Maths 1 - Exam Notes

1. Parametric eq^n for line spanned by vector and eq^n of plane spanned by vectors

Parametric eq^n for line spanned by vector $u = (1, -2, 4, 0)$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} t \\ -2t \\ 4t \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x = t \\ y = -2t \\ z = 4t \\ w = 0 \end{cases}$$

Eq^n of plane spanned by $u = (-1, 2, 3)$ $v = (1, 1, 0)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = k_1 \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{cases} x = -k_1 + k_2 \\ y = 2k_1 + k_2 \\ z = 3k_1 \end{cases}$$

$K_1 = \frac{z}{3}$ $\qquad K_2 = y - \frac{2}{3}z$

$x = -\frac{1}{3}z + y - \frac{2}{3}z$

$x = y - z$

$x - y + z = 0$ = eq^n of plane

2. Rank And Nullity of A matrix

$$\begin{pmatrix} 6 & -2 & -4 \\ -3 & 1 & 2 \end{pmatrix} \quad \begin{matrix} r_1 \div 2 \\ r_2 + r_1 \end{matrix} \Rightarrow \begin{pmatrix} 3 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$r_1 \div 3 \Rightarrow \begin{pmatrix} 1 & -1/3 & -4/3 \\ 0 & 0 & 0 \end{pmatrix}$$

Rank(A) = 1 $\qquad$ = number of leading ones

Nullity(A) = 3 - 1 = 2 $\qquad$ columns - rank.

b: $\begin{pmatrix}6&-2&-4\\-3&1&2\\1&0&-1\end{pmatrix}$ $r_1 \div 2$ r_2+r_1 $\Rightarrow$ $\begin{pmatrix}3&-1&-2\\0&0&0\\1&0&-1\end{pmatrix}$

$\Rightarrow \begin{pmatrix}1&0&-1\\3&-1&-2\\0&0&0\end{pmatrix}$ r_2-3r_1 $\begin{pmatrix}1&0&-1\\0&-1&1\\0&0&0\end{pmatrix} \Rightarrow \begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix}$

Rank (b) $= 2$

Nullity (b) $= 3-2 = 1$

3 Basis and Dimen for row, column and null space of matrix

$A = \begin{pmatrix}0&2&-4&2\\0&-3&6&-3\end{pmatrix} \Rightarrow \begin{pmatrix}0&1&-2&1\\0&0&0&0\end{pmatrix}$

Rank (A) $=$ dim $\overset{row}{(A)}$ $=$ dim col (A) $= 1$

Nullity (A) $= 4-1 = 3$ $=$ col(n) $-$ rank

Basis for row space $=$ Span $\{(0,1,-2,1)\}$

Basis for column space $=$ Span $\left\{\begin{pmatrix}2\\-3\end{pmatrix}\right\}$

Basis for null space

$\begin{pmatrix}0&1&-2&1\\0&0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ $\quad y-2z+w=0 \quad w = 2z-y$ $\quad x = s_1$, $y = s_3$, $w = s_4$

$\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}s_1\\ \cancel{2s_3}\ s_3\\ s_4\\ 2s_4-s_3\end{pmatrix}$ $\quad \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\-1\end{pmatrix} + \begin{pmatrix}0\\0\\1\\2\end{pmatrix}$

Null (A) $=$ Span $\{(1,0,0,0), (0,1,0,-1), (0,0,1,2)\}$

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3. continued

b $\begin{pmatrix} 0 & 1 \\ 0 & -2 \\ -2 & 1 \end{pmatrix} \to \begin{pmatrix} -2 & 1 \\ 0 & 1 \\ 0 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

Rank(A) = ~~dim~~ dim row(A) = dim col(A) = 2

Nullity(A) = 2 − 2 = 0

Basis for row space Row(A) = Span $\{(1, -\frac{1}{2}), (0, 1)\}$

Basis for col space Col(A) = Span $\left\{\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right\}$

$\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad x - \frac{1}{2}y = 0$

$y = 0 \Rightarrow x = 0$

Null(A) = Span $\{0, 0\}$

4. Least Square Approximate Solution to System

rewrite system

$\begin{cases} x = -1 \\ y = 1 \\ x + 4y = 0 \end{cases}$ $\quad \overset{A}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 4 \end{pmatrix}} \overset{x}{\begin{pmatrix} x \\ y \end{pmatrix}} = \overset{b}{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$

Solve $A^T A x = A^T b$

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 2 & 4 \\ 4 & 17 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 4 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad 2x + 4y = -1$

$9y = 1 \Rightarrow y = \frac{1}{3}$

$2x = -1 - 4y$

$2x = -\frac{7}{3} \quad x = -\frac{7}{6} \quad \Rightarrow (x, y) = (-\frac{7}{6}, \frac{1}{3})$

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5. Eigenvalues and corresponding Eigenvectors

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 1 & 2 & 3 \\ 2 & -1 & -2 \end{pmatrix} \qquad P(\lambda) = \det(\lambda I - A)$$

$$\det\left[\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} -3 & 0 & 0 \\ 1 & 2 & 3 \\ 2 & -1 & -2 \end{pmatrix}\right] = 0$$

$$\det\left[\begin{pmatrix} \lambda+3 & 0 & 0 \\ -1 & \lambda-2 & -3 \\ -2 & 1 & \lambda+2 \end{pmatrix}\right] = (\lambda+3)\left[(\lambda-2)(\lambda+2) - (-3)(1)\right]$$

$$(\lambda+3)\left[\lambda^2 - 4 + 3\right]$$

$$= (\lambda+3)(\lambda^2-1)$$

$$= (\lambda+3)(\lambda+1)(\lambda-1) = 0$$

Eigen value = $\lambda_1 = -3$ $\quad \lambda_2 = -1$ $\quad \lambda_3 = 1$

$\lambda_1 = -3$

$$\left[\begin{pmatrix} -3 & 0 & 0 \\ 1 & 2 & 3 \\ 2 & -1 & -2 \end{pmatrix} - \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}\right]\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{0}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 5 & 3 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{0} \quad \Rightarrow \quad \begin{pmatrix} 1 & 5 & 3 \\ 2 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 5 & 3 \\ 0 & -11 & -5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{0}$$

$x + 5y + 3z = 0$

$-11y - 5z = 0 \qquad 11y = -5z \qquad y = \frac{-5}{11}z$

$x - \frac{25}{11}z + 3z = 0$

$x + \frac{8}{11}z = 0 \qquad x = \frac{8}{11}z$

$z = -11, \quad x = 8, \quad y = 5 \qquad \bar{v}_1 = \begin{pmatrix} 8 \\ 5 \\ -11 \end{pmatrix}$

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5... $\lambda_2 = -1$ $\begin{pmatrix} -2 & 0 & 0 \\ 1 & 3 & 3 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 3 & 3 \\ 2 & -1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & -1 & -1 \\ 1 & 3 & 3 \\ 0 & -7 & -7 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{0} \quad \begin{cases} x = 0 \\ -y - z = 0 \end{cases} \quad y = -z \quad \bar{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$\lambda_3 = 1$ $\begin{pmatrix} -4 & 0 & 0 \\ 1 & 1 & 3 \\ 2 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & -1 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{0}$

$$\begin{cases} -x = 0 \\ -y - 3z = 0 \end{cases} \quad \begin{matrix} x = 0 \\ y = -3z \end{matrix} \quad \bar{v}_3 = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$$

Matrix that diagonalise A.

$$D = P^{-1}AP$$

$$P = \begin{pmatrix} 8 & 0 & 0 \\ 5 & 1 & 3 \\ -11 & -1 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6 Fourier Series

$$f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0 \\ 2x & \text{if } 0 < x \le \pi \end{cases}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{0} 0\,dx + \int_0^{\pi} 2x\,dx$$
$$= \frac{1}{2\pi}\, x^2 \Big|_0^{\pi} = \frac{1}{2\pi}\pi^2 = \pi/2$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{0} 0\,dx + \int_0^{\pi} 2x\cos nx\,dx$$
$$= 1/\pi \left[\frac{2x\sin(nx)}{n} + \frac{2\cos nx}{n^2} \right]_0^{\pi}$$
$$= 1/\pi \left[\left(0 + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{2}{n^2}\right) \right]$$
$$= 1/\pi \left(\frac{(-1)^n - 2}{n^2} \right) = \frac{(-1)^n - 2}{\pi n^2}$$

2.

$b_n = \frac{1}{\pi} \int_{-\pi}^{0} 0 dx + \int_0^{\pi} 2x \sin nx \, dx$

$= \frac{1}{\pi} \left[\frac{2\sin nx}{n^2} - \frac{2x\cos nx}{n} \right]_0^{\pi}$

$= \frac{1}{\pi} \left[\left(0 - \frac{2\pi(-1)^n}{n}\right) - (0-0) \right]$

$= \frac{1}{\pi} \left[-\frac{2\pi(-1)^n}{n} \right] = -\frac{2(-1)^n}{n}$

$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{(-1)^n - 2}{\pi n^2} \right) \cos nx + \left(-\frac{2(-1)^n}{n} \right) \sin(nx) \right]$

7. Gramm Schmdt Proced

$V_1 = U_1 = (0,-1,0)$ $U_1 = (0,-1,0)$ $U_2 = (0,1,1)$ $U_3 = (1,2,0)$

$V_2 = U_2 - \frac{V_1 \cdot U_2}{V_1 \cdot V_1} V_1 = (0,-1,0) - \frac{0+1+0}{0+1+0}(0,-1,0)$

$= (0,1,1) - (-1)(0,-1,0)$

$= (0,0,1)$

$V_3 = U_3 - \frac{V_1 \cdot U_3}{V_1 \cdot V_1} V_1 - \frac{V_2 \cdot U_3}{V_2 \cdot V_2} V_2$

$= (1,2,0) - \frac{0-2+0}{1}(0,-1,0) - \frac{0}{\sim}(0,0,1)$

$= (1,2,0) - (-2)(0,-1,0)$

$= (1,0,0)$

8. Projection of vector

Reflection of $(2,-1)$ about x-axis

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Orthogonal projection $(2,-1)$ to y-axis $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

7.

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8 Image of $(2,-1)$ under rotation through angle $-\pi/4$ about origin

$$\begin{pmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2\sqrt{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} +\sqrt{2} - \frac{\sqrt{2}}{2} \\ \sqrt{2} - \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 \\ -3\sqrt{2}/2 \end{pmatrix}$$

9 linearly independent / Span R^n / Basis of R^n ?

Given $(0,1,1)$ $(1,0,1)$ $(1,1,0)$ $(1,1,1)$

linearly independent?

$$k_1\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + k_3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + k_4\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$k_2 + k_3 + k_4 = 0$
$k_1 + k_3 + k_4 = 0$
$k_1 + k_2 + k_4 = 0$

$\Big\}$ $k_3 + k_4 = -k_2 = -k_1$

$k_1 = k_2$

$\Rightarrow 2k_1 = k_4$ $\quad k_1 = -\frac{1}{2}k_4$

$k_1 + k_3 - 2k_1 = 0$

$k_1 = k_3$

$k_1 = k_2 = k_3 = -\frac{1}{2}k_4$

There is a non zero solution $\therefore$ vectors are linearly dependent i.e. NOT linearly independent

Does it Span $\mathbb{R}^3$?

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = k_1\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + k_3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + k_4\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

if letter = letter does not span R^n

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} k_2 + k_3 + k_4 \\ k_1 + k_3 + k_4 \\ k_1 + k_2 + k_4 \end{pmatrix}$$

There is always a solution $\therefore$ The vectors span $\mathbb{R}^3$.

Basis of $\mathbb{R}^3$?

Vectors are linearly dependent, because any set of n vectors with $n > 3$ is linearly dependent in $\mathbb{R}^3$, hence they do not form a basis in $\mathbb{R}^3$.

2.

10. Find the subset of vectors that form a basis of their span:

$u_1 = (1,0,1)$ $\quad u_2 = (-1,0,-1)$ $\quad u_3 = (2,0,2)$

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 2 & 0 & 2 \end{pmatrix} \xrightarrow{R_2 + r_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix} \xrightarrow{r_3 - 2r_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Subset $\{u_1\}$ forms a basis of their span

$u_1 = (2,1,-1)$ $\quad u_2 = (1,0,1)$ $\quad u_3 = (4,1,1)$

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 4 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 - 2r_1} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} \xrightarrow{R_1 \div 2} \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & -1 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & 3 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Subset $\{u_1, u_2, u_3\}$ form a basis of their span

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$\int x \sin(x)\,dx$ $\qquad$ $\int u \left(\frac{dv}{dx}\right)dx = uv - \int \frac{du}{dx} v\,dx$

$u = x$ $\qquad$ $dv/dx = \sin(x)$

$du/dx = 1$ $\qquad$ $v = -\cos(x).$ $\qquad$ $-x\cos(x) - \int 1(-\cos x)\,dx$

$\int x \sin(x)\,dx$ $\qquad$ $-x\cos(x) + \sin x$

$\int 2x \sin nx$

$u = 2x$ $\qquad$ $\frac{dv}{dx} = \sin(nx)$

$\frac{du}{dx} = 2$ $\qquad$ $v = -\frac{\cos(nx)}{n}$

$= uv - \int \frac{du}{dx} v\,dx$

$-\frac{2x\cos(nx)}{n} - \int -2\frac{\cos(nx)}{n}$

$= -\frac{2x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}$

2.

Fourier Integral Representation

$$f(x) = \int_0^{\infty} A(w)\cos(wx) + B(w)\sin(wx) \, dw$$

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x)\cos(wx)$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} fx \quad \sin x$$