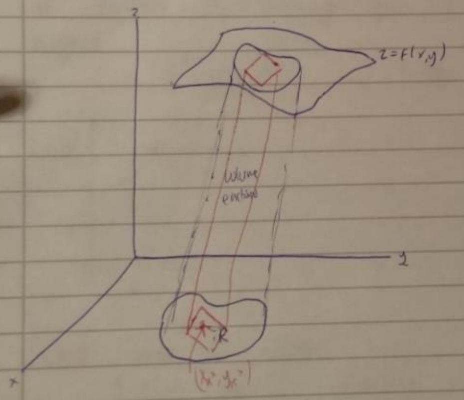
9/10/13 # CHAPTER 3 DOUBLE INTEGRALS

Volume of an enclosed region

Consider $z = f(x,y)$ that lies above some region R on the xy-plane.

How do we find the volume enclosed between the surface and the xy plane?



We break the region R up into infintesmal areas ΔA_k around (x_k^*, y_k^*) and integrate over these areas.

We define the double integral in this region by

$$\iint f(x,y)\,dA = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

we can calculate the volume between the region R and the surface $z = f(x,y) \geq 0$ as

$$V = \iint_R f(x,y)\,dA.$$

If $f(x,y) \geq 0$ does not hold for all (x,y) in R, we find the difference of the volume above and below the xy plane called the net signed volume. Note $f(x,y)$ looks like a "height".

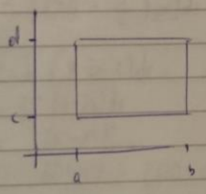
Maths

11/10/13. **Double Integrals over Rectangle Region**

We had $V = \iint_R f(x,y)\,dA$

What does it mean to integrate over a region R? If R is a rectangle with $a \le x \le b$, and $c \le y \le d$ we integrate with these limits and area element $dA = dx\,dy$ and we can perform partial definite integration by treating y as a constant and integrating over x or vice-versa giving integrals:

$$\int_a^b f(x,y)\,dx \quad , \quad \int_c^d f(x,y)\,dy$$



Example: Evaluate the integrals
$\int_0^1 x^3 y\,dx$ and $\int_0^1 x^3 y\,dy$

Solution: $\int_0^1 x^3 y\,dx = y\int_0^1 x^3\,dx = \frac{x^4 y}{4}\Big|_{x=0}^{1} = \frac{y}{4}$

$\int_0^1 x^3 y\,dx = x^3 \int y\,dx = \frac{x^3 y^2}{2}\Big|_{y=0}^{1} = \frac{x^3}{2}$

Furthermore ~~we~~ ~~we~~ a partial with respect to x can be subsequently integrated with respect to y, or vice versa.

This is called iterated integration and one of the means by which we perform double integrals, called iterated integrals

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$
$$= \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

2.

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$\int_a^b \int_c^d f(x,y)\,dy\,dx$

$= \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$

Example: Calculate $\int_1^2 \int_0^3 (x^2y - 2y)\,dx\,dy$ and $\int_0^3 \int_1^2 (x^2y - 2y)\,dy\,dx$

Solution: $\int_1^2 \int_0^3 (x^2y-2y)\,dx\,dy = \int_1^2 \left[\int_0^3 (x^2y-2y)\,dx\right] dy$

$= \int_1^2 \left[\frac{x^3y}{3} - 2xy\right]_{x=0}^{3} dy$

$= \int_1^2 9y - 6y - (0-0)\,dy$

$= \int_1^2 3y\,dy$

$= \frac{3y^2}{2}\Big|_{y=1}^{2}$

$\frac{3(2)^2}{2} - \frac{3(1)^2}{2} = \frac{9}{2}$

$\int_0^3 \int_1^2 x^2y - 2y\,dy\,dx = \int_0^3 \left[\int_1^2 x^2y - 2y\,dy\right] dx$

$= \int_0^3 \left[\frac{x^2y^2}{2} - y^2\right]_{y=1}^{2} dx$

$= \int_0^3 \left[2x^2 - 4 - \left(\frac{x^2}{2} - 1\right)\right] dx$

$= \int_0^3 \left[\frac{3}{2}x^2 - 3\right] dx$

$= \frac{x^3}{2} - 3x\Big|_{x=0}^{3}$

$\left(\frac{27}{2} - 9\right) - (0-0) = \frac{9}{2}$

The fact that these are the same is due to Fubini's theorem

Let R be the rectangle defined by $a \le x \le b$, $c \le y \le d$. If $f(x,y)$ is continuous on the rectangle, then $\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$
$= \int_a^b \int_c^d f(x,y)\,dy\,dx$

So for rectangular regions only we can exchange the order of integration

3.

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Example: Find the volume enclosed between the surface
$z = 4-x-y$ and the region $R = [0,1] \times [0,2]$
$f(x,y) = $ " "

or $R = \{(x,y): 0 \le x \le 1,\ 0 \le y \le 2\}$
$V = \iint_R f(x,y)\,dA = \iint z\,dA$

Solution: $\int_0^1 \int_0^2 f(x,y)\,dy\,dx$
$= \int_0^1 \left[\int_0^2 (4-x-y)\,dy\right] dx$

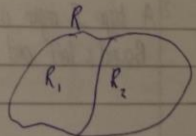
$= \int_0^1 \left[4y - x - \frac{y^2}{2} \Big|_{y=0}^{2} \right] dx$

$= \int_0^1 (8-2x-2)\,dx$
$= \int_0^1 (6-2x)\,dx$
$= 6x - x^2 \Big|_{x=0}^{1}$
$6(1)-(1^2) - (6(0)-(0^2))$
$= 5$ units^3

Verify: $\int_0^2 \left[\int_0^1 (4-x-y)\,dx\right] dy$ verify result is the same if you change the order of
$= \int_0^2 \left[4x - \frac{x^2}{2} - y \Big|_{x=0} \right] dy$ ← integration ←

Properties of Double integration
1 $\iint_R c f(x,y)\,dA = c \iint_R f(x,y)\,dA$ where c is a constant

2 $\iint_R (f(x,y) \pm g(x,y))\,dA = \iint_R (f(x,y))\,dA \pm \iint_R (g(x,y))\,dA$



3 If R_1 and R_2 are regions such that $R = R_1 + R_2$, then eg
$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

Example of (3) $R_1 = \{(x,y): 0 \le x \le 1,\ 0 \le y \le 2)$
$R_2 = \{(x,y): 1 \le x \le 0,\ 0 \le y \le 2)$
Integrate $f(x,y) = x$ over these regions.
$\iint_{R_1} f(x,y) =$

4

$= \int_0^1 \int_0^2 x \, dy dx$

$= \int_0^1 xy |_0^2 \, dx \quad = \int_0^1 2x dx = x^2 |_0^1 \quad = 1$

$\iint_{R_2} f(x,y) dA \quad = \int_1^2 \int_0^2 x \, dy dx$

= Same g range = $\int_1^2 2x \, dx = x^2 |_{x=1}^2 \quad = 4-1 = 3$

Sum of these = $\boxed{4.}$

Instead we use region R $\{(x,y): 0 \leq x \leq 2, 0 \leq y \leq 2\}$

$\iint_R f(x,y) dA \quad = \int_0^2 \int_0^2 x \, dy dx = \int_0^2 2x dx \; = x^2 |_{x=0}^2 \quad = 4-0$

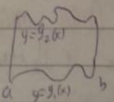
$= \boxed{4}$

DOUBLE INTEGRALS OVER NON RECTANGLE REGIONS

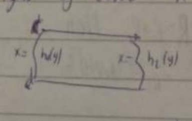
In general we can integrate over a region of arbitrary complexity.

We will consider two types of region.

1. A type I region is bounded to the left and right by vertical lines $x = a$ and $x = b$, And is bound above and below by curves $y = g_1(x)$, $y = g_2(x)$



2. A type II region is bounded below and above by horizontal lines, $y = c$, and $y = d$. Bounded left and right by curves $x = h_1(y)$ and $y = h_2(y)$



In terms of integrals, this Type I $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) dy \, dx$ y first

Type II $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) dx \, dy$

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Non rectangular Regions

Type I region $\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$.

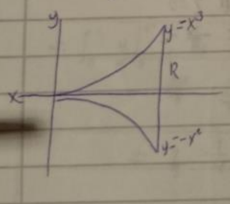
Type II region $\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

In one variable, the limits depend on the other variable

So in a type I region we must integrate with respect to x first since the limits of y depend on x.

Some regions can be both type I and type II regions
For example integrating over the region $x^2+y^2 \leq a^2$ we integrate it as a type I or type II region. $y = \pm\sqrt{x^2-a^2}$ as a limit or $x = \pm\sqrt{a^2-y^2}$ as a limit

Example: Compute the volume of the solid bounded above by $z = xy^2$ and below by R between $y = x^3$ and $y = -x^2$ for $0 \leq x \leq 1$



Solution: This is a type I region

$V = \iint_R xy^2\,dA$

$= \int_0^1 \left[\int_{-x^2}^{x^3} xy^2\,dy \right] dx$

$= \int_0^1 \left[\frac{xy^3}{3} \Big|_{y=-x^2}^{x^3} \right] dx$

$= \int_0^1 \left[\frac{x}{3} (x^9 - (-x^6)) \right] dx$

$\frac{1}{3} \int_0^1 [x^{10} + x^7]\,dx$

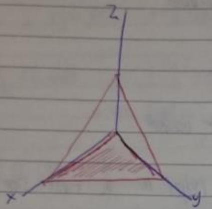
$= \frac{1}{3} \left[\frac{x^{11}}{11} + \frac{x^8}{8} \Big|_{x=0}^{1} \right]$

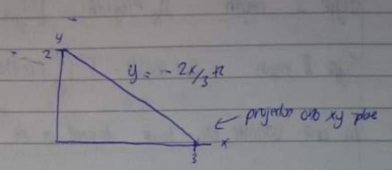
$= \frac{1}{3} \left[\frac{1}{11} + \frac{1}{8} - 0 \right]$

$= \frac{19}{264}$

2

Example: Find the volume of the tetrahedron bounded by the coordinate plane and the plane $z = 6-2x-3y$





Solution: We can treat this as a type I or type II region
Look at the projection. In the xy plane, $z=0$ so we have $0=6-2x-3y$

To find the intersection with the x-axis, set $y=0$

$0 = 6-2x-0 \Rightarrow x=3$

Since $x \geq 0$ (bounded by the coordinate plane) and so $0 \leq x \leq 3$

Again for y, $y \geq 0$ but the upper limit comes from the eqn, $z=0$

$\Rightarrow 6-2x = 3y \Rightarrow y = -\frac{2x}{3}+2 \Rightarrow 0 < y \leq -\frac{2x}{3}+2$

$V = \iint_R (6-2x-3y)\,dA$

$\int_0^3 \int_0^{-2x/3+2} (6-2x-3y)\,dy\,dx$

$= \int_0^3 \left[6y-2xy-\frac{3}{2}y^2\right]_0^{-\frac{2x}{3}+2} dx$

$= \int_0^3 \left[\frac{6}{-\frac{2x}{3}+2} - 2x\left(\frac{-2x}{3}+2\right) - \frac{2}{3}\left(\frac{-2x}{3}+2\right)^2\right] dx$

$= \int_0^3 \left[6-4x+\frac{2x^2}{3}\right] dx$

$= 6x - 2x^2 + \frac{2x^3}{9}\Big|_{x=0}^{3}$

$= 18 - 18 + \frac{2(27)}{9} - 0$

$= 6$

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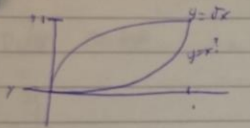
Area As a Double Integral

If we are given a surface $z=f(x,y)$, we can set $z=1$ and then
$V = A.h = A\cdot 1 = A =$

$\iint 1.dA$ = area of R

Area of $R = \iint_R dA$

Example: Calculate the area between the curve $y=x^3$ and $y=\sqrt{x}$ for $0 \le x \le 1$



Solution Since $0 \le x \le 1$ $\sqrt{x} \geq x^3$

$$A = \iint_R dA = \int_0^1 \int_{x^3}^{\sqrt{x}} dy\,dx$$

$$= \int_0^1 y \Big|_{y=x^3}^{\sqrt{x}} dx = \int_0^1 (\sqrt{x} - x^3)\,dx$$

$$= \frac{2x^{3/2}}{3} - \frac{x^4}{4}\Big|_{x=0}^{1}$$

$$= \frac{2}{3} - \frac{1}{4} - 0 = \frac{5}{12}$$

Parametric Surfaces:

Recall that a parametric curve has equation $x=x(t), y=y(t), z=z(t)$ (one parameter t)

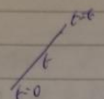
It is also possible to have a parametric surface, which has two parameters, u and v
Because it is a surface there is two independent directions, so we need two variables
This will have equation

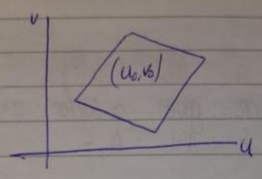
$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v)$$

We still need a reference point, which gives initial position of u and v

u

curve

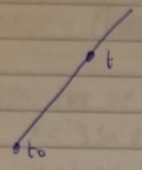


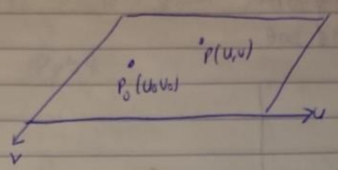


Multi

1.

16/10/13 **Parametric Surfaces**



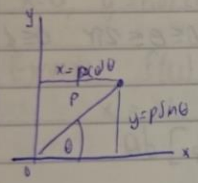


Some useful change of variables:

Rectangular Coordinates: (or cartesian coordinates)

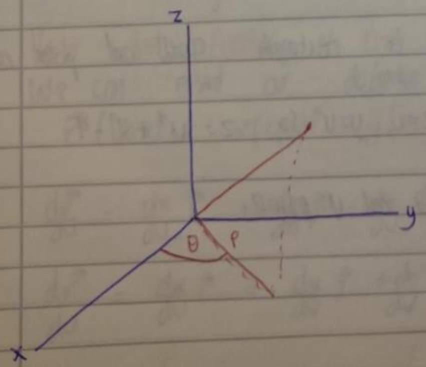
$x = u, \quad y = v$

Polar coordinates $\quad x = \rho\cos\theta \quad y = \rho\sin\theta$



Cylindrical Coords:

$x = \rho\cos\theta \quad y = \rho\sin\theta \quad z$



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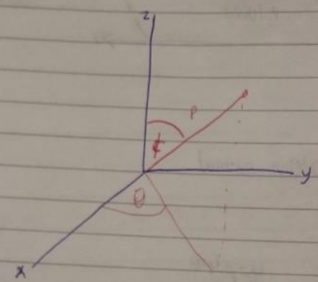
Spherical coords:

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$



The ranges of the parameters are $\rho \geq 0$, $0 \leq \theta \leq 2\pi$ $0 \leq \phi \leq \pi$

In polar coords, a double integral is

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \left[\int_{\rho_1(\theta)}^{\rho_2(\theta)} f(\rho,\theta)\rho\,d\rho \right] d\theta$$

Note that an extra ρ appears which we explain later when we discuss Jacobians

The limits are determined from the definition of polar coords

Example: Change $z = x^2+y^2-9$ to rectangular coords and polar coords

Solution: In rectangular coords $x=u$, $y=v$ so $z = u^2+v^2-9$

In polar coords $x=\rho\cos\theta$ and $y=\rho\sin\theta$

$$z = \rho^2\cos^2\theta + \rho^2\sin^2\theta - 9$$
$$= \rho^2 - 9$$

Example

3.

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Example: Change $x^2+y^2+z^2=9$ to Spherical polar coord

Solution: Here we have $\rho=3 \Rightarrow \rho^2=9$

$x = 3\sin\phi\cos\theta$

$y = 3\sin\phi\sin\theta$

$z = 3\cos\phi$

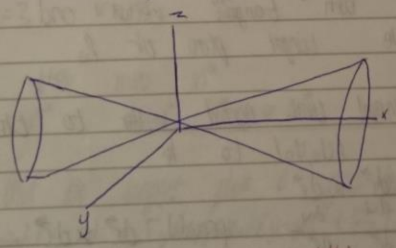
Surfaces of Revolution

If we take a curve $y=f(x)$ and rotate it about the x axis we get a surface in three dimensions

$x=u, \quad y=f(u)\cos v \quad z=f(u)\sin v$

Example $f(u)=v$ we get a double - napped cone

$x=u, \quad y=u\cos v \quad z=u\sin v$



Vector - valued Function of two variables

We can extend our definition of vvf to two parameters

$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$

$\frac{d\vec{r}}{du} = \frac{dx}{du}\vec{i} + \frac{dy}{du}\vec{j} + \frac{dz}{du}\vec{k}$

$\frac{d\vec{r}}{dv} = \frac{dx}{dv}\vec{i} + \frac{dy}{dv}\vec{j} + \frac{dz}{dv}\vec{k}$

4.

Tangent plane to Parametric Surface

Let σ be a parametric surface in three dimensions. A plane is tangent to σ at P_0 if a line through P_0 lies in the plane if and only if it is a tangent line to σ at P_0.

Let $\vec{r}(u,v)$ be a curve on the parametric surface σ and $P_0 = (a,b,c)$ a point on σ with $a = x(u_0, v_0)$,
$b = y(u_0, v_0)$, $c = z(u_0 v_0)$

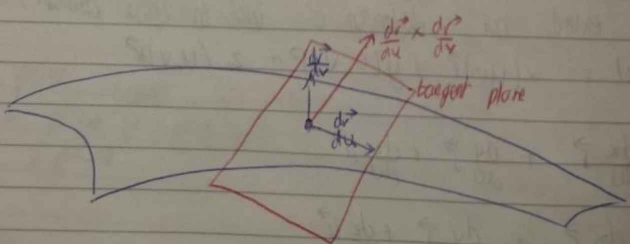
Then: If $\frac{d\vec{r}}{du} \neq 0$, it is tangent to the constant v-curve

If $\frac{d\vec{r}}{dv} \neq 0$ it is tangent to the constant u-curve.

From this, we see $\frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} = 0$ at P_0, then it is orthogonal to both tangent vectors and therefore it is normal to the tangent plane at P_0

We define principal unit normal vector to the surface $\vec{r}(u,v)$ at (u_0, v_0) to be

$$\vec{n} = \frac{\frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv}}{\| \frac{d\vec{r}}{du} + \frac{d\vec{r}}{dv}} \quad \text{provided } \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} \neq 0$$



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Example: Find the equation of the tangent plane for the surface:

$\vec{r}(u,v) = u\vec{i} + 2v^2\vec{j} + (u^2+v^3)\vec{k}$
at the point $(2,2,3)$

Solution: First we find the normal vector

$\frac{d\vec{r}}{du} = \vec{i} + 2u\vec{k}$ $\frac{d\vec{r}}{dv} = 4v\vec{j} + \vec{k}$

$\Rightarrow \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} = -8uv\vec{i} - \vec{j} + 4v\vec{k}$

At the point $(2,2,3)$
$u=2,$ $2v^2=2$ $u^2+v=3$
$\Rightarrow u=2,$ $v=-1$

Substitute these into $\vec{n}$
$\Rightarrow$ $\vec{n} = 16\vec{i} - \vec{j} + 4\vec{k}$

Recall that the tangent plane is given by
$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$

$\Rightarrow$ $16(x-2) + (-1)(y-2) - 4(z-2) = 0$
$16x - y - 4z = 18$

Note that we can use any normal vector to find the plane. Normalising will divide the entire equation by a constant which doesn't change it

2

Summary $S = \sum_{k=1}^{n} \|\frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv}\| \Delta A_k$

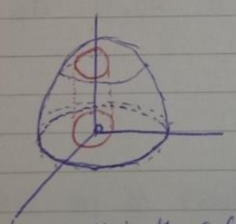
Taking the limit, we get the surface area of a parametric $\vec{r}(u,v)$ over a region R.

$$S = \iint_R dS = \iint_R \|\frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv}\| dA$$

The limits of the integrals come from R.

Example: Find the surface area of the sphere of radius 4 that lies in the cylinder above the xy-plane with base $x^2+y^2=12$

Solution: The sphere has equation $x^2+y^2+z^2=16$



We need to parameterise the surface. We introduce spherical coordinates and on the surface of the sphere $\rho=4$.

$$r(\theta,\phi) = 4\sin\phi\cos\theta\,\vec{i} + 4\sin\phi\sin\theta\,\vec{j} + 4\cos\phi\,\vec{k}$$

We need to find the range of θ and ϕ. θ goes the whole way around the so $0 \le \theta \le 2\pi$ but ϕ runs from the z axis to the intersection of the sphere and cylinder above the xy-plane

On the surface of the cylinder $x^2+y^2=12$

$$\underbrace{x^2+y^2}_{=12}+z^2=16$$

$$12+z^2=16 \qquad z^2=4 \quad z=\pm 2$$

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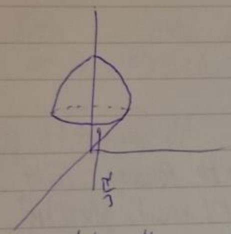
Maths.

So $4\cos\phi = \pm 2$

$\cos\phi = \pm\frac{1}{2}$

$\cos\phi = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ but $\frac{2\pi}{3}$ is below the xy plane

$0 \leq \phi \leq \pi/3$



We now take the derivative of $\vec{r}$

$\vec{r_\theta} = -4\sin\phi\sin\theta\vec{i} + 4\sin\phi\cos\theta\vec{j}$

$\vec{r_\phi} = 4\cos\phi\cos\theta\vec{i} + 4\cos\phi\cos\theta\vec{j} - 4\sin\phi\vec{k}$

The cross product is

$\vec{r_\theta} \times \vec{r_\phi} = = 16\sin^2\phi\cos\theta\vec{i}$

$-16\sin^2\phi\sin\theta\vec{j}$

$+(-16\sin\phi\cos\phi\sin^2\theta - 16\sin\phi\cos\phi + \cos^2\theta\vec{k})$

$= -16\sin^2\phi\cos\theta\vec{i} - 16\sin^2\phi\sin\theta - 16\sin\phi\cos\phi\vec{k}$

$\|\vec{r_\theta} \times \vec{r_\phi}\| = 16\sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi}\cos^2\phi$

$= 16\sqrt{\sin^2\phi} = 16|\sin\phi|$

But for the region we consider $\sin\phi \geq 0$ so $|\sin\phi| = \sin\phi$

The Surface area is then given by

$S = \iint_R 16\sin\phi\, dA$

$= \int_0^{2\pi}\left[\int_0^{\pi/3} 16\sin\phi\, d\phi\right]d\theta$

u

$$= \int_0^{2\pi} \left[-16\cos\phi\right]_{\phi=0}^{\pi/3} d\theta$$

$$= \int_0^{\pi} \left[-16\cos\tfrac{\pi}{3} + 16\cos 0\right] d\theta$$

$$= \int_0^{2\pi} \left[-16(\tfrac{1}{2}-1)\right] d\theta$$

$$= \int_0^{2\pi} 8\, d\theta = 8\theta\Big|_0^{2\pi}$$

$$= 16\pi$$

Surface area for $z = f(x,y)$

If our surface can be written in the following form

$$\vec{r}(x,y) = x(u,v)\vec{i} + y(u,v)\vec{j} + f(x(u,v), y(u,v))\vec{k}$$

and can parameterize $x = u$, $y = v$ we can write

$$\vec{r}(u,v) = u\vec{i} + v\vec{j} + f(u,v)\vec{k}$$

We now have derivatives

$$\vec{r}_u = \vec{i} + \frac{dz}{du}\vec{k}$$

$$\vec{r}_v = \vec{j} + \frac{dz}{dv}\vec{k}$$

The cross product gives $\vec{r}_u \times \vec{r}_v = \vec{k} - \frac{dz}{du}\vec{i} - \frac{dz}{dv}\vec{j}$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{1 + \left(\frac{dz}{du}\right)^2 + \left(\frac{dz}{dv}\right)^2}$$

But, since $x = u$, $y = v$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2}$$

$$\Rightarrow \quad S = \iint_R \sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2}\, dA.$$

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Maths

5.

Example: Find the surface area of the paraboloid $z=x^2+y^2$ below the plane $z=1$.

Solution

$z \geq 0$ because it's the sum of squares

The Region R comes from the intersection of the plane $z=1$ with surface and so $x^2+y^2=1$

$$S=\iint_R \sqrt{1+4x^2+4y^2}\,dA$$

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Example: Find the surface area of the paraboloid $z=x^2+y$ below the plane $z=1$.

Solution: PREVIOUS SHEET →

$$S = \iint \sqrt{1+4x^2+4y^2}\, dA$$

It's easier if we use polar coordinates $x=\rho\cos\theta$ $y=\rho\sin\theta$

$\Rightarrow z=\rho^2$ $(\rho\cos\theta)^2=\rho^2$ $\Rightarrow 0\le\rho\le1$

$$S = \int_0^{2\pi}\left[\int_0^1 \sqrt{1+4\rho^2}\,\rho\, d\rho\right]d\theta$$

$$= 2\pi\int_0^1 \sqrt{1+4\rho^2}\, d\rho$$

$t=\rho^2 \Rightarrow dt = 2\rho d\rho$

$$S = 2\pi\int_0^1 \sqrt{1+4t}\,\frac{dt}{2}$$

$$= 2\pi\left(\frac{1}{2}\right)(1+4t)^{3/2}\left(\frac{1}{4}\right)\Big|_{t=0}^{1}$$

$$= 2\pi\,\frac{1}{12}(5^{3/2}-1) = \frac{\pi}{6}(5\sqrt{5}-1) \approx 5.33$$

only works with $z=f(x,y)$

Lamina

A lamina is a region of space with mass m and a variable density $\delta(x,y)$. The mass is given by

$$m = \iint_R \delta(x,y)\, dA$$

The centre of mass/centre of gravity of the lamina is $(\bar{x},\bar{y})$ where $\bar{x} = \frac{1}{m}\iint_R x\,\delta(x,y)\,dA$

$$\bar{y} = \frac{1}{m}\iint_R y\,\delta(x,y)\,dA$$

For a lamina with constant density, the centre of gravity is called the centroid.

This is the same as finding the centre of mass of an object with non-uniform mass distribution.

Example: Compute the mass and centre of mass of the lamina with density $\delta(x,y) = x^2+y^2$ inside the unit circle.

Solution: $M = \iint_R (x^2+y^2)\,dA \overset{\text{polar coord}}{=} \int_0^{2\pi}\int_0^1 \rho^2 \cdot \rho \,d\rho\, d\theta$ ← unit circle

$2\pi \int_0^1 \rho^3 d\rho = 2\pi \frac{\rho^4}{4}\Big|_{\rho=0}^{1} = \frac{\pi}{2} = M$

$\bar{x} = \frac{1}{M} \iint_R x(x^2+y^2)\,dA$ $\qquad x = \rho\cos\theta \quad y = \rho\sin\theta$

$= \frac{2}{\pi} \int_0^{2\pi}\int_0^1 (\rho\cos\theta)(\rho^2)\,\rho\, d\rho\, d\theta$

$= \frac{2}{\pi}\int_0^{2\pi}\int_0^1 \rho^4 \cos\theta \,d\rho\, d\theta$

$= \frac{2}{\pi}\int_0^{2\pi} \frac{\rho^5}{5}\cos\theta\Big|_{\rho=0}^{1} d\theta$

$= \frac{2}{\pi}\int_0^{2\pi} \frac{1}{5}\cos\theta\, d\theta$

Aside: $\int_0^{2\pi}\cos\theta\, d\theta = \int_0^{2\pi}\sin\theta\, d\theta = 0$ $\qquad \cos(\theta+2\pi) = \cos\theta$, $\sin(\theta+2\pi) = \sin\theta$

$\Rightarrow \frac{2}{\pi}\frac{1}{5}(-\sin\theta)\Big|_{\theta=0}^{2\pi}$

$\frac{2}{5\pi}(\underbrace{-\sin 2\pi}_{=0} + \underbrace{\sin 0}_{=0}) = 0$

$\bar{y} = \frac{1}{M}\iint \underset{\rho\sin\theta}{y}(x^2+y^2)\,dA = 0$