CHAPTER 8

Mathematic Modelling with Differential Equations

(See last sheet)

Murray Semestr 2 JF 2012-2013

CHAPTER

9

INFINATE SERIES

2.

$S = \frac{1}{8}(1+S)$

$\Rightarrow 8S = 1+S$

$\Rightarrow 7S = 1$

$\Rightarrow S = 1/7$

Example: $0.999 = \sum_{n=1}^{\infty} \frac{9}{10^n} = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} \dots$

The Sum has Some value, S:

$S = \frac{9}{10} + \frac{9}{100} + \frac{9}{10^3} + \dots$

$S = \frac{1}{10}\left(9 + \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \dots\right)$

$S = \frac{1}{10}(9 + S)$

$10S = 9 + S$

$9S = 9$

$S = 1$

General Answer: $\sum_{n=1}^{\infty} ar^n = ar + ar^2 + ar^3 + ar^4$

1. The Sum has Some value, S;

$S = ar + ar^2 + ar^3 + \dots$

2. factor out r:

$S = r(a + ar + ar^2 + ar^3 + \dots)$

3. everything after the first term is a copy of S

$S = r(a + \underbrace{ar + ar^2 + ar^3 + \dots}_{S})$

4. Solve for S

$S = ra + rS$

$\Rightarrow S - rS = ra$

$\Rightarrow S = \frac{ra}{(1-r)}$ $\quad S = \frac{ar}{(1-r)}$

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Exam Q3

Sequences and series:

Sequences: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots$ List of numbers

Series: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$ Sum of numbers

Notatation Sequence: $\{a_n\}$
Series: $\sum_{n=1}^{\infty} a_n$ means sum from n to ∞

Example series: $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots$

$\sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$

Problem: What does it mean to add up an infinite collection of numbers?

Geometric series:
$\sum_{n=1}^{\infty} ar^n$ a, r are some factor

e.g. $a=1$, $r = \frac{1}{8}$
$\sum_{n=1}^{\infty} \frac{1}{8^n} = \frac{1}{8^1} + \frac{1}{8^2} + \frac{1}{8^3} + \frac{1}{8^4}$

Value: $\frac{1}{8^n} = \frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} \ldots$

This is equal to some number S: $\sum_{n=1}^{\infty} \frac{1}{8^n} = S$

$S = \frac{1}{8^1} + \frac{1}{8^2} + \frac{1}{8^3} + \ldots$

Factor out 1/8 $S = \frac{1}{8}\Big(1 + \underbrace{\frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + \ldots}_{S}\Big)$

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Examples: $\sum_{n=1}^{\infty} \frac{4}{15^n}$ $a=4$ $r=\frac{1}{15}$

$$S = \frac{ra}{(1-r)} = \frac{4/15}{1-1/15} = \frac{4/15}{14/15} = \frac{4}{14}$$

$\sum_{n=1}^{\infty} \frac{8}{19^n}$ $a=8,$ $r=1/19$ $S=\frac{ar}{(1-r)}$

$$\frac{8/19}{1-1/19} = \frac{8/19}{18/19} = \frac{8}{18}$$

Note: r must be less than 1
Otherwise answer IS WRONG

Example: $\sum_{n=1}^{\infty} 2^n = 2+4+8+\ldots$

$a=1$ $r=2$

$$S = \frac{ra}{(1-r)} \quad \frac{2}{1-2} = -2 \text{ WRONG}$$

Example: $0.999\ldots$ $\sum_{n=1}^{\infty} \frac{9}{10^n} = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3}$

$a=9$ $r=1/10$

$$S = \frac{ar}{(1-s)} \quad \frac{9/10}{9/10} = 1$$

General Series (not geometric)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} \cdots = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} \cdots ?$$ not known yet

To difficult to figure out the value of a series
So, we will look at convergence

Convergence:

$$\sum_{n=1}^{\infty} a_n \nearrow \text{diverges } \infty \quad \searrow s \text{ converges}$$

$\uparrow$
Some particular value

Two main methods:
1. Ratio test
2. Integral test.

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Series:

Examples: (1) $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{10^n}$

Choice: ratio test or Integral test

Integral test: $n^3 \Rightarrow x^3$

$10^n \Rightarrow 10^x$

$(-1)^n \Rightarrow (-1)^x$ not defined for some values of x, value would be complex

e.g. $(-1)^{3/2}$ $((-1)^{1/2})^3 = (i)^3 = -i$

Don't use integral test when:

i. A factorial appears, e.g $(n+1)!$

ii. Powers of negative numbers $(-1)^n, (-2)^n$ ...

Need to use ratio test.

$\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{10^n}$

1. $a_n = \frac{(-1)^n n^3}{10^n}$

2. $a_{n+1} = \frac{(-1)^{n+1}(n+1)^3}{10^{n+1}}$

$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}(n+1)^3}{10^{n+1}} \cdot \frac{10^n}{(-1)^n (n)^3}$

$\frac{a_{n+1}}{a_n} = \frac{(-1)}{10} \frac{(n+1)^3}{n^3}$

$\frac{a_{n+1}}{a_n} = \frac{(-1)}{10} \frac{n^3+3n^2+3n+1}{n^3}$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{-1}{10} \cdot \frac{n^3+3n^2+3n+1}{n^3}$

2.

3. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{10} \cdot \frac{n^3+3n^2+3n+1}{n^3}$

$= \frac{1}{10} \lim_{n\to\infty} \left(\frac{n^3+3n^2+3n+1}{n^3} \right)$ coefficients of highest power

$\Rightarrow \frac{1}{10} \cdot 1 = \frac{1}{10}$

4. Limit was $1/10 < 1$ so the series converges

Example: $\sum_{n=1}^{\infty} \frac{4n^3}{n^4+1}$ (if $(-1)^n$ or $n!$ are not present, we integral test)

Integral test: convert sum to an integral

$\frac{4n^3}{n^4+1} \Rightarrow \frac{4x^3}{x^4+1}$

$\sum_{n=1}^{\infty} \frac{4n^3}{n^4+1} \Rightarrow \int^{\infty} \frac{4x^3}{x^4+1}\,dx$

$\Rightarrow \int_1^{\infty} \frac{4x^3}{x^4+1}\,dx$

$\Rightarrow u = x^4+1$ $\quad \frac{du}{dx} = 4x^3$ $\quad du = 4x^3dx$

limits changed $\Rightarrow \int_2^{\infty} \frac{du}{u} = \left[\ln|u| \right]_2^{\infty}$

$\Rightarrow \ln(\infty) - \ln(2)$

$= \infty$ = diverges

iii Does the following series converge, if so put a bound on its value:

$\sum_{n=1}^{\infty} \frac{3}{n^6}$ Integral test: no $(-1)^n$ or $n!$

2. Bound

Convert sum to integral:

$\frac{3}{n^6} = \frac{3}{x^6}$

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Test if the following series converge or diverge, state a bound if one exists:

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ 2. $\sum_{n=1}^{\infty} \frac{n^2}{e^{-n^3}}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 2^n}{n!}$ 4. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

1. $\frac{1}{n^2+4}$ Integral test

Σ a series with positive terms
f is a positive function decreasing and continuous on $[a, \infty]$ such that $U_k = f(k)$ for all $k \geq a$
either converge or diverges

$$\int_1^{\infty} \frac{1}{x^2+4}\,dx \Rightarrow \int_1 \frac{1}{x^2+2^2} \Rightarrow \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_1^b$$

$$\lim_{b \to \infty^+} \frac{1}{2}\tan^{-1}\frac{b}{x} - \frac{1}{2}\tan^{-1}\frac{1}{2}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{\pi}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$$

$\Rightarrow$ integral converges $\Rightarrow$ series converges

$\frac{1}{5}, \frac{1}{8}, \frac{1}{13}, \ldots \to 0$

Bound $= \frac{1}{5}$

2. $\sum_{n=1}^{\infty} \frac{n^2}{e^{-n^3}} = \int_1^{\infty} \frac{x^2}{e^{-x^3}}$ $u = -x^3$

integral test

$\frac{du}{dx} = -3x^2$

$du = -3x^2 dx$

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$$\sum_{n=1}^{\infty} \frac{3}{n^6} \Rightarrow \int_1^{\infty} \frac{3}{x^6}\,dx \Rightarrow \int_1^{\infty} 3x^{-6}\,dx$$

$$\Rightarrow \left[\frac{3x^{-5}}{-5}\right]_1^{\infty}$$

$$\Rightarrow \left[\frac{-3}{5} \cdot \frac{1}{x^5}\right]_1^{\infty}$$

$$= -\frac{3}{5}\frac{1}{(\infty)^5} + \frac{3}{5}\frac{1}{1^5}$$

$$0 + 3/5$$

$$= 0 \Rightarrow 3/5 \text{ converge}$$

Bound

$$\sum_{n=1}^{\infty} \frac{3}{n^6} \leq a_1 + \int_1^{\infty} \frac{3}{x^6}\,dx$$

$$\Rightarrow a_1 : \frac{3}{n^6} = \frac{3}{1^6} = 3$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{3}{n^6} \leq 3 + 3/5$$

NOT ON EXAM!

Uses of Series

1. Expansions of functions

examples (a) Taylor series

$$f(x) = \sum_{n=1}^{\infty} a_n x^n$$

(b) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin(nx) + b_n \cos(nx)$

2.

$\Rightarrow \frac{1}{3}\int_1^{\infty} e^{-u} du \Rightarrow \frac{1}{3}[-e^{-u}]$

$\lim_{b \to \infty^+} \left[\frac{1}{3}e^{x^3}\right]_1^b$

$= \lim_{b \to \infty^+} \left(\frac{1}{3}e^{b^3} - \frac{1}{3}e\right) = \infty$

both integral and level converge

Ratio test: $\sum u_k$ a series

$e = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k}$

i. $e < 1 \Rightarrow$ conv

ii. $e > 1 \Rightarrow$ div

iii. $e = 1 \Rightarrow$ not enough info

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 2^n}{n!}$

$\lim_{k \to \infty} \frac{u_{k+1}}{u_k} = \lim_{k \to \infty} \frac{\cancel{(-1)^{k+2}} (k+1)^2 2^k k!}{(k+1)! \, \cancel{2^{k+1}} \, k^2 2^k}$

$\Rightarrow \lim_{k \to \infty} \frac{(-1)(2)(\cancel{k+1})^2}{(\cancel{k+1}) \, k^2}$

$\rightarrow \left|\frac{(-2)(k+1)}{k^2}\right|$ modul $= \lim_{k \to \infty} \left|\frac{2(k+1)}{k^2}\right|$

~~~~ ~~~~

$= 0 < 1 \Rightarrow$ diverge

Bound: $n = 1$ $\frac{(-1)^{1+1}(1)^2(2)^1}{1!}$

$= 2 =$ upper

lower $\Rightarrow$ (2) sub in to equ

$= -8$
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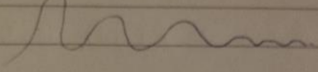
4. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$



$\lim_{k\to\infty} \frac{u_{k+1}}{u_k} \quad \frac{(-1)^{k+1}}{\ln(k+1)} \cdot \frac{\ln(k)}{(-1)^k}$

$\frac{-1 \ \ln(k)}{\ln(k+1)}$

$\lim_{k\to\infty} \frac{\ln(k)}{\ln(k+1)}$ $\qquad \log a > \log b$ when $a > b$

$\Rightarrow \ln(k) > \ln(k+1)$

$\forall\, k \geq 2$

$\Rightarrow \lim_{k\to\infty} \left| \frac{\ln(k)}{\ln(k+1)} \right| \Rightarrow$ as $k \to \infty$

$< 1 \Rightarrow$ converges

Upper bound 1st term lower $= 2^{nd}$

~~Upper~~ $\cancel{\frac{(-1)}{\ln(2)}} =$

~~Upper~~ $\frac{(-1)^2}{\ln(2)} = \frac{1}{\ln(2)}$

Upper

Lower $\frac{(-1)^3}{\ln(3)} = \frac{-1}{\ln 3}$