17/10/13

MA2E01: Problem Set 3

Due at the end of the tutorial, 15-17 October.

1. Consider the function

$$f(x, y, z) = y \ln(x + y + z),$$

at the point $P = (-4, 5, 0)$.

(a) Find the unit vector in the direction in which f increases fastest at P.

(b) Find the unit vector in the direction in which f decreases fastest at P. Sketch the projection of the vector in the xy-plane, and in the xz-plane.

(c) Find the rate of change along these directions at the point P.

2. Consider the function defined by

$$F(x, y, z) = z - \sqrt{25 - x^2 - y^2}.$$

For $F(x, y, z) = 0$ it defines a surface. Find

(a) The gradient and rate of change of $F(x, y, z)$ at the point $P = (3, 0, 4)$.

(b) The equation of the tangent plane of the surface defined by $F(x, y, z) = 0$ at $P = (3, 0, 4)$.

3. For the surface

$$z = f(x, y) = \sqrt{x^2 y - x + \sin(2x - y)},$$

find the equations of the tangent plane and normal line at the point $(1, 2, 1)$.

4. Locate all relative minima, relative maxima and saddle points for the function

$$f(x, y) = x^2 + xy + y^2 - 6x\,,$$

and determine what types of point they are.

2

Summary $S = \sum_{K=1}^{n} \| \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} \| \Delta A_K$

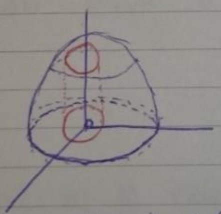
Taking the limit, we get the surface area of a parametric $\vec{r}(u,v)$ over a region R.

$$S = \iint_R dS = \iint_R \| \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} \| dA$$

The limits of the integrals come from R.

Example: Find the surface area of the sphere of radius 4 that lies in the cylinder above the xy-plane with base $x^2+y^2=12$

Solution: The sphere has equation $x^2+y^2+z^2=16$



We need to parameterise the surface. We introduce spherical coordinates and on the surface of the sphere $\rho=4$.

$$r(\theta,\phi) = 4\sin\phi\cos\theta\,\vec{i} + 4\sin\phi\sin\theta\,\vec{j} + 4\cos\phi\,\vec{k}$$

We need to find the range of θ and ϕ. θ goes the whole way around the so $0 \le \theta \le 2\pi$ but ϕ runs from the z axis to the intersection of the sphere and cylinder above the xy-plane

On the surface of the cylinder $x^2+y^2=12$

$$\underbrace{x^2+y^2}_{=12}+z^2=16$$

$$12+z^2=16 \qquad z^2=4 \quad z=\pm 2$$

DAVID WETTBRECHT 12700644

16/10/13 Maths Tutorial Week 4 Problem set 3

8/15

1. a $F(x,y,z) = y\ln(x+y+z)$ at point $P = (-4,5,0)$

unit vector

$\nabla F|_{(-4,5,0)} = \frac{df}{dx} i + \frac{df}{dy} j + \frac{df}{dz} k$

$\frac{1}{(x+y+z)} i + y\frac{1}{(x+y+z)} j + \ln(x+y+z) j + \frac{1}{(x+y+z)} k$

$\frac{1}{(-4+5+0)} i + \frac{5}{(-4+5+0)} j + \ln(-4+5+0) j + \frac{1}{(-4+5+0)} k$

$i + 5j + 0j + k$

$\cancel{5} i + 5j \; \cancel{5}K$

$\sqrt{5^2+5^2+5^2}$

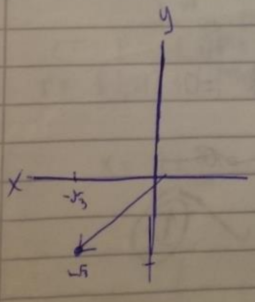
$\sqrt{75} = +5\sqrt{3}$

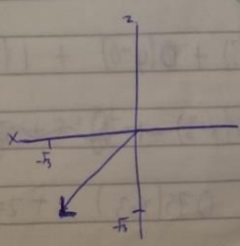
$\frac{5\sqrt{3}}{5} i + \frac{5\sqrt{3}}{5} j + \frac{5\sqrt{3}}{5} k$ $\quad \frac{5}{5\sqrt{3}} i + \frac{5}{5\sqrt{3}} j + \frac{5}{5\sqrt{3}} k$

$\sqrt{3} i + \sqrt{3} j + \sqrt{3} k$ 0.5

b. Unit vector where decreasing fastest = negative of above unit vector =

$-\sqrt{3} i - \sqrt{3} j - \sqrt{3} k$ 0.5





DAVID WETHERELL   12300644

c. Rate of change $= +5\sqrt{3}$ shown in part A.

2 a $F(x,y,z) = z - \sqrt{25-x^2-y^2}$

$\nabla F|_{(3,0,4)} \quad \frac{df}{dx} i + \frac{df}{dy} j + \frac{df}{dz} k$

$z - \frac{1}{2}(25-x^2-y^2)^{-\frac{1}{2}}(-2x)$

$\left(\cancel{z} + \frac{x}{\sqrt{25-x^2-y^2}}\right) i + \left(\cancel{z} \frac{+y}{\sqrt{25-x^2-y^2}}\right) J + \left(1 \cancel{-\sqrt{25-x^2-y^2}}\right) K$

Sub in at $\overset{x\,y\,z}{(3,0,4)}$ $\quad \frac{\cancel{z} + 3}{\sqrt{25-3^2-0^2}} i + \frac{\cancel{z}+0}{\sqrt{16}} K + 1$

$\left(\cancel{z}\frac{+3}{4}\right) i + \cancel{\frac{4}{4}}\cancel{K} J + K$

= gradient $\cancel{\left(\frac{4+3}{4}\right) i + \left(2+0\right) J + K}$

$\frac{3}{4} i + 0J + K$

rate of change $\cancel{\sqrt{\left(4+\frac{3}{4}\right)^2 + (2)^2 + (1)^2}} = \sqrt{(3/4)^2+0^2+1^2} = 1.25$

$\cancel{\sqrt{27.5625}}$ = ~~equation~~ = rate of change

b $F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$

$(x,y,z) = (3,0,4)$

$0.75(x-3) + 0(y-0) + 1(z-4) = 0$

$\cancel{4.75(x-3) + 2y + z - 4 = 0}$

$0.75(x-3) + z = 4$

(1) DAVID WEITBRECHT 1230064

Math 17/10/13

3 $z = f(x,y) \quad \sqrt{x^2y - x + \sin(2x-y)}$

tangent plane: $F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0) = 0$

$$\frac{df}{dx} = (x^2y - x + \sin(2x-y))^{-\frac{1}{2}} \cdot 2xy - 1 + 2\cos(2x-y)$$

$$= \frac{-1(2xy - 1 + 2\cos(2x-y))}{2\sqrt{x^2y - x + \sin(2x-y)}}$$

$$\frac{df}{dy} = (x^2y - x + \sin(2x-y))^{-\frac{1}{2}} \cdot (x^2 - x - \cos(2x-y))$$

$$= \frac{-1(x^2 - x - \cos(2x-y))}{2\sqrt{x^2y - x + \sin(2y-y)}}$$

(1,2,1)

$$\frac{2(1)(2) - 1 + 2\cos(2(1)-2)}{\sqrt{1^2(2) - (1) + \sin(2(1)-2)}}(x-1) + \frac{1^2 - 1 - \cos(2(1)-2)}{\sqrt{1^2(2) - 1 + \sin(2(1)-2)}} + z(z-1)$$

-1 0.5

$$\frac{+5}{2}(x-1) \quad +10(y-2) + \cancel{2}1(z-1) = 0$$ = tangent plane

$$= -\frac{5}{2}(x-1) + \cancel{(z-1)} = 0$$

→ Normal line equation:

$= (-f_x(x_0,y_0) - f_y(x_0,y_0), 1)$

$\left(\frac{-5}{\sqrt{3}}, \frac{+1}{0}, 1\right)$

$z=1$

$\frac{+5}{2}(x-1) = t$

$r = p + \nabla F \cdot t$ ($n \cdot t$)

$r = (1,2,1) + t\left(-\frac{5}{2}, 0, 1\right)$

$x = 1 + \frac{5}{2}t \quad y = 2 + 0t \quad z = 1 + t$ 0.5

DAVID WETTBRECHT 12700644

4 $f(x,y) = x^2 + xy + y^2 - 6x$

$f_x = 2x + y - 6$ $\quad f_y = x + 2y$

$f_{xx} = 2$ $\quad f_{yy} = 2$

$D = 2(2) - 1^2$

$\frac{df}{dx}\left(\frac{df}{dy}\, x^2 + xy + y^2 - 6x\right)$

$\frac{df}{dx}\; x + 2y = 1$

$D = f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2$

$D = 2(2) - 1^2$

$4 - 1 = 3$

$D > 0$ $\quad f_{xx}(x_0,y_0) > 0$

therefore:

$f(x,y)$ has a relative minimum at (x_0,y_0) $(4,-2)$

2

Critical point

$2x + y - 6 = 0$

$2(-2y) + y - 6 = 0$

$-3y = 6$

$\boxed{y = -2}$ $\boxed{x = 4}$

$(4,-2)$

$x + 2y = 0$

$x = -2y$