3. Tutorial 9 week 9

Tutorial 9: MA1E01

The Indefinite Integral and Area Under a Curve

1. Evaluate the following indefinite integrals:

   (a) $\int x^{17} dx$

   (b) $\int \sqrt[3]{x^2} dx$

   (c) $\int \frac{1}{x^{5/6}} dx$

   (d) $\int \frac{10 - y + 4\sqrt{y}}{y^{3/4}} dx$

   (e) $\int (x^3 - \sin x)\, dx$

2. Solve the second-order initial-value problem by integrating both sides of the equation twice,

$$\frac{d^2y}{dx^2} = x + \cos x, \quad y(0) = 1, \, y'(0) = 2.$$

3. Evaluate the following indefinite integrals by an appropriate substitution:

   (a) $\int (3x - 7)^{11} dx$

   (b) $\int x^3 \sqrt{5 + x^4} dx$

   (c) $\int \frac{\sin(1/x)}{3x^2} dx$

   (d) $\int \sin^3 2\theta \, d\theta$

   (e) $\int \sin^n(a + bx) \cos(a + bx) dx$ for n a positive integer and $b \neq 0$.

4. Express the following sums in closed form:

   (a) $\sum_{k=1}^{n} \frac{7k}{n}$

   (b) $\sum_{k=1}^{n} (3 - 2k)^2$

   (c) $\sum_{k=1}^{n} \left(\frac{5}{n} - \frac{2k}{n}\right)$

5. From class, the definition of the area under a non-negative curve $y = f(x)$ over $[a, b]$ where the interval is divided into n equal sub-intervals of width $\Delta x = (b-a)/n$ is

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

where x_k^* can be any point in the interval $[x_{k-1}, x_k]$.

(a) Taking x_k^* to be the right endpoints of the sub-intervals, find the area under the curve $y = x^2$ over the interval $[-1, 2]$.

(b) Taking x_k^* to be the left endpoints of the subintervals, show that the area under $y = x^3$ over the interval $[0, b]$ is $b^4/4$.

$a+(n-1)\Delta x$ in left.

$a=0$
$b=b$

$(n-1)\Delta x$

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Maths Tutorial 9

1 a $\int x^{17} dx = \frac{x^{18}}{18} + c$ ✓

b $\int x^{\frac{3}{2}} = \frac{2x^{5/2}}{5}$ $\frac{2x^{5/2}}{5}$

c $\int x^{-\frac{5}{6}} = 6x^{\frac{1}{6}}$ ✓ $= \frac{3}{4} x^{4/3}$

d $\frac{10 - y + 4}{y^{3/4}}$

$10y^{-3/4} - y^{\frac{1}{4}} + 4y^{-\frac{3}{4}} = 40y^{1/4} - \frac{4}{5} y^{5/4} + 16^{\frac{1}{4}} + c$

e $-\frac{x^4}{4} + \cos x$

2 $\int \frac{d^2y}{dx^2} \, dx = \int x + \cos x \, dx$

$\int \frac{d^2y}{dx^2} = \int x + \cos x \quad dx$

$\frac{dy}{dx} = \frac{1}{2}x^2 + \sin x + c \quad y'(0) = 2 \quad \frac{1}{2}(0)^2 + \sin 0 = 2$

$c = 2$

$\int \frac{dy}{dx} = \int \frac{1}{2}x^2 + \sin x + 2$

$f(x) = \frac{x^3}{6} - \cos x + 2x + c$

$y(0) = 1 \quad \frac{0^3}{6} - \cos 0 + 2(0) + c = 1$

$-1 + c = 1$

$c = 2$

$f(x) = \frac{x^3}{6} + \cos x + 2x + 2$

3 $\int (3x-7)^{11} dx$ $\quad u = (3x-7)$

$\int u^{11} dx$ $\quad \frac{du}{dx} = 3dx \quad dx = \frac{1}{3} du$

$\frac{u^{12}}{12} du \quad \frac{3(3x-7)^{12}}{12} = \frac{1}{4}(3x-7)^{12}$

$\frac{1}{3} \int u^{11} du = \frac{1}{3} \cdot \frac{1}{2} u^{12} = \frac{1}{30}(3x-7)^{12} + c$

week 11

2.

3b $\int x^3 \sqrt{5+4x^4}$ $\qquad u = 5+4x^4$

$\int x^3 (5+4x^4)^{\frac{1}{2}}$ $\qquad \frac{du}{dx} = \cancel{16x^3}$

$\cancel{du = 16x^3 dx}$ $\qquad du = 4x^3 dx$

$\cancel{dx = \frac{1 du}{16x^3}}$ $\qquad dx = \frac{1}{4x^2} du$

$= \int \cancel{x^3} (u)^{1/2} \left(\frac{1}{4\cancel{x^2}}\right) du$

$\frac{1}{4} \int u^{1/2} du$

$\frac{1}{4} \cdot \frac{u^{3/2}}{3/2}$

$\frac{1}{4} \cdot \frac{2}{3} (5+4x^4)^{3/2}$

c $\int \frac{\sin(1/x)}{3x^2} dx$ $\qquad u = \frac{1}{x}$ $\qquad \frac{du}{dx} = -x^{-2}$

$\int \frac{\sin(u)}{3x^2} \cdot (-x^2) du$ $\qquad dx = \frac{1}{-x^{-2}} \cdot du$

$-\frac{1}{3} \int \sin(u) du$

$-\frac{1}{3} (-\cos u)$

$\frac{1}{3} \cos\left(\frac{1}{x}\right) + C$

d $\int \sin^3(2\theta)$

$\int \sin 2\theta \cdot \sin^2 2\theta$

$\int \sin 2\theta \cdot (1 - \cos^2 2\theta)$

$\int \sin 2\theta - \sin 2\theta \cos^2 2\theta$ $\qquad u = \cos 2\theta$ $\qquad du = -\sin 2\theta (2)$

$\frac{1}{2} \cos 2\theta$ $\qquad -\frac{1}{2} \int \sin 2\theta \cdot u^2 (\sin 2\theta\, du)$ $\qquad d\theta = -\frac{1}{2} \sin 2\theta$

$-\frac{1}{2} \int u^2 du$

$= -\frac{1}{2} \cos 2\theta - \frac{1}{6} \cos^3 2\theta + C$

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3e $\int \sin^2(a+bx)\cos(a+bx)\,dx$

$u = \sin(a+bx)$

$du = b\cos(a+bx)\,dx$

$dx = \frac{1}{b\cos(a+bx)}\,du$

$\int u^n \cos(a+bx) \cdot \frac{1}{b\cos(a+bx)}\,du$

$\frac{1}{b}\int u^n\,du$

$\frac{1}{b}\cdot\frac{1}{n+1}u^{n+1}+C$

$\frac{1}{b(n+1)}\cdot\sin^{n+1}(a+bx)+C$

4a $\sum_{k=1}^{n} \frac{7k}{n}$

$\frac{7}{n}\sum_{k=1}^{n} k$

$= \frac{7}{n}\left(\frac{n}{2}(n+1)\right)$

b. $\sum_{k=1}^{n}(3-2k)^2$

$\sum_{k=1}^{n}(9-12k+4k^2)$

$\sum 9 - \sum 12k \quad \sum + 4k^2$

$\sum 9 - 12\sum k + 4\sum k^2$

$9n - 12\left(\frac{n}{2}(n+1)\right) + 4\frac{n}{6}(n+1)(2n+1)$

$= 9n + 6n(n+1)\left[-1 + 9(2n+1)\right]$

$9n + 6n(n+1)(18n+8)$

$9n + 12n(n+1)(9n+4)$

4c $\sum_{n=1}^{n} \left(\frac{5}{n} - \frac{24}{n}\right.$

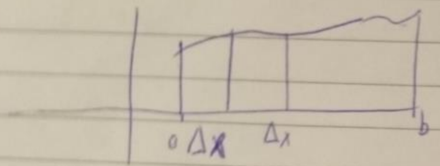
$\sum \frac{5}{n} - \sum \frac{24}{n}$

$\frac{5}{n}(n) - \frac{2}{n} \cdot \frac{9}{2}(n+1)$

$5 - n - 1$

$4 - n$

5.



$a = -1$

$b = 2$ right endpoint

$X_k^* = x_n \quad [x_{k-1}, x_k]$

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*)(\Delta x)$$

$\Delta x = \frac{b-a}{n} = \frac{2+1}{n} = 3n$

$f(x_k^*) = (x_k^*)^2 = x_k^2$

$x_n = a + k\Delta x$

start interval

$x_k^2 = (a + k\Delta x)^2$

$= \left(-1, +\frac{6k}{n}, k\frac{3}{n}\right)^2$

$(-1)^2 - 6\frac{k}{2} + k\left(\frac{3}{n}\right)^2$

$1 - \frac{6}{n}k + \frac{9}{n^2}k^2$

$= f(x_k^*)$

$\lim_{n \to \infty} \sum_{k}^{n} \left(\left(1 - \frac{6}{n}k + \frac{9}{n^2}k^2\right)\frac{3}{n}\right)$

$\sum \frac{3}{n} - \frac{18}{n^2}k + \frac{27}{n^3}k^2$

$\sum \frac{3}{n} - \frac{18}{n^2}\sum k + \frac{27}{n^3}\sum k^2$

$\frac{3}{n}(n) - \frac{18}{n^2}\left(\frac{n}{2}(n+1)\right) + \frac{27}{n^3}\left(\frac{n}{6}(n+1)(2n+1)\right)$

$= 3 - \frac{9}{n}\, n+1 + \frac{1}{n^2} \cdot \frac{9}{2}(n+1)(2n+1)$

$\lim_{n \to 0} 3 - 9 - \frac{9}{n} + \text{''} \quad \text{''}$

$- 6 - \frac{9}{n} + \text{''} \ldots$

$3 - 0 + 0$

$= 3$