MA1E01: Chapter 2 Summary

Limits and Continuity

Definitions

- **Limits (Epsilon-Delta formalism)**: Let $f(x)$ be defined for all x in some open interval containing a, with the exception that $f(x)$ need not be defined at a. We write

$$\lim_{x \to a} f(x) = L$$

  if, given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta.$$

  Similar definitions hold for one-sided limits; for example, for the right-sided limit $x \to a^+$, we can drop the absolute value on $x - a < \delta$ since the left-hand side is always positive for $x \to a^+$.

- **Limits at** $+\infty$: Let $f(x)$ be defined for all x in some infinite open interval extending in the positive x-direction. We write

$$\lim_{x \to \infty} f(x) = L$$

  if, given any number $\epsilon > 0$, we can find a positive number N such that

$$|f(x) - L| < \epsilon \qquad \text{if} \qquad x > N.$$

  If this limit exists, then $y = L$ is a *horizontal asymptote*.

- **Limits at** $-\infty$: Let $f(x)$ be defined for all x in some infinite open interval extending in the negative x-direction. We write

$$\lim_{x \to -\infty} f(x) = L$$

  if, given any number $\epsilon > 0$, we can find a negative number N such that

$$|f(x) - L| < \epsilon \qquad \text{if} \qquad x < N.$$

  If this limit exists, then $y = L$ is a *horizontal asymptote*.

- **Infinite Limits**: Let $f(x)$ be defined for all x in some open interval containing a, with the exception that $f(x)$ need not be defined at a. We write

$$\lim_{x \to a} f(x) = \infty \ (-\infty)$$

  if, given any positive (negatiive) number M, we can find a number $\delta > 0$ such that

$$f(x) > M \qquad \text{if} \qquad 0 < |x - a| < \delta.$$
$$< M$$

  Then $x = a$ is called a *vertical asymptote*.

- **Continuity**: A function is said to be continuous at $x = c$ provided

$$\begin{array}{ll} 1. & f(c) \text{ is defined.} \\ 2. & \lim_{x\to c} f(x) \text{ exists.} \\ 3. & \lim_{x\to c} f(x) = f(c). \end{array}$$

  This definition can be extended to intervals by considering the continuity of each point of the interval.

Theorems

- **Existence of a limit**: The two-sided limit of $f(x)$ at $x = a$ exists if and only if the two one-sided limits exist and are equal, i.e.,

$$\lim_{x\to a} f(x) = L \quad \Longleftrightarrow \quad \lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x).$$

- **Limit Laws**: If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$, then
  1. $\lim\limits_{x\to a}[f(x) \pm g(x)] = L_1 \pm L_2$
  2. $\lim\limits_{x\to a}[f(x)g(x)] = L_1L_2$
  3. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L_1}{L_2}$ for $L_2 \neq 0$
  4. $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L_1}$ for $L_1 > 0$ whenever n even.

- **Continuity of Compositions**: If the function g is continuous at c and the function f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c. Hence, if g is continuous everywhere and f is continuous everywhere, then $f \circ g$ is continuous everywhere.

- **Continuity of inverse functions**: If f is a one-to-one function that is continuous at each point of its domain, $\mathcal{D}(f)$, then f^{-1} is a continuous function of each point of its domain, i.e., f^{-1} is continuous at each point of $\mathcal{R}(f)$.

- **Intermediate Value Theorem**: If f is continuous on a closed interval $[a, b]$, and k is any number between $f(a)$ and $f(b)$ inclusive, then there is at least one number x in the interval $[a, b]$ such that $f(x) = k$, i.e., f must take on every value between $f(a)$ and $f(b)$ as x varies from a to b.

- **Corollary of IVT**: If f is continuous on $[a, b]$ and if $f(a)$ and $f(b)$ are non-zero and have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).

- **Squeezing Theorem**: Let f, g and h be functions satisfying

$$g(x) \leq f(x) \leq h(x)$$

for all x in some open interval containing the number c, witht he possible exception that the inequality need not hold at c. If g and h have the same limit as x approaches c, say

$$\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x)$$

then f has this limit as x approaches c,

$$\lim_{x \to c} f(x) = L.$$

- **Important trigonometric limits**: We can use the squeezing theorem to prove the following important limits (provided x is measured in radians)

  1. $\displaystyle \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x}{\sin x} = 1.$
  2. $\displaystyle \lim_{x \to 0} x \sin(1/x) = 0.$

Miscellaneous Results

- **Properties of continuous functions**: If f and g are continuous at c, then

  1. $f \pm g$ is continuous at c.
  2. $f \cdot g$ is continuous at c.
  3. f/g is continuous at c, provided $g(c) \neq 0$.

- **Continuity of polynomials/rationals**: Polynomials are everywhere continuous. Rational functions are everywhere continuous except the points where the denominator is zero, which correspond to vertical asymptotes.

- **Continuity of absolute value**: The absolute value of a continuous function is continuous.

- **Continuity of trigonometric functions**: The trigonometric functions $\sin x$ and $\cos x$ are everywhere continuous. Other trigonometric functions ($\tan x$, $\csc x$, $\sec x$, $\cot x$ etc) will have an infinite number of discontinuities whenever the denominator is zero.