TRANSPORTATION

Transportation Problem

Arises frequently in planning for the distribution of goods and services from several supply locations to several demand locations

Objective is to minimize the cost of shipping goods from origin to destinations

Foster Generators example, Transportation of a product from three plants to 4 distribution centres

Origin	Plant	Production Capacity
1	Cleveland	5000
2	Bedford	6000
3	York	2500
	Total	13500

Distributed to:

Destination	Destination name	Demand
1	Boston	6000
2	Chicago	4000
3	St. Louis	2000
4	Lexington	1500
		13500

We need to determine how much of its production should be shipped from each plant to each distribution centre

Transportation costs per unit for Foster Generators transportation problem:

Origin	Destination: Boston	Chicago	St. Louis	Lexington
Cleveland	3	2	7	6
Bedford	7	5	2	3
York	2	5	4	5

2.

- LP can be used to solve problem.
- We used double subscripted decision variables, x_{11} denoted number of units shipped from origin 1 to destination 1.

- Because the objective of the transportation problem is to minimize the total transportation cost, we can use the cost in 3rd table :

Transportation cost for units shipped from Cleveland $= 3x_{11} + 2x_{12} + 7x_{13} + 6x_{14}$

" Bedford $= 7x_{21} + 5x_{22} + 2x_{23} + 3x_{24}$

" York $= 2x_{31} + 5x_{32} + 4x_{33} + 5x_{34}$

The sum of these expressions provides the objective function showing the total transportation cost for Foster Generators

Transportation problems need constraints because each origin has a limited supply and each has a demand requirement. Supply constraints first.

Capacity at Cleveland is 5000 units

The total number of units shipped from the cleveland is $x_{11} + x_{12} + x_{13} + x_{14}$

$\Rightarrow x_{11} + x_{12} + x_{13} + x_{14} \leq 5000$ Cleveland Supply

$\Rightarrow x_{21} + x_{22} + x_{23} + x_{24} \leq 6000$ Bedford Supply

$x_{31} + x_{32} + x_{33} + x_{34} \leq 2500$ York Supply

With four demands, constraints are needed to ensure that destination demand is satisfied.

$\Rightarrow x_{11} + x_{21} + x_{31} + x_{41} = 6000$ Boston demand

$x_{12} + x_{22} + x_{32} + x_{42} = 4000$ Chicago demand

$x_{13} + x_{23} + x_{33} + x_{43} = 2000$ St. Louis demand

$x_{14} + x_{24} + x_{34} + x_{44} = 1500$ Lexington demand

Add O.F the constraints to create L.P model

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Problem Variation

1. Total supply not equal to total demand
2. Maximisation of objective coal.
3. Route capacities or route minimum
4. Unacceptable route

Total supply not equal to total demand

- If total supply exceed total demand, no modification in the L.P ~~formulation~~ formulation is necessary
- Excess supply will appear as slack in the LP Solution.
- Slack for any particular origin can be interpreted as the unused supply or amount not shipped from the origin

- If total supply is <u>less</u> than total demand it will ~~have~~ not have a feasible solution
- We add a dummy ~~origin~~ origin with a supply equal to the difference between total demand and supply
- A zero cost is assigned to each arc leaving the dummy origin so that the value of the optimal solution will represent the shipping cost for the units actually shipped (no shipments actually from dummy origin).

When the optimal solution is implemented, the destinations showing shipments being received from the dummy origin will be the destinations experiencing a shortfall, or unsatisfied demand

Maximisation Objective Function

- In some problem, the objective is to maximise profit or revenue
- Using the values for profit or revenue per unit as coefficients in the objective function, we simply maximise rather than a minimisation linear program.
- This change does not affect the constraints.

4

Route Capacities or Route Minimums

- Formulation can also accomodate capacities or minimum quantities for one or more routes.
- Constraint in form of $x_{31} \leq 1000$ for max capacity
  $x_{22} \geq 200$ for minimum capacity

Unacceptable Routes

- Establishing a route from every origin to every destination may not be possible
- To handle this situation, we simply drop the corresponding arc from the network and remove the corresponding variable from the LP formulation
- We remove " x_{12} for example and solve model ensuring route x_{12} is not used

Known as Capacitated transportation problem when route has a min or max capacity: $x_{32} \leq$ " $x_{12} \geq$ "

ASSIGNMENT PROBLEM

- jobs to machines, agent to tasks etc.
- We look for a set of assignments that will optimise a stated objective min cost, min time or max profit.
- A distinguishing feature of the assignment problem is that One agent is assigned to one and only one task.

Example Fowle Marketing. Received request for market study for 3 new clients. Company must assign one leader to each client.
- 3 individuals available
- Realised that the time required to complete each study will depend on the experience and ability of each project leader.
- 3 projects have approximately the same priority
- Wants to minimise total number of days to complete all 3 projects.

5.

- Management must first consider all possible leader-client assignments and estimate project completion time
- With 3 leaders and 3 clients, nine assignment alternatives are possible

completion days :

Project leader	Client 1	2	3
1. Terry	10	15	9
2. Clare	9	18	5
3. Mac	6	14	3

The assignment problem is a special case of the transportation problem, in which all supply and demand values equal 1, and amount shipped is either 0 or 1.

- LP can be formulated
- Use double subscripts x_{12} = project leader 1 to client 2
- We define the decision variables for Fowle's assignment problem as:

$$x_{ij} \begin{cases} 1 & \text{if project leader } i \text{ is assigned to client } j \\ 0 & \text{otherwise} \end{cases}$$

where $i = 1,2,3$ $\quad j = 1,2,3$

We develop completion time expressions

Days required for Terry's assignment : $10x_{11} + 15x_{12} + 9x_{13}$
Days " Clare's " : $9x_{21} + 18x_{22} + 5x_{23}$
Days " Mac's " : $6x_{31} + 14x_{32} + 3x_{33}$

The sum of completion times for the 3 leaders will provide total days required to complete the 3 assignments. Objective function is:

$$\text{Min}: 10x_{11} + 15x_{12} + 9x_{13} + 9x_{21} + 18x_{22} + 5x_{23} + 6x_{31} + 14x_{32} + 3x_{33}$$

The constraints for the assignment problem reflect the conditions that each project leader can be assigned to at most one client, and that client must have one assigned project leader

6.

$x_{11} + x_{12} + x_{13} \leq 1$ Terry's assignment

$x_{21} + x_{22} + x_{23} \leq 1$ Carl's assignment

$x_{31} + x_{32} + x_{33} \leq 1$ Mac's assignment

$x_{11} + x_{21} + x_{31} = 1$ Client 1

$x_{12} + x_{22} + x_{32} = 1$ Client 2

$x_{13} + x_{23} + x_{33} = 1$ Client 3

Combine to get model with 9 variables and 6 constraints.

Solution	client	days
Terry	2	15
Carle	3	5
Mac	1	6
	Total	26

Problem Variations

Ass problem can be viewed as special case of transportation problem, the problem variations that may arise in an assignment premblem are parallel to those of transportation problem.

1. Total number of agents (supply) not equal to total number of tasks (demand)
2. A maximization ob.
3. Unacceptable assignments

- If more agents than tasks, extra agents available.
- More tasks than agents, infeasible ⇒ add dummy agent
- O.F cost for dummy would be zero

TRANSHIPMENT PROBLEM

- An extension of transportation problem, in which intermediate nodes referred to as transhipment nodes are added for locations such as warehouses
- Shipments may be made between any pair of the three general types of nodes: origin nodes, transhipment nodes and destination nodes

7.

- Supply available at each origin is limited, and demand is specified
- Objective is to determine how many units should be shipped over each arc in the network so that all destinations demands are satisfied with the minimum possible transportation cost.

Example: Ryan Electronics. Production facilities in Denver and Atlanta
- Components produced at either facility may be shipped to 2 different regional warehouses Kansas or Louisville.
- From regional warehouses the firm supplies outlets in Detroit, Miami, Dallas and New Orleans

Supply and demand shown

plant	warehouse: Kansas	Louisville
Denver	2	3
Atlanta	3	1

warehouse	retail outlet: Detroit	Miami	Dallas	New Orleans
Kansas	2	6	3	6
Louisville	4	4	6	5.

- We need a constraint for each node and variable for each arc.
- Let X_{ij} denote the number of units shipped from node i to node j.
- Because supply at denver plant is 600 units amount shipped is ≤ 600

$X_{13} + X_{14} \leq 600$ Denver Supply

and $X_{23} + X_{24} \leq 400$ Atlanta Supply

- Constraints corresponding to 2 transhipment nodes
- for node 3 (Kansas), we must guarantee that number of units shipped out must equal number of units shipped into warehouse:

Number of units shipped out of node 3 = $X_{35} + X_{36} + X_{37} + X_{38}$

and units shipped into node 3 = $X_{13} + X_{23}$

We obtain

Problem 15

$$\text{Min } 2x_{1} + 3x_{14} + 3x_{23} + 1x_{24} + 2x_{35} + 6x_{36} + 3x_{37} + 6x_{38} + 4x_{45} + 4x_{46} + 6x_{47} + 3x_{48}$$

ST:

$$\left.\begin{array}{l} x_{13} + x_{14} \leq 600 \\ x_{23} + x_{24} \leq 400 \end{array}\right\} \text{origin node constraint}$$

$$\left.\begin{array}{l} -x_{13} - x_{23} + x_{35} + x_{36} + x_{37} + x_{38} = 0 \\ -x_{14} - x_{24} + x_{45} + x_{46} + x_{47} + x_{48} = 0 \end{array}\right\} \text{Transhipment node constraint}$$

$$\left.\begin{array}{l} x_{35} + x_{45} = 200 \\ x_{36} + x_{46} = 150 \\ x_{37} + x_{47} = 350 \\ x_{38} + x_{48} = 300 \end{array}\right\} \text{Destination node constraint}$$

Problem Variation

1. Total Supply not equal total demand
2. Max objective function
3. Route capacities or route minimal
4. Unacceptable route

Transportation Her role

	Waterford	~~From~~ TO Sligo	Athlone	Supply
Dublin	9	11	8	200
Cork	6	15	10	150
Galway	7	6	6	100
Demand	180	140	130	

Check for balance: Supply = 200+150+100 = 450

Demand = 180 + 140 + 130 = 450

Problem is balanced, no need for dummy supply or dummy demand

Formulation

Min Z: $9x_{11} + 11x_{12} + 8x_{13} + 6x_{21} + 15x_{22} + 10x_{23} + 7x_{31} + 6x_{32} + 6x_{33}$

ST: $x_{11} + x_{12} + x_{13} = 200$ supply dublin

$x_{21} + x_{22} + x_{23} = 150$ supply cork

$x_{31} + x_{32} + x_{33} = 100$ supply galway

$x_{11} + x_{21} + x_{31} = 180$ demand waterford

$x_{12} + x_{22} + x_{32} = 140$ demand Sligo

$x_{13} + x_{23} + x_{33} = 130$ demand athlone

All variables ≥ 0

Three different solutions:

- North West Corner
- Least Cost
- Vogel's Approximation Method VAM

Northwest Corner

- Starting at the top left corner allocate the least of the row supply or the column demand to the cell
- Subtract the amount allocated to the cell from the row and column total
- If the column total is now zero move to next cell on the right. If it is the row total that is zero, then move down to the cell below
- Allocate an amount to the new cell as in step (1) and repeat until all demand and supply is allocated

2

1

	Waterford	Sligo	Athlone	Supply	
Dublin	9(180)	11	8	200-180=20	colum =0
Cork	6	15	10	150	move right one
Galway	7	6	6	100	
Demand	180-180=0	140	130		

2

	waterford	Sligo	athlone	Supply	
Dublin	9(180)	11(20)	8	20-20=0	row=0
Cork	6	15	10	150	down 1
Galway	7	6	6	100	
Demand	0	140-20=120	130		

3

	waterford	Sligo	athlone	Supply	
Dublin	9(180)	11(20)	8	0	colum=0
Cork	6	15(120)	10	150-120=30	over 1
Galway	7	6	6	100	
Demand	0	120-120=0	130		

4

	waterford	Sligo	athlone	Supply	
Dublin	9(180)	11(20)	8	0	row=0
Cork	6	15(120)	10(30)	30-30=0	down 1
Galway	7	6	6	100	
Demand	0	0	130-30=100		

5

	waterford	Sligo	athlone	Supply
Dublin	9(180)	11(20)	8	0
Cork	6	15(120)	10(30)	0
Galway	7	6	6(100)	100-100=0
Demand	0	0	100-100=0	

Finished

3

North West Corner - Initial table:

	Waterford	Sligo	Athlone	Supply
Dublin	9(180)	11(20)	8	200
Cork	6	15(120)	10(30)	150
Galway	7	6	6(100)	100
Demand	180	140	130	

LEAST COST METHOD

- Allocate as much as possible to the cell with the lowest cost.
- Adjust row and column totals like NWC method
- From the remaining cells whose column has some free demand and whose row has some free supply select the with lowest cost.
- Repeat until all has been allocated

1.

	Waterford	Sligo	Athlone	Supply
Dublin	9	11	8	200
Cork	6(150)	15	10	150-150=0
Galway	7	6	6	100
Demand	180-150=30	140	130	

2.

	Waterford	Sligo	Athlone	Supply
Dublin	9	11	8	200
Cork	6(150)	15	10	0
Galway	7	6(100)	6	100-100=0
Demand	30	140-100=40	130	

3.

	Waterford	Sligo	Athlone	Supply
Dublin	9	11	8(130)	200-130=70
Cork	6(150)	15	10	0
Galway	7	6(100)	6	0
Demand	30	40	130-130=0	

4

	waterford	Sligo	Athlone	supply
4 Dublin	9(30)	11	8(130)	70-30=40
Cork	6(150)	15	10	0
Galway	7	6(100)	6	0
Demand	30-30=0	40	0	

	waterford	Sligo	Athlone	supply
5 Dublin	9(30)	11(40)	8(130)	40-40=0
Cork	6(150)	15	10	0
Galway	7	6(100)	6	0
Demand	0	40-40=0	0	

Least cost initial table:

	waterford	Sligo	athlone	SUPPLY
Dublin	9(30)	11(40)	8(130)	200
Cork	6(150)	15	10	150
Galway	7	6(100)	6	100
DEMAND	180	140	130	

VOGEL'S APPROXIMATION METHOD (VAM)

- This attempts to identify the greatest advantage of a cell over the next best cell in its row or column
- This method generally yields a close to optimum starting solution and is superior to LCM and NWC method

Steps in VAM

- Calculate a penalty for each row and column by subtracting the smallest cost element in the row or column from the next smallest cost from the same row or column
- Identify the row or column with the greatest penalty. Allocate as much as possible to the cell with the least cost in this row or column

5.

- Adjust the free demand and supply totals and cross out the row or column demand or supply is entirely used up.

- If both the row and the column have no supply or demand left only cross out one

- If all columns and rows have been satisfied then we have an initial solution. Otherwise recalculate the penalties for uncrossed out rows and columns without counting rows and columns with zero free demand or supply.

Penalties for Vam (1)

Row 1 9-8=1
Row 2 10-6=4
Row 3 7-6=1
Column 1 7-6=1
Colum 2 11-6 =5 * Greater penalty!
Colum 3 8-6 =2

	Waterford	Sligo	Athlone	Supply
Dublin	9	11	8	200
Cork	6	15	10	150
Galway	7	6(100)	6	100-100=0
Demand	180	140-100=40	130	

Penalties for VAM (2)

Row 3 Satisfied (0)
Row 1 9-8 =1
Row 2 10-6=4* Choose arbitrarily - choose column 2
Column 1 7-6 =1
Colum 2 15-11 =4*
Colum 3 10-8 =2

6

	Waterford	Sligo	Athlone	Supply
Table VAM 2 Dublin	9	11(40)	8	200-40=160
Cork	6	15	10	150
Galway	7	6(100)	6	0
Demand	180	40-40=0	130	

Penalties for VAM 3

Row 3 and Column 2 satisfied

Row 1 9-8=1

Row 2 10-6=4*

Column 1 9-6 =3

Column 3 10-8=2

	Waterford	Sligo	Athlone	Supply
Table VAM 3 Dublin	9	11(40)	8	160
Cork	6(150)	15	10	150-150=0
Galway	7	6(100)	6	0
Demand	180-150=30	0	130	

Checking for Optimality

- The initial table must have M+N-1 cells occupied to enable a valid solution to be found. M=rows N=columns

- If it doesn't, then a zero amount is put into empty cells until the above condition is satisfied

- There is no rule for which cells but a low cost one if min problem and high cost if max problem

7

Stepping Stone technique

- looks at each of the unused cells in turn to find a better route
- As problem is linear, it uses one unit at a time and keeps a closed loop

MODI - Modified Distribution Method

MODI method is an alternative to the Stepping Stone technique and is often less cumbersome for larger problems

In method we have a variable U_i associated with each row and a similar V_j associated with each column

For every route in the table which is being used, the unit cost of that route is equal to the sum of the U value of the row and the V value of the column

- Hence $U_i + V_j = C_{ij}$ for a cell which is being used
- We calculate U_i and V_j values by assigning an arbitrary value to one of them and then solve the remaining equation $C_{ij} = U_i + V_j$ for the occupied cells
- We can predict the cost change for the unused cell to be: cost improvement for unused cell = $C_{ij} - U_i - V_j$.
- Therefore when we have calculated the U_i and V_j values we can go on to calculate the cost improvement potentials for each of the unused cells
- The arbitrary value we usually assign is 0 to U_1
- MODI used after performing an initial Basic feasible solution like least

8.

	Waterford	Sligo	Athlone	Supply
Dublin	180^{9}	20^{11}	0^{8}	200
Cork	* 0^{6}	120^{15}	30^{10}	150
Galway	0^{7}	0^{6}	100^{6}	100
DEMAND	180	140	130	

$R_1 + K_1 = 9$ Let $R_1 = 0 \Rightarrow K_1 = 9$

$R_2 + K_1 = 6$ $R_2 = -3$

$R_3 + K_1 = 7$ $R_3 = -2$

$R_1 + K_2 = 11$ $K_2 = 11$

$R_2 + K_2 = 15$

$R_3 + K_2 = 16$

$R_1 + K_3 = 8$

$R_2 + K_3 = 10$

$R_3 + K_3 = 6$

Set up eq^n for occupied cells (non-zero)

$R_1 + K_1 = 9$ Let $R_1 = 0 \Rightarrow K_1 = 9$ Calculate saving by $R_1 + K_3$ $8 - R_1 - K_3 = 2$

$R_1 + K_2 = 11$ $K_2 = 11$ $I = C_{ij} - R_i - K_j \Rightarrow$ $6 - R_2 - K_1 = -7$

$R_2 + K_2 = 15$ $R_2 = 4$ $7 - R_3 - K_1 = -2$

$R_2 + K_3 = 10$ $K_3 = 6$ Select $6 - R_3 - K_2 = -5$

$R_3 + K_3 = 6.$ $R_3 = 0$ largest negative

Move as much in $R_2 K_1$ within a closed path.

	Waterford	Sligo	Athlone	Supply
Dublin	$180 - 120^{9}$	$20 + 120^{11}$	0^{8}	200
Cork	$+120^{6}$	$120 - 120^{15} = 0$	30^{10}	150
Galway	0^{7}	0^{6}	100^{6}	100
Demand	180	140	130	

Non-zero eqns: $\cancel{R_1 + K_1 = 9}$

$R_2 + K_1 =$

9

①

		$K_1 = 9$	$K_2 = 11$	$K_3 = 6$	
		W	S	A	Supply
$R_1 = 0$	D	180 −120 9	20 +120 11	0 8	200
$R_2 = 4$	C	0 +120 6	120 −120 15	30 10	150
$R_3 = 0$	G	0 7	0 6	100 6	100
	Demand	180	140	130	

C → W   6 − ~~4~~ 4 − 9 = ~~−13~~ −7

G → W   7 − 0 − 9 = −2

G → S   6 − 11 − 0 = −5

D → A   8 − 6 − 0 = 2

②

		$K_1 = 9$	$K_2 = 11$	$K_3 = 13$	
		W	S	A	Supply
$R_1 = 0$	D	60 −30 9	140 11	0 +30 8	200
$R_2 = -3$	C	120 +30 6	0 15	30 −30 10	150
$R_3 = -5$	G	0 7	0 6	100 6	100
	Demand	180	140	130	

G → W   7 − −5 − 9 = 3

C → S   15 − 11 − −3 = 7

G → S   6 − −5 − 11 = 0

D → A   8 − 0 − 13 = −5

3.

		$K_1 = 9$	$K_2 = 2$	$K_3 = 8$	
		W	S	A	Supply
$R_1 = 0$	D	30 9	140 11	30 8	200
$R_2 = -3$	C	150 6	0 15	0 10	150
$R_3 = -2$	G	0 7	0 6	100 6	100
	demand	180	140	130	

G → W   7 − 9 − −2 = 0

C → S   15 − −3 − 2 = 16

G → S   6 − −2 − 2 = 6

C → A   10 − −3 − 8 = 5

Optimal because all values are positive, no more savings to be made!

Degeneracy

- A Degenerate transportation table will have less than $M+N-1$ cells with flow greater than zero,

want to only be able a closed path in table

- We therefore enter one or more zeros until a legitimacy table is formed
- No rules for entering zeros. Choosing a low cost cell is a good idea

Maximisation

If max problem we chose the cell with the greatest positive value when setting up the table and also when using Stepping Stone or MODI methods

Multiple Optimum Solutions

- Identified by the existence of cells in the optimum solution, which have a cost improvement potential of zero
- This means that using such a route will lead to rearrangement of the solution without changing the total cost

Prohibited Routes

- In this case we allocate a very large cost to the prohibited route
- Referred to as big M method
- Even if the initial ~~optimal~~ solution uses this route, the algorithm to find the optimum solution will alter the solution to use a different route