\sum_{i=0}^{\infty}\frac{(i+j)!}{i!j!}(c_f)^{i+1}\\
&= \sum_{j=0}^{k-1}p^j\frac{c_f}{(1-c_f)^{j+1}}\\
&\le \frac{c_f}{1-p-c_f}\\
&< 1.
\end{aligned}$$

Therefore, according to [25pp. 26-31], we have

Theorem 1 *Under (A1) and (A2), (7) has one and only one bounded solution for* $\{E[T^k]/k!, k = 1,2,\ldots,\}$*, and furthermore, if* $\{z_k^K; k = 1,2,\ldots,K\}$ *is the solution of the following finite system of linear equations*

$$z_k^K = \beta_k p^k + \sum_{i=0}^{K-k-1}\alpha_i\left(\sum_{j=0}^{k-1}C_{i+j}^i p^j\right)z_{k+i+1}^K, \quad k = 1,2,\ldots,K,$$

then $\{z_k^K; k = 1,2,\ldots,K\}$ *is uniformly bounded, i.e.,* $|z_k^K| < M$ *for some constant* $M > 0$*, and*

$$\frac{E[T^k]}{k!} = \lim_{K\to\infty} z_k^K, \quad k = 1,2,\ldots.$$

Theorem 1 provides a method based on which we can truncate the infinite system of linear equations by the first K terms to produce an approximate solution for the moments of the system time. However, in what follows, we present a different way to calculate the moments of the system time based on the MacLaurin series expansion, which usually produces much better results.

Suppose we can write $E[T^k]$ as

$$\frac{E[T^k]}{k!} = \sum_{m=k}^{\infty} t_{km}p^m. \tag{8}$$

To establish differentiability and analyticity of $E[T^k]$ with respect to p at $p = 0$, we can apply similar methods in [19, 20]. Substituting (8) into (7) and comparing the coefficients of p^m on both sides, we have

$$t_{km} = \begin{cases} \beta_k, & \text{if } m = k;\\ \sum_{j=0}^{m-k-1}\sum_{i=0}^{m-k-j-1}\alpha_i C_{i+j}^i t_{(k+i+1)(m-j)}, & \text{if } k < m \le 2k;\\ \sum_{j=0}^{k-1}\sum_{i=0}^{m-k-j-1}\alpha_i C_{i+j}^i t_{(k+i+1)(m-j)}, & \text{if } m > 2k. \end{cases} \tag{9}$$

t_{km} can be calculated recursively based on (9). $E[T^k]$ can then be obtained based on t_{km} either via MacLaurin series approximation (8) or Padé approximation (e.g., see [2, 35]). The following result guarantees that there exists $p_0 > 0$ such that the solutions obtained based on the MacLaurin series approximation (MSA) and Padé approximation (PA) converge to $E[T^k]/k!$ for $0 \le p \le p_0$.

Theorem 2 *Under (A1) and (A2),* $E[T^k]$ *is analytic at* $p = 0$*, or more precisely, there exists* $p_0 > 0$ *such that for* $0 \le p \le p_0$ *(8) holds and* t_{km} *can be calculated based on the recursive equation (9)*

Proof. We first show $|t_{km}| \le (c_t)^m$, where c_t is any constant satisfying

$$c_t > \max\left(c_A, \frac{1}{1-2c_f}\right).$$

It is clear that $t_{km} = E[A^k]/k! \le (c_A)^k \le (c_t)^k$ for $m = k$. Assuming that $|t_{kn}| \le (c_t)^n$ holds for $n \le m$ and $k \le n$, then based on (9), we have for $k < m+1$

$$\begin{aligned}
|t_{k(m+1)}| &\le \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} C_{i+j}^i (c_f)^{i+1}(c_t)^{m+1-j} \\
&= \sum_{j=0}^{\infty}(c_t)^{m+1-j}\frac{c_f}{(1-c_f)^{j+1}} \\
&= (c_t)^{m+1}\frac{c_t c_f}{c_t(1-c_f)-1} \\
&\le (c_t)^{m+1}.
\end{aligned}$$

By induction, we have proven that $|t_{km}| \le (c_t)^m$ holds for any m. Therefore, $\sum_{m=k}^{\infty} t_{km}p^m$ converges for $0 \le p \le p_0 < 1/c_t$ and is bounded by $1/(1-p_0c_t)$. It is easy to verify that $\{\sum_{m=k}^{\infty} t_{km}p^m, k = 1, 2, \ldots,\}$ also satisfies (7). Since (7) has only one bounded solution, (8) holds for $0 \le p \le p_0$. This completes our proof. □

If A is exponentially distributed with rate λ, i.e., $f(x) = \lambda e^{-\lambda x}$ for $x \ge 0$, then (9) can be simplified as:

$$t_{km} = \begin{cases} \beta_k, & \text{if } k \le m \le 2k; \\ 0, & \text{if } m = 2k+1; \\ \sum_{j=0}^{k-1}\sum_{i=0}^{m-k-j-1} \alpha_i C_{i+j}^i t_{(k+i+1)(m-j)}, & \text{if } m > 2k+1. \end{cases} \tag{10}$$

First, we show $t_{km} = \beta_k$ for $k \le m \le 2k$ by induction. It is easy to verify that $t_{km} = \beta_k$ for $m = k, k+1$. Assume that for $k \le m \le k+n, 0 \le n < k$, we have $t_{km} = \beta_k$. Then,

$$\begin{aligned}
t_{k(k+n+1)} &= \sum_{j=0}^{n}\sum_{i=0}^{n-j} \alpha_i C_{i+j}^i t_{(k+i+1)(k+n+1-j)} \\
&= \sum_{0\le i+j\le n} \alpha_i C_{i+j}^i \beta_{k+i+1} \\
&= \beta_k \sum_{0\le i+j\le n} C_{i+j}^i (-1)^i \\
&= \beta_k \sum_{l=0}^{n}\sum_{i+j=l} C_{i+j}^i (-1)^i \\
&= \beta_k \sum_{l=0}^{n}\sum_{i=0}^{l} C_l^i (-1)^i \\
&= \beta_k + \sum_{l=1}^{n}(1-1)^l \\
&= \beta_k.
\end{aligned}$$

Note that the third equality is due to $\alpha_i = \lambda(-\lambda)^i$ and $\beta_k = 1/\lambda^k$. Second, we show $t_{k(2k+1)} = 0$, which follows from

$$\begin{aligned} t_{k(2k+1)} &= \sum_{j=0}^{k-1}\sum_{i=0}^{k-j} \alpha_i C_{i+j}^i t_{(k+i+1)(2k+1-j)} \\ &= \sum_{j=0}^{k}\sum_{i=0}^{k-j} \alpha_i C_{i+j}^i t_{(k+i+1)(2k+1-j)} - \alpha_0 t_{(k+1)(k+1)} \\ &= \beta_k - \beta_k \\ &= 0. \end{aligned}$$

The third term in (10) remains the same as in (9).

We now introduce the multi-point Padé approximation (m-PA) method, which was first used in [14] for the traditional G/G/1 queue. The crux of the m-PA method is its use of the heavy-traffic limits of the moments of the system time (at $p = 1$), i.e., $\lim_{p\to 1} E[T^k]$. In fact, for the correlated queue considered here, we have (e.g., see [13])

$$\lim_{p\to 1}(1-p)E[T] = 0. \tag{11}$$

The heavy-traffic limit (11) can be further refined. According to [5], if A is exponentially distributed, then

$$E[T] = E[A](1-p)\sum_{j=1}^{\infty}\frac{p^j}{1-p^j}. \tag{12}$$

As $p \to 1$

$$\sum_{j=1}^{\infty}\frac{p^j}{1-p^j} \approx \frac{1}{1-p}\ln\left(\frac{1}{1-p}\right),$$

therefore, we have

$$\lim_{p\to 1}\frac{E[T]}{-\ln(1-p)} = E[A],$$

i.e., $E[T] = O(-\ln(1-p))$ (while for the traditional G/G/1 queue, $E[T] = O(1/(1-p))$). For generally distributed A, we conjecture[1]:

$$\lim_{p\to 1}\frac{E[T]}{-\ln(1-p)} = \frac{E[A^2]}{2E[A]}. \tag{13}$$

The m-PA method uses both the derivatives at $p = 0$ and the heavy traffic limit at $p = 1$; for more details the reader is referred to [14]. For the heavy traffic limit, we can either use (11) or (13). However, our extensive numerical experiments show that both produce very similar and good results (see the numerical results presented in Section 4). Equally interesting, our numerical experiments also show that the results of m-PA are not very sensitive to the values of the heavy-traffic limits (more discussions will be provided on this in Section 4).

Finally, we discuss how to apply our method to more general cases of the correlated queue. We first consider the case in which p is not a constant but a random variable. In this case, we can simply replace p^j with $E[p^j]$ in (7). When p is a constant, we use p as a parameter in applying the MSA and PA methods; however we can no longer do that in this case. Instead, we introduce a scale parameter θ in the service time, i.e., let $S = \theta p A$. Then (7) becomes

$$\frac{E[T^k]}{k!} = \beta_k E[p^k]\theta^k + \sum_{i=0}^{\infty}\alpha_i\left(\sum_{j=0}^{k-1} C_{i+j}^i E[p^j]\theta^j\right)\frac{E[T^{k+i+1}]}{(k+i+1)!}. \tag{14}$$

[1] The verification of (13) is beyond the scope of this paper, however, it is an interesting future research topic.

Let

$$\frac{E[T^k]}{k!} = \sum_{m=k}^{\infty} t_{km}\theta^m, \tag{15}$$

similar to (9) we have

$$t_{km} = \begin{cases} \beta_k E[p^k], & \text{if } m = k; \\ \sum_{j=0}^{m-k-1}\sum_{i=0}^{m-k-j-1} \alpha_i C_{i+j}^i E[p^j] t_{(k+i+1)(m-j)}, & \text{if } k < m < 2k; \\ \sum_{j=0}^{k-1}\sum_{i=0}^{m-k-j-1} \alpha_i C_{i+j}^i E[p^j] t_{(k+i+1)(m-j)}, & \text{if } m \geq 2k. \end{cases} \tag{16}$$

For the most general case $S = pA + B$, we also introduce a scale parameter θ in the service time, i.e., $S = \theta(pA + B)$, and (14) becomes

$$\frac{E[T^k]}{k!} = \sum_{l=0}^{k} b_n \beta_{k-l} E[p^{k-l}]\theta^k + \sum_{i=0}^{\infty}\alpha_i \sum_{j=0}^{k-1} C_{i+j}^i E[p^j]\theta^j \sum_{l=0}^{k-j-1} b_l \frac{E[T^{k+i+1-l}]}{(k+i+1-l)!}\theta^l, \tag{17}$$

where $b_l = E[B^l]/l!$. Let

$$\frac{E[T^k]}{k!} = \sum_{m=k}^{\infty} t_{km}\theta^m,$$

then we have

$$t_{km} = \begin{cases} \sum_{l=0}^{k} b_l \beta_{k-l} E[p^{k-l}], & \text{if } m = k; \\ \sum_{j=0}^{m-k-1}\sum_{i=0}^{m-k-j-1}\sum_{l=0}^{k-j-1} \alpha_i C_{i+j}^i E[p^j] b_l t_{(k+i+1-l)(m-j-l)}, & \text{if } k < m < 2k; \\ \sum_{j=0}^{k-1}\sum_{i=0}^{m-k-j-1}\sum_{l=0}^{k-j-1} \alpha_i C_{i+j}^i E[p^j] b_l t_{(k+i+1-l)(m-j-l)}, & \text{if } m \geq 2k. \end{cases} \tag{18}$$

Note that (18) reduces to (16) when $B = 0$ and (18) reduces to the traditional G/G/1 queue when $p = 0$.

3 The departure process

In this section, we consider the departure process. We want to calculate the moments and covariances of the inter-departure times. Let D_n be the inter-departure time between customers n and $n + 1$. For simplicity, we will focus on the simplest case: p is a constant and $B = 0$. The derivations and results for this case can be easily extended to more general cases as we have demonstrated in the previous section for the system time. Hence we will not repeat them here. Since

$$D_n = \max(A_n - T_n, 0) + S_{n+1} = \max(A_n - T_n, 0) + pA_n, \tag{19}$$

where A_n and T_n are independent of each other, and in steady-state

$$D \stackrel{d}{=} \max(A - T, 0) + pA, \tag{20}$$

where A and T are also independent of each other, we have

$$\frac{E[D^k]}{k!} = \sum_{j=0}^{k} \frac{E[(\max(A-T,0))^j \cdot (pA)^{k-j}]}{j!(k-j)!}.$$

Note

$$\begin{aligned}(\max(A - T, 0))^j &= (-1)^j(T - A - \max(T - A, 0))^j \\ &= (-1)^j((T - A)^j - W^j),\end{aligned}$$

for $i = 1, 2, \dots, k$, therefore,

$$\frac{E[D^k]}{k!} = \frac{p^k E[A^k]}{k!} + \sum_{j=1}^{k} \frac{p^{k-j}}{(k-j)!} \frac{E[A^{k-j}(\max(A-T,0))^j]}{j!}$$
$$= \frac{p^k E[A^k]}{k!} + \sum_{j=1}^{k} \frac{p^{k-j}}{(k-j)!} \frac{(-1)^j}{j!} \left(E[A^{k-j}(T-A)^j] - E[A^{k-j}W^j] \right). \quad (21)$$

$E[A^{k-j}(T-A)^j]$ and $E[A^{k-j}W^j]$ in (21) can be calculated as follows:

$$E[A^{k-j}(T-A)^j] = j! \sum_{i=0}^{j} (-1)^{j-i} \frac{E[A^{k-i}]}{(j-i)!} \frac{E[T^i]}{i!}, \quad (22)$$

$$E[A^{k-j}W^j] = E\left[\int_0^T x^{k-j}(T-x)^j f(x)dx\right]$$
$$= E\left[\int_0^T x^{k-j}(T-x)^j \sum_{i=0}^{\infty} \frac{\alpha_i}{i!} x^i dx\right]$$
$$= j! \sum_{i=0}^{\infty} \frac{\alpha_i}{i!} (i+k-j)! \frac{E[T^{k+i+1}]}{(k+i+1)!}. \quad (23)$$

Substituting (22) and (23) into (21), we have

$$\frac{E[D^k]}{k!} = \beta_k(1+p)^k + \sum_{j=1}^{k} p^{k-j} \left(\sum_{i=1}^{j} (-1)^i C_{k-i}^{j-i} \beta_{k-i} \frac{E[T^i]}{i!} - (-1)^j \sum_{i=0}^{\infty} \alpha_i C_{i-j+k}^{i} \frac{E[T^{k+i+1}]}{(k+i+1)!} \right). \quad (24)$$

Suppose $E[D^k]/k!$ can be written as

$$\frac{E[D^k]}{k!} = \sum_{m=0}^{\infty} d_{km} p^m.$$

It is clear from (24) that we can obtain $E[D^k]$ based on the moments of the system time, on the other hand, we can also calculate $E[D^k]$ more directly by the MSA and PA methods as follows. Comparing the coefficients of p^m in both sides of (24), we obtain

$$d_{km} = \begin{cases} \beta_k, & \text{if } m = 0; \\ C_k^m \beta_k + \sum_{j=k-m+1}^{k} \sum_{i=1}^{m-k+j} (-1)^i C_{k-i}^{j-i} \beta_{k-i} t_{i(m+j-k)}, & \text{if } 0 < m \le k; \\ \sum_{j=1}^{k} \sum_{i=1}^{j} (-1)^i C_{k-i}^{j-i} \beta_{k-i} t_{i(m+j-k)} - \sum_{j=0}^{m-k-1} \sum_{i=0}^{m-k-j-1} (-1)^{k-j} \alpha_i C_{i+j}^{i} t_{(k+i+1)(m-j)}, & \text{if } k < m \le 2k; \\ \sum_{j=1}^{k} \sum_{i=1}^{j} (-1)^i C_{k-i}^{j-i} \beta_{k-i} t_{i(m+j-k)} - \sum_{j=0}^{k-1} \sum_{i=0}^{m-k-j-1} (-1)^{k-j} \alpha_i C_{i+j}^{i} t_{(k+i+1)(m-j)}, & \text{if } m > 2k. \end{cases} \quad (25)$$

With d_{km}'s being calculated based on (25), we can then obtain $E[D^k]$ based on the MSA or PA method. Furthermore, we note that as $p \to 1$ the inter-departure times approach to i.i.d. service times, therefore,

$$\lim_{p\to 1} \frac{E[D^k]}{k!} = \beta_k. \tag{26}$$

With this heavy traffic limit, we can also use the m-PA method to obtain $E[D^k]$.

We next discuss how to calculate the covariances of the inter-departure times. We will focus on $Cov(D_1, D_n)$. First, we note

$$D_n = T_{n+1} - T_n + A_n,$$

therefore,

$$\begin{aligned} Cov(D_1, D_n) &= E[D_1 D_n] - E[D_1]E[D_n] \\ &= 2E[T_1T_n] - E[T_1T_{n-1}] - E[T_1T_{n+1}] + E[A_1T_{n+1}] - E[A_1T_n], \end{aligned} \tag{27}$$

for $n \geq 2$. In deriving (27), we have assumed the queue is in steady-state, hence $E[T_n] = E[T_{n+1}] = E[T]$, $E[T_2T_{n+1}] = E[T_1T_n]$, $E[T_2T_n] = E[T_1T_{n-1}]$, and $E[T_1A_n] = E[T_2A_n] = E[T]E[A]$. Based on (27), we only need to calculate $E[T_1T_n]$ and $E[A_1T_n]$ for $n \geq 2$.

Similar to (6), for $n > 2$, we have

$$\begin{aligned} \frac{E[T_1T_n^k]}{k!} &= \frac{E[T_1(W_n + S_n)^k]}{k!} \\ &= \beta_k p^k E[T] + \sum_{j=0}^{k-1} p^j \frac{E[T_1 W_n^{k-j} A_{n-1}^j]}{(k-j)!j!} \\ &= \beta_k p^k E[T] + \sum_{j=0}^{k-1} \frac{p^j}{j!(k-j)!} E\left[\int_0^{T_{n-1}} T_1(T_{n-1}-x)^{k-j} x^j f(x)dx\right] \\ &= \beta_k p^k E[T] + \sum_{j=0}^{k-1} \frac{p^j}{j!(k-j)!} E\left[\int_0^{T_{n-1}} T_1(T_{n-1}-x)^{k-j} x^j \sum_{i=0}^{\infty} \frac{\alpha_i}{i!} x^i dx\right] \\ &= \beta_k p^k E[T] + \sum_{j=0}^{k-1} p^j \sum_{i=0}^{\infty} \alpha_i C_{i+j}^i \frac{E[T_1 T_{n-1}^{k+i+1}]}{(k+i+1)!}, \end{aligned} \tag{28}$$

and

$$\begin{aligned} \frac{E[A_1T_n^k]}{k!} &= \frac{E[A_1(W_n + S_n)^k]}{k!} \\ &= \beta_k p^k E[A] + \sum_{j=0}^{k-1} p^j \frac{E[A_1 W_n^{k-j} A_{n-1}^j]}{(k-j)!j!} \\ &= \beta_k p^k E[A] + \sum_{j=0}^{k-1} \frac{p^j}{j!(k-j)!} E\left[\int_0^{T_{n-1}} A_1(T_{n-1}-x)^{k-j} x^j f(x)dx\right] \\ &= \beta_k p^k E[A] + \sum_{j=0}^{k-1} \frac{p^j}{j!(k-j)!} E\left[\int_0^{T_{n-1}} A_1(T_{n-1}-x)^{k-j} x^j \sum_{i=0}^{\infty} \frac{\alpha_i}{i!} x^i dx\right] \\ &= \beta_k p^k E[A] + \sum_{j=0}^{k-1} p^j \sum_{i=0}^{\infty} \alpha_i C_{i+j}^i \frac{E[A_1 T_{n-1}^{k+i+1}]}{(k+i+1)!}. \end{aligned} \tag{29}$$

In particular, for $n = 2$, we have

$$\frac{E[T_1 T_2^k]}{k!} = \beta_k p^k E[T] + \sum_{j=0}^{k-1} p^j \sum_{i=0}^{\infty} \alpha_i (k+i+2) C_{i+j}^i \frac{E[T^{k+i+2}]}{(k+i+2)!}, \tag{30}$$

$$\frac{E[A_1 T_2^k]}{k!} = (k+1)\beta_{k+1} p^k + \sum_{j=0}^{k-1} p^j \sum_{i=0}^{\infty} \alpha_i (i+j+1) C_{i+j}^i \frac{E[T^{k+i+2}]}{(k+i+2)!}. \tag{31}$$

Therefore, $E[T_1 T_n^k]/k!$ and $E[A_1 T_n^k]/k!$ can be obtained recursively via (28)-(31) for $n \geq 2$ and $k \geq 1$.

Let

$$\frac{E[T_1 T_n^k]}{k!} = \sum_{m=0}^{\infty} t_{km}^{(n)} p^m \quad \text{and} \quad \frac{E[A_1 T_n^k]}{k!} = \sum_{m=0}^{\infty} a_{km}^{(n)} p^m.$$

(Note $t_{km}^{(1)} = (k+1) t_{(k+1)m}$.) Then it immediately follows from (28)-(31) that

$$t_{km}^{(2)} = \begin{cases} 0, & \text{if } m < k+1; \\ \beta_1 \beta_k, & \text{if } m = k+1; \\ \beta_k t_{1(m-k)} + \sum_{j=0}^{k-1} \sum_{i=0}^{m-k-j-2} \alpha_i (k+i+2) C_{i+j}^i t_{(k+i+2)(m-j)}, & \text{if } m > k+1; \end{cases} \tag{32}$$

$$a_{km}^{(2)} = \begin{cases} 0, & \text{if } m < k; \\ (k+1)\beta_{k+1}, & \text{if } m = k; \\ 0, & \text{if } m = k+1; \\ \sum_{j=0}^{k-1} \sum_{i=0}^{m-k-j-2} \alpha_i (i+j+1) C_{i+j}^i t_{(k+i+2)(m-j)}, & \text{if } m > k+1; \end{cases} \tag{33}$$

$$t_{km}^{(n)} = \begin{cases} 0, & \text{if } m < k+1; \\ \beta_1 \beta_k, & \text{if } m = k+1; \\ \beta_k t_{1(m-k)} + \sum_{j=0}^{k-1} \sum_{i=0}^{m-k-j-1} \alpha_i C_{i+j}^i t_{(k+i+1)(m-j)}^{(n-1)}, & \text{if } m > k+1; \end{cases} \tag{34}$$

$$a_{km}^{(n)} = \begin{cases} 0, & \text{if } m < k; \\ \beta_1 \beta_k, & \text{if } m = k; \\ \sum_{j=0}^{k-1} \sum_{i=0}^{m-k-j-1} \alpha_i C_{i+j}^i a_{(k+i+1)(m-j)}^{(n-1)}, & \text{if } m > k. \end{cases} \tag{35}$$

Suppose

$$Cov(D_1, D_n) = \sum_{m=0}^{\infty} d_m^{(n)} p^m.$$

Based on (27), we have

$$d_m^{(n)} = \begin{cases} 0, & \text{if } m = 0; \\ 2t_{1m}^{(n)} - t_{1m}^{(n-1)} - t_{1m}^{(n+1)} + a_{1m}^{(n+1)} - a_{1m}^{(n)}, & \text{if } m > 0. \end{cases} \tag{36}$$

For $n = 2$, we have

$$d_m^{(2)} = \begin{cases} 0, & \text{if } m = 0; \\ 2t_{1m}^{(2)} - 2t_{2m} - t_{1m}^{(3)} + a_{1m}^{(3)} - a_{1m}^{(2)}, & \text{if } m > 0. \end{cases} \tag{37}$$

For the heavy-traffic limit, we have

$$\lim_{p \to 1} Cov(D_1, D_2) = 0.$$

Therefore, MSA, PA, m-PA can all be applied to calculating $Cov(D_1, D_2)$.

4 Numerical Results

In this section, we numerically investigate the efficiency of the methods proposed in the previous two sections. We have four methods available for calculating the moments of the system time and the moments and covariances of the inter-departure times: 1) solving an infinite system of linear equations (LE); 2) using polynomial functions based on MacLaurin series approximation (MSA); 3) using rational functions based on Padé approximation (PA); 4) using rational functions based on multi-point Padé approximation (m-PA). However, in most cases LE and MSA only work in some small neighborhood of p as indicated by Theorems 1 and 2, and PA and m-PA usually provide much better results. Therefore, in our numerical study here, we focus on PA and m-PA. Furthermore,

1. For $E[T]$, we use both (11) and (13) as the heavy-traffic limit in the m-PA method, which are denoted as m-PA^1 and m-PA^2, respectively.
2. For $E[T^2]$, we only use the following heavy traffic limit

$$\lim_{p \to 1} (1-p)^2 E[T^2] = 0. \tag{38}$$

3. For the inter-departure times, we only consider $E[D^2]$ and $Cov(D_1, D_2)$. In addition to the PA and m-PA methods, we also present the numerical results for $E[D^2]$ and $Cov(D_1, D_2)$ obtained directly based on the moments of the system time.

We present three numerical examples: 1) $f(x)$ is an exponential distribution with mean 1; 2) $f(x)$ is a two-stage Erlang distribution with mean 2; 3) $f(x)$ is a hyper-exponential distribution with

$$f(x) = \frac{1}{2}\lambda_1 e^{-\lambda_1 x} + \frac{1}{2}\lambda_2 e^{-\lambda_2 x},$$

where $\lambda_1 = 1, \lambda_2 = 2$. In all three examples, we assume p is a constant and $B = 0$, and we set p to five different values 0.1, 0.3, 0.5, 0.7, and 0.9, ranging from low-traffic to heavy-traffic. For the PA and m-PA methods, we use the first 80 derivatives at $p = 0$, and obviously, our results can be improved if more derivatives are used. We compare our numerical results with simulation results (except for $E[T]$ in Example 1 for which the analytical formula is available). The simulation results (mean $\pm$ standard deviation) are obtained based on 40 replications and 1,000,000 customers per replication (40,000,000 customers for $p = 0.9$) for each setting.

The numerical results for the three examples are presented in Tables 1-3. For simulation results, the standard deviation based on 40 replications is provided in parenthesis. For Tables 1-3, we make the following remarks:

1. The m-PA method produces very good results in all cases, and the PA method produces good results when p is small ($p \leq 0.5$) and doesn't work well when p is large ("—" in Tables 1-3 indicates that the corresponding result is either very poor or doesn't converge at all).
2. The results on $E[T]$ from m-PA^1 and m-PA^2 indicate that the m-PA method works well by using both heavy-traffic limits. In fact, we also applied two different heavy-traffic limits to calculating $E[T^2]$ and obtained similar results (since they are very similar to those on $E[T]$, we didn't provide them here).
3. We also investigated the effect of the value of the heavy-traffic limit at the right-side of (11) on the results of the m-PA, we found that the results are extremely insensitive to the value.

5 Conclusion

In this paper, we study a type of correlated queue, in which service times are dependent on inter-arrival times. By using the method of MacLaurin series expansion, we first obtain an infinite system of linear equations for the moments of the system time based on which we develop several methods to calculate the moments of the system time. We then establish relationships between the moments of the system time and the moments and covariances of the inter-departure times, which allow us to further calculate the moments and covariances of the inter-departure times. Finally, We conduct numerical experiments to validate our approach. Our numerical results show that the multi-point Padé approximation works extremely well. One possible future research direction is to extend our method in this paper to other types of correlated queues.

Declarations

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Conflicts of interest/Competing interests

None.

Availability of data and material

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Not applicable.

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Table 1 A is exponential

p		0.1	0.3	0.5	0.7	0.9
$E[T]$	PA	0.1101	0.3968	0.8033	1.4269	2.3343
	m-PA[1]	0.1101	0.3968	0.8033	1.4269	2.7086
	m-PA[2]	0.1101	0.3968	0.8033	1.4269	2.7086
	Analytical	0.1101	0.3968	0.8033	1.4269	2.7086
$E[T^2]$	PA	0.0222	0.2526	0.9298	2.6492	4.2598
	m-PA	0.0222	0.2526	0.9298	2.6492	8.5008
	Simulation	0.0222 (0.0000)	0.2525 (0.0005)	0.9297 (0.0022)	2.6473 (0.0065)	8.5022 (0.0048)
$E[D^2]$	Using $E[T^k]$	2.1982	2.5556	2.8034	2.8562	—
	PA	2.1982	2.5556	2.8034	2.8562	2.4510
	m-PA	2.1982	2.5556	2.8034	2.8562	2.5418
	Simulation	2.1987 (0.0033)	2.5529 (0.0042)	2.8034 (0.0045)	2.8577 (0.0045)	2.5419 (0.0007)
$Cov(D_1, D_2)$	Using $E[T^k]$	-0.0811	-0.1501	-0.1367	—	—
	PA	-0.0811	-0.1501	-0.1367	-0.0806	-0.0177
	m-PA	-0.0811	-0.1501	-0.1367	-0.0806	-0.0171
	Simulation	-0.0810 (0.0013)	-0.1501 (0.0013)	-0.1367 (0.0013)	-0.0806 (0.0013)	-0.0171 (0.0002)

Table 2 A is Erlang

p		0.1	0.3	0.5	0.7	0.9
$E[T]$	PA	0.2032	0.6641	1.2570	2.9528	—
	m-PA[1]	0.2032	0.6641	1.2570	2.1049	3.7302
	m-PA[2]	0.2032	0.6641	1.2570	2.1049	3.7246
	Simulation	0.2032 (0.0001)	0.6642 (0.0006)	1.2569 (0.0012)	2.1047 (0.0016)	3.7277 (0.0008)
$E[T^2]$	PA	0.0610	0.6108	2.0472	—	—
	m-PA	0.0610	0.6108	2.0470	5.3626	15.5660
	Simulation	0.0610 (0.0001)	0.6110 (0.0008)	2.0468 (0.0030)	5.3626 (0.0067)	15.5077 (0.0055)
$E[D^2]$	Using $E[T^k]$	6.4262	7.2698	7.8616	—	—
	PA	6.4262	7.2698	7.8744	11.3544	—
	m-PA	6.4262	7.2698	7.8744	8.0064	7.2414
	Simulation	6.4282 (0.0072)	7.2714 (0.0097)	7.8750 (0.0099)	8.0062 (0.0065)	7.2582 (0.0012)
$Cov(D_1, D_2)$	Using $E[T^k]$	-0.2036	-0.4729	-0.4761	—	—
	PA	-0.2036	-0.4729	-0.4783	-1.9802	—
	m-PA	-0.2036	-0.4729	-0.4783	-0.2985	-0.0662
	Simulation	-0.2033 (0.0024)	-0.4737 (0.0034)	-0.4775 (0.0036)	-0.2982 (0.0031)	-0.0660 (0.0005)

Table 3 A is hyper-exponential

p		0.1	0.3	0.5	0.7	0.9
$E[T]$	PA	0.0844	0.3141	1.9047	—	—
	m-PA^1	0.0844	0.3141	0.6540	1.1972	2.2974
	m-PA^2	0.0844	0.3141	0.6540	1.1972	2.4906
	Simulation	0.0844 (0.0001)	0.3141 (0.0004)	0.6540 (0.0010)	1.1984 (0.0025)	2.3747 (0.0009)
$E[T^2]$	PA	0.0144	0.01718	—	—	—
	m-PA	0.0144	0.01718	0.6606	1.9692	6.2270
	Simulation	0.0144 (0.0000)	0.1720 (0.0005)	0.6606 (0.0017)	1.9736 (0.0066)	6.7376 (0.0040)
$E[D^2]$	Using $E[T^k]$	1.3808	1.6330	—	—	—
	PA	1.3808	1.6330	0.6318	—	—
	m-PA	1.3808	1.6330	1.8090	1.8492	1.6452
	Simulation	1.3966 (0.0027)	1.6338 (0.0037)	1.8090 (0.0035)	1.8574 (0.0042)	1.6420 (0.0005)
$Cov(D_1, D_2)$	Using $E[T^k]$	-0.0527	-0.0914	—	—	—
	PA	-0.0527	-0.0914	—	—	—
	m-PA	-0.0527	-0.0914	-0.0804	-0.0464	-0.0132
	Simulation	-0.0528 (0.0009)	-0.0915 (0.0010)	-0.0802 (0.0010)	-0.0466 (0.0010)	-0.0096 (0.0001)