Maths Paper 1 2014

1 a $g \circ f(x) = g(f(x)) = \sin(x^3) = \sin x^3$ ✓

$f \circ g(x) = f(g(x)) = (\sin x)^3$ ✓

b If two functions f and g satisfy $g(f(x)) = x$ for all $x \in D(f)$
and if $f(g(x)) = x$ for all $x \in D(g)$
then f and g are inverse functions ✓

c $y = \frac{2}{x-1}$ $x = \frac{2}{y-1}$

$y-1(x) = 2$

$y-1 = \frac{2}{x}$

$y = \frac{2}{x} + 1 = f^{-1}(x)$ ✓

d For function f(x) to be continuous at a the following three conditions must occur
- The function f(x) is defined at a. ✓
- The limit exists of f(x) as x approaches a
- The value of the function and the value of the limit a are the same

e. $|x| = \begin{cases} x \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -x \text{ if } x < 0 \end{cases}$

Clearly on interval $(-\infty, 0), (0, +\infty)$
Must show continuous at $x = a$ $(x=0)$

$\lim_{x \to 0} |x| = f(0) = 0$

$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$ $\frac{|h|}{h} = +1$ ✓

$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} -x = 0$ $\frac{h}{h} = -1$

$|x|$ is ~~continuous~~ at $x=0$

dNE

2 a i $y = xe^x \ln x \sin x$ $\qquad \frac{d}{dx} \ln x \sin x \quad \ln x \cos x + \frac{\sin x}{x}$

$u \qquad v$

$xe^x \qquad \ln x \sin x \qquad\qquad xe^x \quad xe^x + e^x$

$u\frac{dv}{dx} + v\frac{du}{dx}$

$xe^x\left(\ln x\cos x + \frac{\sin x}{x}\right) + \ln x \sin x\left(xe^x + e^x\right)$ ✓

ii. $y = \ln\left(\sin\sqrt{x^2+1}\right)$

$= \dfrac{\cos\left((x^2+1)^{\frac{1}{2}}\right)\frac{1}{2}(x^2+1)(2x)}{\sin\left(\sqrt{x^2+1}\right)}$ ✓

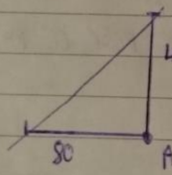
iii. $x^2 + x^2y^3 + xy = 1$

$2x + 3x^2y^2\frac{dy}{dx} + 2xy^3 + x\frac{dy}{dx} + 1 = 0$

$\frac{dy}{dx}\left(3x^2y^2 + x\right) = -1 - 2xy^3 - 2x$

$\frac{dy}{dx} = \dfrac{-1-2xy^3-2x}{3x^2y^2+x}$ ✓

2b



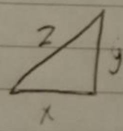
$\frac{dy}{dt} = \frac{45}{80} \qquad \frac{dy}{dt} = \frac{90}{160}$

$3(\text{hours}) = \frac{135}{240} = 0.5625$

2c ~~$y = x^4 - 4x^3 + 4x^2$ max/min $4x^3 + 12x^2 + 8x = 0$~~

$4x(x^2+3x+2)$

2h



At $t = 3 \qquad z^2 = y^2 + x^2$

$2z\frac{dz}{dt} = 2y\frac{dy}{dt} + 2x\frac{dx}{dt}$ $\qquad \frac{dx}{dt} = 80 \quad \frac{dy}{dt} = 45$

at $t = 3$ $x = 240$ $y = 135$

$z = \sqrt{x^2+y^2} = 275.36$

$2(275.36)\frac{dz}{dt} = 240(80) + 135(45$

$= 25275$

$\frac{dz}{dt} = 91.7884$ km/hr.

Marks 1 2009

$x = 26$ $\quad f(x) = \sqrt[3]{x}$

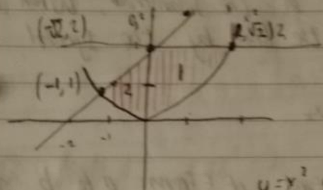
3a $\quad x_0 = 27$ $\quad f'(x) = \dfrac{1}{3(x)^{2/3}}$

$f(x) = f(x_0) + f'(x_0)(x - x_0)$

$\sqrt[3]{27} + \dfrac{1}{3(27)^{2/3}}(26-27)$

$3 + \left(\dfrac{-1}{27}\right) = \dfrac{80}{27} = 2.962962962$ ✓

b



$y = x^2 \quad 2 = x^2$

$y = 2 \quad x = \sqrt{2}$

$y = x^2$ and $y = x+2$ $\quad$ area =

$x^2 = x+2$

$x^2 - x - 2 = 0$ $\quad \int_{-1}^{0} x+2-x^2\,dx$ +

$(x+1)(x-2)$

area 1 $= \int_0^{\sqrt{2}} 2\,dx \quad \int_0^{\sqrt{2}} x^2\,dx$ $\quad x = -1 \quad x = 2$ $\quad \int_0^{\sqrt{2}} 2-x^2\,dx$

$[2x]_0^{\sqrt{2}} - \left[\dfrac{x^3}{3}\right]_0^{\sqrt{2}}$ $\quad = \dfrac{7}{6} + 2\sqrt{2} - \dfrac{2\sqrt{2}}{3}$

$2\sqrt{2} - \dfrac{2\sqrt{2}}{3} = \dfrac{4\sqrt{2}}{3} + \dfrac{7}{6} = 3.05228475$

area 2 $= \int_{-1}^{0} x+2 - \int_{-1}^{0} x^2$

$\int_{-1}^{0} \dfrac{x^2}{2} + 2x - \int_{-1}^{0} \dfrac{x^3}{3}$

$\dfrac{1}{2} + 2(-1) = -1.5 \quad + \left(-\dfrac{1}{3}\right)$

$= +1.5 - \frac{1}{3} = 7/6$

c. $\int_1^3 x^3\sqrt{x^4+4}$ $\quad u = x^4+4$

$dx = 4x^3\,du$

$\dfrac{1}{4}\int \sqrt{u}$

$\dfrac{1}{4}\dfrac{u^{3/2}}{3/2} = \dfrac{2u^{3/2}}{12} = \left[\dfrac{(x^4+9)^{3/2}}{6}\right]_1^3$ $\quad 15 - 5/3 = \dfrac{40}{3}$

$= \dfrac{1}{6}\left[(90)^{3/2} - (1)^{3/2}\right]$

3i A function f is said to be integrable on a finite closed interval $[a, b]$ if the limit

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

exists and does not depend on the choice of partitions or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

which is called the definite integral of f from a to b. The numbers a and b are called the lower limit of integration and the upper limit of integration. $f(x)$ is called the integrand

c Part one: If f is continuous on an interval $[a, b]$ and F is any anti-derivative then

$$\int_a^b f(x)dx = F(A) - F(B)$$

Part two: If f is continuous on an interval, f has a antiderivative on that interval. In particular, if a any point in the interval then the funct F defined by

$$F(x) = \int_a^x f(t)dt$$ is and antiderivative of f.

That is $F'(x) = f(x)$, for each x in the interval

i.e.

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

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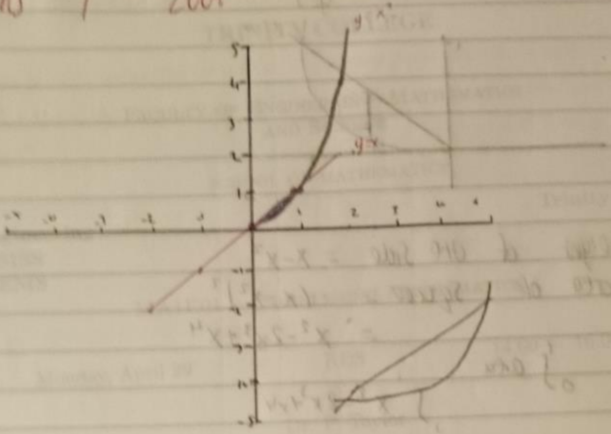
4b



The height of $x - x^2$ so we have area above the $= (x-x^2)^2$

volume $= \int_0^1 (x-x^2)^2\,dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{1}{30}$

$$\cancel{\int_0^1 \pi \left((x^2)^2 - (x)^2\right) dx}$$
$$\cancel{\pi \int_0^1 x^4 - x^2\, dx}$$
$$\cancel{= \pi \left[\frac{x^5}{5} - \frac{x^3}{3}\right]_0^1}$$
$$\cancel{\frac{1}{5} - \frac{1}{3}}$$

$$\pi \int_0^1 (\sqrt{y})^2 - (y^2)\,dy$$
$$\pi \int_0^1 y - y^2$$
$$= \pi \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$
$$\pi \left[\frac{1}{2} - \frac{1}{3}\right]$$

By Slicing!

$$= \frac{1}{6}\pi \text{ units}$$

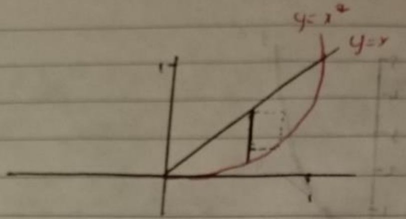
4d $S = \int_0^4 f(y)\, 2\pi \sqrt{1 + (f'(y))^2}$ $\quad y = x^2 \quad x = \sqrt{2} \quad x = y^{1/2}$

$2\pi \int_0^2 \sqrt{y} \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$ $\quad u = 4y+1 \quad du = 4dy \quad \frac{1}{2\sqrt{y}}$

$\pi \int \sqrt{4y+1}\, dy$

$= \frac{\pi}{4}\left(\frac{2}{3}u^{3/2}\right) = \frac{\pi}{4}\left(\frac{2}{3}(4y+1)^{3/2}\right) = \frac{9}{2}\pi - \frac{1}{6}\pi = \frac{13}{3}\pi$

4a



height of one side $= x - x^2$

area of square $= (x - x^2)^2$

$= x^2 - 2x^3 + x^4$

$\int_0^1$ area

$\int_0^1 x^2 - 2x^3 + x^4$

$= \left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1$

$= \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{1}{30}$ units?

4c length $= \int_0^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_0^1 \sqrt{1 + \left(\frac{3}{2}(x)^{\frac{1}{2}}\right)^2}$

$\int_0^1 \sqrt{1 + \frac{9}{4}x}\ dx$ $\qquad u = 1 + \frac{9}{4}x$

$du = \frac{9}{4}dx$

$= \left[\frac{3\left(1 + \frac{9}{4}x\right)^{3/2}}{2}\right]_0^1$ $\qquad \frac{4}{9}\int_0^1 \sqrt{u}$

$8.7885 - \frac{3}{2}$ $\qquad \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]$

7.2885

$= \frac{8}{27}\left[\left(1 + \frac{9}{4}\right)^{3/2} - 1\right]$