Discrete-Event Stochastic Systems with Copula Correlated Input Processes

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ABSTRACT
In this paper, we develop a new method based on copulas to model correlated inputs in discrete-event stochastic systems. We first define a type of correlated stochastic process, called the copula correlated process (CCP), which we then use to model correlated inputs for discrete-event stochastic systems. In general, it is very difficult to analyze discrete-event stochastic systems with correlated inputs. However, we show that discrete-event stochastic systems with CCPs can be discretized and approximated by discrete-event stochastic systems with discrete correlated stochastic process (DCPPs), which are equivalent to discrete-event stochastic systems driven by Markov-modulated processes and are much easier to analyze. An illustrative queueing example is provided to demonstrate how our method works.

KEYWORDS
Discrete-event stochastic systems; Correlated processes; Copulas;
Markov-modulated processes

1. Introduction

Most discrete-event stochastic systems in the literature assume that stochastic inputs (e.g., event clock times) are independent and identically distributed. However, it has been observed that in many applications the inputs could be correlated, such as the inter-arrival times of customers in queueing networks (*1*, *2*), and demands in inventory systems (*3*, *4*). Very few studies have been done on discrete-event stochastic systems with correlated inputs in the literature. One difficulty is how to model correlated inputs. There are two commonly used methods to model correlated inputs: one is based on the Auto Regressive Moving Average (ARMA) process, (e.g., see (*5–9*)), and the other is based on the Markov-modulated process (e.g., see (*10–13*)). However, the ARMA process can only capture linear dependence in correlation, hence it is quite restrictive in capturing other important properties of correlated processes. **There are other forms of dependence between random inputs that cannot be captured by correlation alone but may be quite important in some applications. For example, the tail dependence, which measures the probabilistic relationship between extreme values of two random variables, is found to be very important in some contexts (e.g. (*14*), (*15*)). In these contexts, copulas, which can model different dependence structures including various tail dependence become appropriate tools.** On the other hand, in the Markov-modulated processes, it is difficult to separate the effects of correlation from those of marginal distributions. In addition, the Markov-modulated process is often over-parameterized for statistical purposes.

To overcome these shortcomings, (*16*) proposed the idea to transform a normal autoregressive process into the desired univariate time-series input process which they assumed to have an Autoregressive-To-Anything (ARTA) distribution. They also developed another similar transformation-based model, Normal-To-Anything (NORTA) for random vectors. Based on the work of (*16*), (*17*, *18*) developed the method of Vector-Autoregressive-To-Anything (VARTA). Guassian copula is actually the key ingredient in these transformation-based models. (*15*) illustrated the shortcomings of VARTA and extended it by use more general copulas. Copula-based multivariate time-series input models, which include VARTA as a special case, allow better fitting for correlated inputs in large-scale stochastic simulation. For more detailed discussions on the advantages of using copulas to model dependence structures of inputs and the review of the techniques to construct copula-based input models, the reader is referred to (*19*) and (*20*). The works so far have mainly focused on developing good fitting and sampling algorithms for copula-based time-series input processes, but very little has been done on the analysis of stochastic systems with copula-based time-series input processes. In general, it is extremely difficult to analyze stochastic systems with correlated inputs and simulation is often considered as the only viable method. The aim of our work here is to explore possible analytical methods to study these systems.

In this paper, we consider discrete-event stochastic systems with copula-based input processes in the framework of generalized semi-Markov processes (GSMP). We use copula-based input processes to model correlated event clock times in a GSMP. Our main idea is to discretize copula-based input processes and then analyze the corresponding GSMP with discrete copula-based clock times. It turns out that these discrete copula-based processes are equivalent to Markov-modulated processes, hence the corresponding GSPM can be viewed as a GSMP driven by Markov-modulated input processes. For example, for a queueing system, it essentially becomes a queueing system with Markov-modulated inter-arrival times and service times. We can then use some of the existing methods developed for stochastic systems driven by Markov-modulated processes in the literature to analyze these systems, e.g., see (*4*, *21*).

There are a couple of related works in the literature that study the relationship between copulas and Markov processes. (*22*) is the first that recognizes that copulas have certain Markov properties. It shows that copulas satisfy certain conditions which are equivalent to the Chapman-Kolmogorov equations for Markov processes, based on which it proposes a new way to construct Markov processes. Recently, (*23*) uses copulas to capture the functional temporal correlation of Markov-modulated poisson processes. In our paper, we study the relationship between discrete copulas and Markov processes. By linking discrete copula correlated processes to Markov-modulated processes, we make it possible to apply previous results on discrete-event stochastic systems driven by Markov-modulated processes in this new setting.

The main contributions of this paper can be summarized as follows:

- It incorporates copula-based input process in the framework of generalized semi-Markov process to model correlated event clock times and analyze the performance of these systems.
- It shows that discrete-event stochastic systems with general copula correlated processes can be discretized and approximated by discrete-event stochastic systems with discrete copula correlated processes, which are equivalent to discrete-event stochastic systems driven by Markov-modulated processes.
- It provides the G/G/1 queue as an illustrative example to demonstrate how our method works.

The rest of the paper is organized as follows. In Section 2, we introduce both continuous and discrete copulas, establish their relationships to Markov processes, and define both copula-based processes. We show how a continuous copula can be discretized and approximated by a discrete copula and establish the convergence of the discrete copula to the corresponding continuous copula. Some examples are also provided. In Section 3, we present the GSMP framework with copula correlated clock times. In Section 4, we provide the G/G/1 queue as an illustrative example. Finally, Section 5 contains a conclusion and some discussions.

2. Copulas and Markov processes

2.1. *Copulas*

Definitions of copulas

Suppose $\mathbf{u} = (u_1, ..., u_N) \in [0,1]^N$.

Definition 2.1. An N-dimensional copula is a function $C(\mathbf{u}) : [0,1]^N \to [0,1]$, which has the following properties:

(i) If any one of u_i's $= 0$, then $C(\mathbf{u}; \theta) = 0$;
(ii) $C(1, ..., 1, u_i, 1, ..., 1) = u_i,\ i = 1, ..., N$;
(iii) For any $\mathbf{v} = (v_1, ..., v_N) \in [0,1]^N,\ v_i \le u_i,\ i = 1, ..., N$, we have

$$\sum_{j_1=1}^{2} \cdots \sum_{j_N=1}^{2} (-1)^{j_1+\cdots+j_N} C(u_{1,j_1}, ..., u_{N,j_N}; \theta) \ge 0,$$

with $u_{i,1} = v_i$ and $u_{i,2} = u_i$.

The second condition implies that the marginal distribution of the copula is a uniform distribution over $[0,1]$. The first and third conditions guarantee that the copula is a multi-dimensional distribution function consistent with the axiom of probability. In the case of $N = 2$, the third condition is equivalent to the following rectangle condition:

$$C(u_1, u_2) - C(u_1, v_2) - C(v_1, u_2) + C(v_1, v_2) \ge 0,$$

for any $v_i \le u_i$, $i = 1, 2$.

Sklar's theorem is central to the theory of copulas as it elucidates the role played by copulas in the relationship between the multivariate joint distribution function and the corresponding marginal distributions.

Theorem 2.2. *(***Sklar's theorem***, (24)) Let $H(\cdot)$ be a joint distribution function of N-dimensional random variables $X_1, X_2, \ldots, X_N$, with margins $F_1(x_1), F_2(x_2), ..., F_n(x_N)$, respectively. Then there exists an N-dimensional copula C such that for all $\boldsymbol{x} \in \mathbb{R}^N$,*

$$H(\boldsymbol{x}) = C(F_1(x_1), F_2(x_2), ..., F_N(x_N)). \tag{1}$$

If $F_1, F_2, ..., F_N$ are continuous, then C is unique. Otherwise, C is uniquely determined *on $RanF_1 \times ... \times RanF_N$, where $RanF_i$ denotes the range of F_i ($i = 1, ..., N$). Conversely, if C is an N-dimensional copula and $F_1(x_1), F_2(x_2), ..., F_N(x_N)$ are distribution functions, then the function $H(\cdot)$ defined by (1) is a joint distribution function for a vector of N random variables with marginal distributions $F_1(x_1), F_2(x_2), ..., F_N(x_N)$.*

We say an N-dimensional copula C is the copula of $X_1, X_2, \ldots, X_N$ if it satisfies (1). Based on Sklar's theorem, a joint distribution function can be linked to any marginal distribution via a copula. For a copula $C(\cdot)$ that is absolutely continuous with respect to its arguments, the density of the joint distribution is given as follows:

$$h(\boldsymbol{x}) = c(F_1(x_1), ..., F_N(x_N)) \prod_{i=1}^{N} f_i(x_i),$$

where

$$c(\mathbf{u}) = \frac{\partial^N C(\mathbf{u})}{\partial u_1 \cdots \partial u_N},$$

and f_i is the density function of F_i.

Simulation of copulas

A general method to generate samples for a given copula is by using conditional sampling. Let $U_1, ..., U_N$ have joint distribution function C. Then, the conditional distribution of U_k given the values of $U_1, ..., U_{k-1}$ is given by

$$\begin{aligned} C_k(u_k|u_1, ..., u_{k-1}) &= P\{U_k \leq u_k | U_1 = u_1, ..., U_{k-1} = u_{k-1}\} \\ &= \frac{\partial^{k-1} C(u_1, ..., u_k, 1, ..., 1)}{\partial u_1 ... \partial u_{k-1}} \Big/ \frac{\partial^{k-1} C(u_1, ..., u_{k-1}, 1, ..., 1)}{\partial u_1 ... \partial u_{k-1}}. \end{aligned}$$

Therefore, a random vector $\mathbf{X} = (X_1, ..., X_N)^T$ with marginal distributions F_i ($i = 1, \ldots, N$) and copula function C can be generated as follows.

- Generate a random variate U_1 from $U(0, 1)$;
- Generate a random variate U_2 from $C_2(\cdot|U_1)$;
- ...
- Generate a random variate U_N from $C_N(\cdot|U_1, ..., U_{N-1})$;
- Return $\mathbf{X} = (F_1^{-1}(U_1), ..., F_N^{-1}(U_N))$.

To sample U_k from $C_k(\cdot|U_1, ..., U_{k-1})$ requires sampling $U \sim U(0, 1)$ and calculating $U_k = C_k^{-1}(U|U_1, ..., U_{k-1})$ (usually by numerical root-finding). Conditional sampling approach provides a general way to simulate dependent uniform random numbers from a given copula, but in general, the inverse function cannot be calculated analytically. Usually, this procedure is computationally intensive, especially when N is large. However, there are more efficient methods to sample some special families of copulas, e.g., Archimedean copulas (*25*). Since this is beyond scope of this paper, we won't discuss this further.

Copulas and Markov processes

Suppose C is an N-dimensional copula and D is an M-dimensional copula, then their

$\star$-product is defined by $C \star D$: $[0,1]^{N+M-1} \to [0,1]$ via

$$C \star D(u_1, \dots, u_{N+M-1}) = \int_0^{u_N} \partial_N C(u_1, \dots, u_{N-1}, u) \partial_1 D(u, u_{N+1}, \dots, u_{N+M-1}) du,$$

where $\partial_N C(u_1, \dots, u_N) = \partial C / \partial u_N$ and $\partial_1 D(u_1, \dots, u_M) = \partial D / \partial u_1$. It can be shown that $C \star D$ is an $(N + M - 1)$-dimensional copula (see (*22*)). When C and D are both 2-dimensional copulas, we define their $*$-product $C * D$: $[0,1]^2 \to [0,1]$ as

$$C * D(u_1, u_2) = \int_0^1 \partial_2 C(u_1, u) \partial_1 D(u, u_2) du.$$

It is not difficult to see that $\star$- and $*$- products are related by

$$C * D(u_1, u_2) = C \star D(u_1, 1, u_2).$$

Therefore, $C * D$ is also a 2-dimensional copula.

We now present the following results that relate $\star$- and $*$- product to Chapman-Kolmogorov equations and Markov processes.

Theorem 2.3. *(22) Suppose $\{X_t\}$ is a discrete-time stochastic process and C_{st} is the copula of X_s and X_t $(1 \le s < t)$.*

(i) The Chapman-Kolmogorov equations hold for the transition probabilities of $\{X_t\}$ if and only if

$$C_{st} = C_{sr} * C_{rt},$$

for all $1 \le s < r < t$.

(ii) $\{X_t\}$ is a Markov process if and only if for all integers $k \ge 1$ and for all $1 \le t_1 < \dots < t_k$,

$$C_{t_1 \cdots t_k} = C_{t_1 t_2} \star C_{t_2 t_3} \star \dots \star C_{t_{k-1} t_k},$$

where $C_{t_1 \cdots t_k}$ is the copula of $X_{t_1}, \dots, X_{t_k}$.

Based on Theorem 2.3 (ii), a stationary Markov process can be uniquely determined by the marginal distribution of X_t and the copula of X_t and X_{t+1}. Therefore, we define

Definition 2.4. A stationary discrete-time Markov process $\{X_t\}$ is called a copula correlated process (CCP) with marginal distribution F and 2-dimensional copula C if F is the marginal distribution of X_t and C is the copula of X_t and X_{t+1}, i.e., $C(F(x_t), F(x_{t+1}))$ is the joint distribution of X_t and X_{t+1}.

It is clear that if $\{X_t\}$ is a copula-based process with marginal distribution F and copula C, then it can be generated as

$$X_{t+1} = F^{-1}(C_2^{-1}(U_{t+1} | F(X_t))), \qquad t = 1, 2, ..., \tag{2}$$

where $C_2(\cdot | U)$ is the conditional distribution of copula C, i.e.,

$$C_2(v|u) = C_2(u, v) = Pr\{V \le v | U = u\} = \partial C(u, v) / \partial u,$$

and $\{U_t\}$ are independent and identically distributed $U(0,1)$ random variables. **More specifically, the copula correlated process $\{X_t\}$ with marginal distribution F and 2-dimensional copula C linking X_t and X_{t+1} can be generated as follows:**

- **Generate a random variate U_1 from $U(0,1)$;**
- **Generate a random variate U_2 from $C_2(\cdot|U_1)$;**
- **...**
- **Generate a random variate U_N from $C_2(\cdot|U_{N-1})$;**
- **...**
- **Return $\mathbf{X} = (F^{-1}(U_1), ..., F^{-1}(U_N), ...)$.**

X_t can be viewed as a Markov process with state space $\mathcal{X}$, where $\mathcal{X}$ is the range of F. In what follows, we assume that state space $\mathcal{X}$ is continuous. Let Σ be a σ-algebra of subsets of $\mathcal{X}$, then for any $[a,b] \subset \mathcal{X}$, the kernel $K(x,[a,b])$: $\mathcal{X} \times \Sigma \to [0,1]$ is defined by

$$K(x,[a,b]) = C_2(F(x),F(b)) - C_2(F(x),F(a)).$$

2.2. *Discrete Copulas*

Definition of discrete copulas

(*26*) introduced discrete copulas, which can be viewed as copulas with *uniform* discrete marginals. Instead of $[0,1]$, discrete copulas are defined on

$$I_n = \left\{0, \frac{1}{n}, ..., \frac{n-1}{n}, 1\right\}.$$

For ease of exposition, in the rest of this subsection, we shall focus on 2-dimensional copulas, though most of our results can be extended to N-dimensional copulas.

Definition 2.5. (*26*) A function $C_{n,m} : I_n \times I_m \to [0,1]$ is called a discrete copula if it satisfies the following conditions:

(i) $C_{n,m}\left(\frac{i}{n}, 0\right) = C_{n,m}\left(0, \frac{j}{m}\right) = 0;$

(ii) $C_{n,m}\left(\frac{i}{n}, 1\right) = \frac{i}{n};\ C_{n,m}\left(1, \frac{j}{m}\right) = \frac{j}{m};$

(iii) $C_{n,m}\left(\frac{i}{n}, \frac{j}{m}\right) + C_{n,m}\left(\frac{i-1}{n}, \frac{j-1}{m}\right) - C_{n,m}\left(\frac{i-1}{n}, \frac{j}{m}\right) - C_{n,m}\left(\frac{i}{n}, \frac{j-1}{m}\right) \geq 0,$

for all $i \in \{0,1,...,n\}$ and $j \in \{0,1,...,m\}$.

In this paper, we only consider the case $m = n$ and use C_n and I_n^2 instead of $C_{n,n}$ and $I_n \times I_n$, respectively, for notational simplicity when there is no confusion. For a copula C, if

$$C_n\left(\frac{i}{n}, \frac{j}{n}\right) = C\left(\frac{i}{n}, \frac{j}{n}\right), \qquad i,j = 0,1,...,n,$$

then we say C_n is the n-th order discretization of C. For any copula C and its associated

n-th order discretization C_n, (*27*) prove that

$$\lim_{n\to+\infty} C_n\left(\frac{\lfloor nu\rfloor}{n}, \frac{\lfloor nv\rfloor}{n}\right) = \lim_{n\to+\infty} C\left(\frac{\lfloor nu\rfloor}{n}, \frac{\lfloor nv\rfloor}{n}\right) = C(u,v), \tag{3}$$

where $\lfloor x\rfloor$ denotes the integer part of $x \in \mathbb{R}$. That is to say, any copula is the limit of its n-th order discretization. In the rest of the paper, we use $C_n(i,j)$ instead of $C_n\left(\frac{i}{n}, \frac{i}{n}\right)$ for simplicity when there is not confusion.

Discrete copulas and Markov processes

(*26*) provided a representation of discrete copulas by using bistochastic matrices. An $n \times n$ matrix $P_n = (p_n(i,j))$ is called bistochastic if all entries are non-negatives, and the sum of the entries of every row and column is equal to 1, i.e., $\forall k \in \{1, ..., n\}$,

$$\sum_{i=1}^{n} p_n(i,k) = \sum_{j=1}^{n} p_n(k,j) = 1.$$

Theorem 2.6. *(26) For a function $C_n : I_n^2 \to [0,1]$, the following are equivalent:*

(i) C_n is a discrete copula.
(ii) There is a bistochastic matrix $P_n = (p_n(i,j))$ such that $\forall i,j \in \{0,1,2,...,n\}$

$$C_n\left(i,j\right) = \frac{1}{n}\sum_{k=1}^{i}\sum_{m=1}^{j} p_n(k,m).$$

In the proof of Theorem 2.6 in (*26*), it is shown that the bistochastic matrix $P_n = (p_n(i,j))$ can be constructed as

$$p_n(i,j) = n\left(C_n\left(i,j\right) - C_n\left(i-1,j\right) - C_n\left(i,j-1\right) + C_n\left(i-1,j-1\right)\right). \tag{4}$$

We sometimes drop n in $p_n(i,j)$ and $P_n(i,j)$ for notational simplicity.

We now extend $*$-product to discrete copulas and also discuss its interpretation in the context of Markov processes. Denote $\mathbb{C}_n$ as the set of all discrete copulas on I_n^2. Suppose $C_n, D_n \in \mathbb{C}_n$, then $C_n * D_n$ is defined as

$$\left(C_n * D_n\right)(i,j) = \sum_{k=1}^{n} n\left[C_n\left(i,k\right) - C_n\left(i,k-1\right)\right]\cdot\left[D_n\left(k,j\right) - D_n\left(k-1,j\right)\right],$$

for $i,j \in \{1,2,...,n\}$.

Theorem 2.7. *if $C_n, D_n \in \mathbb{C}_n$, then $C_n * D_n \in \mathbb{C}_n$.*

Proof. It is easy to know that for all $i,j \in \{0,1,...,n\}$,

$$\left(C_n * D_n\right)(i,0) = 0,$$

and

$$(C_n * D_n)(i,1) = \sum_{k=1}^{n} (C_n(i,k) - C_n(i,k-1)) = C_n(i,1) = 1/n.$$

Similarly, we have $(C_n * D_n)(0,j) = 0$ and $(C_n * D_n)(1,j) = j/n$, therefore, we now only need to show that $C_n * D_n$ satisfies (iii) in Definition 3. Note

$$\begin{aligned}
& (C_n * D_n)(i,j) + (C_n * D_n)(i-1,j-1) - (C_n * D_n)(i-1,j) - (C_n * D_n)(i,j-1) \\
= \; & n \sum_{k=1}^{n} ([C_n(i,k) - C_n(i,k-1)] \cdot [D_n(k,j) - D_n(k-1,j)] \\
& + [C_n(i-1,k) - C_n(i-1,k-1)] \cdot [D_n(k,j-1) - D_n(k-1,j-1)] \\
& - [C_n(i-1,k) - C_n(i-1,k-1)] \cdot [D_n(k,j) - D_n(k-1,j)] \\
& - [C_n(i,k) - C_n(i,k-1)] \cdot [D_n(k,j-1) - D_n(k-1,j-1)]) \\
= \; & n \sum_{k=1}^{n} ([C_n(i,k) - C_n(i,k-1) - C_n(i-1,k) + C_n(i-1,k-1)] \\
& \quad \cdot [D_n(k,j) - D_n(k-1,j) - D_n(k,j-1) + D_n(k-1,j-1)]) \geq 0.
\end{aligned}$$

This completes the proof. □

Similar to Theorem 2.3 (i), for discrete copulas we have

Theorem 2.8. *Let $\{X_t\}$ be a discrete-time stochastic process with finite state space $\{1,2,...,n\}$. Let C_{st} be the copula of X_s and X_t ($1 \leq s < t$). Then the following are equivalent:*

(i) The transition probabilities of $\{X_t\}$ satisfy the Chapman-Kolmogorov equations

$$Pr\{X_t = j | X_s = i\} = \sum_{l=1}^{n} Pr\{X_t = j | X_r = l\} Pr\{X_r = l | X_s = i\},$$

for all $i,j \in \{1,2,...,n\}$ and for all $1 \leq s < r < t$.

(ii) $C_{st} = C_{sr} * C_{rt}$, for all $1 \leq s < r < t$.

Proof. First, we show $(ii) \to (i)$. Notice that if $C_{st} = C_{sr} * C_{rt}$, then for all $i,j \in \{1,2,...,n\}$

$$C_{st}(i,j) = \sum_{k=1}^{n} n \cdot \left[C_{sr}(i,j) - C_{sr}(i,k-1)\right]\left[C_{rt}(k,j) - C_{rt}(k-1,j)\right].$$

According to Theorem 2.6, there exists a bistochastic matrix $P_{st} = (p_{st}(i,j))$ with

$$\begin{aligned}
p_{st}(i,j) &= n(C_{st}(i,j) - C_{st}(i-1,j) - C_{st}(i,j-1) + C_{st}(i-1,j-1)) \\
&= n^2 \sum_{k=1}^{n} ([C_{sr}(i,j) - C_{sr}(i,k-1)]\,[C_{rt}(k,j) - C_{rm}(k-1,j)] \\
&\qquad - [C_{sr}(i,j) - C_{sr}(i,k-1)]\,[C_{rt}(k,j) - C_{rt}(k-1,j)] \\
&\qquad - [C_{sr}(i,j) - C_{sr}(i,k-1)]\,[C_{rt}(k,j) - C_{rt}(k-1,j)] \\
&\qquad + [C_{sr}(i,j) - C_{sr}(i,k-1)]\,[C_{rt}(k,j) - C_{rt}(k-1,j)]) \\
&= n^2 \sum_{k=1}^{n} [C_{sr}(i,j) - C_{sr}(i-1,k) - C_{sr}(i,k-1) + C_{sr}(i-1,k-1)] \\
&\qquad \cdot [C_{rt}(k,j) - C_{rt}(k-1,j) - C_{rt}(k,j-1) + C_{rt}(k-1,j-1)] \\
&= \sum_{k=1}^{n} p_{sr}(i,k) p_{rt}(k,j),
\end{aligned}$$

i.e.,

$$Pr\{X_t = j | X_s = i\} = \sum_{l=1}^{n} Pr\{X_t = j | X_r = l\} Pr\{X_r = l | X_s = i\}.$$

Conversely, if the Chapman-Kolmogorov equations hold, then we have

$$\begin{aligned}
& \sum_{k=1}^{n} n \cdot [C_{sr}(i,k) - C_{sr}(i,k-1)]\,[C_{rt}(k,j) - C_{rt}(k-1,j-1)] \\
= \; & \frac{1}{n} \left[\sum_{l=1}^{i} \sum_{q=1}^{k} p_{sr}(l,q) - \sum_{l=1}^{i} \sum_{l=1}^{k-1} p_{sr}(l,q) \right] \cdot \left[\sum_{l=1}^{k} \sum_{q=1}^{j} p_{rt}(l,q) - \sum_{l=1}^{k-1} \sum_{q=1}^{j} p_{rt}(l,q) \right] \\
= \; & \sum_{k=1}^{n} \frac{1}{n} \left[\sum_{l=1}^{i} p_{sr}(l,k) \right] \left[\sum_{q=1}^{j} p_{rm}(k,j) \right] \\
= \; & \frac{1}{n} \sum_{l=1}^{i} \sum_{q=1}^{j} \sum_{k=1}^{n} p_{sr}(l,k) \cdot p_{rt}(k,q) \\
= \; & \frac{1}{n} \sum_{l=1}^{i} \sum_{q=1}^{j} p_{st}(l,q) = C_{st}(i,j).
\end{aligned}$$

This completes our proof. □

(26) use a counter-example to show that $C_n * D_n \neq (C * D)_n$ in general. However, in the following theorem, we prove that $C_n * D_n$ equals to $C * D$ when $n \to \infty$.

Theorem 2.9. *Suppose C_n and D_n are the n-th order discretizations of copulas C and D, respectively, then*

$$\lim_{n \to +\infty} C_n * D_n \left(\frac{\lfloor nu \rfloor}{n}, \frac{\lfloor nv \rfloor}{n} \right) = C * D(u,v).$$

Proof. Suppose $t_i \in [\frac{i-1}{n}, \frac{i}{n}]$ for $i = 1, ..., n$, then we have

$$
\begin{aligned}
& C * D(u, v) \\
= & \int_0^1 \partial_2 C(u, t) \partial_1 D(t, v) dt \\
= & \lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} \partial_2 C(u, t_i) \partial_1 D(t_i, v) \\
= & \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} n \left[C\left(u, \frac{i}{n}\right) - C\left(u, \frac{i-1}{n}\right) \right] \cdot n \left[C\left(\frac{i}{n}, v\right) - C\left(\frac{i-1}{n}, v\right) \right] \\
= & \lim_{n \to +\infty} \sum_{k=1}^{n} n \left[C_n\left(\frac{\lfloor nu \rfloor}{n}, \frac{i}{n}\right) - C\left(\frac{\lfloor nu \rfloor}{n}, \frac{i-1}{n}\right) \right] \\
& \cdot \left[C\left(\frac{i}{n}, \frac{\lfloor nv \rfloor}{n}\right) - C\left(\frac{i-1}{n}, \frac{\lfloor nv \rfloor}{n}\right) \right] \\
= & \lim_{n \to +\infty} C_n * D_n \left(\frac{\lfloor nu \rfloor}{n}, \frac{\lfloor nv \rfloor}{n}\right).
\end{aligned}
$$

The second equation holds based on the definition of Riemann integral, the third equation holds based on the definition of derivatives, the forth equation holds based on (3), and the first and last equations hold due to the definition of $*$-product. $\square$

2.3. *Discretization of copula correlated processes*

We now consider how to discretize a copula-based process $\{X_t\}$ with marginal distribution F and 2-dimensional copula C. Suppose that C_n is the n-th order discretization of C. If $\{X_{t,n}\}$ is a discrete copula-correlated process (DCCP) with marginal distribution F and copula C_n, then we say it is the n-th order discretization of $\{X_t\}$. It is clear that the joint distribution of $X_{t,n}$ and $X_{t+1,n}$ is given by

$$C_n \left(\frac{\lceil nF(x_{t,n}) \rceil}{n}, \frac{\lceil nF(x_{t+1,n}) \rceil}{n} \right).$$

As discussed in Section 2.1, for Markov process $\{X_{t,n}\}$ with state space $\mathcal{X}$, the corresponding kernel $K_n(x, [a, b]) : \mathcal{X} \times \Sigma \to [0, 1]$ is defined by

$$K_n(x, [a, b]) = \begin{cases} \sum\limits_{j=l+1}^{m-1} p_n(k, j) + p_n(k, m)[nF(b) - (m-1)] + p_n(k, l)[l - nF(a)], & \text{if } l < m; \\ \\ p_n(k, l)[nF(b) - nF(a)], & \text{if } l = m, \end{cases}$$

where $k = \lceil nF(x) \rceil$, $l = \lceil nF(a) \rceil$, $m = \lceil nF(b) \rceil$, and $p_n(i, j)$'s are defined by (4).

$\{X_{t,n}\}$ can also be viewed as a Markov-modulated process with the underlying Markov process $\{J_{t,n}\}$ which has state space $\{1, 2, ...n\}$ and transition probability matrix $(p_n(i, j))$ defined by (4). Given that $J_{t,n} = i$, the conditional probability distribution function of $X_{t,n}$ is

$$F(x|J=i) = \begin{cases} nF(x)-(i-1), & \text{if } x \in \left(F^{-1}((i-1)/n), F^{-1}(i/n)\right]; \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, this implies that $X_{t,n} \in \left(F^{-1}((i-1)/n), F^{-1}(i/n)\right]$.

We note that only the copula is discretized and the marginal distribution is modified accordingly but not discretized, and finally the state space of the process still remains the same. The corresponding discrete copula correlated process $\{X_{t,n}\}$ can be generated as follows:

- **Generate a random variate U_1 from $U(0,1)$ and obtain corresponding $X_1 = F^{-1}(U_1)$ and the underlying state $J_{1,n} = \lceil nU_1 \rceil$;**
- **Generate $J_{2,n}$ based on $J_{1,n}$ and transition probability matrix $p_n(i,j)$; obtain corresponding $U_2 = (J_{2,n}-1)/n + U \cdot n$, with U being generated from $U(0,1)$ and $X_2 = F^{-1}(U_2)$**
- **...**
- **Return $\mathbf{X} = (X_1, X_2, ...)$.**

Proposition 2.10. *If $C_2(u,v) = \partial C/\partial u$ satisfy the following Lipschitz conditions $\forall u, u', v, v' \in I$*

$$\begin{aligned} |C_2(u,v) - C_2(u',v)| &\leq L|u-u'|, \\ |C_2(u,v) - C_2(u,v')| &\leq L|v-v'|, \end{aligned}$$

where $L > 0$ is some constant, then

$$K_n(x,[a,b]) \Rightarrow K(x,[a,b]) \quad \text{as} \quad \text{n} \to \infty.$$

Proof. Based on the Lipschitz conditions, we have

$$\begin{aligned} p_n(k,j) &= n\left[C\left(\frac{k}{n},\frac{j}{n}\right) - C\left(\frac{k-1}{n},\frac{j}{n}\right) - C\left(\frac{k}{n},\frac{j-1}{n}\right) + C\left(\frac{k-1}{n},\frac{j-1}{n}\right)\right] \\ &= n\int_{(k-1)/n}^{k/n}\left[C_2\left(u,\frac{j}{n}\right) - C_2\left(u,\frac{j-1}{n}\right)\right] du \leq L/n, \end{aligned}$$

for $1 \leq j \leq n$,

$$
\begin{aligned}
& \left| \sum_{j=l+1}^{m-1} p_n(k,j) - \left[C_2\left(F(x), \frac{m-1}{n}\right) - C_2\left(F(x), \frac{l}{n}\right) \right] \right| \\
= & \left| n \int_{(k-1)/n}^{k/n} \left[C_2\left(u, \frac{m-1}{n}\right) - C_2\left(u, \frac{l}{n}\right) \right] du \right. \\
& \left. - \left[C_2\left(F(x), \frac{m-1}{n}\right) - C_2\left(F(x), \frac{l}{n}\right) \right] \right| \\
= & \left| n \int_{(k-1)/n}^{k/n} \left[C_2\left(u, \frac{m-1}{n}\right) - C_2\left(F(x), \frac{m-1}{n}\right) \right] du \right. \\
& \left. - n \int_{(k-1)/n}^{k/n} \left[C_2\left(u, \frac{l}{n}\right) - C_2\left(F(x), \frac{l}{n}\right) \right] du \right| \\
\leq & \left| n \int_{(k-1)/n}^{k/n} \left[C_2\left(u, \frac{m-1}{n}\right) - C_2\left(F(x), \frac{m-1}{n}\right) \right] du \right| \\
& + \left| n \int_{(k-1)/n}^{k/n} \left[C_2\left(u, \frac{l}{n}\right) - C_2\left(F(x), \frac{l}{n}\right) \right] du \right| \\
\leq & \ 2L/n,
\end{aligned}
$$

and

$$
\left| \left[C_2\left(F(x), \frac{m-1}{n}\right) - C_2\left(F(x), \frac{l}{n}\right) \right] - \left[C_2\left(F(x), F(b)\right) - C_2\left(F(x), F(a)\right) \right] \right| \leq 2L/n.
$$

Putting the above three inequalities together, we have

$$
|K_n(x, [a,b]) - K(x, [a,b])| \leq 6L/n.
$$

This completes the proof. $\square$

Denote $(\mathcal{M}, \Omega) = (\mathcal{X}, \Sigma)^\infty$ as the infinite product space of stochastic process $\{X_t\}$. Given a probability measure ω on Σ and a Markov kernel K on $(\mathcal{X}, \Sigma)$, then there exists a probability measure $\mathcal{P}$ on $(\mathcal{M}, \Omega)$ such that $\{X_t\}$ is a Markov chain over the probability space $(\mathcal{M}, \Omega)$ with state space $\mathcal{X}$, initial distribution ω, and transition kernel K ((*28*)). Similarly, for Markov kernels K_n, we can also have the corresponding measure $\mathcal{P}_n$ on $(\mathcal{M}, \Omega)$.

Theorem 2.11. *If the Lipschitz conditions in Proposition 2.10 hold, then $\mathcal{P}_n \Rightarrow \mathcal{P}$ on $(\mathcal{M}, \Omega)$.*

Proof. From Proposition 2.10, we know that for any $\epsilon > 0$, there exists an N_1 such that when $n > N_1$, $|K_n(x, [a,b]) - K(x, [a,b])| \leq \epsilon$. Furthermore, for any sequence $\{x_m\} \in \mathcal{X}$ satisfying $x_m \to x$, then these exists N_2 such that when $m > N_2$, $x_m \in ((k-1)/n, n/k]$. Hence, when $n > N_1$ and $m > N_2$, we have

$$
\left| K_n(x_{n'}, [a,b]) - K(x, [a,b]) \right| = \left| K_n(x, [a,b]) - K(x, [a,b]) \right| \leq \epsilon.
$$

Then based on the remarks following Theorem 4 in (*28*), we can conclude that $\mathcal{P}_n \Rightarrow \mathcal{P}$. $\square$

Theorem 2.11 guarantees that $\{X_{t,n}\}$, the n-th order discretization of $\{X_t\}$, converges to $\{X_t\}$ (weakly). In what follows, we provide some examples.

Example 2.12. The Farlie-Gumbel-Morganstern family of copulas (FGM copulas)

FGM copula is defined as

$$C_F(u,v;\theta) = uv(1+\theta(1-u)(1-v)), \tag{5}$$

where $-1 \leq \theta \leq 1$ is a correlation parameter. **It is easy to check that $C_F(u,v;\theta)$ in Equation (5) satisfies the requirements in Definition 2.1 if $\theta \in [-1,1]$. Due to its simple analytical form, FGM distributions have been widely used, e.g., see (*24*)**. We the n-th order discretization of FGM copula by $C_{F,n}$, then its transition matrix is given by $P_{F,n} = (p_{F,n}(i,j))$:

$$p_{F,n}(i,j) = \frac{n^2 + \theta(1-2i+n)(1-2j+n)}{n^3}. \tag{6}$$

Let $P_{FF,n} = P_{F,n} * P_{F,n} = (p_{FF,n}(i,j))$. We have

$$\begin{aligned} p_{FF,n}(i,j) &= \frac{\theta^2\big(i(2-4j)+2j-1\big)}{3n^5} + \frac{2\theta^2(i+j-1)}{3n^4} \\ &\quad + \frac{2\theta^2\big(i(2j-1)-j\big)}{3n^3} - \frac{2\theta^2(i+j-1)}{3n^2} + \frac{3+\theta^2}{3n}, \end{aligned}$$

and

$$\begin{aligned} & C_{F,n} * C_{F,n}\left(\frac{i}{n},\frac{j}{n}\right) \\ = & \sum_{k=1}^{n} n\left[\frac{ik}{n^2}\left(1+\theta\left(1-\frac{i}{n}\right)\left(1-\frac{k}{n}\right)\right) - \frac{i(k-1)}{n^2}\left(1+\theta\left(1-\frac{i}{n}\right)\left(1-\frac{k-1}{n}\right)\right)\right] \\ & \cdot\left[\frac{kj}{n^2}\left(1+\theta\left(1-\frac{k}{n}\right)\left(1-\frac{j}{n}\right)\right) - \frac{(k-1)j}{n^2}\left(1+\theta\left(1-\frac{k-1}{n}\right)\left(1-\frac{j}{n}\right)\right)\right] \\ = & -\frac{i^2j^2\theta^2}{3n^6} + \frac{ij(i+j)\theta^2}{3n^5} + \frac{ij(ij-1)\theta^2}{3n^4} + \frac{i(-i-j)j\theta^2}{3n^3} + \frac{ij(3+\theta^2)}{3n^2}. \end{aligned}$$

Based on $C_{F,n} * C_{F,n}$, we can further calculate its corresponding bistochastic matrix $\bar{P}_{FF,n}$ and verify that $\bar{P}_{FF,n} = P_{FF,n}$, as we have shown in Theorem 2.8. In addition,

$$C_F * C_F(u,v) = \frac{1}{3}uv(3+(u-1)(v-1)\theta^2),$$

therefore,

$$
\begin{aligned}
&\lim_{n\to\infty} C_{F,n} * C_{F,n}\left(\frac{\lfloor nu\rfloor}{n}, \frac{\lfloor nv\rfloor}{n}\right) \\
= \;& -\frac{u^2v^2\theta^2}{3n^2} + \frac{uv(u+v)\theta^2}{3n^2} + \frac{u^2v^2\theta^2}{3} - \frac{uv\theta^2}{3n^2} + \frac{u(-u-v)v\theta^2}{3} + \frac{uv(3+\theta^2)}{3} \\
= \;& \frac{1}{3}uv(3+(u-1)(v-1)\theta^2) \\
= \;& C_F * C_F(u,v),
\end{aligned}
$$

as shown in Theorem 2.9.

Let $\{X_t\}$ be a copula-based process with marginal distribution $F(x) = 1 - exp(-\lambda x)$ and FGM copula C_F linking adjacent inputs as defined by Definition 2.4. If we choose $n = 3$ and $\theta = 0.8$ in FGM copula, then the 3rd order discretization Markov process has transition matrix

$$
P_{F,3} = \begin{pmatrix} 0.452 & 0.333 & 0.215 \\ 0.333 & 0.333 & 0.333 \\ 0.215 & 0.333 & 0.452 \end{pmatrix}.
$$

Example 2.13. Gaussian Copulas

For a given covariance matrix $R \in [-1,1]^{2\times 2}$, the Gaussian copula with R can be written as

$$
C_G(u,v) = \Phi_R(\Phi^{-1}(u), \Phi^{-1}(v)),
$$

where Φ^{-1} is the inverse cumulative distribution function of a standard normal and Φ_R is the joint cumulative distribution function of a bivariate normal distribution with mean vector zero and covariance matrix R. **It should be noted that since the Auto Regressive Moving Average model cannot be applied directly to model those processes whose marginal distribution is not normal. (*29*) and (*30*) developed the idea to transfer an AR(1) process into corresponding uniform autocorrelated random variables, and applied an inverse transformation method to generate process with given marginal distribution. This transformation-based model is identical to Gaussian copula based model in this example.** While there is no simple analytical formula for the copula function $C_G(u,v)$, we can numerically calculate $C_G(u,v)$ for any given (u,v) and the corresponding transition matrix of its n-th order discretization $P_{G,n} = (p_{G,n}(i,j))$. For example, suppose

$$
R = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}
$$

and $n = 3$, then we have

$$
P_{G,3} = \begin{pmatrix} 0.549 & 0.311 & 0.140 \\ 0.311 & 0.378 & 0.311 \\ 0.140 & 0.311 & 0.549 \end{pmatrix}.
$$

3. Discrete-event stochastic systems with copula correlated input processes

In this section, we introduce discrete-event stochastic systems with copula-based input processes. We use the framework of the generalized semi-Markov process (GSMP). A GSMP can be thought of as a formal mathematical model of the sample path of a discrete-event stochastic system (*31*, *32*). To specify a GSMP, we define the following elements:

S	a countable set of states;
$\mathcal{E}$	a finite set of events;
$\mathcal{E}(s), s \in S$	the event list (the set of possible events) in state s;
$p(s'; s, e)$	the probability of jumping to s' when event e occurs in state s;
$F_e, e \in \mathcal{E}$	the distribution function of clock samples for event e.

The GSMP jumps from one state to another (possibly the same one) upon the occurrence of events. The events in $\mathcal{E}(s)$ are said to be *active* in state s. At any point in time, each active event has associated with it a clock, representing the time remaining until the event is scheduled to occur. The next event, and the time until it occurs, are determined by the clock with the smallest value. If event e occurs in state s the process moves to a new state s' with probability $p(s'; s, e)$. The set of active events typically changes along with the state. Clocks for any old events which remain active continue to run in the new state. New clocks are initialized for all newly active events, *and* for the event which just occurred, if it is also active in the new state. The initial value of each new clock is a random variable with the distribution F_e for the associated event e.

An example of a GSMP is a single-server queue, where the state is the number of customers in the system $S = \{0, 1, 2, ...\}$, and the events are arrivals and service completions $\mathcal{E} = \{\alpha, \beta\}$ where α denotes an arrival and β denotes a departure (service completion). $\alpha \in \mathcal{E}(s)$ for all s and $\beta \in \mathcal{E}(s)$ for $s > 0$; $p(s+1; s, \alpha) = 1$ and $p(s-1; s, \alpha) = 1$, with all other routing probabilities zero; and F_α and F_β are the probability distribution functions of the inter-arrival time and the service time, respectively. The dynamics of the GSMP mimic a sample path of such a system.

To give a precise formulation of the evolution of a GSMP, we introduce some more notation, displayed here for ease of reference:

$\{X(e,k)\}$		for $e \in \mathcal{E}$ and $k = 1, 2, \cdots$, $X(e,k)$ is the k-th new clock time for e;
$\{U(e,k)\}$		for $e \in \mathcal{E}$ and $k = 1, 2, \cdots$, independent uniform random variables on $[0,1]$, $U(e,k)$ is the routing indicator at the k-th occurrence of e;
e_l	$=$	the l-th event;
S_l	$=$	the l-th state;
t_l	$=$	the epoch of the l-th event;
τ_l	$=$	the holding time in S_l, the time between t_{l-1} and t_l;
T_l	$=$	$\{T_l(e) : e \in \mathcal{E}(S_l)\}$, where $T_l(e)$ is the remaining clock time for e at t_l;
$N(e,l)$	$=$	number of instances of e among $e_1, \cdots, e_l$;
$\mathcal{N}(s'; s, e)$	$=$	$\mathcal{E}(s') \setminus (\mathcal{E}(s) - \{e\})$, the set of new events following a transition from s to s' triggered by event e;
$\mathcal{O}(s'; s, e)$	$=$	$\mathcal{E}(s') \cap (\mathcal{E}(s) - \{e\})$, the set of old events at such a transition.

Traditionally, it is assumed that $X(e,k)$'s are independent random variables with each $X(e,k)$ having distribution F_e ($k = 1, 2, \ldots$), i.e., clock times are independent of each other, and furthermore they are identically distributed for each event. However, later we will introduce copula correlated clock times that allow clock times to be correlated. We first construct the GSMP as follows: Fix an initial state S_0 and set $T_0(e) = X(e,1)$ for each $e \in \mathcal{E}(S_0)$. Let e_1 be the element of $\mathcal{E}(S_0)$ with the smallest $T_0(e)$, and set $N(e_1,1) = 1$ and $N(e,1) = 0$ for all other e. Set $\tau_1 = \min_{e\in\mathcal{E}[S_0]} T_0(e) = T_0(e_1)$. For $l = 1, 2, \cdots$, sample S_l from $p(\cdot; S_{l-1}, e_l)$ based on $U(e_l, N(e_l, l))$, and denote

$$S_l = \phi(S_{l-1}, e_l, U(e_l, N(e_l, l)))$$

where ϕ is a mapping: $S \times \mathcal{E} \times [0,1] \to S$, satisfying

$$P(\phi(s, e, U) = s') = p(s'; s, e)$$

for any $s \in S$ and $e \in \mathcal{E}(S)$, where U is a uniform random variable on $[0,1]$. We now set

$$\begin{aligned}
T_l(e) &= \begin{cases} X(e, N(e,l)+1), & \text{if } \ e \in \mathcal{N}(S_l; S_{l-1}, e_l), \\ T_{l-1}(e) - \tau_l, & \text{if } \ e \in \mathcal{O}(S_l; S_{l-1}, e_l); \end{cases} \\
\tau_{l+1} &= \min_{e\in\mathcal{E}(S_l)} T_l(e); \\
e_{l+1} &= \arg\min_{e\in\mathcal{E}(S_l)} T_l(e); \\
N(e, l+1) &= \begin{cases} N(e,l)+1, & \text{if } \ e = e_{l+1}, \\ N(e,l), & \text{otherwise.} \end{cases}
\end{aligned}$$

(In defining e_l, $l \geq 1$, use an arbitrary, fixed rule to break ties.) Setting $t_0 = 0$, we have

$$t_l = \sum_{i=1}^{l} \tau_l, \ l = 1, 2, \ldots,$$

and the GSMP is defined by setting

$$Z_t = S_l, \quad \text{for } t \in [t_{l-1}, t_l),$$

i.e., Z_t is the state of the GSMP at time t. Clearly, if all the clock times are independent of each other, then the GSMP is completely determined by the discrete-time, general state space Markov process $\{(S_l, T_l), l = 1, 2, \ldots\}$.

We now introduce copula correlated clock times. Suppose for each $e \in \mathcal{E}$, $\{X(e,k)\}$ is a copula-based process with marginal distribution F_e and copula C_e, i.e., $X(e,k)$ and $X(e,k+1)$ are correlated according to copula C_e. Of course, we can also introduce additional correlations for clock times among different events, for example, by using copulas. However, for ease of exposition, we assume here that clock times for different events are independent of each other. Then $\{X(e,k)\}$ can be generated via (2). For a single-server queue, both the service times and inter-arrival times can be copula-based processes. We should point out that, strictly speaking, when $\{X(e,k)\}$ is a copula correlated process, $\{(S_l, T_l), l = 1, 2, \ldots\}$ is no longer a Markov process, though we still call it a GSMP with copula correlated clock times. Of course, if we expand $\{(S_l, T_l), l = 1, 2, \dots\}$ to include $\{X(e, N(e, l)), e \in \mathcal{E}\}$, then it again becomes a Markov process.

Finally, we consider the discretization of a GSMP with copula correlated clock times $\{X(e, k)\}$. If we replace $\{X(e, k)\}$ with its n-th order discretization $\{X_n(e, k)\}$ as the clock times in the GSMP, then we call the corresponding GSMP with copula correlated clock times $\{X_n(e, k)\}$ the n-th order discretization of the original GSMP. The n-th order discretization of a GSMP with copula correlated clock times $\{X_n(e, k)\}$ can also be viewed as a GSMP whose clock times are Markov-modulated random variables. Similar to the convergence result in Theorem 2.11, we can also show that under some Lipschitz conditions the n-th order discretization GSMP converges weakly to the original GSMP; see (*31*). Various discrete-event stochastic systems with Markov-modulated input processes have been studied in the literature (e.g., (*21*) and (*11*) for queueing systems with Markov-modulated service time or inter-arrival time, (*33*) and (*4*) for (s, S) inventory systems with Markov-modulated demands). Hence, methodologies developed in these works can be used to analyze GSMPs with Markov-modulated clock times.

A key advantage of using copulas to model clock times is that it can easily capture correlations of clock times. This is particularly useful in those studies where we are interested in the impact of these correlations on system performance measures. For example, for queueing networks, we are often interested in the impact of correlations among inter-arrival times and service times on the average waiting time of customers, e.g., see (*1*).

4. An Illustrative example

In this section, we consider a FCFS $G/G/1$ queue with correlated service times. Let $\{A_i\}$ be the inter-arrival times and $\{V_i\}$ be the service times. We assume that $\{A_i\}$ are independent and identically distributed exponential random variables with distribution function $F_\alpha(x) = 1 - e^{-\lambda x}$ and $\{V_i\}$ are copula correlated with exponential marginal distribution $F_\beta(x) = 1 - e^{-\mu x}$ and copula C. By modeling the service times as a copula correlated process, we can easily study the effect of the correlation of the service times on the performances of the queue (e.g., the average queue length and the average waiting time). We consider three cases for C: Gaussian copula, Clayton copula and Frank copula. Gaussian copula (C_G) has already been discussed in Example 2.13.

Clayton copula ($C_C(u, v)$) and Frank copula ($C_F(u, v)$) are two popular Archimedean copulas which are defined as

$$C_C(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta},$$

and

$$C_F(u, v) = -\frac{1}{\theta} ln\left(1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1}\right),$$

respectively. Clayton copula can be used to model asymmetric dependence and it also has lower tail dependence, which is suited for applications in which two adjacent outcomes are likely to experience low values together.

Gaussian copula can be used to model symmetric dependence and it contains no tail dependence. Frank copula exhibits symmetric dependence in both tails, but its dependence is weaker in both tails and stronger at the center compared to Gaussian copula. More details about the dependence structures of different copulas can be found in (*24*).

The autocorrelation ρ between two adjacent service times V_i and V_{i+1} is given by

$$\rho = \frac{\int_{I^2} F_\beta^{-1}(u) F_\beta^{-1}(v) dC(u,v) - (E[V])^2}{Var(V)},$$

where V denotes a generic service time.

We use simulation to obtain the average waiting time of a customer (here we define the waiting time of a customer as the total time it spends in the system) for the queue with $\lambda = 0.4$ and $\mu = 0.9$ (the length of each simulation is 10^7 customers). In Figure 1, we plot the average waiting time with respect to ρ for the three input service time processes (Gaussian copula, Clayton copula, and Frank copula). We have the following observations

- The average waiting time increases with respect to the autocorrelation in all three cases. **Correlation can substantially affect the performance measure (the average waiting time in this example), and therefore should be taken into consideration when analyzing systems in the presence of correlated inputs.**
- **For the three different types of copulas we consider here, even when the autocorrelation and the marginal distribution are the same, the average waiting time can differ significantly, especially when the autocorrelation is large.** This implies that the autocorrelation alone cannot fully capture all dependent properties, and we may have to use different types of copulas to fit data with different dependent structures, which is usually hard to achieve when using other models, such as those based on linear correlation.

Next, we use discretization to approximate the copula correlated service process and the corresponding queue. We use FGM copula C_F as our illustrative. Let $\{V_{i,n}\}$ be the n-th order discretization of $\{V_i\}$, with marginal distribution F_β and copula $C_{F,n}$. Then the queue with inter-arrival times $\{A_i\}$ and service times $\{V_{i,n}\}$, denoted as $G/G_n/1$, is the n-th order discretization of the original queue. In Figure 2, we plot simulation results for the average waiting time of $G/G_n/1$ with $n = 2, 5, 10$ and $G/G/1$ with respect to θ (again, the length of each simulation is 10^7 customers). It is clear from Figure 2 that the average waiting time of $G/G_n/1$ approaches that of $G/G/1$ as n increases, i.e., $G/G_n/1$ becomes a good approximation of $G/G/1$ as n becomes large enough.

Finally, **we show how the MacLaurin series method can be used to analyze $G/G_n/1$ queue.** As we pointed out in Section 3 that the n-th order discretization of a GSMP can be viewed as a GSMP whose clock times are Markov-modulated processes. In our current example, $G/G_n/1$ is in fact equivalent to the following queue with Markov-modulated service times: Let $\{J_i, i \geq 0\}$ be a Markov chain with state space $\{1, 2, ..., n\}$ and transition probability matrix $P = (p_{ij})$ $(i, j \in \{1, 2, ...n\})$, where p_{ij} is defined by (4). If $J_i = j$, then the ith customer has service time with distribution $F_{\beta,j}(x) = nF_\beta(x) - (j-1)$ if $x \in (F_\beta^{-1}((i-1)/n), F_\beta^{-1}(i/n))$ and 0 otherwise.

(*21*) apply the following Maclaurin series method (MSM), which is first proposed by (*34*), to calculate the moments of the waiting time for queues with Markov-modulated

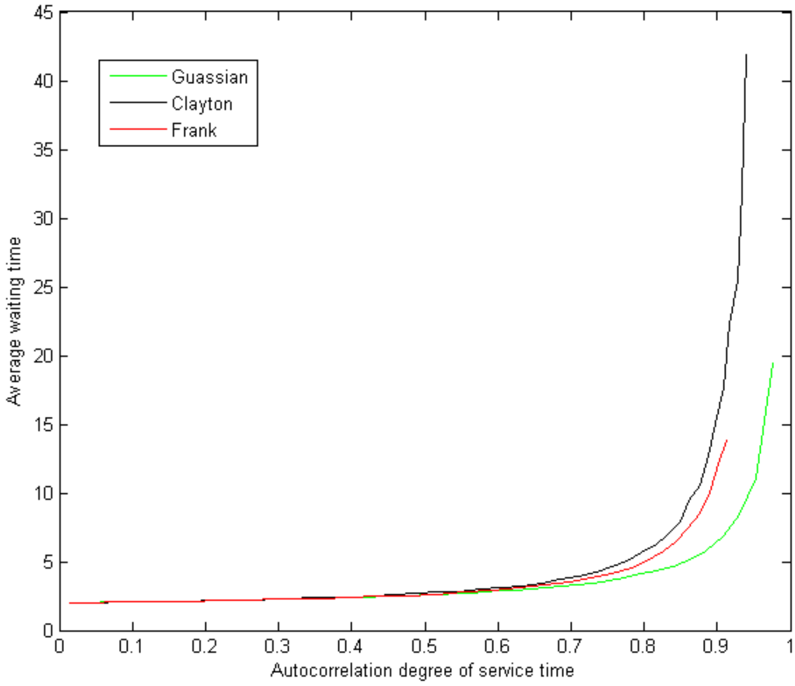

Figure 1. The average waiting time vs. the autocorrelation of the service times for queues with Frank, Clayton, and Gaussian copula correlated service times

service times. Let $f_\alpha(x)$ be the density function of the inter-arrival time, i.e., $f_\alpha(x) = F'_\alpha(x)$, and assume

$$f_\alpha(x) = \sum_{m=0}^{\infty} \frac{f_\alpha^{(m)}(0)}{m!} x^m \quad \text{for all} \quad x \in [0, \infty),$$

where $f_\alpha^{(m)}$ is the m-th derivative of $f_\alpha(x)$ at $x = 0$, and in our example $f_\alpha^{(m)}(0) = (-1)^m \lambda^{m+1}$. Denote T as the waiting time of a customer, then its moments can be calculated as (see (*21*) for details)

$$\frac{E[T^k]}{k!} = \sum_{j=1}^{n} \sum_{m=1}^{\infty} a_{jkm},$$

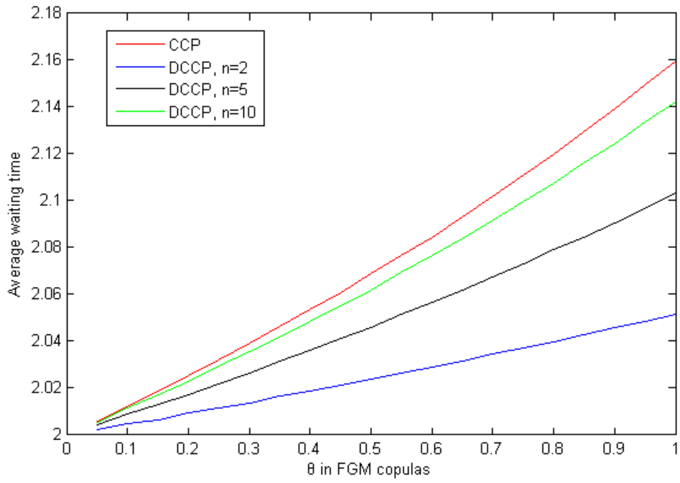

Figure 2. Comparison of the average waiting times of $G/G_n/1$ and $G/G/1$

where

$$
\begin{aligned}
a_{jkm} &= \begin{cases} 0, & \text{if } m < k, \\ \beta_{jk0}, & \text{if } m = k, \\ \sum\limits_{i=1}^{k} \beta_{jki} b_{ji,m+l-k}, & \text{if } m > k, \end{cases} \\
b_{jkm} &= \begin{cases} 0, & \text{if } m \le k, \\ \sum\limits_{l=1}^{n} \sum\limits_{n=0}^{m-k-1} \alpha_{ljn} a_{l,n+k+1,m}, & \text{if } m > k, \end{cases} \\
\alpha_{ljm} &= p(l,j) f_{\alpha}^{(m)}(0), \\
\beta_{jkl} &= \frac{E[V_{i+1}^{k-l} | J_i = j]}{(k-l)!}.
\end{aligned}
$$

Therefore, we can use MSM to calculate the average waiting time of the $G/G_n/1$ queue. In the following numerical experiments, we use $\sum_{j=1}^{n} \sum_{m=1}^{M} a_{jkm}$ with $M = 20$ to approximate $E[T^k]/k!$.

In Table 1, we present the average waiting time of $G/G_n/1$ calculated based on MSM for $n = 2, 5, 10$, along with simulation results. To better compare with MSM, simulation results are obtained based on 100 replications and each replication has 10^7 customers. The simulation results are presented in the form of "mean $\pm$ standard deviation". Computation time (in seconds) is also provided to compare computational efficiency between MSM and simulation. As shown in Table 1, MSM produces very good results and is computationally more efficient than simulation.

This example shows that after discretizing, the existing methods availabe to analyze discrete-event systems with Markov-modulated input processes, such as MSN, may be applicable.

$G/G/1$	$G/G_n/1$		
Simulation (Time)	n	MSM (Time)	Simulation (Time)
$\theta = 0.2$			
2.0254 ± 0.0020 (444.03)	2	2.0115 (0.134)	2.0115 ± 0.0017 (376.22)
	5	2.0221 (0.186)	2.0219 ± 0.0014 (387.52)
	10	2.0244 (0.281)	2.0243 ± 0.0016 (447.29)
$\theta = 0.5$			
2.0690 ± 0.0020 (388.66)	2	2.0304 (0.129)	2.0302 ± 0.0017 (388.57)
	5	2.0593 (0.176)	2.0590 ± 0.0021 (442.02)
	10	2.0659 (0.329)	2.0660 ± 0.0016 (446.46)
$\theta = 0.8$			
2.1199 ± 0.0023 (423.10)	2	2.0514 (0.138)	2.0513 ± 0.0017 (330.72)
	5	2.1029 (0.178)	2.1030 ± 0.0021 (414.64)
	10	2.1147 (0.255)	2.1145 ± 0.0022 (494.51)

Table 1. Numerical results based on both MSM and simulation

5. Conclusion

By using copula-based processes, we propose a new way to model correlated input processes for discrete-event stochastic processes in the framework of GSMP. In general, it is difficult to analyze discrete-event stochastic systems with copula-based inputs, however, we show that they can be discretized and approximated by discrete-event stochastic systems with discrete copula-based inputs, which are equivalent to discrete-event stochastic systems driven by Markov-modulated processes. We show that the discretized systems converge to the original systems. Therefore, we can use the existing methods (e.g., the MSM method) developed to analyze discrete-event stochastic systems driven by Markov-modulated processes in our study. Finally, through a simple queueing example, we demonstrate how our method works in detail.

One major shortcoming of our method is its increasing computation complexity as the order of its discritization (n) increases. Therefore, having computationally efficient methods to analyze large-size dicrete-event stochastic systems driven by Markov-modulated processes is crucial . This is one future research direction related to our work here. The second possible future research direction is to study the effects of correlations on various discrete-event stochastic systems now we can model correlations better and separate them from marginal distributions.

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