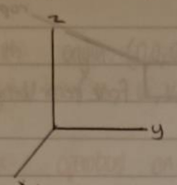
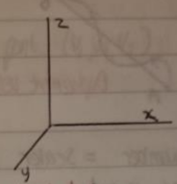
Chapter 11 Three Dimensional Space: Vectors





Right handed Space — ALWAYS USED

Left handed Space

Region can be split into octants
Positive octant ⇒ all points positive

Distance between two point in 3D
$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

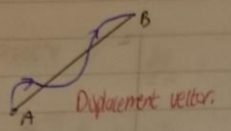
example: distance between $(2,3,-1)$ and $(4,-1,3)$
$d = \sqrt{(4-2)^2 + (3--1)^2 + (-1-3)^2} = \sqrt{36} = 6$

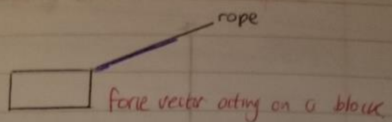
Standard equation of circle in 2d: $(x-x_0)^2 + (y-y_0)^2 = r$ (x_0,y_0) centre r-radius
in 3d: $(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$ (x_0,y_0,z_0) centre r-radius.
$\Rightarrow (x+1)^2 + y^2 + (z+4)^2 = 5$ ⇒ Sphere centre $(-1,0,-4)$ radius $\sqrt{5}$

example: Find centre and radius of sphere $x^2+y^2+z^2-2x-4y+8z+17=0$
Complete the squares: $(x^2-2x)+(y^2-4y)+(z^2+8z)+17=0$
$(x^2-2x+1)+(y^2-4y+4)+(z^2+8z+16) = -17+21$
$(x-1)^2 + (y-2)^2 + (z+4)^2 = 4$
centre $(1,2,-4)$ radius 2

Theorem: $x^2+y^2+z^2+Gx+Hy+Iz+J=0$
represents a sphere, a point or has no graph

Vectors

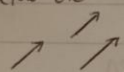




real number = Scaler

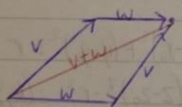
Two vectors v and w are considered equal if they have the same length and same direction, we write v=w

Geometrically 2 vectors are equal if they are translations of each other; vectors below are equal even though they are in different positions



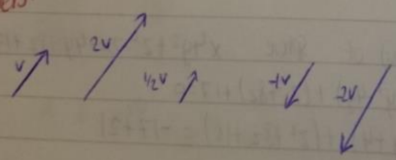
$\vec{AB}$ represents a vector with start point A and terminal point B.
Vector with length zero is a zero vector

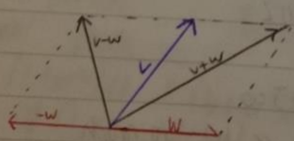
If v and w are vectors then the sum v+w =



v+w = w+v
0+v = v+0 = v

Scalers:





Chapter 11 3.

Vector with origin $(0,0,0)$ and terminal point (v_1, v_2, v_3)

$$v = (v_1, v_2, v_3)$$

Arithmetic operations on vectors

If $v = (v_1, v_2, v_3)$ and $w(w_1, w_2, w_3)$

$v + w = (v_1+w_1, v_2+w_2, v_3+w_3)$

$v - w = (v_1-w_1, v_2-w_2, v_3-w_3)$

$kv = (kv_1, kv_2, kv_3)$ where k is a scalar

Example: $v = (-2,0,1)$ $w = (3,5,-4)$

$v+w = (-2+3, 0+5, 1-4) = (1,5,-3)$

$3v = (-6,0,3)$

$-w = (-3,-5,4)$

$w-2v = (3,5,-4) - (-4,0,2) = (7,5,-6)$

Vector not with initial point at origin

$P_1(x_1, y_1)$ $P_2(x_2, y_2)$

$$\overrightarrow{P_1P_2} = (x_2-x_1, y_2-y_1)$$

Example: $A(0,-2,5)$ $B(3,4,-1)$

$\overrightarrow{AB}$ $(3-0, 4--2, -1-5) = (3,6,-6)$

Rules for any vectors u, v, w with any scalars k, L.

$u+v = v+u$

$(u+v)+w = u+(v+w)$

$u+0 = 0+u = u$

$u+(-u) = 0$

$k(Lu) = (kL)u$

$k(u+v) = ku+kv$

$(k+L)u = ku+Lu$

$1u = u$

Norm of a vector

Distance between initial and terminal point of a vector v is called the length, norm or magnitude of v denoted by $\|v\|$

norm of a vector $v(v_1, v_2, v_3)$ is given by

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$\|10v\| = 10\,\|v\| \qquad \|kv\| = |k|\,\|v\|$$

$$\|-2v\| = 2\|v\|$$

Unit vector

Vector with length 1. $\underset{x\text{-axis}}{i = (1,0,0)}$ $\underset{y\text{-axis}}{j = (0,1,0)}$ $\underset{z\text{-axis}}{k = (0,0,1)}$

examples: $(2,-3,4) = 2i - 3j + 4k$

$(0,3,0) = 3j$

$(3i + 2j - k) - (4i - j + 2k) = -i + 3j + k$

$\|i + 2j - 3k\| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$

Normalising a vector

Finding a vector u that has the same direction as some given nonzero vector v. Done by multiplying v by the reciprocal of it's length

$$u = \frac{1}{\|v\|}v = \frac{v}{\|v\|}$$

This is a unit vector with same direction as v, the same because $k = \frac{1}{\|v\|}$ is a positive scaler and length is one:

$$\|u\| = \|kv\| = |k|\,\|v\| = k\,\|v\| = \frac{1}{\|v\|}\,\|v\| = 1$$

This is called normalising a vector.

Example: Find unit vector that has same direction as $v = 2i + 2j - k$

$$\|v\| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3.$$

unit vector u in the same direction is:

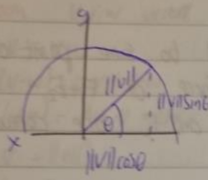
$$u = \frac{1}{3}v = \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$

5 Vectos

Vectos determind by length and angle

If v is a nonzero vector originating at origin of xy plane and θ is angle from positive x axis to radial line through v, then x component is $\|v\|\cos\theta$ and y is $\|v\|\sin\theta$

$$v = \|v\|(\cos\theta, \sin\theta) \quad \text{or} \quad v = \|v\|(\cos\theta\, i), \|v\|(\sin\theta\, j)$$



In special case of unit vector
$u = (\cos\theta, \sin\theta)$ or $u = \cos\theta\, i + \sin\theta\, j$

Example: find vector of length 2 and angle $\pi/4$

$$v = 2\cos\tfrac{\pi}{4} i + 2\sin\tfrac{\pi}{4} j = \sqrt{2}\, i + \sqrt{2}\, j$$

Example: find angle that vector $v = -\sqrt{3}i + j$ makes with positive x-axis

normalize $v = \dfrac{v}{\|v\|} = \dfrac{-\sqrt{3}\, i + j}{\sqrt{(-\sqrt{3})^2 + (1)^2}} = \dfrac{\sqrt{3}}{2} i + \dfrac{1}{2} j$

thus $\cos\theta = \dfrac{\sqrt{3}}{2}$ and $\sin\theta = \dfrac{1}{2}$ we can conclude $\theta = 5\pi/6$

In above example we have coefficient before $\cos\theta$ but since it is a unit vector it is 1.

Vectors determind by length and a vector in same direction

Example: Find components (points) of v given vector of V of length $\sqrt{5}$ along the line $A(0,0,4)$ to $B(2,5,0)$

Find $\overrightarrow{AB} = (0,0,4) - (2,5,0) = (2,5,-4)$

Normalize $\|\overrightarrow{AB}\| = \sqrt{2^2+5^2+(-4)^2} = 3\sqrt{5}$

Normalize to get unit vector $\dfrac{\overrightarrow{AB}}{\|AB\|} = \dfrac{2}{3\sqrt{5}}, \dfrac{2}{3\sqrt{5}}, \dfrac{-4}{3\sqrt{5}}$

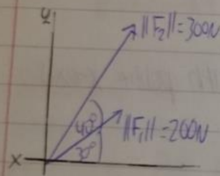
$v = \|u\|\left(\frac{\vec{AB}}{\|AB\|}\right) = \sqrt{5}\left(\frac{2}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}, \frac{-4}{3\sqrt{5}}\right) = \left(\frac{2}{3}, \frac{5}{3}, \frac{-4}{3}\right)$

first we find unit vector of AB so we can find new vector v with length 5 in direction of $\vec{AB}$.

Resultant of two concurrent forces

If two forces (vectors) F_1 and F_2 are applied to same point of an object, they have effect of having single force of F_1+F_2.
F_1+F_2 called the resultant and forces F_1 and F_2 are concurrent indicating they are applied at same point.

Example: Find magnitude of resultant and angle θ with positive x-axis



F_1 makes angle of 30°
F_2 make $30+40 = 70°$

Given $\|F_1\| = 200N \Rightarrow F_1 = 200(\cos 30, \sin 30) = (100\sqrt{3}, 100)$
and $F_2 = 300(\cos 70, \sin 70) = (300\cos 70°, 300\sin 70°)$

Resultant $F_1+F_2 = (100\sqrt{3} + 300\cos 70°, 100 + 300\sin 70°)$
$= 100(\sqrt{3} + 3\cos 70°, 1 + 3\sin 70°) \approx (275.8, 381.9)$

Magnitude $\|F\| = 100\sqrt{(\sqrt{3}+3\cos 70°)^2 + (1+3\sin 70°)^2} \approx 471N$

Denote θ as angle F makes with positive x-axis and initial point F is at origin
Using earlier formula

$\|F\| \cos\theta = 100\sqrt{3} + 300\cos 70°$

get $\cos^{-1}\theta \approx 54.2°$

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Definition of a dot product:

If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$

dot product $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$

$(1,-3,4) \cdot (1,5,2) = 1(1) + (-3)(5) + 4(2) = -6$

$(2i + 3j) \cdot (-3i + 2j) = 2(-3) + 3(2) = 0$

Theorem

u, v, w vectors k - scalar

$u \cdot v = v \cdot u$

$u \cdot (v \cdot w) = u \cdot v + v \cdot w$

$k(u \cdot v) = ku \cdot v = u \cdot kv$

$v \cdot v = \|v\|^2$

$0 \cdot v = 0$

Angle between vectors

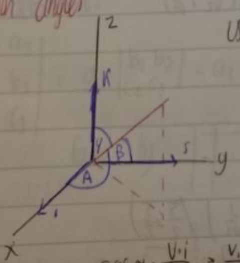
Given vector v and u if θ is angle between them $0 < \theta < \pi$

$\cos\theta = \dfrac{u \cdot v}{\|u\|\|v\|}$ comes from cosine rule

When write in form $u v = \|u\|\|v\| \cos\theta$ $u \cdot v$ has same sign as $\cos\theta$ so we can tell size of angle

$u \cdot v > 0$ acute $u \cdot v = 0$ $90°$ $u v < 0$ obtuse

Direction angles



Using above formula we can find angle between vectors and i, j, k called the direction angle of v and the cosines are called direction cosines

$v = v_1 i + v_2 j + v_3 k$

$\cos\alpha = \dfrac{v \cdot i}{\|v\|\|i\|} = \dfrac{v_1}{\|v\|}$ $\cos\beta = \dfrac{v \cdot j}{\|v\|\|j\|} = \dfrac{v_2}{\|v\|}$ $\cos\gamma = \dfrac{v \cdot k}{\|v\|\|k\|} = \dfrac{v_3}{\|v\|}$

comes from fact $\|i\|$ j and $k = 1$ and $i, j, k = 1$

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The direction cosine of vector $v = v_1 i + v_2 j + v_3 k$ can be computed by normalizing v and reading off $v/\|v\|$

$$\frac{v}{\|v\|} = \frac{v_1}{\|v\|} i + \frac{v_2}{\|v\|} j + \frac{v_3}{\|v\|} k = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, K$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Example: Find direction cos of $v = 2i - 4j + 4k$

normalize $\|v\| = \sqrt{4+16+16} = 6$ $\quad \frac{v}{\|v\|} = \frac{1}{3} i - \frac{2}{3} j + \frac{2}{3} K$

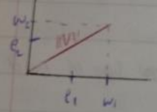
$\cos\alpha = \frac{1}{3}$ $\quad \cos\beta = -\frac{2}{3}$ $\quad \cos\gamma = \frac{2}{3}$

71° $\quad 132^\circ$ $\quad 48^\circ$

Decomposing vectors into orthogonal components

orthogonal → perpendicular

Decomposing a vector into a sum of 2 orthogonal vectors, e_1 and e_2 are orthogonal unit vectors. we want to express $V = w_1 + w_2$



$$v = (v \cdot e_1)e_1 + (v \cdot e_2)e_2$$

If angle θ is less than $\pi/2$ then

$v \cdot e_1 = \|v\| \cos\theta$ and $v \cdot e_2 = \|v\| \sin\theta$

$$\Rightarrow v = (\|v\| \cos\theta)e_1 + (\|v\| \sin\theta)e_2$$

Example: $v = (2,3)$ $\quad e_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\quad e_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Scaler components of v along e_1 and e_2 are

$$v \cdot e_1 = 2\left(\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) = \frac{5}{\sqrt{2}}$$

$$v \cdot e_2 = 2\left(-\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

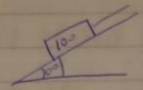
Vector components are $(v \cdot e_1)e_1 = \frac{5}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(\frac{5}{2}, \frac{5}{2}\right)$

$(v \cdot e_2)e_2 = \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)$

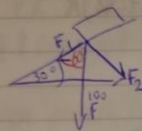
Notice $(v \cdot e_1)e_1 + (v \cdot e_2)e_2 = (2,3) = V$

In above exampe we know e_1 and e_2 but not angles

example



block weighing 100lb being pulled up 30° slope



$\|F_1\| = \|F\| \cos 60° = 100(\tfrac{1}{2}) = 50lb$

$\|F_2\| = \|F\| \sin 60° = 100(\tfrac{\sqrt{3}}{2}) = 86.6lb$

Orthogonal projection

Vector component of v along e_1 and e_2 are also called orthogonal projections of v on e_1 and e_2 denoted by

$$\text{proj}_{e_1} v = (v \cdot e_1) e_1$$

Orthogonal projection of v on arbitrary nonzero vector b can be obtained by normalising b:

$$\text{proj}_b v = \left(v \cdot \frac{b}{\|b\|}\right)\left(\frac{b}{\|b\|}\right)$$

Cross Product

determinants are functions which assign numerical value to square array of numbers

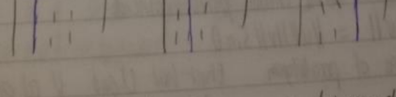
If a_1, a_2, b_1, b_2 are real numbers 2x2 determinant is

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

3x3 determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} - a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

found by



Theorem

- If two rows in array of a determinant are the same, value of determinant is 0
- Interchanging 2 rows in array of a determinant multiplies it's value by -1

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If $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$ in 3d then cross product $u\times v$ is

$$u\times v = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j - \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

or $u\times v = (u_2v_3 - u_3v_2)i - (u_1v_3 - u_3v_1)k - (u_1v_2 - u_2v_1)k$

Example $u=(1,2,-2)$ $v=(3,0,1)$

$$u\times v = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} i - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} k = 2i - 7j - 6k$$

- Cross product only for 3d
- Cross product of two vectors is 0 vector.

$u\times v = -(v\times u)$

$u\times(v+w) = u\times v + u\times w$

$(u+v)\times w = u\times w + v\times w$

$k(u\times v) = (ku)\times v = u\times(kv)$

$u\times 0 = 0\times u = 0$

$u\times u = 0.$

$i\times j = k$ $\quad j\times k = i$ $\quad k\times i = j$

$j\times i = -k$ $\quad k\times j = -i$ $\quad i\times k = -j$.

Geometric properties

$u\cdot(u\times v)=0$ $\quad u\times v$ is orthogonal to u

$v\cdot(u\times v)=0$ $\quad u\times v$ is orthogonal to v

Theorem

u and v are non zero vectors in 3d, θ the angle between the vectors when they are positioned so their initial points coincide

1. $\|u\times v\| = \|u\|\|v\|\sin\theta$
2. The area of parallelogram that has u and v as adjacent sides is
$$A = \|u\times v\|$$
3. $u\times v = 0$ if and only if u and v are parallel vectors, if and only if they are scalar multiples of each other

Example find area of triangle with $P_1(2,2,0)$, $P_2(-1,0,2)$ and $P_3(0,4,5)$

Area of triangle is half area of parallelogram

$\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3} = (-3,-2,2)$ and $(-2,2,5)$

$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (-10,5,-10)$

$A = \frac{1}{2} \| (-10,5,10) \| = \frac{15}{2}$

Scalar triple products

if $u = (u_1, u_2, u_3)$ $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$

$u \cdot (v \times w)$ is called the scalar triple product

$$u \cdot (v \times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

proof $u \cdot (v \times w) = u \cdot \left[\begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} i \ \ldots j \ \ldots k \right]$

$= \begin{vmatrix} v_2 & v_3 \\ w_2 & v_3 \end{vmatrix} u_1 \ \ldots \ \| u_2 \ \ldots \ \| u_3$

Example: $u = 3i - 2j - 5k$ $v = i + 4j - 4k$ $w = 3j + 2k$

$$u \cdot (v \times w) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 49$$

Theorm vollum of 3d parallelogram (parallelepiped) with u, v, w adjacent edges is $V = | u \cdot (v \times w) |$

$u \cdot (v \times w) = 0$ if and only if u v and w lie in the same plane

The dot and cross product in a scalar triple product can be interchanged.

$u \cdot (v \times w) = w \cdot (u \times v) = v \cdot (w \times u)$

$u \cdot v = \|u\| \|v\| \cos\theta$

$\| u \times v \| = \|u\| \|v\| \sin\theta$

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11.5 parametric equation of lines

Theorm

The line in 2 Space that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the non zero vector $v = (a, b) = ai + bj$ has parametric equations:

$$x = x_0 + at \quad y = y_0 + bt.$$

The line in 3d that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the non zero vector $v = (a, b, c) = ai + bj + ck$ has parametric equations:

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct.$$

c is length t represent time

Example: Find parametric equation passing through $(1, 2, -3)$ parallel to $4i + 5j - 7k$

$x = 1 + 4t \quad y = 2 + 5t \quad z = -3 - 7t$

$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct.$

Example $L_1 = \; x = 1 + 4t \quad y = 5 - 4t \quad z = -1 + 5t$

$L_2 = \; x = 2 + 8t \quad y = 4 - 3t \quad z = 5 + t$

parallel? intersect?

L_1 is parallel to $(4i - 4j + 5k)$ $\quad L_2$ parallel to $(8i - 3j + 1k)$

not parallel, neither is scalar multiple of each other.

Intersect at point (x_0, y_0, z_0).

$x_0 = 1 + 4t_1 \quad y_0 = 5 - 4t_1 \quad z_0 = -1 + 5t_1$

$x_0 = 2 + 8t_2 \quad y_0 = 4 - 3t_2 \quad z_0 = 5 + t_2$

$1 + 4t_1 = 2 + 8t_2$

$5 - 4t_1 = 4 - 3t_2$

$-1 + 5t_1 = 5 + t_2$ $\quad$ Solve by linear equation

From first 2 equs $\quad 6 = 6 + 5t_2 \Rightarrow t_2 = 0$ $\quad$ Sub back in $t_1 = \frac{1}{4}$

these values do not satisfy third equation $\Rightarrow$ do not intersect.

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1 Line Segment

Want line segment between 2 points $P_1(2,4,-1)$ $P_2(5,0,7)$

$\vec{P_1P_2}$ $P_2 - P_1 = (3,-4,8)$ parallel to 2 points

$3i - 4j + 8k$

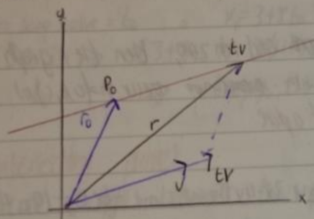
$x = 2 + 3t \quad y = 4 - 4t \quad z = -1 + 8t$

With these equation P_1 corresponds to $t=0$, and P_2 corresponds to $t=1$

$x = 2+3t, \quad y = 4-4t \quad z = -1+8t \quad (0 \le t \le 1)$

Vector equation of line

can write $r = (x,y,z)$ $r_0 = (x_0, y_0, z_0)$ $v = (a,b,c)$



Yield equation $r = r_0 + tv$ called vector equation of line

Example: Find equation of line in 3d that passes through $P_1(2,4,-1)$ and $P_2(5,0,7)$

Vector $\vec{P_1P_2} = P_2 - P_1 = (3,-4,8)$ is parallel vector v.

Use origin point either P_1 or P_2 as r_0

$\Rightarrow (x,y,z) = (2,4,-1) + t(3,-4,8)$

11.6 Planes determined by a point and a normal vector

Vector perpendicular to a plane is called a normal plane to the plane

We want to find equation of plane passing through $P_0(x_0,y_0,z_0)$ and perpendicular to vector $n = (a,b,c)$ Define vectors r_0 and r

$r_0(x_0,y_0,z_0)$ $r = (x,y,z)$

The plane consists of those points $P(x,y,z)$ for which vector $r-r_0$ is orthogonal to n: or expressed as equation:

$$n \cdot (r-r_0) = 0$$

$$\Rightarrow (a,b,c) \cdot (x-x_0,\ y-y_0,\ z-z_0) = 0$$

$$\Rightarrow a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

Point normal form of equation of a plane

Example: Find eq^n of plane passing through $(3,-1,7)$ perpendicular to vector $n = (4,2,-5)$

equation is $4(x-3) + 2(y+1) - 5(z-7) = 0$

$= 4x + 2y - 5z + 25 = 0$

Theorem: If a, b, c and d are constants and non zero, then the graph or eq^n $ax + by + cz + d = 0$ is plane with ~~[illegible]~~ vector $n = (a,b,c)$ as a normal

called general form of equation of a plane

Example Determine whether the planes ~~$3x+4$~~ $3x - 4y + 5z = 0$ and $-6x + 8y - 10z - 4 = 0$ are parallel

They are parallel if and only if their normals are parallel vectors

$n_1 = (3,-4,5)$ $\quad n_2 = (-6,8,-10)$

Since n_2 is a scalar multiple of n_1 the normals are parallel and hence so are the planes

Example: Find equation of plane through points $P_1(1,2,-1)$ $P_2(2,3,1)$ and $P_3(2,-3,3)$

Since P_1, P_2 and P_3 lie on plane, $\overrightarrow{P_1P_2} = (1,1,2)$ and $\overrightarrow{P_1P_3}(2,-3,3)$ are parallel

$$\underset{\text{normal}}{\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}} = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 2 & -3 & 3 \end{vmatrix} = 9i + j - 5k$$

This is the normal and point P_1

$$9(x-1) + (y-2) - 5(z+1) = 0$$

$$9x + y - 5z - 16 = 0$$

15.

Example: determine whether the line $x=3+8t$, $y=4+5t$ $z=-3-t$ is parallel to plane $x-3y+5z=12$

Vector $v=(8,5,-1)$ is parallel to line, vector $N=(1,-3,5)$ is normal to the plane
For line and plane to be parallel, vectors v and n must be orthogonal
$v \cdot n = (8)(1)+(5)(-3)+(-1)(5) = -12$
this is nonzero, $\Rightarrow$ not parallel

EXAMPLE: find intersection of $x=3+8t$, $y=4+5t$ $z=-3-t$ and plane $x-3y+5z=12$
(x_0,y_0,z_0) point of intersection
$x_0-3y_0+5z_0=12$
$t=$ some value $=t_0$ $= x_0=3+8t_0$ $y_0=4+5t_0$ $z_0=-3-t_0$
Sub in $\Rightarrow (3+8t_0)-3(4+5t_0)+5(-3-t_0)=12$
$t_0=-3 \Rightarrow (x_0,y_0,z_0)=(-21,-11,0)$

Intersecting planes

Acute angle between two planes θ is

$$\cos\theta = \frac{|n_1 \cdot n_2|}{\|n_1\|\,\|n_2\|}$$

Example find acute angle between $2x-4y+4z=6$ $6x+2y-3z=4$
$n_1=(2,-4,4)$ $n_2=(6,2,-3)$

$$\cos\theta = \frac{|n_1 \cdot n_2|}{\|n_1\|\,\|n_2\|} = \frac{|-8|}{\sqrt{36}\sqrt{49}} = \frac{4}{21} \quad \theta = 79°$$

Example: find intersection for line L and intersection of planes above ↑
$V = n_1 \times n_2 = (2,-4,4)\times(6,2,-3) = (4,30,28)$
V is orthogonal to n_1, it is parallel to first plane, V is orth to $n_2 \Rightarrow$ parallel to 2^{nd} plane
V is parallel to line L intersection of two planes

We observe L must intersect xy-plane at $z=0$ since $V\cdot(0,0,1)=28\neq0$
$z=0 \Rightarrow 2x-4y=6$ $\quad x=1,\ y=-1$ $\quad P(1,-1,0)$ point on L
$6x+2y=4$ $\quad$ vector eqn $= (x,y,z)=(1,-1,0)+t(4,30,28)$

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Distance problems including planes

Theorem: The distance D between a point $P_0(x_0, y_0, z_0)$ and the plane $ax+by+cz+d=0$ is

$$D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

Example: Find distance between $(1,-4,-3)$ and plane $2x-3y+6z+1=0$

$$D = \frac{|(2)(1)+(-3)(-4)+6(-3)+1|}{\sqrt{2^2+(-3)^2+6^2}} = \frac{|-3|}{7} = \frac{3}{7}$$

Example planes $x+2y-2z=3$ and $2x+4y-4z=7$ are parallel since normal $(1,2,-2)$ and $(2,4,-4)$ are parallel vectors. Find distance

Select an arbitrary point in one of the planes $y=z=0$ in eqn $x+2y-2z=3$ $P_0=(3,0,0)$ distance to $2x+4y-4z=7$ is

$$\frac{|(2)(3)+4(0)+(-4)(0)-7|}{\sqrt{2^2+4^2+(-4)^2}} = \frac{1}{6}$$

Example: Find distance between lines $L_1: x=1+4t,\ y=5-4t,\ z=-1+5t$
$x=2+8t,\ y=4-3t,\ z=5+t$

P_1 and P_2 are parallel plane containing L_1 and L_2 we calculate distance from point in P_1 to the plane P_2. Since L_1 lies in P_1 we can easily find a point by substituting in any value for t, $t=0$ gives point $(1,5,-1)$

Find eq^n of P_2 $u_1=(4,-4,5)$ parallel to L_1 $u_2(8,-3,1)$ parallel to L_2 Find cross product which will be normal to normal to both P_1 and P_2

$$n = u_1 \times u_2 = \begin{vmatrix} i & j & k \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix} = 11i+36j+20k$$

Using this normal and point $Q_2(2,4,5)$ ($t=0$ in L_2) we obtain eq^n for P_2

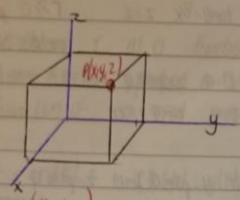
$11(x-2)+36(y-4)+20(z-5)=0$

$11x+36y+20z-266=0$

$$\text{Distance} = \frac{|(11)(1)+(36)(5)+(20)(-1)-266|}{\sqrt{11^2+36^2+20^2}} = \frac{95}{\sqrt{1817}}$$

17.

Rectangular

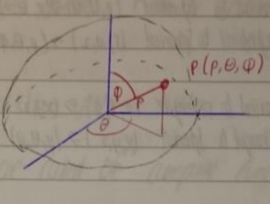


(x, y, z)

Spherical

(ρ, θ, φ)

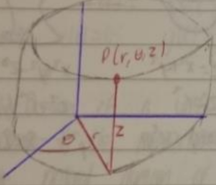
$(\rho \geq 0, \ 0 \leq \theta \leq 2\pi \quad 0 \leq \varphi \leq \pi)$



Cylindrical coordinates

(r, θ, z)

$(r \geq 0, \ 0 \leq \theta \leq 2\pi)$



In rectangular coordinates surfaces represented by $x = x_0$ $y = y_0$ $z = z_0$
where x_0, y_0 and z_0 are constant are planes perp to yz plane, xz plane and xy plane

In cylindrical form the surface represented by $r = r_0$ $\theta = \theta_0$ and $z = z_0$
where r_0, θ_0 and z_0 are constant

In Spherical $\rho = \rho_0$, $\theta = \theta_0$ and $\varphi = \varphi_0$

Formula

conversion

cylindrical to rectangular $(r,\theta,z) \to (x,y,z)$ $x = r\cos\theta$ $y = r\sin\theta$ $z = z$

Rectangular to cylindrical $(x,y,z) \to (r,\theta,z)$ $r = \sqrt{x^2+y^2}$ $\tan\theta = y/x$ $z = z$

$r \geq 0$ $\rho \geq 0$

$0 \leq \theta \leq 2\pi$

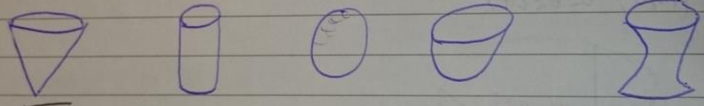
$0 \leq \varphi \leq \pi$

Spherical to cylindrical $(\rho,\theta,\varphi) \to (r,\theta,z)$ $r = \rho\sin\varphi$ $\theta = \theta$ $z = \rho\cos\varphi$

cylindrical to Spherical $(r,\theta,z) \to (\rho,\theta,\varphi)$ $\rho = \sqrt{r^2+z^2}$ $\theta = \theta$ $\tan\varphi = r/z$

Spherical to rectangular $(\rho,\theta,\varphi) \to (x,y,z)$ $x = \rho\sin\varphi\cos\theta$ $y = \rho\sin\varphi\sin\theta$ $z = \rho\cos\varphi$

rectangular to Spherical $(x,y,z) \to (\rho,\theta,\varphi)$ $\tan\theta$ $\rho = \sqrt{x^2+y^2+z^2}$, $\tan\theta = y/x$, $\cos\varphi = z/\sqrt{x^2+y^2+z^2}$



	cone	cylinder	Sphere	paraboloid	hyperboloid
rectangular :	$z = \sqrt{x^2+y^2}$	$x^2+y^2=1$	$x^2+y^2+z^2=1$	$z = x^2+y^2$	$x^2+y^2-z^2=1$
cylindrical :	$z = r$	$r = 1$	$z^2 = 1-r^2$	$z = r^2$	$z^2 = r^2-1$
spherical :	$\varphi = \pi/4$	$\rho = \csc\varphi$	$\rho = 1$	$\rho = \cos\varphi\csc^2\varphi$	$\rho^2 = -\sec 2\varphi$

12 Vector Valued Functions

$r = (x, y, z) = (t, t^2, t^3) = ti + t^2j + t^3k$

We obtain r as a function of parameter t, that is, $r = r(t)$
Since function produces a vector, $r = r(t)$ defines r as a vector valued function of a real variable

$r = r(t) = (x(t), y(t), z(t)) = x(t)i + y(t)j + z(t)k$

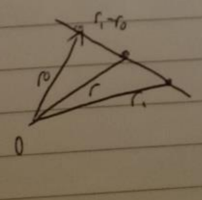
The functions $x(t)$, $y(t)$ and $z(t)$ are called the component functions or components of $r(t)$.

Example: component functions of
$r(t) = (t, t^2, t^3) = ti + t^2j + t^3k$
are $x(t) = t$ $y(t) = t^2$ and $z(t) = t^3$

The Domain of a VVF $r(t)$ is set of all allowable inputs for t.
Domain is the intersection of the natural domains of the component functions, called the natural domain of $r(t)$

Example: Find natural domain of $r(t) = (\ln|t-1|, e^t, \sqrt{t}) = (\ln|t-1|i + e^tj + \sqrt{t}k)$
$x(t) = \ln|t-1|$ $y(t) = e^t$ $z(t) = \sqrt{t}$
$(-\infty, 1) \cup (1, +\infty)$ $(-\infty, +\infty)$ $[0, +\infty)$
Domain is $0 \le t < 1$ or $t > 1$

Vector form of a line segment
$r = r_0 + tv \Rightarrow r = r_0 + t(r_1 - r_0)$ or $r = (1-t)r_0 + tr_1$



Formula

conversion

Cylindrical to rectangular $(r,\theta,z) \to (x,y,z)$ $x = r\cos\theta$ $y = r\sin\theta$ $z = z$

Rectangular to cylindrical $(x,y,z) \to (r,\theta,z)$ $r = \sqrt{x^2+y^2}$ $\tan\theta = y/x$ $z = z$

$r \geq 0$ $\rho \geq 0$

$0 \leq \theta \leq 2\pi$

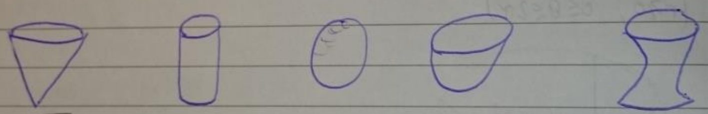
$0 \leq \varphi \leq \pi$

Spherical to cylindrical $(\rho,\theta,\varphi) \to (r,\theta,z)$ $r = \rho\sin\varphi$ $\theta = \theta$ $z = \rho\cos\varphi$

cylindrical to Spherical $(r,\theta,z) \to (\rho,\theta,\varphi)$ $\rho = \sqrt{r^2+z^2}$ $\theta = \theta$ $\tan\varphi = r/z$

Spherical to rectangular $(\rho,\theta,\varphi) \to (x,y,z)$ $x = \rho\sin\varphi\cos\theta$ $y = \rho\sin\varphi\sin\theta$ $z = \rho\cos\varphi$

rectangular to Spherical $(x,y,z) \to (\rho,\theta,\varphi)$ $\rho = \sqrt{x^2+y^2+z^2}$, $\tan\theta = y/x$, $\cos\varphi = z/\sqrt{x^2+y^2+z^2}$



	cone	cylinder	Sphere	paraboloid	hyperboloid
rectangular :	$z = \sqrt{x^2+y^2}$	$x^2+y^2=1$	$x^2+y^2+z^2=1$	$z = x^2+y^2$	$x^2+y^2-z^2=1$
cylindrical :	$z = r$	$r = 1$	$z^2 = 1-r^2$	$z = r^2$	$z^2 = r^2-1$
Spherical :	$\varphi = \pi/4$	$\rho = \csc\varphi$	$\rho = 1$	$\rho = \cos\varphi\csc^2\varphi$	$\rho^2 = -\sec 2\varphi$

2.

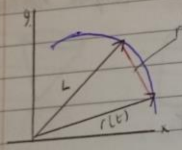
Limits and Continuity

Limit as radius of vector r(t) approach its limit is vector L

$$\lim_{t \to a} r(t) = L$$

Let r(t) be a vvf defined for all t in some open interval containing number a, except r(t) need not be defined at a.

$\lim_{t \to a} r(t) = L$ if and only if $\lim_{t \to a} \| r(t) - L \| = 0$



$\| r(t) - L \|$ is distance between terminal points for vectors r(t) and L when positioned at same initial point

Example $r(t) = t^2 i + e^t j - 2\cos \pi t \, k$

$$\lim_{t \to 0} = \left(\lim_{t \to 0} t^2\right) i + \left(\lim_{t \to 0} e^t\right) j + \left(\lim_{t \to 0} -\cos \pi t\right) k$$

$$= 0i + 1j - 2k = j - 2k$$

vvf r(t) is continuous at t = a if

$$\lim_{t \to a} = r(a)$$

Derivative

Example: $r(t) = t^2 i + e^t j - (2\cos \pi t) k$

$$r(t) = \frac{d}{dt}(t^2) i + \frac{d}{dt}(e^t) j - \frac{d}{dt}(2\cos \pi t) k$$

$$= 2t\, i + e^t j + (2\pi \sin \pi t) k$$

3.

Tangent lines to graphs of vvf

Let P be a point on graph of vvf $r(t)$, let $r(t_0)$ be the radius vector from the origin to P. If $r'(t_0)$ exists and $r'(t_0) \neq 0$, then we call $r'(t_0)$ a tangent vector to the graph of $r(t)$ at $r(t_0)$ and we call the line through P that is parallel to the tangent vector the tangent line to graph of $r(t)$ at $r(t_0)$

Let $r_0 = r(t_0)$ and $v_0 = r'(t_0)$

Tangent line to the graph of $r(t)$ at r_0 is given by vector eqⁿ

$$r = r_0 + t v_0$$

initial point + time (parallel vector)

Example: Find parametric eqⁿ of tangent line to circular helix

$x = \cos t \quad y = \sin t \quad z = t$

Where $t = t_0$, we result to find tangent line at point $t = \pi$

Vector eqn $r(t) = \cos t\, i + \sin t\, j + t\, k$

$r_0 = r(t_0) = \cos t_0\, i + \sin t_0\, j + t_0 k$

$v_0 = r'(t_0) = (-\sin t_0) i + \cos t_0\, j + k$

Vector eqⁿ of tangent line at $t = t_0$ = $r = r_0 + t v_0$

$r = \cos t_0\, i + \sin t_0\, j + t_0 k + t[-\sin t_0\, i + \cos t_0\, j + k]$

$= (\cos t_0 - t\sin t_0)\, i + (\sin t_0 + t\cos t_0) j + (t_0 + t) k$

$x = \cos t_0 - t\sin t_0 \qquad y = \sin t_0 + t\cos t_0 \qquad z = t_0 + t$

$t = \pi \quad = \quad x = -1 \quad y = -t \quad z = \pi + t$

4

Derivative of dot and cross products

$$\frac{d}{dt}[r_1(t) \cdot r_2(t)] = r_1(t) \cdot \frac{dr_2}{dt} + \frac{dr_1}{dt} \cdot r_2(t)$$

$$\frac{d}{dt}[r_1(t) \cdot r_2(t)] = r_1(t) \times \frac{dr_2}{dt} + \frac{dr_1}{dt} \times r_2(t)$$

Theorem: If $r(t)$ is a differentiable vector valued function in 2d or 3d and $\|r(t)\|$ is constant for all t, then

$$r(t) \cdot r'(t) = 0$$

that is, $r(t)$ and $r'(t)$ are orthogonal vectors for all t.
because of constant we get a zero.

Definite Integrals of vector valued functions

$$\int_a^b r(t)dt = \left(\int_a^b x(t)dt\right)i + \left(\int_a^b y(t)dt\right)j + \left(\int_a^b z(t)dt\right)k$$

Example $r(t) = t^2 i + e^t j + (2\cos \pi t)k$

$\int_0^1 r(t)dt = \left(\int_0^1 t^2 dt\right)i + \left(\int_0^1 2\cos\pi t\right)j - \left(\int_0^1 2\cos\pi t\right)k$

$= \frac{t^3}{3}\Big|_0^3 i + e^t\Big|_0^1 j - \frac{2}{\pi}\sin\pi t\Big|_0^1 k$

$= \frac{1}{3}i + (e-1)j$

Antiderivatives of vector valued function

$\int r(t)dt = R(t) + C$ $\quad$ C represents an arbitrary constant vector.

Example $\int (2ti + 3t^2 j)dt = \left(\int 2t\,dt\right)i + \left(\int 3t^2 dt\right)j$

$= (t^2 + C_1)i + (t^3 + C_2)j$

$= (t^2 i + t^3 j) + (C_1 i + C_2 j) = t^2 i + t^3 j + C$

Where $C = C_1 i + C_2 j$ is an arbitrary vector constant of integration

5.

Example: Find $r(t)$ given $r'(t) = (3, 2t)$ and $r(1) = (2, 5)$

Integrating $r'(t)$ to obtain $r(t)$ $\Rightarrow$

$r(t) = \int r'(t)\,dt = \int (3, 2t)\,dt = (3t, t^2) + C$

where C is a vector constant of integration.

To find value of C, sub in $t=1$ and use the given value of $r(1)$:

$r(1) = (3, 1) + C = 2, 5$

So $C = (-1, 4)$. Thus

$\Rightarrow r(t) = (3t, t^2) + (-1, 4) = (3t - 1, t^2 + 4)$

12.3 Change of parameter, Arc length

Smooth parametrization

We say that a curve represented by $r(t)$ is smoothly parametrised by $r(t)$, or that $r(t)$ is a smooth function of t if $r'(t)$ is continuous and $r'(t) \neq 0$, for any allowable value of t.

Example: Is $r(t) = a\cos t\, i + a\sin t\, j + ct\, k$ $(a>0, c>0)$ smooth?

$r'(t) = -a\sin t\, i + a\cos t\, j + c\, k$

The components are continuous functions, and there is no value of t for which all three of them are zero, so $r(t)$ is a smooth function

Example: $r(t) = t^2 i + t^3 j$

$r'(t) = 2t\, i + 3t^2 j$

Components are continuous functions but both equal to zero if $t=0$.
So $r(t)$ is <u>not</u> a smooth function

Arc length from the vector viewpoint

Arc length of parametric curve

$x = x(t)$, $y = y(t)$, $z = z(t)$ $(a \le t \le b)$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$r(t) = x(t)i + y(t)j + z(t)k$$

It follows

$$\frac{dr}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$$

and hence $\left\|\frac{dr}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

Theorem If C is the graph in 2d or 3d of a smooth vector valued function $r(t)$, then its arc length L from $t=a$ to $t=b$ is

$$L = \int_a^b \left\|\frac{dr}{dt}\right\| dt$$

Example: Find arc length from $t=0$, $t=\pi$ $x=\cos t$, $y=\sin t$, $z=t$

$r(t) = \cos t\, i + \sin t\, j + t\, k = (\cos t, \sin t, t)$ then

$r'(t) = (-\sin t, \cos t, 1)$ and $\|r'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$

$$L = \int_0^{\pi} \left\|\frac{dr}{dt}\right\| dt = \int_0^{\pi} \sqrt{2}\, dt = \sqrt{2}\pi$$

Arc length as a parameter.

In 3d coordinate is $(x(s), y(s), z(s))$

Curve C is given by parametric equations $x = x(s)$, $y = y(s)$, $z = z(s)$.

A parametric representation of a curve with arc length as the parameter is called an arc length parameterization of the curve

Example: Find arc length parameterization of circle $x^2 + y^2 = a^2$ with counterclockwise orientation and $(a, 0)$ as a reference point.

7.

The circle with counterclockwise orientation can be represented by parametric eq^n $x = a\cos t$, $y = a\sin t$ $(0 \le t \le 2\pi)$

t can be interpreted as the angle in radian measure from positive x-axis to radius from origin. If we take the positive direction for measuring the arc length to be counterclockwise, and we take $(a, 0)$ to be ref point, then s and t are related by

$s = at$ or $t = s/a$. $\quad s = L = \int_0^t \|v(u)\| du$

Making this change of variable in original parameter, and noting s increases from 0 to $2\pi a$ as t increases from 0 to 2π yields

$x = a\cos(s/a)$ $\quad y = a\sin(s/a)$ $\quad 0 \le s \le 2\pi a$

Change of parameter

A change of parameter in a vector valued function $r(t)$ is a substitution $t = g(\tau)$ that produces a new vf $r(g(\tau))$ having the same graph as $r(t)$ but possibly traced differently as parameter τ increases

Example $r(u) = 2\cos u\, i + 2\sin u\, j + u k$ where u is some parameter replacing t in order to avoid poor notation in integral. We use $t_0 = 0$ as ref point. $\left\|\frac{dr}{du}\right\| = \sqrt{5}$ and so $s = \int_0^t \sqrt{5}\, du = \sqrt{5}t$

gives us $r(s) = 2\cos\left(\frac{s}{\sqrt{5}}\right)i + 2\sin\left(\frac{s}{\sqrt{5}}\right)j + \frac{s}{\sqrt{5}}k$

If $g(\tau)$ is smooth, then change is called a smooth change of parameter.
If $dt/d\tau > 0$ for all τ then it is a positive change of parameter.
If $dt/d\tau < 0$ for all τ then it is negative change of parameter

8.

Finding arc length parameterizations

Theorem: Let C be the graph of a smooth VVF $r(t)$ in 2d or 3d and let $r(t_0)$ be any point on C. Then the following formula defines a positive change of parameter from t to s, where s is an arc lenght parameter having $r(t_0)$ as its reference point.

$$s = \int_{t_0}^{t} \left\| \frac{dr}{du} \right\| du$$

Example: find the arc length parameterization of the circular helix $r = \cos t\, i + \sin t\, j + t k$ ref point $= r(0) = (1,0,0)$ and same orientation as helix

Replacing t by u in r for integration purposes and taking $t_0 = 0$ we obtain

$$r = \cos(u) i + \sin u j + u k$$

$$\frac{dr}{du} = (-\sin u)\, i + \cos(u) j + k$$

$$\left\| \frac{dr}{du} \right\| = \sqrt{(-\sin u)^2 + \cos^2 u + 1} = \sqrt{2}$$

$$s = \int_0^t \left\| \frac{dr}{du} \right\| du = \int_0^t \sqrt{2}\, du = \sqrt{2}\, u \Big|_0^t = \sqrt{2} t$$

Thus $s = \sqrt{2} t \;=\; t = s/\sqrt{2}$

$$\Rightarrow r = \cos\left(\frac{s}{\sqrt{2}}\right) i + \sin\left(\frac{s}{\sqrt{2}}\right) j + \frac{s}{\sqrt{2}} k$$

Example Find arc length parameterization of line $x = 2t+1$, $y = 3t-2$ that has same orientation as the given line and $(1,-2)$ ref point

The line passes through the point $(1,-2)$ and parallel to $v = 2i + 3j$. To find arc length par we only rewrite given equations using $v/\|v\|$ rather than v to determine the direction and replace t by s

Since $\frac{v}{\|v\|} = \frac{2i+3j}{\sqrt{13}} \Rightarrow \frac{2}{\sqrt{13}} i + \frac{3}{\sqrt{13}} j$

parametric eq"s are $x = \frac{2}{\sqrt{13}} s + 1 \qquad y = \frac{3}{\sqrt{13}} s - 2$

Properties of arc length parameterization

Theorem:

If C is graph of smooth vector v.f. $r(t)$ in 2d or 3d where t is a general parameter, and if s is the arc length parameter for C defined by formula (10) then for every value of t the tangent vector has length

$$\left\|\frac{dr}{dt}\right\| = \frac{ds}{dt}$$

If C is the graph of a smooth v.f. $r(s)$ in 2d or 3d where s is an arc length parameter, then for every value of s, the tangent vector to C has length $\left\|\frac{dr}{ds}\right\| = 1$ let t = s in above formula

If C is graph of smooth v.f. $r(t)$ in 2d or 3d and if $\|dr/dt\| = 1$ for every value of t, then for every value of t_0 in the domain of r, the parameter $s = t - t_0$ is an arc length parameter that has ref point at point on C where $t = t_0$

12.4 Unit tangent normal and binormal vectors

Unit tangent vectors

By normalizing $r'(t)$ we obtain a unit vector $T(t) = \frac{r'(t)}{\|r'(t)\|}$ that is tangent to C and points in direction of increasing parameter. We call $T(t)$ the unit tangent vector to C at t.

Example find unit tangent vector to graph of $r(t) = t^2 i + t^3 j$ at point $t = 2$

$$r'(t) = 2t i + 3t^2 j$$

We get

$$T(2) = \frac{r'(2)}{\|r'(2)\|} = \frac{4i + 12j}{\sqrt{160}} = \frac{4i + 12j}{4\sqrt{10}} = \frac{1}{\sqrt{10}} i + \frac{3}{\sqrt{10}} j$$

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Unit normal vectors

- If a vector vf has constant norm then $r(t)$ and $r'(t)$ are orthogonal vectors
- In particular $T'(t)$ is perpendicular to the tangent line to C at t, so we say $T'(t)$ is normal to C at t.
- $T(t)$ has constant norm 1, so $T(t)$ and $T'(t)$ are orthogonal vectors

If $T'(t) \neq 0$ $\quad N(t) = \dfrac{T'(t)}{\|T'(t)\|}$

We call $N(t)$ the principle unit normal vector to C at t or unit normal vector

$$T(t) = \frac{r't}{\|r't\|} \qquad N(t) = \frac{T'(t)}{\|T'(t)\|}$$

skipped next bit NOT NEEDED

⇓

12.6 Motion Along A Curve

velocity, acceleration and speed

If $r(t)$ is the position function of a particle moving along a curve in 2d or 3d then the instantaneous velocity, instantaneous acceleration and instantaneous speed of the particle at time t are:

$$\text{velocity} = v(t) = \frac{dr}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$$

$$\text{accelerations} = a(t) = \frac{dv}{dt} = \frac{d^2r}{dt^2} = \frac{d^2x}{dt^2}i + \frac{d^2y}{dt^2}j + \frac{d^2z}{dt^2}k$$

$$\text{Speed} = \|v(t)\| = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

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Curvate of a curve C is

$$\kappa(s) = \left\| \frac{dT}{ds} \right\| = \| r''(s) \|$$

$$\text{or} \quad \kappa(t) = \frac{\| T'(t) \|}{\| r'(t) \|}$$