Maths Semester 2 2013 JF

# CHAPTER 6

## Exponential, Logaritmic and Inverse trigonometric functions

① week 1

14/1/13 Maths Exponential and Logarithmic functions:

Exponential Functions:

i. Exponentials of integers eg. $2^1, 4^7, 8^0$

ii. Exponential of rational numbers:
Rational Numbers: Any number which can be written as a fraction $\frac{m}{n}$

Symbol for rational numbers = $Q$
Exponentials : $3^{4/5}$

Example: $3^{1/2} = a$
$a^2 = 3$, $3^{1/2} = a$

Based of rule of exponential:

i. $a^n \cdot a^m = a^{m+n}$

ii. $\frac{a^n}{a^m} = a^{n-m}$

iii. $(a^n)^m = a^{mn}$

Example: $3^{1/2} = a$
$a^2 = a.a = 3^{1/2} . 3^{1/2} = 3^{1/2 + 1/2} = 3^1 = 3$

How do I know such a number exists?
example: $4^{1/5} = b$
Meaning: $b^5 = 4$

All fractions like $1/m$:
$4^{1/m} = b$
$b^m = 4$

General fractions:
example: $3^{4/5} = b$
$b^5 = 3^4$

2.

Any function:

$3^{n/m} = b$

$b^m = 3^n$

Exponentiation of real numbers:

Real numbers: Includes numbers which cannot be written as fractions, eg $\pi$. $\pi \neq \frac{m}{n}$ for any choice of m or n

Symbol for real numbers $\mathbb{R}$

Example $3^{\pi} = b$

Definition: All real numbers can be approximated to any accuracy by rational numbers

eg. $\pi$ : $\frac{3}{1}$, $\frac{22}{7}$ ....

To define $3^{\pi} = b$ $3^{3/1}$, $3^{22/7}$ ....

This sequence will approach some number

This number is $3^{\pi}$

So for any real number x.

x: $\frac{m_1}{n_1}$, $\frac{m_2}{n_2}$ $\frac{m_3}{n_3}$ .... $\rightarrow$ getting closer to x.

$4^{\wedge}$, $4^{\frac{m}{n}}$, $4^{m_2/n_2}$, $4^{m_3/n_3}$, ... getting closer to $4^x$

Now I can define exponentials of any number

Two problems:

1. How do I know the sequence to converg?

$4^{\frac{m}{n}}$, $4^{\frac{m_2}{n_1}}$ .... $\longrightarrow$ ?

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ii. Powers of negative numbers

$(-4)^{1/2} = b$

$b^2 = -4$

Problem: No real numbers square to a negative number. DOESN'T EXIST

Enlarge algebra to include these numbers (complex numbers)

Exponential functions:

$f(x) = 2^x$ — Fixed base number.

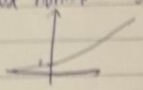
$f(x) = \pi^x$ — Variable power.

$f(x) = 49^x$

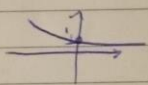
Four Properties:

i. When $x = 0$, all functions equal 1 eg. $2^0 = 1$, $\pi^0 = 1$, $9^0 = 1$

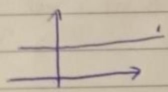
ii. If the base number $b > 1$, then they increase to the right $x \to \infty$



iii. If $0 < b < 1$, they decrease as $x \to \infty$



iv. If $b = 1$, constant



$f(x) = 1^x$

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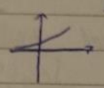
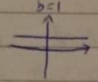
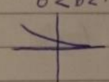
i  ii  iii 



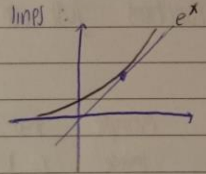


Domain: Well defined for all values of $x$ $(-\infty, +\infty)$ $(\mathbb{R})$

Range $\Rightarrow$ All possible outputs
$\Rightarrow$ Range $(0, \infty)$ (doesn't output 0)

Useful property: $b^{-\infty} = 0$ $(b>1)$ no number "$\infty$"
$b^{-\infty} = 0$ is just a shorthand for $\lim_{x \to -\infty} b^x = 0$

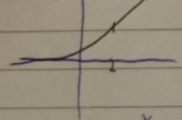
Special exponential
$f(x) = e^x$
$e = 2.718282$ Euler's number
Slope of tangent lines:





Slope of tangent line is equal to value of function

Example:



slope of tangent line at $x=2$
$= e^2$

$e^x$ is the only function for which this is true

Applications: Slope of tangent line tells you the rate of change
How fast the function is changing at that point.

Slope of tangent lines: Derivative $\frac{df}{dx}$

$e^x$ is the only function which increases proportional to its value

Usual practice: $2^x$, $3^x$ or any $b^x$ function is normally written in terms of $e^x$

This required the logarithm function

## Logarithm functions:

Inverse function of the exponential function

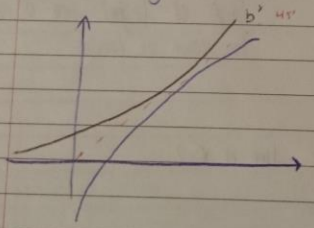
Example $\log_b(b^x) = x$

$$x \xrightarrow{exp} b^x \xrightarrow{\log_b} x$$

Inverse

## Graph of $\log_b(x)$:

When drawing inverse functions just use mirror image




Example: $x=0$, $b^x=1$

Inverse: $x=1$, $\log_b(x)=0$

$\log_b(x)$: b is known as base and $(b>0)$

## Rules of log functions:

| exp | log |
|---|---|
| i. $b^x b^y = b^{x+y}$ | i. $\log_b(xy) = \log_b(x) + \log_b(y)$ |
| ii. $\frac{b^x}{b^y} = b^{x-y}$ | ii. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$ |
| iii. $(b^x)^y = b^{xy}$ | iii. $r\log_b(x) = \log_b(x^r)$ |

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Natural Log:

$Log_e(x)$, base is $e = 2.718282$

known as natural logarithm or $\ln(x)$

Inverse of $e^x$: $\ln(e^x) = x$

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Value of $\ln(x)$:                     Inverse function  $f(x) = a \to b$

$f^{-1}(y) = b \to a$

i. $\ln(1) = 0$   $e^0 = 1$

ii. $\ln(e) = 1$   $e^1 = e$

iii. (not strictly correct) $\ln(0) = -\infty$  $(e^{-\infty} = 0)$

Derivative of $\ln(x)$

Chain rule: Function $f(u(x))$

Derivative $\frac{d\, f(u(x))}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$

Example: $\cos e^x$   $f(u) = \cos(u)$

$u(x) = e^x$

Derivative $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$

$\frac{df}{du} = -\sin(u)$

$\frac{du}{dx} = e^x$

$\frac{df}{dx} = -\sin(u)e^x$

$= -\sin(e^x)(e^x)$

Use the formula  i $\ln(e^x) = x$

ii Take the derivative of both sides

$\frac{d}{dx}[\ln(e^x)] = \frac{d}{dx} x$

$$\frac{d}{dx}\left[\ln(e^x)\right] = 1$$

iii. Chain rule

$f(u) = \ln(u)$

$u(x) = e^x$

$\frac{df}{du} = \frac{d}{du}\ln(u)$ } left as function of u

$\frac{du}{dx} = e^x = u$

$$\frac{d}{dx}\left[\ln(e^x)\right] = \left(\frac{d}{du}\ln(u)\right)u$$

iv. Substitute back into original formula:

$$\left(\frac{d}{du}(\ln(u))\right)u = 1$$

$$\Rightarrow \frac{d}{du}\ln(u) = \frac{1}{u}$$ (true in u true in x)

Completed the usual formula for integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$n = -1 \quad \int x^{-1}\,dx = \frac{x^0}{0} + C$$

$$n = -1 = \int \frac{1}{x}\,dx = \ln(x) + C$$

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Limits Q.1

Meaning of: i. $\frac{1}{0}$ → for every real number $x$, the following is t...

ii. $\frac{0}{0}$

$x \cdot 0 = 0$

Inverse numbers $4$, $\frac{1}{4}$

$5$, $\frac{1}{5}$ etc

These obey $4 \cdot \frac{1}{4} = 1$

$5 \cdot \frac{1}{5} = 1$

Hence $0 \cdot \frac{1}{0} = 1$

This is impossible, any number multiplied by 0 is zero.

$\frac{1}{0}$ does not exist "undefined".

Call whatever number that equal c.

$0/0 = c$

Take any other fraction

$\frac{4}{5} = c$ $\quad 4 = c.5$

Always true

Use the other form on $0/0$

$0 = c.0$

True for every number: $0/0$ is just another way to write a number.

Examples i. $\lim_{x \to 0} \frac{\cos(x)}{x}$

$= \frac{\cos(0)}{0}$

$= \frac{1}{0}$

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ii. $\lim_{x \to 0} \frac{\sin(x)}{x}$ $(=1)$

$= \frac{\sin(0)}{0}$

$\frac{0}{0}$ the answer, so number

Standard nonel

i. $1/0$ is undefined or ill defined form

All expressions like this are ill-defined unless numerator is zero

$3/0$, $4/0$, $\pi/0$

ii. $0/0$, the indeterminate form (does) not determine the answer

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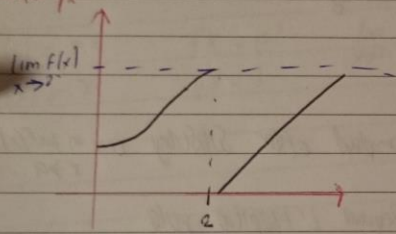
## Evaluating limits

The limit is the real number an expression approaches as $x$ approaches some value $x_0$.

It is not $f(x_0)$ itself.

### Example:




$\lim_{x \to 2} f(x) \neq f(2)$

### Three Types of limits:

i. $\lim_{x \to a^-} f(x)$ limit as $x=a$ is approached from smaller values of $x$

ii. $\lim_{x \to a^+} f(x)$ limit as $x=a$ is approached from larger values of $x$

iii. If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ one can use $\lim_{x \to a} f(x)$.

No need to mention direction

### Calculating Limits - Six types

Based on what happens when you directly substitute limit value.

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1. Resut is Simply a real number (not $\frac{0}{0}, \frac{1}{0}, \frac{\infty}{\infty}, \infty, 0$ etc)

Example: $\lim_{x \to 2} \frac{x^2+2x+8}{x+2} =$

Substitute $x=2$ $= \frac{16}{4} = 4$

2 Result is $\frac{n}{0}$, where $n$ is some number.

Example: $\lim_{x \to 0} \frac{\cos(x)}{x} = \frac{\cos(0)}{0} = \frac{1}{0}$

The limit is undefined

Both type 1 and 2 are completed after Substituting $a$ in $\lim_{x \to a} f(x)$ into $f(x)$

3 Result is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ Requires L'Hospitals rule

L'Hospitals rule: If a limit, $\lim_{x \to a} \frac{f(x)}{g(x)}$ results in $\frac{0}{0}$ or $\frac{\infty}{\infty}$

after Substitution, then use: $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ Both expressions have Same limits

Example: $\lim_{x \to 0} \frac{\sin(x)}{x} = \frac{\sin(0)}{0} = \frac{0}{0}$

L'Hospital's rule; $f(x) = \sin x$ $f'x = \cos x$

$g(x) = x$ $g'x = 1$

using $\lim_{x \to} \frac{f'(x)}{g'(x)}$ $\frac{\cos(0)}{1} = \frac{1}{1} = 1$ $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

Example: $\lim_{x \to 2} \frac{x^2-4}{x-2} = \frac{0}{0}$

$f(x) = x^2-4$ $f'(x) = 2x$ $\lim_{x \to 2} \frac{2x}{1} = 4$

$g(x) = x-2$ $g'(x) = 1$

$\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$

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Common Error

Do not take the derivative of the entire expression

$\frac{d}{dx}\left(\frac{\sin(x)}{x}\right) = \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}$ wrong way!

4. $\lim_{x \to \infty} \frac{e^x}{x} = \frac{e^\infty}{\infty} = \frac{\infty}{\infty}$

$f(x) = e^x$ $\quad f'x = e^x$ $\quad \lim_{x \to \infty} \frac{e^x}{1} = \infty$ lim is undefined (same as $\frac{1}{0}$)

$g(x) = x$ $\quad g'x = 1$

2

Any limit of this type:
Keep applying L'Hôpital's rule until you get a number ( $\frac{1}{2}$, $\frac{0}{4}$, etc ) or an indeterminate form ( $\frac{1}{0}$, $\frac{3}{0}$ ... )

Note: L'Hôpital's rule only works if the result is $\frac{0}{0}$ or $\frac{\infty}{\infty}$

TYPE 4 Result is $(0 \cdot \infty)$

Example: $\lim_{x \to 0} x \ln(x) \Rightarrow (0)\ln(0) = 0 \cdot -\infty$ $\quad \ln(0) = -\infty$

Method: Change expression into one where L'hôpital's rule can be used

$$\lim_{x \to a} f(x) \cdot g(x) \Rightarrow \lim_{x \to a} \frac{g(x)}{1/f(x)}$$

Example: $\lim_{x \to 0} x \ln(x) \Rightarrow \lim_{x \to 0} \frac{\ln(x)}{1/(x)}$

$$\lim_{x \to 0} \frac{\ln(0)}{1/0} = \frac{-\infty}{\infty}$$

Use L'hôpital's rule: $f(x) = \ln x \qquad f'(x) = \frac{1}{x}$

$g(x) = 1/x \qquad g'(x) = -1/x^2$

Replace functions: $\lim_{x \to 0} \frac{\frac{1}{x}}{-1/x^2} = \lim_{x \to 0} \frac{1}{x} \cdot -x^2 = \lim_{x \to 0} -x = 0$

1. $\lim_{x \to 0} f(x) \cdot g(x)$

$\lim_{x \to 0} \frac{g(x)}{1/f(x)}$

2. Now good a L'hôpital's rule, $\frac{0}{0}$ $\frac{\infty}{\infty}$

Sometimes if you have $\frac{f(x)}{1/g(x)}$ change back to $f(x) \cdot g(x)$

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$\frac{0}{0}, \frac{\infty}{\infty}$ type limits (continued)

Example:

$\lim_{x \to 0^-} \frac{\tan(x)}{x^2} \xrightarrow{x=0} \frac{\tan(0)}{0^2} = \frac{0}{0}$

L'Hôpital's rule; $f(x) = \tan(x)$ $\quad f'(x) = 1 + \tan^2(x)$

$g(x) = x^2$ $\quad g'(x) = 2x$

$\lim_{x \to 0} \frac{1+\tan^2 x}{2x} = \frac{1+\tan^2(0)}{2(0)} = \frac{1}{0}$ undefined

Example:

$\lim_{x \to \infty} \frac{x}{e^x} \Rightarrow \frac{\infty}{e^\infty} = \frac{\infty}{\infty}$

Rules for $e^\infty = \infty$

$e^{-\infty} = 0$

$\frac{1}{\infty} = 0$

$f'x = 1$

$g'x = e^x$

$= \lim_{x \to \infty} \frac{1}{e^x} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$

Example:

$\lim_{x \to 0} \frac{1-\cos(x)}{x^2} = \frac{1-\cos(0)}{0^2} = \frac{1-1}{0} = \frac{0}{0}$

$f(x) = 1 - \cos(x)$ $\quad f'(x) = \sin(x)$

$g(x) = x^2$ $\quad g'(x) = 2x$ $\quad \lim_{x \to 0} \frac{\sin(x)}{2(x)} = \frac{\sin(0)}{2(0)} = \frac{0}{0}$

Just apply l'Hôpital's rule again

$f(x) = \sin x$ $\quad f'(x) = \cos(x)$ $\quad = \lim_{x \to 0} \frac{\cos x}{2} = \frac{\cos(0)}{2} = \frac{1}{2}$

$g(x) = 2x$ $\quad g'(x) = 2$

$\lim_{x \to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$

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Type 6 $0^{\infty}, \infty^{0}, 1^{\infty}$

Type 5 $\infty - \infty$

Type 4 $0 \cdot \infty$

Type 3 $\frac{0}{0}, \frac{\infty}{\infty}$

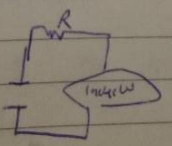
## Useful Expressions

1. $1/\infty = 0$
2. $1/0 = \infty$
3. $e^{\infty} = \infty$
4. $e^{-\infty} = 0$
5. $\ln(1) = 0$
6. $\ln(0) = \infty$
7. $\ln(\infty) = \infty$
8. $\cos(0) = 1$
9. $\sin(0) = 0$
10. $\tan(0) = 0$
11. $\cos\left(\frac{\pi}{2}\right) = 0$
12. $\sin\left(\frac{\pi}{2}\right) = 1$
13. $\tan\left(\frac{\pi}{2}\right) = \infty$

## Applications of limits:

Often in applied maths real objects tend to be near some "extreme case".

example




current $= I(t) = \frac{V}{R}\left(1 - e^{-t\left(\frac{R}{L}\right)}\right)$

V = initial current
R = resistance
L = inductor
t = time

2.

$$\lim_{x \to \infty} \frac{\ln(\cancel{1} + \frac{1}{x})}{1/x} \overset{x=\infty}{\Rightarrow} \ln(1 + \tfrac{1}{\infty})$$

$$= \frac{\ln(1)}{0} = \frac{0}{0}$$

$\downarrow$

$\lim_{x \to \infty} \frac{\ln(1+\frac{1}{x})}{1/x}$ is type 3 <u>Apply L'Hôpital's rule</u>

$f(x) = \ln(1 + 1/x) \qquad f'(x) = \frac{1}{1+1/x} \cdot -\frac{1}{x^2}$

$g(x) = 1/x \qquad g'(x) = -1/x^2$

$$\lim_{x \to \infty} \left( \frac{\frac{1}{1+1/x} \cdot \cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{-1/x^2}} \right) \qquad \lim_{x \to \infty} \frac{1}{1+1/x} = \frac{1}{1+1/\infty} = 1$$

Remember: The logarithm has been taken, so this is the limit of $\ln(f(x)^{g(x)})$, <u>not</u> $f(x)^{g(x)}$

<u>If</u> $\lim_{x \to a} \ln(f(x)^{g(x)}) = c$

<u>Then</u> $\lim_{x \to a} f(x)^{g(x)} = e^c$

Hence: $\lim_{x \to \infty} (1 + \frac{1}{x})^x = e^1 = e$

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Type 5 $\infty - \infty$ is the rule $\left(\frac{a}{0} - \frac{a}{0}\right)$

$$\lim_{x \to 0} \left( \frac{1}{x} - \frac{1}{\sin(x)} \right) \overset{x=0}{\Rightarrow} \frac{1}{0} - \frac{1}{\sin(x)} = \frac{1}{0} - \frac{1}{0} = \infty - \infty$$

Method: Combine the fraction

$$\frac{1}{x} - \frac{1}{\sin(x)} = \frac{\sin(x) - x}{x \sin x}$$

$$\lim_{x \to 0} \frac{\sin(x) - x}{x \sin(x)} \overset{x=0}{=} \frac{\sin(0) - 0}{0 \sin(0)} = \frac{0 - 0}{0 \cdot 0} = \frac{0}{0}$$

L'Hopital's rule.

$f(x) = \sin(x) - x$ $\quad f'(x) = \cos x - 1$

$g(x) = x \sin x$ $\quad g'(x) = \sin x + x \cos x$

$$\lim_{x \to 0} \frac{\cos x - 1}{\sin x + x \cos(x)} \Rightarrow \frac{\cos(0) - 1}{\sin(0) + (0)\cos(0)} = \frac{0}{0}$$

Again. L'hopital's rule.

$f(x) = \cos x - 1$ $\quad f'x = -\sin x$

$g(x) = \sin x + x \cos(x)$ $\quad g'x = +\cos x + \cos x - x \sin x = 2\cos x - x \sin x$

$$\lim_{x \to 0} \frac{-\sin x}{2\cos(x) - x \sin(x)} \overset{x=0}{\Rightarrow} \frac{-\sin(0)}{2\cos(0) - 0 \sin(0)} \quad \frac{0}{2} = 0$$

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Type 6: Result is $0^0$, $\infty^0$, $1^\infty$

Example: $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x \overset{x\to\infty}{\Rightarrow} \left(1+\frac{1}{\infty}\right)^\infty = 1^\infty$

The function usually looks like $f(x)^{g(x)}$

In the example: $f(x) = \left(1+\frac{1}{x}\right)$ $g(x)=x$

Take the logarithm of the expression:

Example: $\left(1+\frac{1}{x}\right)^x$

$= \ln\left[\left(1+\frac{1}{x}\right)^x\right]$

$= x\ln\left(1+\frac{1}{x}\right)$

In general i $f(x)^{gx}$

ii $\ln\left(f(x)^{g(x)}\right)$

iii $g(x)\ln(f(x))$

Now take: $\lim_{x\to\infty} x.\ln\left(1+\frac{1}{x}\right)$

$\overset{x=\infty}{\Rightarrow}$ $\infty\cdot\ln\left(1+\frac{1}{\infty}\right)$

$= \infty.\ln(1) = \infty.0$

converted into type 4.

$\lim_{x\to\infty} x.\ln\left(1+\frac{1}{x}\right)$ is type 4

Type 4: $f(x).g(x) \Rightarrow \frac{g(x)}{1/f(x)}$

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Explanation of L'Hôpital's rule:

- Not actually L'Hôpital's rule
- Purchased from Bernoulli

- The rule only works when the answer is $\frac{0}{0}$ or $\frac{\infty}{\infty}$

- Only in this situation do the original function and the derivative have the same limit.

Example:

$$\lim_{x \to 2} \frac{x^2}{x} = 2$$

Use derivatives

$$\lim_{x \to 2} \frac{2x}{1} = 4$$

Derivative gave different limit (wrong)

Reason:

"Every" function 'is' basically an infinitely long polynomial

$$\frac{f(x)}{g(x)} \quad \frac{f_0 + f_1 x + f_2 x^2 + \dots}{g_0 + g_1 x + g_2 x^2 + \dots}$$

$$x \to 0 \quad \frac{f(x)}{g(x)} = \frac{f_0}{g_0}$$

What if $f_0, g_0 = 0$: $\frac{f(x)}{g(x)} = \frac{f_1 x + f_2 x^2 + f_3 x^3 + \dots}{g_1 x + g_2 x^2 + g_3 x^3 + \dots} \xrightarrow{x=0} \frac{f_1}{g_1}$

Derivative $\frac{f(x)}{g(x)}$ $\frac{f_1 + 2f_2 x + 3f_3 x^2 + \dots}{g_1 + 2g_2 x + 3g_3 x^2 \dots} \xrightarrow{x=0} \frac{f_1}{g_1}$

If $f_0, g_0 \neq 0$ The derivative 'deletes' $f_0, g_0$.

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11/13 Maths Week 3

# Examples of limits

i $\lim_{x \to 1} \frac{\ln(x)}{x-1} \overset{x=1}{\Rightarrow} \frac{\ln(1)}{1-1} = \frac{0}{0}$

L'Hopitals rule

$f(x) = \ln(x)$ $\quad f'(x) = 1/x$

$g(x) = x-1$ $\quad g'(x) = 1$

$\lim_{x \to 1} \frac{1/x}{1} = \lim_{x \to 1} \frac{1}{x} = \frac{1}{1} = 1$

ii $\lim_{x \to (\frac{\pi}{2})^-} (\cancel{\cot(x)} - \tan(x))$ $\quad \cot(x) = \frac{1}{\cos x}$

$\frac{1}{\cos(x)}$

$\lim_{x \to (\frac{\pi}{2})^-} \left(\frac{1}{\cos(x)} - \tan(x)\right) \Rightarrow \left(\frac{1}{\cos\frac{\pi}{2}} - \tan\frac{\pi}{2}\right)$

$\Rightarrow \frac{1}{0} - \infty = \infty - \infty$

COMBINE THE FRACTIONS $\frac{1}{\cos x} - \tan(x)$

$\frac{1}{\cos x} - \frac{\sin x}{\cos x}$

$= \frac{1 - \sin x}{\cos x}$ $\quad \lim_{x \to (\frac{\pi}{2})^-} \frac{1 - \sin x}{\cos x} = \frac{1 - \sin\frac{\pi}{2}}{\cos\frac{\pi}{2}} = \frac{1-1}{0} = \frac{0}{0}$

L'HOPITALS RULE

$f(x) = 1 - \sin x$ $\quad f'(x) = -\cos x$

$g(x) = \cos x$ $\quad g'(x) = -\sin x$

$\lim_{x \to \frac{\pi}{2}} \frac{-\cos(x)}{-\sin(x)} = \lim_{x \to \frac{\pi}{2}} \frac{\cos(x)}{\sin(x)} = \frac{0}{1} = \cancel{\infty}\ 0$

The limit is ~~undefined~~

2.

iii. $\lim_{x \to 0} (1 + \sin(4x))^{\frac{1}{\tan(x)}}$ $f(x)^{g(x)}$ i. $f(x) = 1 + \sin 4x$

$g(x) = \frac{1}{\tan x}$

$\overset{x=0}{\Rightarrow} (1 + \sin(0))^{\frac{1}{\tan(0)}} = (1)^{\frac{1}{0}} = 1^{\infty}$ (Use logarithm function)

Take the Logarithm

$\ln\left[(1+\sin 4x)^{\frac{1}{\tan}}\right]$

$\Rightarrow \frac{1}{\tan(x)} \ln(1 + \sin(4x))$

$\overset{x=0}{\Longrightarrow} \frac{\ln(1+\sin(0))}{\tan(0)} = \frac{\ln(1)}{0} = \frac{0}{0}$

$\ln x \overset{f'(x)}{=} = \frac{1}{x}$

L'HOPITAL RULE:

$f(x) = \ln(1 + \sin 4x)$ $f'(x) = \frac{4\cos(x)}{1+\sin 4x}$

$g(x) = \tan x$ $g'(x) = 1 + \tan^2(x)$

$\lim_{x \to 0} \frac{\left(\frac{4\cos 4x}{1+\sin 4x}\right)}{1+\tan^2 x}$ $\overset{x=0}{\Longrightarrow} = \frac{\frac{4}{1}}{1} = 4$

Not the limit of the original function

Limit: $e^4$

iv. $\lim_{x \to 0^+} x^x$ $\overset{x=0}{\Longrightarrow} 0^0$ (use Logarithm function)

LOGARITHM:

$\ln(x^x) = x\ln(x)$

$\lim_{x \to 0} x \ln x = 0$ (already done)

Actual limit $e^0 = 1$

4

$$\lim_{t \to \infty} \frac{V}{R}\left(1 - e^{-t/\tau_R}\right) = \frac{V}{R}$$

$$\Rightarrow \quad I = \frac{V}{R}$$

Limits give you some "extreme case" that is mathematically simpler.

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## Explanation of L'Hôpital's rule:

- Not actually L'Hôpital's rule
- Purchased from Bernoulli

- The rule only works when the answer is $\frac{0}{0}$ or $\frac{\infty}{\infty}$

- Only in this situation do the original function and the derivative have the same limit.

**Example:**

$$\lim_{x \to 2} \frac{x^2}{x} = 2$$

Use derivatives

$$\lim_{x \to 2} \frac{2x}{1} = 4$$

Derivative gave different limit (wrong)

**Reason:**

"Every" function 'is' basically an infinitely long polynomial

$$\frac{f(x)}{g(x)} \quad \frac{f_0 + f_1 x + f_2 x^2 + \ldots}{g_0 + g_1 x + g_2 x^2 + \ldots}$$

$$x \to 0 \quad \frac{f(x)}{g(x)} = \frac{f_0}{g_0}$$

What if $f_0, g_0 = 0$: $\frac{f(x)}{g(x)} = \frac{f_1 x + f_2 x^2 + f_3 x^3 + \ldots}{g_1 x + g_2 x^2 + g_3 x^3 + \ldots} \overset{x \to 0}{\Rightarrow} \frac{f_1}{g_1}$

Derivative $\frac{f(x)}{g(x)}$ $\frac{f_1 + 2f_2 x + 3f_3 x^2 + \ldots}{g_1 + 2g_2 x + 3g_3 x^2 \ldots} \overset{x \to 0}{\Rightarrow} \frac{f_1}{g_1}$

If $f_0, g_0 \neq 0$ The derivative 'deletes' $f_0, g_0$.

① 

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$\lim\limits_{x\to(\frac{\pi}{2})^-} \left(\dfrac{1}{\cos(x)} - \tan(x)\right) \Rightarrow \left(\dfrac{1}{\cos\frac{\pi}{2}} - \tan\frac{\pi}{2}\right)$

$\Rightarrow \dfrac{1}{0} - \infty \quad = \infty - \infty$

COMBINE THE FRACTIONS $\dfrac{1}{\cos x} - \tan(x)$

$\dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x}$

$= \dfrac{1-\sin x}{\cos x} \qquad \lim\limits_{x\to(\frac{\pi}{2})^-} \dfrac{1-\sin x}{\cos x} = \dfrac{1-\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}} = \dfrac{1-1}{0} = \dfrac{0}{0}$

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The limit is ~~[illegible]~~

2.

iii. $\lim_{x\to 0} (1+\sin(4x))^{\frac{1}{\tan(x)}}$ $f(x)^{g(x)}$ $\therefore f(x) = 1+\sin 4x$

$g(x) = \frac{1}{\tan x}$

$\overset{x=0}{\Rightarrow} (1+\sin(0))^{\frac{1}{\tan(0)}} = (1)^{\frac{1}{0}} = 1^{\infty}$ (use logarithm function)

Take the Logarithm

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Not the limit of the original function

Limit: $e^4$

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LOGARITHM:

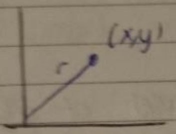
$\ln(x^x) = x\ln(x)$

$\lim_{x\to 0} x\ln x = 0$ (already done)

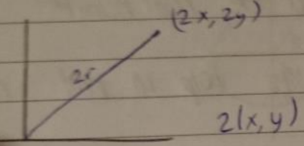
Actual limit $e^0 = 1$

1/2/13 Week 3 Mum

Multiplication Rule for the plane




Multiplication by a real number




A real number is just a point on the horizontal axis, so:

$\underbrace{(a, 0)}_{\text{real number}} . (c . d) = (ac, ad)$

Note: $(2, 8) . (2, 3) = (2.2, 8.3) = (4, 24)$

Any point can be written as:

i) $(x, y)$
ii) Can add points $\Rightarrow (x, 0) + (0, y)$
iii) $\Rightarrow x(1, 0) + y(0, 1)$
iv. $(1, 0) \Rightarrow 1$
$(0, 1) \Rightarrow i$
v. $x + iy$

So Multiplication: $(a + bi)(c + di)$
$= (a.c) + (bc + ad)i + (bd) \underset{?}{ii}$

Many Choices Possible

$i^2 = 0$
$i^2 = -1$

The choice $i^2 = -1$ is known as the complex number.

($i^2 = 0$, Grassman number)

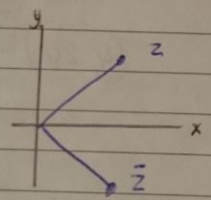
With the choice $i^2 = -1$ the basic rules of algebra are the same as the rules of the real numbers.

The complex numbers can be used to help in real number calculations.

Conjugation:

General complex number, $z = x + iy$.

Conjugate $z^*$ or $\bar{z} = x - iy$




Lenght of complex number:

Given by: $z^* z = (x-iy)(x+iy) = x^2 + y^2 = r^2$

Lenght $r = \sqrt{z^* z}$

Example: $2+3i$ $\quad z^* z = (2-3i)(2+3i) = 4+9 = 13$

$\sqrt{13} = r$.

Polar form of complex numbers.

Take a complex exponential $e^{i\phi}$ $\quad \phi$ is a real number.

Take the conjugate: $e^{-i\phi}$

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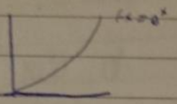
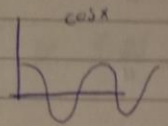
Final notes about limits:

i. $x \to \infty$ of $\sin(x)$ $\cos(x)$




$\lim_{x \to \infty} e^x = \infty$ (undefined)

$\lim_{x \to \infty} \cos(x)$ undefined

$\infty \Rightarrow$ usually refers to something which is undefined due to unbounded growth

$\lim_{x \to \infty} \cos(x)$ undefined as the function keep oscilating

ii. Limits as $x \to \infty$ of ratio of polynomials
(important in the sequences and series part of course) (Third question)

a Numerator has a higher power

$\lim_{x \to \infty} \frac{x^3 + 3x^2}{6x^2 + 4x^3} = \infty$ (undefined) Polynomials are sums of powers of x.

$\lim_{x \to \infty} \frac{x^4 + 2x}{9x^2 + 2} = \infty$ (undefined)

can be checked using L'Hopital's rule or just looking at growth of the two functions

b Denominator has a higher power

$\lim_{x \to \infty} \frac{4x^2 + 6x}{x^5 + 2x^3} = 0$

$\lim_{x \to \infty} \frac{2x^2 + 4}{y^3 + 2y} = 0$

(2)

c Same power in both (highest power)

$$\lim_{x \to \infty} \frac{3x^3 + 2x^2}{4x^3 + 4x} = \frac{3}{4}$$

$$\lim_{x \to \infty} \frac{2x^2 + 4}{9x^2 + 3} = \frac{2}{9}$$

All of these can be checked by L'Hopital's rule Not practical

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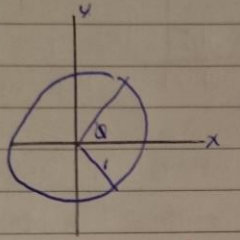
Distance from the origin: $z^* z = e^{-i\phi} . e^{i\phi} = e^{(-i\phi + i\phi)} = e^0 = 1$

$\Rightarrow$ Numbers like $e^{i\phi}$ are at a distance 1 from the origin.

$e^{i\phi}$



$\phi$ is the angle from the x-axis

To get numbers of other lengths just multiply by a constant.

$$z = \underbrace{r}_{\text{Length}}\underbrace{e^{i\phi}}_{\text{Distance}=1}$$

$r$ = distance from origin

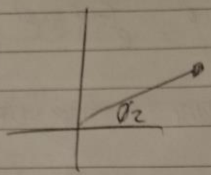
$\phi$ = angle from x-axis

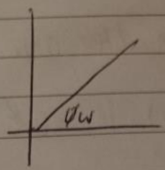
Multiplying complex numbers

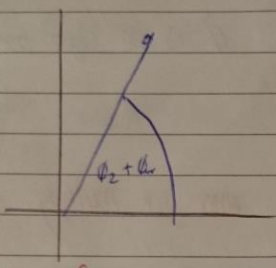
$$z = r_z e^{i\phi_z}$$
$$w = r_w e^{i\phi_w}$$

$$z . w = (r_z e^{i\phi_z})(r_w e^{i\phi_w})$$
$$= r_z r_w e^{i\phi_z} . e^{i\phi_w}$$
$$= r_z r_w e^{i(\phi_z + \phi_w)}$$

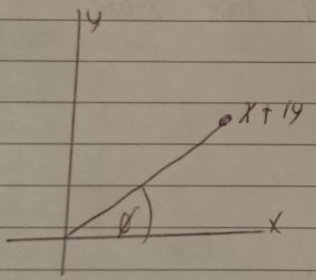
Multiply lengths and add angles




One other form




$$x = r\cos(\phi)$$
$$y = r\sin(\phi)$$

$$z = x + iy$$
$$= r\cos(\phi) + i\,r\sin(\phi)$$

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# Complex Numbers

Motivation: Solving the polynomial

$x^3 - 15x^2 + 81x - 75 = 0$

The roots of this can be found by multiplying the roots of: $x^2 - 10x + 40 = 0$

Factorised this quadratic even though it wasn't considered to be sensible

$x^2 - 10x + 40 = (x + (5 + \sqrt{-15}))(x + (5 - \sqrt{-15}))$

Multiply using normal rules $(5+\sqrt{-15})(5-\sqrt{-15})$

$= 25 - 5\sqrt{-15} + 5\sqrt{-15} - \sqrt{-15}\sqrt{-15}$

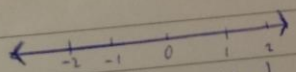
$= 25 - (-15)$

$= 40$

Complex numbers: Geometry and algebra

Real numbers: Numbers used in everyday calculation Rules of multiplication and addition well known

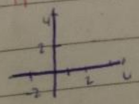
What are the real numbers geometrically? They're a line




Algebraic object (multiplication, addition),
Geometric (line).

What if I try to do the same to a plane?

plane:




(4,3), (2,6) etc

Geometric object
Algebraic $(4,6) + (2,3) = (6,9)$

Multiplication: $(2,2)\cdot(1,4) = ?$

Guess - multiply each part $(2,2)\cdot(1,4) \stackrel{?}{=} (2,8)$
- NOT wrong but useless
- This behaves nothing like normal multiplication from
- Not practical the real numbers

Couldn't use it to help with calculations of real numbers

Need a multiplication rule that behaves like the real number's one

This rule can be found by thinking about the geometry