1. Sensitivity Analysis Simplex

Study of how changes in the coefficients of a LP affect the optimal solution

Objective Function coefficients

Sensitivity analysis for an objective function coef involves placing a range on the coef's value called the range of optimality

As long as the actual value of the objective function coefficient is within the range of optimality, the current basic feasible solution will remain optimal

The range of optimality for a basic variable defines the objective function coef values for which that variable will remain part of the current O BFS.

The range of optimality for a non basic variable defines the objective function coef values for which that variable remains non basic

In computing the range, only one coef is allowed to change at a time

Example hightech problem. Max $50x_1 + 40x_2$ $\quad x_1$ = desktop

ST $\quad 3x_1 + 5x_2 \le 150$ assembly time $\quad x_2$ = portable

$1x_2 \le 20$ portable display

$8x_1 + 5x_2 \le 300$ warehouse cap $\quad x_1, x_2 \ge 0$

Final Simplex tableau		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	50	40	0	0	0	
x_2	40	0	1	8/25	0	-3/25	12
s_2	0	0	0	-8/25	1	3/25	8
x_1	50	1	0	-5/25	0	5/25	30
z_j		50	40	14/5	0	26/5	1980
$c_j - z_j$		0	0	-14/5	0	-26/5	

- Optimal Solution because all $(c_j - z_j) \leq 0$.
- However if a change in one of O.F. coefs were to cause one or more of $(c_j - z_j)$ values to become positive current Solution would no longer be optimal; more simplex iterations required to reach sol.

The range of optimality for an objective function coefficient is determined by those coefficient values that maintain

$$c_j - z_j \leq 0 \quad \text{for all values of } j.$$

Example: compute range of optimality for c_1, profit contribution per deskpro. Using c_1 instead of 50 for O.F. coef of x_1 we get:

Basis	c_B	x_1	x_2	s_1	s_2	s_3	
		c_1	40	0	0	0	
x_2	40	0	1	8/25	0	-3/25	[illegible]
s_2	0	0	0	-8/25	1	3/25	12
x_1	c_1	1	0	-5/25	0	5/25	8
z_j		c_1	40	$\frac{64-c_1}{5}$	0	$\frac{c_1-24}{5}$	30
$c_j - z_j$		0	0	$\frac{c_1-65}{5}$	0	$\frac{24-c_1}{5}$	$480+30c_1$

Current Solution will Remain optimal as long as values of c_1 result in all $c_j - z_j \leq 0$.

From column S_1 we have $\frac{c_1 - 64}{5} \leq 0$ ①

and from S_3 $\frac{24 - c_1}{5} \leq 0$ ②

rewriting ① we get $c_1 \leq 64$
rewriting ② we have $c_1 \geq 24$

Because c_1 must satisfy both ① and ② range of optimality for c_1 is:
$24 \leq c_1 \leq 64$

- Suppose an increase in material cost reduced the profit contribution per unit for the deshipe to £30 from £50.
- The range of optimality indicated that the current solution ($x_1 = 30$, $x_2 = 12$, $S_1 = 0$, $S_2 = 8$, $S_3 = 0$) is still optimal
- To verify this solution, let us recompute the final simplex after reducing value of c_1 to 30

we have simply set $c_1 = 30$ everywhere it appears in previous table

		x_1	x_2	S_1	S_2	S_3	
basis	c_B	30	40	0	0	0	
x_2	40	0	1	8/25	0	-3/25	12
S_2	0	0	0	-8/25	1	3/25	8
x_1	30	1	0	-5/25	0	5/25	30
z_j		30	40	34/5	0	6/5	~~1980~~ 1380
$c_j - z_j$		0	0	-34/5	0	-6/5	

~~Note that only changes are in S_1 column~~

Because $c_j - z_j \leq 0$ for all variables, solution $x_1 = 30$, $x_2 = 12$, $S_1 = 0$, $S_2 = 8$, $S_3 = 0$ is still optimal. That is, optimal solution with $c_1 = 30$ is the same as the optimal solution with $c_1 = 50$
- Note decrease in profit from 1980 to 1380

What is X_1 coef $C_1 = 20$? It is outside optimality range:

Basis	C_B	X_1 20	X_2 40	S_1 0	S_2 0	S_3 0	
X_2	40	0	1	8/25	0	-3/25	12
S_2	0	0	0	-8/25	1	3/25	8
X_1	20	1	0	-5/25	0	5/25	30
	Z_J	20	40	44/5	0	-4/5	1080
	$C_J - Z_J$	0	0	-44/5	0	4/5	

- Current sol ($x_1 = 30$, $x_2 = 12$, $S_1 = 0$, $S_2 = 8$, $S_3 = 0$) is no longer optimal because entry in $C_J - Z_J$ row for S_3 is ~~zero~~ >0
- (simplex) another iteration required, result is $X_1 = 16\frac{2}{3}$ and $X_2 = 20$

At the endpoints of the range, the corresponding variable is a candidate for entering the basis if it currently out or for leaving the basis if it is currently in it.

Procedure for computing range of optimality for C_1 can be used for any basic variable

- Procedure for nonbasic variables is easy because only corresponding column in $C_J - Z_J$ row is changed
- Example swap C_{S_1} for coef of S_1 in O.F.

Basis	C_B	X_1 50	X_2 40	S_1 C_{S_1}	S_2 0	S_3 0	
X_2	40	0	1	8/25	0	-3/25	12
S_2	0	0	0	-8/25	1	3/25	8
X_1	50	1	0	-5/25	0	5/25	30
Z_J		50	40	14/5	0	26/5	1980
$C_J - Z_J$		0	0	$C_{S_1} - 14/5$	0	-26/5	

$C_J - Z_J$ row with C_{S_1} = $C_{S_1} - 14/5 \le 0$

$\Rightarrow C_{S_1} \le 14/5$

5.

- Therefore, as long as O.F. coef for S_1 is less than or equal to 14/5 the current solution is optimal.
- With no lower bound on how much the coef may be decreased we write range of optimality for $C_{S_1} \le 14/5$

- Same approach works for all non-basic variables

- In a max problem the range of optimality has no lower limit and upper limit is given by Z_J.
- Thus range of optimality for the O.F coef of any non basic (=0) variable is given by $C_J \le Z_J$.

Assume computing range of optimality for C_K coef of X_K in a max problem. X_K may refer to one of original decision variable, a slack or surplus variable.

Steps to compute the range of optimality

1. Replace numerical value of OF coef ~~var~~ for X_K with C_K everywhere it appears in final simplex tableau
2. Recompute $C_J - Z_J$ for each non basic variable (if X_K is nonbasic only nessecery to recompute $C_K - Z_K$).
3. Requiring that $C_J - Z_K \le 0$, solve each inequality for any upper or lower bound on C_K. If 2 or more upper bounds found, smaller of these is upper bound on range of optimality. If 2 or more lower, larger is lower bound on range of optimality.
4. If original problem was min problem converted to max problem in order to simplex, multiply inequalities in step 3 by -1 and change direction of inequalities to obtain the ranges of optimality for the original min problem

Right hand Side Values

We can interpret the RHS (the b_i's) as resources available

The improvement in the value of the optimal solution per unit increase in a constraints rhs value is called a dual price (Shadow price) Found in Z_J row.

In hightech LP, Z_J values for 3 slack variables are $14/5$, 0, $26/5$

- Dual price for constraint 1, S_1, assembly time is €2.80
  S_2 - portable display = 0
  S_3 - warehouse capacity = €5.20

- Dual price of €5.20 shows us that more warehouse space will have biggest positive impact on hightech's profit

- Slack variables in BFO.S will have a Z_J value of 0 implying dual price of zero for corresponding constraint

- Consider slack variable S_2, a basic variable in solution, Since $S_2 = 8$, there is 8 units unused, management will pay zero for an extra unit of display because 8 left over.

- Nonbasic slack variable, S_1. We determined that current solution will remain optimal as long as OF coef for S_1 (C_{S_1}) stays in following range $C_{S_1} \leq 14/5$

- This implies that variable S_1 should not be increased from its current value of 0 unless it is worth more than $14/5$ = €2.80 to do so.
- If additional units are available, hightech should be willing to pay up to €2.80 per unit.
- Similar interpretation for other Z_J value of each of non basic slack variable

7. SA.

With (≥) constraint, value of dual price will be less than or equal to zero because a one unit increase in the value of the rhs cannot be helpful, makes it more difficult to satisfy constraint.

- For a max problem value of O.S is expected to ~~increase~~ decrease when the rhs of (≥) constraint is increased
- Dual price gives the amount of expected improvement - negative number (decrease).

- As a result, dual price for (≥) constraint is given by the negative of the z_j entry for corresponding surplus variable in the optimal simplex table.

constraint type	Dual price given by:
$\leq$	z_j value for slack variable associated with constraint.
$\geq$	negative of z_j value, for surplus variable associated with constraint.
$=$	z_j value for artificial variable associated with the constraint.

We ignore (=) constraint as artificial variable is dropped.

- We convert min problem to max by multiplying OF by -1.
- Dual price is given by the same z_j values because improvement for a min problem is a decrease in optimal value

Example

Min $2x + 3x$

ST: $1x_1 \geq 125$ demand for product A

$1x_1 + 1x_2 \geq 350$ total production

$2x_1 + 1x_2 \leq 600$ processing time

$x_1, x_2 \geq 0$

Basis	C_B	x_1	x_2	s_1	s_2	s_3	
		-2	-3	0	0	0	
x_1	-2	1	0	0	1	1	250
x_2	-3	0	1	0	-2	-1	100
s_1	0	0	0	1	1	1	125
z_j		-2	-3	0	4	1	-800
$c_j - z_j$		0	0	0	-4	-1	

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Constraint	Constraint type	dual price
Demand product A	$\geq$	0
Total production	$\geq$	-4
Processing time	$\leq$	1

- Constraint one not binding - dual price 0.
- Constraint 2, cost of increasing total production is £4 per unit.
- Constraint 3, per unit value of additional processing time is £1.

Range of feasibility (right hand side ranges)

- Z_j now can be used for dual price and as a result, predict the change in value of objective function corresponding to unit change in a b_i
- Valid as long as $b_i \geq 0$ and solution is feasible

- highlander problem, changing value time available from 150 to 160 hours. With dual price of £2.80 for assembly time, we expect increase in value of $10(2.80) = 28$.

$\Rightarrow$

Basis	C_B	X_1	X_2	S_1	S_2	S_3	
		50	40	0	0	0	
X_2	40	0	1	8/25	0	-3/25	15.2
S_1	0	0	0	-8/25	1	3/25	4.8
X_1	50	1	0	-5/25	0	5/25	28
Z_j		50	40	14/5	0	26/5	2008
$C_j - Z_j$		0	0	-14/5	0	-26/5	

Solution is feasible, $b_i \geq 0$. Only last column of simplex changed. Obtained by adding 10 times first four entries in S_1 column to the last column in tableau.

$$\text{New Solution} = \overset{\text{old soln}}{\begin{bmatrix} 12 \\ 8 \\ 30 \\ \hline 1980 \end{bmatrix}} + 10 \begin{bmatrix} 8/25 \\ -8/25 \\ -5/25 \\ \hline 14/5 \end{bmatrix} = \begin{bmatrix} 15.2 \\ 4.8 \\ 28.0 \\ 2008 \end{bmatrix}$$

- Coef in S_1 tell how many units of each of current basic variable will be driven out of solution if one unit of variable S_1 is brought into the solution

- Bringing one unit of S_1 into solution is the same as reducing the availability of assembly time (decreasing b_1) by one unit, increasing b_1 by one unit has opposite effect

- Entries in S_1 column can be interpreted as the changes in the value of the current basic variables corresponding to a one unit increase in b_1

- Change in value of OF corresponding to 1 unit increase in b_1 is given by value of z_j in that column (dual price)

Given a change in b_1 of Δb_1 the new values for the basic variables are in brackets:

$$\begin{bmatrix} x_2 \\ s_2 \\ x_1 \end{bmatrix} = \overset{\text{old}}{\begin{bmatrix} 12 \\ 8 \\ 30 \end{bmatrix}} + \Delta b_1 \begin{bmatrix} 8/25 \\ -8/25 \\ -5/25 \end{bmatrix} = \begin{bmatrix} 12 + 8/25\,\Delta b_1 \\ 8 - 8/25\,\Delta b_1 \\ 30 - 5/25\,\Delta b_1 \end{bmatrix}$$

As long as value of new basic variables ≥ 0, current basis will remain feasible and therefore optimal

$$12 + 8/25\,\Delta b_1 \geq 0$$
$$8 - 8/25\,\Delta b_1 \geq 0$$
$$30 - 5/25\, b_1 \geq 0$$

Solving for Δb_1 $\Rightarrow$

$$\Delta b_1 \geq -37.5$$
$$\Delta b_1 \leq 25$$
$$\Delta b_1 \leq 150$$

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Because all 3 inequalities must be satisfied, most restrictive limits on b_1 must be satisfied.

$$-37.5 \le b_1 \le 25$$

Initial amount of assembly time = 150 therefore $b_1 = 150 + \Delta b_1$

$\Rightarrow \quad 150 - 37.5 \le 150 + \Delta b_1 \le 150 + 25$

Replacing $150 + \Delta b_1$ with b_1 we get $112.5 \le b_1 \le 175$.

Range of feasibility indicates that as long as the availability assembly time is between 112.5 and 175 hours, the current optimal basis will remain feasible.

For $\le$ constraint RHS ≥ 0.

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} + \Delta b_i \begin{bmatrix} a_{1s} \\ a_{2s} \\ \vdots \\ a_{ms} \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

current solution

column of final Simplex corresponding to slack variable associated with constraint i $S_i = 1$.

For $\ge$ constraint with RHS ≥ 0

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix} - \Delta b_i \begin{bmatrix} a_{1s} \\ a_{2s} \\ \vdots \end{bmatrix} \ge \begin{bmatrix} 0 \\ \vdots \end{bmatrix}$$

current sol.

column corresponding to slack variable for constraint i.

Don't bother with $=$ constraint

Q2. Max $x_1 + 2x_2$

ST: $x_1 + 5x_2 \le 10$

$2x_1 + 6x_2 \le 16$ $\quad x_1, x_2 \geq 0$

Standard form $x_1 + 2x_2$

$x_1 + 5x_2 + S_1 = 10$

$2x_1 + 6x_2 + S_2 = 16$

		x_1	x_2	s_1	s_2		$\frac{b_i}{a_{is}}$
Basis	C_B	1	2	0	0		
S_1	0	1	5	1	0	10	$10/5 = 2$
S_2	0	2	6	0	1	16	$16/6 = 2.66$
Z_J		0	0	0	0	0	
$C_J - Z_J$		1	2	0	0		

Initial $S_1 = 10$, $S_2 = 14$ $\quad x_1 = 0$, $x_2 = 0$.

x_2 enters, S_1 leaves went 0. $\quad$ row 2 − 6 row 1

row ÷ 5.

		x_1	x_2	s_1	s_2		$\frac{b_i}{a_{is}}$
Basis	C_B	1	2	0	0		
x_2	2.	$1/5$	1	$1/5$	0	2	$2/0.2 = 10$ ✓ row 2 −6(row 1)
S_2	0	$4/5$	0	$-6/5$	1	4	$4/0.8 = 5$
Z_J.		$2/5$	2	$2/5$	0	4.	
$C_J - Z_J$.		$3/5$	0	$-2/5$	0		

$x_1 = 0$ $\quad x_2 = 2$ $\quad S_1 = 0$ $\quad S_2 = 4$ $\quad$ feasible ✓.

x_1 enters, S_2 leaves went 0

row 2 × 5/4 =

1/5

1 row − $\frac{1}{5}$ row 2.

		x_1	x_2	S_1	S_2	
Basis	C_B	1	2	0	0	
x_2	2	0	1	$1/10$	$-1/4$	3.
x_1	1	1	0	$-1/2$	$5/4$	5
Z_j		1	2	$-3/10$	$3/4$	11
$C_j - Z_j$		0	0	$-3/10$	$-3/4$	

$x_1 = 5$, $Y_2 = 3$, $S_1 = 0$, $S_2 = 0$ Final Solution

6.

Basis	C_B	X_1 5	X_2 20	X_3 25	S_1 0	S_2 0	S_3 0		b_i/a_{ij}
S_1	0	2	1	0	1	0	0	40	—
S_2	0	0	2	1	0	1	0	30	$\frac{30}{1} = 30$
S_3	0	3	1	$-\frac{1}{2}$	0	0	1	15	15/ —
Z_J		0	0	0	0	0	0	0	
$C_J - Z_J$		5	20	25	0	0	0		

C basis is $S_1 = 40$, $S_2 = 30$, $S_3 = 15$.
~~Does not correspond to the origin values Should be 0, 0, 0~~
S_1, S_2, S_3 is origin.

D value is zero ✓

e. X_3 enter, S_2 leave ✓

F. 30 units, value will be $30 \times 25 = 750$ ✓

row 3 + $\frac{1}{2}$ row 2.

g

Basis	C_B	X_1 5	X_2 20	X_3 25	S_1 0	S_2 0	S_3 0		b_i/a_{ij}
S_1	0	2	1	0	1	0	0	40	$40/2 = 20$
X_3	25	0	2	1	0	1	0	30	—
S_3	0.	3	2	0	0	$\frac{1}{2}$	1	30.	$30/3 = 10$
Z_J		0	50	25	0	25	0	750	
$C_J - Z_J$		5	-30	0	0	-25	0		

X_1 enters S_3 leave. want $\begin{matrix}0\\0\\1\end{matrix}$

divide row 3 by 3
row 1 − 2(row 3.

		x_1	x_2	x_3	s_1	s_2	s_3	
Basis	C_B	5	20	25	0	0	0	
s_1	0	0	$-1/3$	0	1	$-1/3$	$-2/3$	20
x_3	25	0	2	1	0	1	0	30
x_1	5	1	$2/3$	0	0	$1/6$	$1/3$	10
z_j		5	$160/3$	25	0	$155/6$	$5/3$	800
$c_j - z_j$		0	$-100/3$	0	0	$-155/6$	$-5/3$	

$x_1 = 10$, $x_2 = 0$, $x_3 = 30$, $s_1 = 20$, $s_2 = 0$, $s_3 = 0$ $p = 800$

Q11 Max $2x_1 + 8x_2$

ST: $3x_1 + 9x_2 \le 45$

$2x_1 + 1x_2 \ge 12$ $\quad x_1, x_2 \ge 0$

Standard Form $2x_1 + 8x_2$

ST: $3x_1 + 9x_2 + S_1 = 45$

$2x_1 + 1x_2 - S_2 + a_2 = 12$

$x_1, x_2, S_1, S_2, a_2 \ge 0$

		x_1	x_2	S_1	S_2	a_2		
Basis	C_B	2	8	0	0	$-M$		b/c_{ols}
S_1	0	3	9	1	0	0	45	$45/3 - 15$
a_2	$-M$	2	1	0	-1	1	12	$12/2 = 6$
Z_J		$-2M$	$-M$	0	M	$-M$	0	
$C_J - Z_J$		$2+2M$	$8+M$	0	$-M$	0		

X_1 enters, a_2 leaves. want $\begin{smallmatrix}0\\1\end{smallmatrix}$

row 2 $\div$ 2.

row 1 row 1 $- 3$(row 2)

note.

		x_1	x_2	S_1	S_2	a_2		
Basis	C_B	2	8	0	0	$-M$		
S_1	0	0	$15/2$	1	$+3/2$	$-3/2$	27	$27/15/2 = 3.6$
X_1	2	1	$1/2$	0	$-1/2$	$1/2$	6	$6/0.5 = 12$
Z_J		2	1	0	-1		12.	
$C_J - Z_J$		0	7	0	1			

X_2 enters, S_1 leaves want $\begin{smallmatrix}0\\1\end{smallmatrix}$

row 1 $\div$ $15/2$

row 2 $-$ $\frac{1}{2}$ row 1

		x_1	x_2	s_1	s_2	
Basis	C_B	2	8	0	0	
x_2	8	0	1	$15/2$	$1/5$	$18/5$
x_1	2	1	0	$-15/4$	$3/5$	$21/5$
Z_j		2	8	$105/2$	$14/5$	$186/5$
$C_j - Z_j$		0	0	$-105/2$	$-14/5$	

$x_1 = 21/5$, $x_2 = 18/5$ $s_1 = 0$, $s_2 = 0$

Simplex Method

- Can solve LP problems with 1000's of variables and constraints

Example

Hightec make 2 models: Deskpro and portable

- Deskpro profit = £50, Portable = 40
- max production hours = 150 hours
- Deskpro needs 3 hours, Portable 5 hours
- less than 20 units of Portable produced
- 300 space available, deskpro requires 8 portable 5.

X_1 = number of Deskpro's X_2 = number of portables

$$\text{Max } 50x_1 + 40x_2$$

$$\begin{aligned}
\text{ST.} \quad 3x_1 + 5x_2 &\le 150 \quad \text{Assembly time} \\
1x_2 &\le 20 \quad \text{portable display} \\
8x_1 + 5x_2 &\le 300 \quad \text{Warehouse capacity} \\
x_1, x_2 &\geq 0
\end{aligned}$$

Adding a slack variable: Note $\le$ changes to $=$

$$\begin{array}{lllll}
\text{Max} & 50x_1 + 40x_2 + 0S_1 + 0S_2 + 0S_3 & & & 17.1 \\
\text{ST:} & 3x_1 + 5x_2 + 1S_1 & = 150 & & 17.2 \\
 & 1x_2 + 1S_2 & = 20 & & 17.3 \\
 & 8x_1 + 5x_2 + 1S_3 & = 300 & & 17.4 \\
 & x_1, x_2, S_1, S_2, S_3 \geq 0 & & & 17.5
\end{array}$$

Whenever a system of simultaneous linear equations has more variables than equations, we can expect an infinite number of solutions

A basic solution can be obtained by selecting any two variables to equal 0, then solve for remaining variables

linear programming problem consisting of n variables and m linear equations.

Basic Solution: Set $n-m$ of the variables equal to zero and solve the m linear constraint equations for the remaining m variables

2.

We refer the n-m variables set equal to zero as the non basic variables and the remaining m variables as the basic variables.

Basic Feasible Solution.

- A basic feasible solution is a basic solution that also satisfies the
- non-negativity conditions.

e.g. set $x_1=0$ and $x_2=0$ $\Rightarrow$ $S_1=150$, $S_2=20$, $S_3=300$

- We have determined that the "hynkeh" problem does have an optimal basic feasible solution.
- Simplex method is an iterative procedure for moving from one basic feasible solution (extreme point) to another until the O.S. is reached

Tableau Form

- Purpose of tableau is to provide an initial basic feasible solution
- LP with all $\leq$ constraints solved by setting the decision variables $(x_1 x_2)$ equal to zero and solve for values of slack ~~constraints~~ variables.
- This sets slack variables equal to RHS value of the constraint ~~variable~~ equation.
- $x_1=0$ $x_2=0$ $S_1=150$, $S_2=20$, $S_3=300$ $\Rightarrow$ Initial basic feasible solution

Properties ① For each constraint eq^n the coef of one of the m basic variables in that eq^n must be 1, and coef of all remaining basic variables in eq^n must be 0.

②. The coef for each basic variable must be 1 in only one constraint eq^n

RHS must be non negative

Non negative RHS and Standard form = tableau form

Step ① Formulate problem

② Set up Standard form by adding/~~subtracting~~ slack and/or subtracting surplus variables

③ Set up tableau form

3. Simplex

Setting up initial Simplex tableau

C_J = objective function coefficient for variable J.

b_i = right hand side value for constraint i.

a_{ij} = coefficient associated with variable j in constraint i.

illustrated by:

c_1	c_2	...	c_n	
a_{11}	a_{12}		a_{1n}	b_1
a_{21}	a_{22}		a_{2n}	b_2
⋮	⋮		⋮	
a_{m1}	a_{m2}		a_{mn}	b_m

For High tech problem:

50	40	0	0	0	
3	5	1	0	0	150
0	1	0	1	0	20
8	5	0	0	1	300

C row = row of objective function coefficients

b column = column of right hand side values of the constraint equations

A matrix = m rows and n columns of coefficients of the variables in the constraint eq^ns.

c row	
A matrix	b column

To remind us we write the variable associated with each column directly above the column

X_1	X_2	S_1	S_2	S_3	
50	40	0	0	0	
3	5	1	0	0	150
0	1	0	1	0	20
8	5	0	0	1	300

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1. Where for each basic variable (here $S_1 S_2 S_3$) a corresponding column has a 1 in the only non zero position such columns known as unit columns or unit vectors

2. A row of the tableau is associated with each basic variable.

⇒ In example row 1 is associated with basic variable S_1 because this row has a 1 in the unit column corresponding to S_1
- Thus value of S_1 is given by RHS value $b_1 : S_1 = b_1 = 150$
- Similar $S_2 = b_2 = 20$ $S_3 = b_3 = 300$

To move from an initial basic feasible solution to a better basic feasible solution, the simplex method must generate a new basic feasible solution that yields a better value for the O.F

We select one of the current non basic variables (x_1, x_2) to be made basic, and one of the basic variables (S_1, S_2, S_3) to be made non-basic

We add two new columns to the tableau for computational convenience
Basis: list of current basic variables
C_B : Corresponding objective function coefficient for each basic variable

⇒

Basis	C_B	x_1	x_2	S_1	S_2	S_3	
		50	40	0	0	0	
S_1	0	3	5	1	0	0	150
S_2	0	0	1	0	1	0	20
S_3	0	8	5	0	0	1	300

Basis column is wrote in order of ~~of~~ Basis to show that $S_1 = 150$ etc *

3. Simplex

To find out if we can improve O.F by moving to a new basic feasible solution we add two new rows to the bottom of the tableau.
First row Z_J represents decrease in value of O.F, that will result if one unit of the variable corresponding to the j^{th} column of the A matrix is brought into the basis.

The second row $C_J - Z_J$ represents the net change in the value of the objective function if one unit of the variable corresponding to the j^{th} column of the A matrix is brought into the solution.
We refer to the $C_J - Z_J$ row as the net evaluation row.

How entries in Z_j row are computed
We consider increasing value of non-basic variable X_1 by one unit, from $X_1 = 0$ to ~~X_1~~ $X_1 = 1$

In order to make this change and continue to satisfy the constraint equations, the values of some of the other variables will have to be changed.
simplex method requires necessary changes be made to basic variables only.

In first constraint we have $3X_1 + 5X_2 + 1S_1 = 150$
current basic variable is S_1. Assuming X_2 remains a non basic variable with a value of 0, if X_1 is increased in value by 1, then S_1 must be decreased by 3 for the constraint to be satisfied.

Similarly if we were to increase X_1 by 1 (and keep $X_2 = 0$) we see from 2^{nd} and 3^{rd} eqns that although S_2 would not decrease, S_3 would decrease by 5.

The coefficients in the X_1 column indicate the amount of decrease in current basic variables when non basic variable X_1 is increased from 0 to 1.

In general all the column coefficients can be interpreted this way
Eg. wants nonbasic X_2 increased from 0 to 1 $\Rightarrow$ S_1 -5 S_2 -1 S_3 -5.

To compute the value in Z_j row (the decrease in value of O.F when x_j is increased by one) we form the sum of the products obtained by multiplying the elements in the C_B column by the corresponding elements in the j^{th} column of the A matrix.

$\Rightarrow Z_1 = 0(3) + 0(0) + 0(8) = 0$ S_1(column J value) $+ S_2(J) + S_3 \ldots = 0$

$Z_2 = 0(5) + 0(1) + 0(5) = 0$

$Z_3 = 0(1) + 0(0) + 0(0) = 0$

$Z_4 = 0(0) + 0(1) + 0(0) = 0$

$Z_5 = 0(0) + 0(0) + 0(1) = 0$

Because the objective function coefficient of X_1 is $C_1 = 50$, $C_1 - Z_1$ is $50 - 0 = 50$. The net result of bringing one unit of X_1 into the current basis will be an increase in profit of £50.

Hence in the net evaluation row corresponding to X_1 we enter 50. In same manner we can calculate the $C_j - Z_j$ value for remaining vars.

		X_1	X_2	S_1	S_2	S_3	
Basis	C_B	50	40	0	0	0	
S_1	0	3	5	1	0	0	150
S_2	0	0	1	0	1	0	20
S_3	0	8	5	0	0	1	300
Z_j	0	0	0	0	0	0	0 ← value of the objective function
$C_j - Z_j$		50	40	0	0	0	

Computed by multiply objective function coef's in the C_B column by the corresponding value of the basic variables shown in the last column of the tableau $0(150) + 0(20) + 0(300) = 0$

Initial Simplex tableau now complete. Initial basic feasible solution ($X_1 = 0$, $X_2 = 0$, $S_1 = 150$, $S_2 = 20$ and $S_3 = 300$) has profit of 0. $C_j - Z_j$ or net evaluation row has values that will guide us in improving the solution

7 Simplex

Improving the Solution

From net evaluation row we see each unit of x_1 increase profit by 50 and x_2 by 40. Because x_1 causes the largest per unit increase we choose it as the variable to bring into the basis. We must determine which of current basic variables to move non-basic

In computing z_j values we noted each of coef's in x_1 column indicates amount of decrease in corresponding basic variable that would result from increasing x_1 by one unit. Considering the first row we see that every unit of x_1 produced will use 3 hours of assembly time, thus reducing S_1 by 3

In current solution, $S_1 = 150$ and $x_1 = 0$. Considering this row only, the max possible value of x_1 can be calculated by

$$3x_1 = 150 \Rightarrow x_1 = 50$$

If x_1 is 50 (and x_2 remains a nonbasic variable with a value of 0), S_1 will have to be reduced to zero in order to satisfy the first constraint.

Considering 2nd row $0x_1 + 1x_2 + 1s_2 = 20$, coef of x_1 is 0. Increasing x_1 will not have any affect on S_2, increasing x_1 cannot drive the basic variable S_2 to zero

With 8 as coef of x_1 in 3rd row, every unit we increase x_1 will cause a decrease of 8 units in $S_3 \Rightarrow S_3 = 300$

$$8x_1 = 300$$

x_1 cannot be larger than 37.5.

Considering the three rows simultaneously, row 3 is the most restrictive. Producing 37.5 units of x will force corresponding slack variable to become nonbasic $\Rightarrow S_3 = 0$

We must update the table accordingly.

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Eliminating negative Right hand side Variables:

- Values on rhs cannot be negative
- What if Mylach wants to have specified x_4 less than or equal to x_1 after selling 5 units of x_1

$\Rightarrow -x_1 + x_2 \leq -5$

Because constraint has neg RHS, we multiply both side by -1 and change direction or equality sign to achieve + rhs

$\Rightarrow \cancel{-x_1 + x_2 \geq -5} \quad x_1 - x_2 \geq +5.$

Tableau form can be created by subtracting a surplus variable and adding an artificial variable

For $\geq$, multiply by -1 create a $\leq$

Summary of steps to create tableau form

1. If constraint has negative RHS, multiply by -1
2. For $\leq$ constraints, add a slack variable. Coef of S.V. is assigned zero in objective function coef. Slack variable becomes basic.
3. For $\geq$ constraints, subtract a surplus variable and then add an artificial variable. Coef of surplus variable assigned value of zero, Coef of A.V. assigned value of -M. artificial variable becomes basic variable in initial BFS
4. For (=) add an artificial variable to obtain tableau form. Coef of A.V assigned value of -M. A.V. becomes one of basic variables in IBFS.

Example: Convert to tableau form

Max $6x_1 + 3x_2 + 4x_3 + 1x_4$

S.T. $-2x_1 - \frac{1}{2}x_2 + 1x_3 - 6x_4 = -60$

$1x \quad + 1x_3 + \frac{2}{3}x_4 \leq 20$

$-1x_2 - 5x_3 \quad \leq -50$

$x_1, x_2, x_3, x_4 \geq 0.$

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To eliminate negativity in constraint 1 and 3 multiply by -1

$\Rightarrow$ Max $6x_1 + 3x_2 + 4x_3 + 1x_4$

ST: $2x_1 + \frac{1}{2}x_2 - 1x_3 + 6x_4 = 60$

$1x_1 + 1x_3 + \frac{2}{3}x_4 \leq 20$

$1x_2 + 5x_3 \geq 50$ $\quad x_1, x_2, x_3, x_4 \geq 0$

Apply Step 4 to C_1, Step 2 to C_2 and Step 3 to C_3

$\Rightarrow$ ~~Max $6x_1$ $+3x_2$~~ $+4$

Basis	C_B	x_1	x_2	x_3	x_4	S_2	S_3	a_1	a_3	
		6	3	4	1	0	0	-M	-M	
a_1	-M	2	1/2	-1	6	0	0	1	0	60
S_2	0	1	0	1	2/3	1	0	0	0	20
a_3	-M	0	1	5	0	0	-1	0	1	50
Z_J		-2M	-3/2M	-4M	-6M	0	M	-M	-M	-110M
$C_J - Z_J$		6+2M	3+3/2M	4+4M	1+6M	0	-M	0	0	

Pivot element x_4 will enter a_1 will leave the basis

Solving a minimization problem

- In max problem we choose entering variable to basis of largest $C_J - Z_J$.
- For minimization, reverse rule and find smallest value of $C_J - Z_J$
- We stop iterating when every value in $C_J - Z_J$ row is zero or positive

- Second approach is fact that any min problem can be changed in max problem by multiplying objective function by -1
- Solving resulting maximization problem will produce optimal solution to min problem

- Set up initial tableau and iterate as usual
- Value of objective function will be negative so multiply by -1 to get answer

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Special cases

Infeasibility, unboundedness, alternate optimal solution, degeneracy

INFEASIBILITY

- No solution can be found to LP that satisfies all constraints including the non negativity constraints

Example: Max $50x_1 + 40x_2$

ST:

$3x_1 + 5x_2 \leq 150$ assembly time

$1x_2 \leq 20$ portable display

$8x_1 + 5x_2 \leq 300$ warehouse space

$x_1 + x_2 \geq 50$ min. total production

$x_1, x_2 \geq 0$

Two iterations of simplex table produced:

		x_1	x_2	s_1	s_2	s_3	s_4	a_4	
Basis	C_B	50	40	0	0	0	0	$-M$	
x_2	40	0	1	8/25	0	-3/25	0	0	12
s_2	0	0	0	-8/25	1	3/25	0	0	8
x_1	50	1	0	-5/25	0	5/25	0	0	30
a_4	$-M$	0	0	-3/25	0	-2/25	-1	1	8
z_j		50	40	$\frac{70+3M}{25}$	0	$\frac{130+2M}{25}$	M	$-M$	$1480-8M$
		0	0	$\frac{-70-3M}{25}$	0	$\frac{-130-2M}{25}$	$-M$	0	

$C_j - Z_j \leq 0$ for all variables, according to optimality criterion it should be optimal solution

- Not feasible because artificial variable $a_4 = 8$ appears in the solution.
- The solution $x_1 = 30$, $x_2 = 12$ = total production 42 units instead of 50 from con 4.
- Fact that artificial variable is in solution at value $a_4 = 8$, tell us final solution violated fourth constraint $(x_1 + x_2 \geq 50)$ by 8 units
- $s_1, s_3 = 0$ constraints are binding, because these are used up we cannot satisfy the other 2 constraints
- If more time/space ~~work~~ overtime is not available, many need to relax production by 8 unit

We recognise infeasibility when one or more of artificial variables remain in the final solution at a positive value

LP problem with all ≤ constraints and non-negative RHS will always have a solution because of absence of artificial variables

Unbounded

LP is unbounded if value of the solution may be made infinitely large without violating any constraint

The way we recognise the unbounded situation is that all the $\bar{a}_{ij}$ are less than or equal to zero in the column associated with the entering variable

Example: Max $20x_1 + 10x_2$

ST $1x_1 \geq 2$

$x_2 \leq 5$

$x_1, x_2 \geq 0$

Subtract surplus variable from 1^{st} constraint and add slack variable to constraint 2^{nd}

Add artificial a_1 to first constraint

- After bringing in x_1 and removing a_1 at first iteration ⇒

		x_1	x_2	s_1	s_2	
Basis	C_B	20	10	0	0	
x_1	20	1	0	-1	0	2
s_2	0	0	1	0	1	5
Z_j		20	0	-20	0	40
$C_j - Z_j$		0	10	20	0	

- Because s_1 has largest positive $(C_j - Z_j)$ value, we can increase value of objective function most rapidly by bringing s_1 into basis.
- But $\bar{a}_{13} = -1$ and $\bar{a}_{23} = 0$, we can't form ratio $b_i / \bar{a}_{i3}$ for any $\bar{a}_{i3} > 0$ because no values are > 0

Rules for selecting a nonbasic variable to be made basic and for selecting a current basic variable to be made nonbasic.

Criterion for entering a new variable into the Basis

Look at net evaluation row ($C_J - Z_J$) and select the variable to enter the basis that will cause the largest per unit improvement in OF.

Criterion for removing a variable from the current Basis (Minimum Ratio test)

Suppose the incoming basic variable corresponds to column J in the A matrix portion of the Simplex tableau for each row i, compute the ratio b_i / a_{iJ} for each a_{iJ} greater than zero. The basic variable that will be removed from the basis corresponds to the minimum of these ratios. In case of tie we follow convention of selecting the variable that corresponds to the uppermost of the tied rows.

Add extra row to tableau showing b_i / a_{iJ} ratio

Basis	C_B	X_1	X_2	S_1	S_2	S_3		$\frac{b_i}{a_{iJ}}$
		50	40	0	0	0		
S_1	0	3	5	1	0	0	150	$150/3 = 50$
S_2	0	0	1	0	1	0	20	—
S_3	0	8	5	0	0	1	300	$300/8 = 37.5$
Z_J		0	0	0	0	0	0	
$C_J - Z_J$		50	40	0	0	0		

We see $C_1 - Z_1 = 50$ is the largest positive value in $C_J - Z_J$ row. X_1 is selected to become new basic variable.

Checking ratios for values greater than 0 we find $b_3/a_{31} = 37.5$ is the minimum. Current basic variable associated with row 3 (S_3) is variable selected to leave basis.

A_{31} is the pivot element. Column and row are pivot row and pivot column.

To improve current solution ($X_1 = 0$, $X_2 = 0$, $S_1 = 150$, $S_2 = 20$, $S_3 = 300$) we should increase X_1 to 37.5.

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$37.5x_1$ result in profit of $50(37.5) = 1875$. S_3 will be reduced to zero, x_1 will become new basic variable, replacing S_3 in the previous basis.

Cal

17.5 CALCULATING THE NEXT TABLEAU

We want to update simplex tableau to hold the same form

elementary row operations

1. Multiply any row (equation) by a nonzero number.
2. Replace any row (eq^n) by the result of adding or subtracting a multiple of another row (eq^n) to it

These operations will not change the solution, it will change the coef of variables and RHS

Objective is to transform the system of constraint equations into a form that makes it easy to identify the new basic feasible solution

We must perform the elementary row operations so we transform the column for the variable entering the basis into a unit column

Notation

portion of simplex tableau that initially contained a_{ij} as $\bar{A}$

portion of tableau that contained b_i as $\bar{b}_i$

elements in $\bar{A}$ are $\bar{a}_{ij}$

elements in $\bar{b}$ are $\bar{b}_i$

avoid confusion when coef's change!

Goal is to transform column in $\bar{A}$ portion of table corresponding to x to a unit column

$\bar{a}_{11} = 0$

$\bar{a}_{21} = 0$

$\bar{a}_{31} = 1$

15.

We add artificial variable a_4 to tableau to get an initial basic feasible solution
A_4 has nothing to do with high tech, enables set up of tableau
we add a_4 to row 4.

$$3x_1 + 5x_2 + S_1 = 150$$
$$x_1 + S_2 = 20$$
$$8x_1 + 5x_2 + S_3 = 300$$
$$x_1 + x_2 - S_4 + a_4 = 25$$

We can now obtain IBFS by setting n-m variables = 0.

$x_1 = x_2 = S_4 = 0$

$\Rightarrow x_1 = 0,\ x_2 = 0\ S_1 = 150,\ S_2 = 20\ \ S_3 = 300\ \ S_4 = 0,\ a_4 = 25.$

Both requirements of tableau form have been satisfied
Solution not actually feasible for real problem

Only time we need real value solution is in final iteration. To ensure a_4 disappears before this) we assign ~~the~~ a very large negative number to the profit coefficient for artificial variable a_4
say -100000 or -M for handiest

Max $50x_1 + 40x_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4 - Ma_4$

		x_1	x_2	S_1	S_2	S_3	S_4	a_4	
Basis	C_B	50	40	0	0	0	0	-M	
S_1	0	3	5	1	0	0	0	0	150
S_2	0	0	1	0	1	0	0	0	20
S_3	0	8	5	0	0	1	0	0	300
a_4	-M	1	1	0	0	0	-1	1	25
Z_J		-M	-M	0	0	0	M	-M	-25M
$C_J - Z_J$		50+M	40+M	0	0	0	-M	0	

Since 50+M is largest value in $C_J - Z_J$ row, x_1 will become basic
Further calculations show us than x_1 will replace a_4 in the basis

Result of Iteration 1.

Basis	C_B	X_1 50	X_2 40	S_1 0	S_2 0	S_3 0	S_4 0	a_4 -M	
S_1	0	0	2	1	0	0	3	-3	75
S_2	0	0	1	0	1	0	0	0	20
S_3	0	0	-3	0	0	1	8	-8	100
X_1	50	1	1	0	0	0	-1	1	25
Z_J		50	50	0	0	0	0	-50	1250
$C_J - Z_J$		0	-10	0	0	0	50	-M-50	

- With $a_4 = 0$. we have a BFS.
- We can now drop a_4 and it's associated colum from the Simplextabled as soon as it has been eliminated from the basic fecsible solution

⇒ Basis	C_B	X_1 50	X_2 40	S_1 0	S_2 0	S_3 0	S_4 0	
S_1	0	0	2	1	0	0	3	75
S_2	0	0	1	0	1	0	0	20
S_3	0	0	-3	0	0	1	8	100
X_1	50	1	1	0	0	0	-1	25
Z_J		50	50	0	0	0	-50	1250
$C_J - Z_J$		0	-10	0	0	0	50	

Phase II continues now like normal iterations when artificial variables are gone

⇒ Basis	CB	X_1 50	X_2 40	S_1 0	S_2 0	S_3 0	S_4 0	
S_1	0	0	25/8	1	0	-3/8	0	75/2
S_2	0	0	1	0	1	0	0	20
S_4	0	0	-3/8	0	0	1/8	1	25/2
X_1	50	1	5/8	0	0	1/8	0	75/2
Z_J		50	250/8	0	0	50/8	0	1875
$C_J - Z_J$		0	70/8	0	0	-50/8	0	

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- S_4 is entered into basis
- One more iteration required x_2 comes into solution and S_1 is eliminated

$\Rightarrow$

Basis	C_B	x_1	x_2	S_1	S_2	S_3	S_4	
		50	40	0	0	0	0	
x_2	40	0	1	8/25	0	-3/25	0	12
S_2	0	0	0	-8/25	1	3/25	0	8
S_4	0	0	0	3/25	0	2/25	1	17
x_1	50	1	0	-5/25	0	5/25	0	30
Z_J		50	40	14/5	0	26/5	0	1980
$C_J - Z_J$		0	0	-14/5	0	-26/5	0	

Turns out as the modified problem is same as solution for the original problem. One more iteration was needed to eliminate a_4 in phase 1

When all $(C_J - Z_J \leq 0)$ and artificial variables have been eliminated, we have found the optimal solution

If we reach optimality criterion and one or more artificial variables remain in the solution at a positive value, there is no feasible solution to the problem

EQUALITY CONSTRAINT:

- When equality constraint appears, we need to add an artificial variable
- For example if constraint 1 is $6x_1 + 4x_2 - 5x_3 = 30$
- We simply add an artificial variable a_1 to create a BFS

$$\Rightarrow \quad 6x_1 + 4x_2 - 5x_3 + 1a_1 = 30$$

- Now a_1 can be selected as basis for row with RHS value
- Once we have created tableau form by adding an artificial variable to each equality constraint, simplex method proceeds as usual

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Interpreting result of an iteration

Initial basic feasible solution ($x_1=0$, $x_2=0$, $S_1=150$, $S_2=20$, $S_3=300$) profit $=0$

New solution ($x_1 = 75/2$, $x_2=0$, $S = 75/2$, $S_2=20$, $S_3=0$) profit $=1875$

Here warehouse capacity constraint is binding with $S_3=0$

Moving toward a better solution

We need to calculate Z_J and $C_J - Z_J$ for new table

$Z_1 = 0(0) + 0(0) + 50(1) = 50$

$Z_2 = 0(25/8) + 0(1) + 50(5/8) = 250/8$

$Z_3 = 0(1) + 0(0) + 50(0) = 0$

$Z_4 = 0(0) + 0(1) + 50(0) = 0$

$Z_5 = 0(-3/8) + 0(0) + 50(1/8) = 50/8$

Subtracting Z_J from C_J we get

basis	C_B	x_1	x_2	S_1	S_2	S_3		$\bar{b}_i / \bar{a}_{i2}$
		50	40	0	0	0		
S_1	0	0	25/8	1	0	-3/8	75/2	$\frac{75/2}{25/8} = 12$
S_2	0	0	1	0	1	0	20	$20/1 = 20$
x_1	50	1	5/8	0	0	1/8	75/2	$\frac{75/2}{5/8} = 60$
Z_J	50	50	250/8	0	0	50/8	1875	
$C_J - Z_J$		0	70/8	0	0	-50/8		

- We select x_2 to enter the basis next as it has the highest positive coef in the $C_J - Z_J$ row
- To determine which variable will be removed we compute $\frac{b_i}{a_{i2}}$ only compute it $a_{i2} > 0$

With 12 as the minimum ratio, S_1 will leave the basis. Pivot element $\bar{a}_{12} = 25/8$

We will have to transform x_2 column to
$\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}$

12.

We make the change by:

1. Multiply every element in row 1 (pivot row) by $8/25$ in order to make $a_2 = 1$
2. Subtract the new row 1 from row 2 to make $\bar{a}_{22} = 0$
3. Multiply new pivot row by $5/8$ and subtract result from row 3 so $\bar{a}_{32} = 0$.

$\Rightarrow$

		x_1	x_2	s_1	s_2	s_3	
Basis	C_B	50	40	0	0	0	P_0
X_2	40	0	1	$8/25$	0	$-3/25$	12
S_2	0	0	0	$-8/25$	1	$3/25$	8
X_1	50	1	0	$-5/25$	0	$5/25$	30
Z_J		50	40	$14/5$	0	$26/5$	1980
$C_J - Z_J$		0	0	$-14/5$	0	$-26/5$	

$X_2 = 12$, $S_2 = 8$ $X_1 = 30$ Profit $= 40(12) + 0(8) + 50(30) = 1980$

- We must decide to iterate again or not.
- Looking at the net evaluation row, every element is zero or negative

Because $C_J - Z_J$ is less than or equal to 0 for both of the nonbasic variables S_1 and S_3 any attempt to bring a nonbasic variable into the basis at this point will lower value of O.F. Tableau represents the optimum solution

Optimality Criterion

Optimum Solution to a L.P. problem has been reached when all of the entries in the net evaluation row ($C_J - Z_J$) are zero or negative. In such cases, the O.S. is the current basic feasible solution

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Interpreting the optimal Solution

$x_1 = 30$ $x_2 = 12$ $s_1 = 0$ $s_2 = 8$ $s_3 = 0$

Value = 1980

Hightech should produce 30 Deskpro and 12 portable

- Since $S_2 = 8$, 8 portable display units are unused.
- $S_1 = 0$ and $S_3 = 0$, no slack associated, all units used up meaning they are both binding
- If additional assembly time/space is available money should get it.

Summary of the Simplex method

1. Formulate a lp model of the problem
2. Add slack variables to each constraint and change to standard form. Provides us with tableau for all "≤" and nonnegative RHS values
3. Set up initial Simplex tableau
4. Choose the nonbasic variable with largest entry in the net evaluation row to bring into the basis. This variable identifies the pivot column, the column associated with the incoming variable.
5. Choose pivot row which will have smallest ratio of $\frac{b_i}{a_{ij}}$ for $a_{ij} > 0$ when j is the pivot column
6. Perform necessary elementary row operations to convert column into unit column
   - Divide pivot row, subtract/add in other rows to get 0
7. Test for optimality if $c_j - z_j \leq 0$ for all columns, the soln is optimal. If not, iterate again (step 4)

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Tableau form: the general case

When LP contains all $\leq$ and non-neg RHS we simply add a slack variable to each constraint.

Greater than equal constraint $\geq$

Suppose hightech wanted to ensure combined total production for both models to be at least 25 units $\Rightarrow x_1 + x_2 \geq 25$.

$$\Rightarrow \text{Max } 50x_1 + 40x_2$$

S.T.

$3x_1 + 5x_2 \leq 150$ assembly time

$x_2 \leq 20$ portable display

$8x_1 + 5x_2 \leq 300$ warehouse space

$x_1 + x_2 \geq 25$ minimum total production

$x_1, x_2 \geq 0$.

We use three slack variables and one surplus variable to write in standard form

$$\begin{aligned}
\text{Max. } & 50x_1 + 40x_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4 \\
\text{S.T.: } & 3x_1 + 5x_2 + 1S_1 = 150 \\
& x_2 + 1S_2 = 20 \\
& 8x_1 + 5x_2 + 1S_3 = 300 \\
& x_1 + x_2 - 1S_4 = 25 \\
& x_1, x_2, S_1, S_2, S_3, S_4 \geq 0
\end{aligned}$$

Previously for initial basic solution we would set $x_1, x_2 = 0$ giving us $x_1 = 0$, $x_2 = 0$, $S_1 = 150$, $S_2 = 20$, $S_3 = 300$, $S_4 = -25$.

$S_4 = -25$ violate non-negative RHS rule

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Indicates Solution to LP is unbounded because each unit of S_1 that is brought into Solution produces one extra unit of X_1 (Since $\bar{a}_{13}=-1$) and drives zero units out of solution (Since $\bar{a}_{23}=0$)

Because S_1 is a Surplus variable and can be interpreted as the amount of X_1 over the minimum amount required, the Simplex tableau indicates we can introduce as much S_1 as we desire without violating any constraints

Because the objective function coef associated with X_1 is + there will be no upper bound.

Unbounded exists if at some iteration, the simplex tells us to introduce variable J into the solution and all the $\bar{a}_{iJ}$ are less than or equal to zero in the J^{th} colum

Alternative optimal solutions

We can't recognise that a LP has alternative solutions until final simplex is reached. Then if LP has alternative optimal solutions, $C_J - Z_J$ will equal zero for one or more nonbasic variables

Example: Max $30x_1 + 50x_2$

ST: $3x_1 + 5x_2 \leq 150$

$x_2 \leq 20$

$8x_1 + 5x_2 \leq 300$ $\quad x_1, x_2 \geq 0$

final tableau is

Basis	C_B	X_1 30	X_2 50	S_1 0	S_2 0	S_3 0	
X_2	50	0	1	0	1	0	20
S_3	0	0	0	-8/3	25/3	1	200/3
X_1	30	1	0	1/3	-5/3	1	50/3
Z_J		30	50	10	0	0	1500
$C_J - Z_J$		0	0	-10	0	0	

All values in net evaluation row are ≤ 0 indicating optimal solution has been found

$X_1 = 50/3$ $X_2 = 20$ $S_1 = 0$ $S_2 = 0$ $S_3 = 200/3$ $= 1500$

- We see $C_J - Z_J$ value for non basic variable S_2 is equal to zero
$\Rightarrow$ LP may have alternative optimal solution!
- We can introduce S_2 into the basis without changing value of solution.

After introducing S_2 $\Rightarrow$

Basis	C_B	X_1 30	X_2 50	S_1 0	S_2 0	S_3 0	
X_2	50	0	1	8/25	0	-3/25	12
S_2	0	0	0	-8/25	1	3/25	8
X_1	30	1	0	-5/25	0	5/25	30
Z_J		30	50	10	0	0	1500
$C_J - Z_J$		0	0	-10	0	0	

We have different feasible solution, $X_1 = 30$, $X_2 = 12$ $S_1 = 0$ $S_2 = 8$ $S_3 = 0$
Also optimal $C_J - Z_J \leq 0$ for all J.

We can recognise possibility of alternative optimal solution if $C_J - Z_J$ equals zero for one or more of the non basic variables in final simplex tableau

Degeneracy

LP is said to be degenerate if one or more of basic variable $= 0$.

Example: Max $50x_1 + 40x_2$

ST: $3x_1 + 5x_2 \leq 175$

$1x_1 \leq 20$

$8x_1 + 5x_2 \leq 300$

$x_1, x_2 \geq 0$

After 1 itteration:

Basis	C_B	x_1	x_2	S_1	S_2	S_3	
		50	40	0	0	0	
S_1	0	0	25/8	1	0	-3/8	125/2
S_2	0	0	1	0	1	0	20
X_1	50	1	5/8	0	0	1/8	75/2
Z_J		50	250/8	0	0	50/8	1875.
$C_J - Z_J$		0	70/8	0	0	-50/8	

X_2 Should enter, ratios: $\frac{b_1}{a_{12}} = \frac{125/2}{25/8} = 20$

$$\frac{b_2}{a_{22}} = \frac{20}{1} = 20$$

$$\frac{b_3}{a_{32}} = \frac{75/2}{5/8} = 60.$$

First and Second rows tie indicating degenerate Solution
In a tie choose uppermost variable as pivot row. ⇒ S_1 leaves basis

But from tie we see that variable in row$_2$ S_2 will also be driven to zero
Because it doesnt leave basis we will have a basic variable with a value of zero after performing itteration

⇒

Basis	C_B	X_1	X_2	S_1	S_2	S_3	
		50	40	0	0	0	
X_2	40	0	1	8/25	0	-3/25	20
S_2	0	0	0	-8/25	1	3/25	0
X_1	50	1	0	-5/25	0	5/25	25.
Z_J		50	40	70/25	0	130/25	2050
$C_J - Z_J$		0	0	-70/25	0	-130/25	

- Basic feasible Solution but basic variable $S_2 = 0$
- Whenever we have a tie in b_i/a_{ij} next tableau will always have basic variable $= 0$.
- Because we are at optimal Sol, we don't care $S_2 = 0$. However if degeneracy occurs at itteration prior to OS, it is possible for the Simplex to cycle, to alternate between Some Set of values and never reach Solution. Not a big problem just Select uppermost variable

$C_J - Z_J$ | 0 0 -10 0

2013 $\boxed{3}$ Q1

Max. $5000x_1 + 7000x_2$

ST: $3000x_1 + 4000x_2 \le 60000$ cost produce

$5x_1 + 4x_2 \le 100$ time

$x_2 \le 7$ $x_1, x_2 \ge 0$.

Basis	C_B	x_1	x_2	S_1	S_2	S_3		
		5000	7000	0	0	0		
S_1	0	3000	4000	1	0	0	60000	$\frac{60000}{4000}$ 15.
S_2	0	5	4	0	1	0	100	$\frac{100}{4} = 125$
S_3	0	0	1	0	0	1	7	$\frac{7}{1} = 7$
Z_J		0	0	0	0	0	0.	
$C_J - Z_J$		5000	7000	0	0	0		

$x_1 = 0$, $x_2 = 0$, $S_1 = 60000$, $S_2 = 100$, $S_3 = 7$ $P = 0$

x_2 enters S_3 leaves want $\begin{matrix}0\\0\\1\end{matrix}$ row2-4(row3) $r_1 = 4000(\text{row }3)$

Basis	C_B	x_1	x_2	S_1	S_2	S_3		
		5000	7000	0	0	0		
S_1	0	3000	0	1	0	-4000	32000	$\frac{32000}{3000}$ 10.66
S_2	0	5	0	0	1	-4	72	$\frac{72}{5} = 14.4$
x_2	7000	0	1	0	0	1	7	—
Z_J		0	7000	0	0	7000	49000	
$C_J - Z_J$		5000	0	0	0	-7000		

$x_1 = 0$, $x_2 = 7000$, $S_1 = 32000$, $S_2 = 72$, $S_3 = 0$ $P = 49000$

x_1 enters, S_1 leaves want $\begin{matrix}1\\0\\0\end{matrix}$

row 1 ÷ 3000

row 2 − 5(row1

		x_1	x_2	s_1	s_2	s_3	
Basis	C_B	5000	7000	0	0	0	
x_1	5000	1	0	$\frac{1}{3000}$	0	$\frac{-4}{3}$	$32/3$
s_2	0	0	0	$-5/3000$	1	$8/3$	$56/3$
x_2	7000	0	1	0	0	1	7
Z_j		5000	7000	$5/3$	0	$\frac{1000}{3}$	$\frac{307000}{3}$
$C_j - Z_j$		0	0	$-5/3$	0	$-\frac{1000}{3}$	

i $x_1 = 32/3$, $x_2 = 7$, $s_1 = 0$, $s_2 = 56/3$ $s_3 = 0$

ii s_1 contraint 1 binding and s_3 constraint 3, binding
s_2 non binding $56/3$ excess until max reached

Examples sensitivity analysis

Q1 Max $5x_1 + 6x_2 + 4x_3$

ST: $3x_1 + 4x_2 + 2x_3 \le 120$

$x_1 + 2x_2 + x_3 \le 50$

$x_1 + 2x_2 + 3x_3 \ge 30$ $\qquad x_1, x_2, x_3 \ge 0$

Basic	C_B	x_1	x_2	x_3	s_1	s_2	s_3	
		5 c_1	6 c_2	4	0 c_{s_1}	0	0	
s_3	0	0	4	0	-2	7	1	80
x_3	4	0	2	1	-1	3	0	30
x_1	5 c_1	1	0	0	1 c_1	-2 c_1	0	20
Z_J		5 c_1	8	4	1 $-4+c_1$	2 $12-2c_1$	0	
$C_J - Z_J$		0	-2 c_2-8	0	$c_{s_1}-1$ $4+c_1$	-2 $-12+2c_1$	0	

range of optimality for c_1 = coef x_1

$4 + c_1 \le 0$ $\qquad -12 + 2c_1 \le 0$

$-c_1 \le 4$ $\qquad 2c_1 \le 12$ $\qquad c_1 \le 6$ $\qquad$ range $4 \le c_1 \le 6$ ✓

$c_1 \le 4$ $\qquad$ ~~range = (~~

$c_1 \ge 4$

range of optimality for c_2 = $\cancel{c_2 - 8 \le 0}$

$c_2 \le 8$ $\quad$ range $(-\infty, 8)$ ✓

range of optimality for c_{s_1} = $c_{s_1} - 1 \le 0$

$c_{s_1} \le 1$ ✓

2

Q6 Max $10x_1 + 9x_2$ x_1 = standard x_2 = delux

ST: $7/10\,x_1 + 1x_2 \le 630$ cutting + dying time

$1/2\,x_1 + 5/6\,x_2 \le 600$ sewing time

$1x_1 + 2/3\,x_2 \le 708$ finishing time

$1/10\,x_1 + 1/4\,x_2 \le 135$ inspection + packing time

		x_1	x_2	S_1	S_2	S_3	S_4	
basis	C_B	$10\,c_1$	$9\,c_2$	0	0	0	0	
x_2	$9\,c_2$	0	1	$30/16\,c_2$	0	$-21/16\,c_2$	0	252
S_2	0	0	0	$-15/16$	1	$5/32$	0	120
x_1	$10\,c_1$	1	0	$-20/16\,c_1$	0	$30/16\,c_1$	0	540
S_4	0	0	0	$-11/32$	0	$9/64$	1	18
Z_J		10	9	$70/16$	0	$111/16$	0	7668
$C_J - Z_J$		0	0	$-70/16$	0	$-111/6$	0	

A. range of optimality for profit contribution of standard bag X_1

$$(9)\frac{-30}{16} + \frac{20}{16}c_1 \le 0 \qquad \frac{9(-21)}{16} + \frac{30}{16}c_1 \le 0$$

$$\frac{20}{16}c_1 \le \frac{135}{8} \qquad \frac{32}{16}c_1 \le \frac{-189}{16}$$

$$c_1 \le \frac{27}{402}\ 0.671 \qquad c_1 \le -\frac{63}{10}\ \ 6.3$$

$$c_1 \le \frac{27}{402.2} \qquad 6.3 \le c_1 \le 13.5$$

B. range of optimality for the profit contribution of delux bag X_2

$$\frac{-30}{16}c_2 + \frac{200}{16} \le 0 \qquad \frac{21}{16}c_2 - \frac{300}{16} \le 0$$

$$\frac{-30c_2}{16} \le \frac{-200}{16} \qquad \frac{21c_2}{16} \le \frac{300}{16}$$

$$-30c_2 \le -200 \qquad 21c_2 \le 300$$

$$30c_2 \ge 200 \qquad c_2 \le \frac{100}{7}$$

$$c_2 \ge \frac{20}{3}$$

$$\frac{20}{3} \le c_2 \le 100/7$$

$$6.66 \le c_2 \le 14.28$$

3.

Q6 cont. c. If the profit contribution per delux drops to £7 per unit, how will

It will not affect optimality of solution as it is still within range for C_2 $\quad 20/3 \le C_2 \le 100/7$ ... $p = 7164$ ✓

d. below $20/3$ or above $100/7$ ✓

e. It is outside range ⇒ new solution required ✓

7a. range of feasibility for b_1 (cutting and dyeing capacity)

$$\begin{array}{c} \text{old} \\ \begin{bmatrix} 30/16 \\ -15/16 \\ -20/16 \\ -11/32 \\ \hline 70/16 \end{bmatrix} \end{array} + \Delta b_1 \quad \begin{array}{c} \text{old} \\ \begin{bmatrix} 252 \\ 120 \\ 540 \\ 18 \\ \hline 7668 \end{bmatrix} \end{array} + \Delta b_1 \begin{array}{c} 30/16 \\ -15/16 \\ -20/16 \\ -11/32 \\ \hline 70/16 \end{array}$$

$252 + 30/16\, b_1 \ge 0 \qquad b_1 \ge -134.4$

$120 + \Delta b_1(-15/16) \ge 0 \qquad b_1 \le 128$

$540 + \Delta b_1(-20/16) \ge 0 \qquad b_1 \le -432$

$18 + \Delta b_1(-11/32) \ge 0 \qquad b_1 \le 576/11$

$7668 + \Delta b_1(70/16) \ge 0 \qquad b_1 \ge -61344/35$

most relevant are $-134.4 \le b_1 \le 767/11$

original value of constraint = 630

$495.6 \le \Delta b_1 \le 682.36$

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7b range of feasibility for b_2 sewing capacity

old

$$\begin{bmatrix} 252 \\ 120 \\ 540 \\ 18 \\ 7668 \end{bmatrix} + \Delta b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \geq 0$$

$-$

$120 + \Delta b_2 \geq 0$

$- \; b_2 \geq -120$

$-$

$-$

$$b_2 \geq -\cancel{120}\ 480 \leq \Delta b_2 \leq 600$$

$0 \leq \cancel{\Delta b_2} \leq 120$ sewing capacity

c range of feasibility for b_3 finishing capacity

old + new

$$\begin{bmatrix} 252 \\ 120 \\ 540 \\ 18 \\ 7668 \end{bmatrix} + \Delta b_3 \begin{bmatrix} -21/16 \\ 5/32 \\ 30/16 \\ 9/64 \\ 111/16 \end{bmatrix}$$

$252 - \frac{21}{16}\Delta b_3 \geq 0$ $\quad b_3 \leq 192$

$120 + \frac{5}{32}\Delta b_3 \geq 0$ $\quad b_3 \geq -768$

$540 + \frac{30}{16}\Delta b_3 \geq 0$ $\quad b_3 \geq -288$

$18 + \frac{9}{64}\Delta b_4 \geq 0$ $\quad b_4 \geq -128$

$7668 + \frac{111}{16}\Delta b_3 \geq 0$ $\quad b_5 \geq \frac{-40896}{37} = -1105.24$

$$-128 \leq \Delta b_3 \leq 192$$

$$\cancel{7}08 - 128 \leq \Delta b_3 \leq 708 + 192$$

$$580 \leq \Delta b_3 \leq 900$$

d range of feasibility

old

$$\begin{bmatrix} 252 \\ 120 \\ 540 \\ 18 \\ 7668 \end{bmatrix} + \Delta b_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \geq 0$$

$18 + \Delta b_4 \geq 0$

$\Delta b_4 \geq -18$

revenue

$$117 \leq \Delta b_4 \leq 135$$

Because s_3 constraint thee has larger z_5 (dual price of s_3 [illegible]) [illegible]

5.

Q12. Range feasibility for b_1

old

$$\begin{bmatrix} 20 \\ 1 \\ 25 \\ \hline 1600 \end{bmatrix} + \Delta b_1 \begin{bmatrix} 10/3 \\ -2/3 \\ -5/3 \\ \hline 100/3 \end{bmatrix} \qquad \begin{array}{ll} 20 + \frac{10}{3}\Delta b_1 \geq 0 & \Delta b_1 \geq -6 \\ 1 + (-2/3)\Delta b_1 \geq 0 & \Delta b_1 \leq 3/2 \\ 25 - 5/3\,\Delta b_1 \geq 0 & \Delta b_1 \leq 15 \\ 1600 + 100/3\, b_1 \geq 0 & \Delta b_1 \geq -48. \end{array}$$

$$-6 \leq \Delta b_1 \leq 3/2$$

$b_1 = 20$

$$20 - 6 \leq \Delta b_1 \leq 20 + 3/2$$

$$14 \leq \Delta b_1 \leq 21.5.$$

b. feasibility for b_2 (material 2 availability)

old + new

$$\begin{bmatrix} 20 \\ 1 \\ 25 \\ \hline 1600 \end{bmatrix} + \Delta b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{array}{ll} - & \\ 1 + \Delta b_2 \geq 0 & \Delta b_2 \geq -1 \\ - & \\ - & \end{array}$$

$b_2 = 5$ range $5 - 1 \leq \Delta b_2$

$$4 \leq \Delta b_2$$

c. feasibility for $b_3 \Rightarrow S_3$

old + new

$$\begin{bmatrix} 20 \\ 1 \\ 25 \\ \hline 1600 \end{bmatrix} + \Delta b_3 \begin{bmatrix} -20/4 \\ 4/9 \\ 25/9 \\ \hline 400/9 \end{bmatrix} \qquad \begin{array}{ll} 20 - 20/9\, b_3 \geq 0 & \Delta b_3 \leq 9 \\ 1 + 4/9\,\Delta b_3 \geq 0 & \Delta b_3 \geq -\frac{9}{4} \quad -\frac{9}{4} \leq \Delta b_3 \leq 9 \\ 25 + 25/9\,\Delta b_3 \geq 0 & \Delta b_3 \geq -9 \\ 1600 + \frac{400}{9}\Delta b_3 \geq 0 & \Delta b_3 \geq -36 \end{array}$$

$$21 - \frac{9}{4} \leq \Delta b_3 \leq 21 + 4$$

$$18.75. < \Delta b_3 < 30$$

d dual price for Mineral 3 = $400/9 = 44.44$ ✓

Should be willing to pay up to 44.44

Should be willing to pay $18.75 \le b_3 \le 30$.

14 b_1
$$\begin{bmatrix}550\\700\\200\\2850\end{bmatrix} + \Delta b_1 \begin{bmatrix}3/10\\-5/10\\-1/10\\3/10\end{bmatrix}$$

$550 + \frac{3}{10}\Delta b_1 \geq 0$ $\quad \Delta b_1 \geq -\frac{5500}{3}$

$700 - \frac{5}{10}\Delta b_1 \geq 0$ $\quad \Delta b_1 \le 1400$, $\frac{5500}{3} \le \Delta b_1 \le 1400$

$200 - \frac{1}{10}\Delta b_1 \geq 0$ $\quad \Delta b_1 \le 2000$ ✓

$2850 + \frac{3}{10}\Delta b_1 \geq 0$ $\quad \Delta b_1 \geq -9500$

$550 - \frac{5500}{3} \le \Delta b_1 \le 550 + 1400$

Should $-1283.33 \le \Delta b_1 \le 690$

be using $\begin{bmatrix}125\\425\\25\\525\end{bmatrix}$ not $\begin{bmatrix}550\\700\\\vdots\end{bmatrix}$

b_2 dd

$$\begin{bmatrix}550\\700\\200\\2850\end{bmatrix} + \Delta b_2 \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$ $700 + b_2 \geq 0$ $\quad \Delta b_2 \geq -700$

$700 - 700$ b_2

$$\begin{matrix}125\\425\\25\\525\end{matrix} + \Delta b_1 \begin{matrix}0\\1\\0\\0\end{matrix}$$ $425 + \Delta b_1 \geq 0$ $\quad \Delta b_2 \geq -425$

$700 - 425 = 275$