23/10/17 Maths

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~~Foster~~ Problem sheets on web
hand in solutions at end of term

Exam 2hr end of sep. May
90% exam 10% assignment.

Overview
- Vector-valued functions.
- functions of several variables, partial derivatives
- Double Integrals
- Triple Integrals
- Vector calculus.
- Laplace transform

Textbook: calculus like ten(?) 10th edition

Should know.

1 dimension

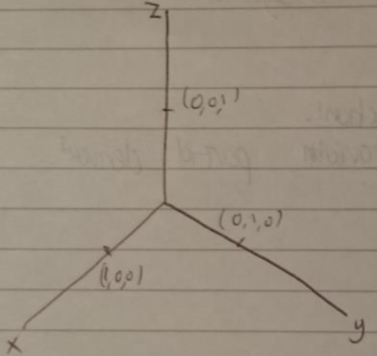
1 Vector

Function $f(t)$

Vector $\vec{v} = (x, y, z)$, text boldface **r**

Another common notation uses unit vectors

Unit vector: a vector of length 1



$\vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k}$

Give position relative to fixed point (origin)

$\vec{0} = (0,0,0)$

We can add vectors

$\vec{r_1} = (1,3,1)$
$\vec{r_2} = (0,3,-2)$

$\vec{r_1} + \vec{r_2} = (1,0,-1)$

The magnitude or length of a vector is

$||\vec{r}|| = \sqrt{x^2+y^2+z^2}$

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We are used to function like

$x(t) = 3t$, $x(t) = x^2$

We define vector-valued function to be function of a real variable with several component function. which depend on a parametric variable, t

$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

In this course we are only concerned with function with values in $\mathbb{R}$ (Real). t is also $\mathbb{R}$ $t \in \mathbb{R}$

The domain of $\vec{r}(t)$ is the set of all values for which $\vec{r}(t)$ is defined. It is denoted $D(r)$

Example: what is the domain of $\vec{r}(t) = t\vec{i} + \sqrt{t}\vec{j} + \ln(t-3)\vec{k}$

Solution: Where is $\vec{r}(t)$ defined. Look at the first component: $f(t) = t$ is defined for any real number, so the domain is $(-\infty, +\infty)$

Next $g(t) = \sqrt{t}$ is defined for non negative values of t
$D(g) = [0, \infty)$

Finally $h(t) = \ln(t-3)$ is defined where the argument is positive
$t-3 > 0 \Rightarrow t > 3$ $D(h) = (3, \infty)$

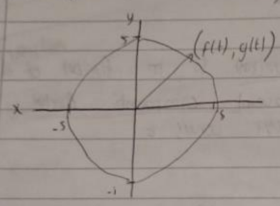
The entire function $\vec{r}(t)$ is therefore only defined on the smallest range i.e. $(3, \infty) = D(r)$

In this course we will only consider function in 2 or 3 dimensions
$\vec{r} \in \mathbb{R}^2$ or $\in \mathbb{R}^3$

Example (2 dimension)

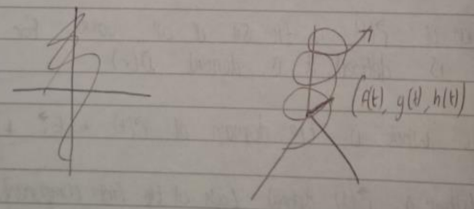
Consider the circle of radius 5.

The equations are $f(t) = 5\cos t$, $g(t) = 5\sin t$



In three dimensions, consider the equations $f(t) = 5\cos t$ $g(t) = 5\sin t$ $h(t) = t$

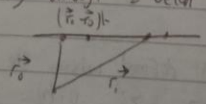
This is a spiral (circular helix)



Vector form of a line segment

Consider how to write a line segment in parametric equations

Consider a line passing through 2 vectors $\vec{r_0}$ and $\vec{r_1}$ which looks like



We could start at $\vec{r_0}$ and move along the line $\vec{r_1}$ by

$$\vec{r} = \vec{r_0} + \vec{v}t$$

$\vec{v}$ is the rate at which our position changes ("velocity") and t describes how "time" has elapsed

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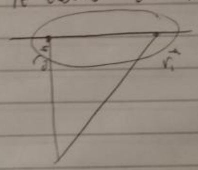
Let $\vec{r_0}$ be at $t=0$, and $\vec{r_1}$ at $t=1$

$\vec{r_1} = \vec{r_0} + \vec{v}(1)$
$\Rightarrow \vec{v} = \vec{r_1} - \vec{r_0}$

and we get $\vec{r} = \vec{r_0} + (\vec{r_1} - \vec{r_0})t$

This is called the two point form of a line

Notice that when $0 \le t \le 1$ it describes the line segment between $\vec{r_0}$ and $\vec{r_1}$



- Monday 10am 2034.
- Wedn 9am Joly
- Friday 10am Rm 2034

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LIMITS AND CONTINUITY

We define the limit of a vector valued function to be
$\lim_{t \to a} \vec{r}(t) = \vec{L}$

if for $\vec{r}(t) = (x(t), y(t), z(t))$

$\lim_{t \to a} x(t) = L_x$

$\lim_{t \to a} y(t) = L_y$

$\lim_{t \to a} z(t) = L_z$

with $\vec{L} = (L_x, L_y, L_z)$

Using this we say a vector-valued function (vvf) is continuous at the point $t = a$ if $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$

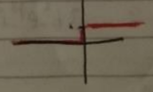
A vvf is continuous if it is continuous at all points in the interval

Example:
The vvf $\vec{r}(t) = (t, t^2, t+t^3)$ this continuous at all points because of $t \to a$, that limit is $\vec{r}(a) = (a, a^2, a+a^3)$

It is continuous for all real values. i.e. on the interval $(-\infty, \infty)$

Consider instead $\vec{r}(t) = (\theta(t), t^2, t+t^3)$, $\theta(t)$ is the heaviside step function, defined by

$\theta(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$



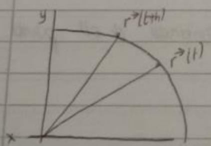
It is continuous on the intervals $[0, \infty)$ and on $(-\infty, 0)$
If we look a the interval $[-2, 0]$, it is discontinuous at $t = 0$

Notice that the limit at $t=0$ is 0 (from below)
But we know that $\theta(0) = 1$
Therefore θt is not continuous and therefore $\vec{r}t$ is not continuous since one of the components is not continuous

DERIVATIVES
We define derivatives of VVF using limits in order to differentiable, the VVF must be continuous, but the converse does not hold

The derivative is defined as $\vec{r}'(t) = \lim_{t \to h} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$
provided the limit exists.

It only exists where the function is continuous



The derivative $\vec{r}'(t)$ is tangent to the curve traced out by $\vec{r}(t)$

It points in the direction of increasing t.

In mechanics, $\vec{v}(t) = \vec{r}'(t)$

Alternative notations $\vec{r}'$ $\quad \frac{d}{dt}\vec{r} \quad \frac{d\vec{r}}{dt}$

Example: $\vec{r}(t) = e^{t^3}t^{-1}\vec{i} + \sqrt{1+t^2}\,\vec{j} - \sin \vec{k}$

$\vec{r}'(t) = 3t^2 e^{t^3}\vec{i} + \frac{t}{\sqrt{1+t^2}}\vec{j} + \cos \vec{k}$

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Rules of differentiation for vvf.

1. $\frac{d}{dt} \vec{i} = 0$ $\quad \vec{i}$ constant

2. $\frac{d}{dt}(k\vec{r}) = k\frac{d}{dt}\vec{r}$ $\quad k$ constant

3. $\frac{d}{dt}(\vec{r_1}+\vec{r_2}) = \vec{r_1}' + \vec{r_2}'$

4. $\frac{d}{dt}(\vec{r_1}-\vec{r_2}) = \vec{r_1}' - \vec{r_2}'$

2-4 $\Rightarrow$ 5. $\frac{d}{dt}(a\vec{r_1}+b\vec{r_2}) = a\vec{r_1}' + b\vec{r_2}'$

a, b constant

6. $\frac{d}{dt}(f(t)\vec{r}) = f'(t)\vec{r} + f(t)\vec{r}'$ $\quad$ when f is any function of t.

$\rightarrow$ chain rule

TANGENT LINE AND TANGENT VECTOR

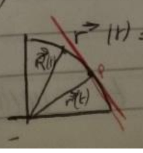
We said earlier that the derivative of a vvf is tangent to the vvf at that point.

We define the tangent line of $\vec{r}(t)$ at t_0 to be the line parallel to the derivative $\vec{r}'(t)$

The equation of the tangent line is

$$\vec{R}(t) = \vec{r_0} + \vec{v_0}t \quad \text{with } \vec{r_0} = \vec{r}(t_0) \quad \vec{v_0} = \vec{r}'(t_0)$$

This is clearly the vector form of a line segment (monday)



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Example:

Find the tangent line of a circular helix with the equation

$$\vec{r}(t) = p\cos t\,\vec{i} + p\sin t\,\vec{j} + ct\,\vec{k}$$

Solution

The derivative is $\vec{r}'(t) = p\sin t\,\vec{i} - p\cos t\,\vec{j} + c\vec{k}$

Lets the tangent line at $t = \pi = t_0$

$$\Rightarrow \vec{r_0} = \vec{r}(\pi) = p\vec{i} + \pi c\vec{k}$$

and $\vec{v_0} = \vec{r}'(\pi) = -p\vec{j} + c\vec{k}$

Therefore the tangent line has equation

$$\vec{R}(t) = (p\vec{i} + \pi c\vec{k}) + (-p\vec{i} + c\vec{k})t$$
$$= p\vec{i} - p\vec{\pi} + c(t+\pi)\vec{k}$$

We define unit tangent vector to be the unit vector pointing along the tangent line

$$\vec{T} = \frac{\vec{v}(t)}{\|\vec{v}(t)\|} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

- Derivative of cross and dot product

Recall that for $\vec{r_1} = (x, y, z)$ and

$$\vec{r_2} = (x_2, y_2, z_2)$$

the dot product is

$\vec{r_1} \cdot \vec{r_2} = x_1x_2 + y_1y_2 + z_1z_2$ and magnitude is given by $\|\vec{r}\| = \sqrt{\vec{r_1}\cdot\vec{r}}$

The cross (or vector) product is $\vec{r_1} \times \vec{r_2} = (y_1z_2 - y_2z_1)\vec{i} + (z_1x_2 - z_2x_1)\vec{j} + (x_1y_2 - x_2y_1)\vec{k}$

with magnitude

$$\|\vec{r_1} \times \vec{r_2}\| = \|\vec{r_1}\| \cdot \|\vec{r_2}\| \sin\theta$$

Where θ is the angle between the vector

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Usefull

Then the derivati

$\frac{d}{dt}(\vec{r_1}\cdot\vec{r_2}) =$

$\frac{d}{dt}(\vec{r_1}\times\vec{r_2})$

Theorem If $\vec{r}(t)$

$\|\vec{r}(t)\|$

i.e. $\vec{r}$

Proof $\|\vec{r}\|^2 = \vec{r}$

$\Rightarrow 0 = \frac{d}{dt}$

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Usefull ~~[illegible]~~ derivatives $\vec{r}_1 \cdot \vec{r}_2 = \vec{r}_2 \cdot \vec{r}_1$

$\vec{r}_1 \times \vec{r}_2 = -\vec{r}_2 \times \vec{r}_1$

Then the derivatives are

$$\frac{d}{dt}(\vec{r}_1 \cdot \vec{r}_2) = \vec{r}_1' \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_2'$$

$$\frac{d}{dt}(\vec{r}_1 \times \vec{r}_2) = \vec{r}_1' \times \vec{r}_2 + \vec{r}_1 \times \vec{r}_2'$$

Theorem: If $\vec{r}(t)$ is a real vvf with constant magnitude $\|\vec{r}(t)\|$ then $\vec{r} \cdot \vec{r}' = 0$

i.e. $\vec{r}'$ is perpendicular to $\vec{r}$ $\quad \vec{r}' \perp \vec{r}$

Proof: $\|\vec{r}\|^2 = \vec{r} \cdot \vec{r}$

$\Rightarrow 0 = \frac{d}{dt}\|\vec{r}\|^2 = \vec{r}' \cdot \vec{r} + \vec{r} \cdot \vec{r}'$

$= 2\vec{r}' \cdot \vec{r}$

$\rightarrow \vec{r} \perp \vec{r}'$

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Thurs 9am Salman M17

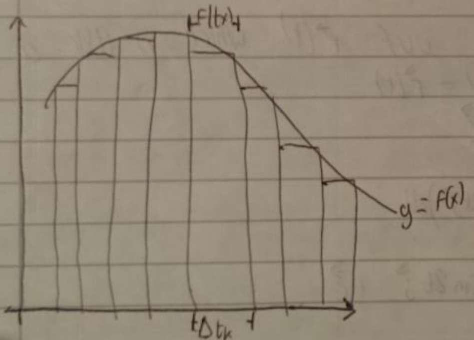
Definite Integrals of Vector-Valued functions

Let $\vec{r}(t)$ be a continuous (not necessarily differentiable) vvf on an interval $a \le t \le b$

We define the definite integral $\int_a^b \vec{r}(t) = \lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} \vec{r}(t_k)\Delta t_k$

$$= \int_a^b x(t)\,dt\,\vec{i} + \int_a^b y(t)\,dt\,\vec{j} + \int_a^b z(t)\,dt\,\vec{k}$$

This can be understood by Riemann Integrals



Rules of Integration:

- $\int_a^b k\vec{r}(t)\,dt = k\int_a^b \vec{r}(t)\,dt$ k, constant

- $\int_a^b \left(c\vec{r_1}(t) + d\vec{r_2}(t)\right)dt = c\int_a^b \vec{r_1}(t)\,dt + d\int_a^b \vec{r_2}(t)\,d$ c,d constant

If the graph of f lies above the graph of g, then the area between the graphs is:

Area $= \int_a^b \left(f(x) - g(x)\right)dx$

2.

The volume of a solid with cross-sectional area $A(x)$ has volume

Volume $= \int_a^b A(x)\,dx$

Example: Find the integral of $\vec{r}(t) = t^3\vec{i} + \sqrt{t}\vec{j} - \sin\frac{\pi t}{2}\vec{k}$ between 0 and 2.

Solution: $\int_0^2 \left(t^3\vec{i} + \sqrt{t}\vec{j} - \sin\frac{\pi}{2}t\vec{k}\right)dt$

$\Rightarrow \left[\frac{1}{4}t^4\vec{i} + \frac{2}{3}t^{3/2}\vec{j} + \frac{2}{\pi}\cos\frac{\pi t}{2}\vec{k}\right]_0^2$

$\Rightarrow \left[\frac{1}{4}(2)^4\vec{i} + \frac{2}{3}(2)^{3/2}\vec{j} + \frac{2}{\pi}\cos\frac{2\pi}{2}\vec{k}\right] - \left[0\vec{i} + 0\vec{j} + \frac{2}{\pi}\vec{k}\right]$

$= 4\vec{i} + \frac{4\sqrt{2}}{3}\vec{j} - \frac{4}{\pi}\vec{k}$

The anti-derivative for a VVF $\vec{r}(t)$ which is itself a VVF $R(t)$ given by $\vec{R}'(t) = \vec{r}(t)$

$\Rightarrow \int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$

$\Rightarrow$ ~~log|t-1|~~ $\int \left(\frac{1}{t-1}\vec{i} + \cos 2t\vec{j}\right)dt$

$\Rightarrow \log|t-1|\vec{i} + \frac{1}{2}\sin 2t\vec{j} + \vec{C}$

Properties:

1. $\frac{d}{dt}\int \vec{r}(t)\,dt = \vec{r}(t)$
2. $\int \vec{r}'(t)\,dt = \vec{r}(t) + \vec{C}$

* THE FUNDAMENTAL THEOREM OF CALCULUS

$\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)$

$\int_2^3 \left(\frac{1}{t-1}\vec{i} + \cos 2t\vec{j}\right)dt$

$\vec{R}(3) - \vec{R}(2)$

$= \left(\log 2\vec{i} + \frac{1}{2}\sin 6\vec{j}\right) - \left(\log 1\vec{i} + \frac{1}{2}\sin 4\vec{j}\right)$

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$= \log 2\vec{i} + \frac{1}{2}(\sin 6 - \sin 4)\vec{j}$

since $\log 1 = 0$

Arc length and Changing Parameter

We say that a curve is smoothly paramed by $\vec{r}(t)$, or that $\vec{r}(t)$ is a smooth function of t and $\vec{r}\,'(t)$ exists, is continuous and $\vec{r}\,'(t) \neq 0$ for all t.

If a function is smooth we can calculate the arc length.

$$L = \int_a^b \left\| \frac{d\vec{r}}{dt} \right\| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example: Find the arc length of

$\vec{r}(t) = 5\cos t\,\vec{i} + 5\sin t\,\vec{j} + t\vec{k}$ from 0 to $\frac{\pi}{2}$

Solution: $\vec{r}\,'(t) = -5\sin t\,\vec{i} + 5\cos t\,\vec{j} + \vec{k}$

$\| \vec{r}\,'(t) \| = \sqrt{(-5\sin t)^2 + (5\cos t)^2 + 1^2} = \sqrt{26}$

$\Rightarrow L = \int_0^{\frac{\pi}{2}} \sqrt{26}\, dt = \sqrt{26}\, \frac{\pi}{2}$

Parameterising curves using arc length.

Select a reference point, P, and we choose an orientation, so one direction is positive and the other negative.

Then the "signed" arc length parameterises the curve via

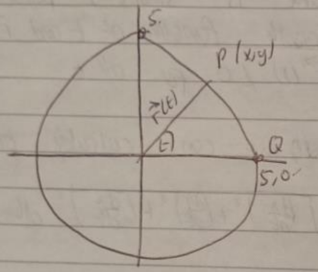
$x = x(s) \quad y = y(s) \quad z = z(s)$

In other words, we view position on the curve as a function of its arc length ("distance") from P. This is known as arc length parameterisation

Example: Find the arc length parameter of $\vec{r}(t)$

$$\vec{r}(t) = 5\cos t\,\hat{i} + 5\sin t\,\hat{j}, \quad 0 \le t \le 2\pi$$

Solution: Here t plays the role of a radian angle measured from Q to P(x,y).



$$s = L = \int_0^t \sqrt{(-5\sin t)^2 + (5\cos t)^2}\, dt = 5t.$$

$$\Rightarrow \quad s = 5t \quad \text{or} \quad t = s/5$$

The circle is now parameterised by
$$x = 5\cos(s/5), \quad y = 5\sin(\tfrac{s}{5}), \quad 0 \le s \le 10\pi$$

Change of Parameter

More generally, we can change the parameter of a curve using the chain rule. If $\vec{r}(t)$ is a vf that is differentiable with respect to t.

We can change parameter to T using $t = g(T)$ where g is differentiable with respect to T. Then ~~$\vec{r}(t)$~~ $\vec{r}(g(T))$ is differentiable with respect to T and we have $\frac{d\vec{r}}{dT} = \frac{d\vec{r}}{dt}\,\frac{dt}{dT}$

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Change of Parameter (continued)

$\frac{d\vec{r}}{dT} = \frac{d\vec{r}}{dt}\frac{dt}{dT}$ where $t = g(T)$

If $g(T) \neq 0$ smooth, this is a smooth change of parameter.

If $\frac{dt}{dT} > 0$ for all T, this is a positive change of parameter

If $\frac{dt}{dT} < 0$ for all T this a negative change of parameter.

If C is a graph of a smooth vf $\vec{r}(t)$, with a reference point $\vec{r}(t_0)$, ~~then~~ the arc lenght parameter is a positive change parameter, given by

$$s = \int_{t_0}^{t} \left\| \frac{d\vec{r}}{du} \right\| du$$

$$= \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

Example: $\vec{r}(u) = 2\cos u\,\vec{i} + 2\cos u\,\vec{j} + u\vec{k}$ u is a parameter. Lets use $t_0 = 0$ as a reference point.

Solution: $\left\|\frac{d\vec{r}}{du}\right\| = \sqrt{5}$

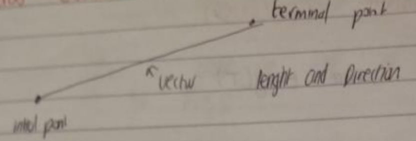
$$\Rightarrow \quad s = \int_{t_0}^{t} \sqrt{5}\, du$$

$$= \sqrt{5}\, u \Big|_0^t = \sqrt{5}t$$

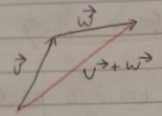
sss letter

$$\vec{r}(s) = 2\cos\left(\frac{s}{\sqrt{5}}\right)\vec{i} + 2\sin\left(\frac{s}{\sqrt{5}}\right) + \frac{s}{\sqrt{5}}\vec{k}$$

Vectors - Overview



Summing Vectors:

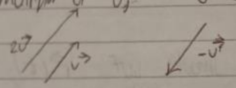


$\vec{v}+\vec{w} = \vec{w}+\vec{v}$

$\vec{v}+\vec{0} = \vec{v}$

If k is a scaler (a number)

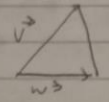
$k\vec{v}$ a multiple of $\vec{v}$ $\vec{v}$ and $k\vec{v}$ are parallel vectors



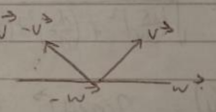
Subtracting Vectors

$\vec{v}-\vec{w} = \vec{v} + (-\vec{w})$

$\vec{v}-\vec{v} = \vec{v} + (-\vec{v}) = \vec{0}$



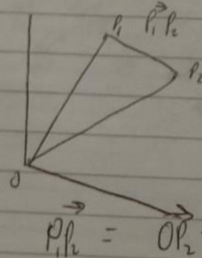
$\vec{v}-\vec{w}$



Components $\vec{v}\ (v_1, v_2, v_3)$

$\vec{v}+\vec{w} = (v_1+w_1, v_2+w_2, v_3+w_3)$

$a\vec{v}+b\vec{w}$ $(av_1+bw_1, av_2+bw_2, av_3+bw_3$



$\vec{OP}$ is the vector pointing from origin to $P(x,y,z)$

$\vec{P_1P_2} = \vec{OP_2} - \vec{OP_1}$

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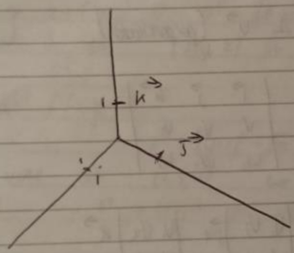
Magnitude (length)

$$\|\vec v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Unit vectors are vectors of length 1.

Special unit vectors $\vec i, \vec j, \vec k$



$\vec i$ (1, 0, 0)
$\vec j$ (0, 1, 0)
$\vec k$ (0, 0, 1)

$$\vec v \quad v_1 \vec i + v_2 \vec j + v_3 \vec k$$

$$a\vec u + b\vec w = (au_1 + bw_1)\vec i + (au_2 + bw_2)\vec j + (au_3 + bw_3)\vec k$$

Normalising a vector $\vec u \quad \dfrac{\vec v}{\|\vec v\|}$ ⇒ unit vector in direction $\vec v$

Dot product.

$\vec u \cdot \vec v = u_1 v_1 + u_2 v_2 + u_3 + v_3$ → Scalar (not a vector)

Properties - $a\vec u \cdot b\vec v = ab\, \vec u \cdot \vec v$

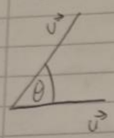
- $\vec u \cdot (\vec v + \vec w) = \vec u \cdot \vec v + \vec u \cdot \vec w$

$\vec 0 \cdot \vec v = 0$

$\vec v \cdot \vec v = (\|\vec v\|)^2$

u

Angle between vectors



$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$

$\vec{u}\cdot\vec{v} = 0 \Rightarrow \vec{u} \perp \vec{v}$ (perpendicular)

Cross product: $\vec{v}\times\vec{w}$ $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\vec{k}$$

$$= (v_2w_3 - v_3w_2)\vec{i} + (v_3w_1 - v_1w_3)\vec{j} + (v_1w_2 - v_2w_1)\vec{k}$$

Properties:
- $\vec{u}\times\vec{v} = -\vec{v}\times\vec{v}$
- $\vec{v}\,(\vec{v}+\vec{w}) = (\vec{v}+\vec{v}) + (\vec{u}+\vec{w})$
- $(\vec{u}+\vec{v})\times\vec{w} = (\vec{u}\times\vec{w}) + (\vec{v}\times\vec{w})$
- $k(\vec{u}\times\vec{v}) = (k\vec{u})\times\vec{v} = \vec{u}\times(k\vec{v})$
- $\vec{u}\times 0 = 0$
- $\vec{u}\times\vec{u} = 0$ parallel vectors have zero cross product.

$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta$