Tutorial 7: MA1E01

Analysis of Functions

1. For each of the following functions, find (*a*) the intervals on which f is increasing, (*b*) the intervals in which f is decreasing, (*c*) the open intervals on which f is concave up, (*d*) the open intervals on which f is concave down, and (*e*) the x-coordinates of all inflection points:

   (a) $f(x) = x^2 - 3x + 8$

   (b) $f(x) = x^4 - 5x^3 + 9x^2$

   (c) $f(x) = \dfrac{x-2}{(x^2 - x + 1)^2}$

2. Show that $x < \tan x$ if $0 < x < \pi/2$. (HINT: Show that the function $f(x) = \tan x - x$ is increasing on $[0, \pi/2)$.)

3. For each of the following functions, locate the critical points and identify which critical points are stationary points:

   (a) $f(x) = 4x^4 - 16x^2 + 17$

   (b) $f(x) = \dfrac{x^2}{x^3 + 8}$

   (c) $f(x) = |\sin x|$

4. For the polynomial $p(x) = x(x^2 - 1)^2$, find

   (a) the coordinates of the x and y-intercepts,

   (b) the stationary points,

   (c) the intervals over which f is increasing and decreasing,

   (d) the intervals over which f is concave up and concave down and

   (e) any inflection points.

   Hence sketch the graph of $p(x)$.

5. For the rational function $f(x) = x^2/(x^2 - 4)$, find

   (a) the symmetries of the function,

   (b) the coordinates of the x and y-intercepts,

   (c) the horizontal and vertical asymptotes,

   (d) the stationary points,

   (e) the intervals over which f is positive and negative,

   (f) the intervals over which f is increasing and decreasing,

   (g) the intervals over which f is concave up and concave down and

   (h) any inflection points.

   Hence sketch the graph of $f(x)$.

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1.a $f(x) = x^2-3x+8$ increasing $f'x > 0$ decreasing $f'x < 0$

$f'(x) = 2x-3 = 0$

$2x = 3$

$x = \frac{3}{2}$

$x > \frac{3}{2}$ increasing $[\frac{3}{2}, \infty)$

or $x < \frac{3}{2}$ $(-\infty, \frac{3}{2}]$ decreasing

$f''(x) = 2 > 0$ concave up for all $x \in (-\infty, \infty)$

inflection point $f''(x) = 0$ but $f''(x) = 2$ no inflection point

b $f(x) = x^4 - 5x^3 + 9x^2$

$f'(x) = 4x^3 - 15x^2 + 18x$

$x(4x^2 - 15x + 18) = 0$

~~$x(4x - 6)(4x - 3)$~~

$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\frac{15 \pm \sqrt{15^2 - 4(4)(18)}}{8}$ $\frac{15 \pm 9}{8}$

2·2 6·3
4·1 9·2
4·1
3·6

When x is positive when $x > 0$

neg when $x < 0$

$4x^2-15x+18 = \frac{x \pm \sqrt{15^2 - 4(4)(18)}}{2(4)}$ (complex number

at $x = 0$ $4x^2 - 15x + 18 > 0$

complex root is above x axis (ie. positive)

$f'(x) > 0$ $\forall x > 0$

< 0 $\forall x < 0$

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b. $f'(x) > 0$ for $x \in (0, \infty)$
$f'(x) < 0$ for $x \in (-\infty, 0)$

$f''(x) = 12x^2 - 30x + 18$
$6(2x^2 - 5x + 3)$
$6(2x - 3)(x - 1)$

$f''(x) = 0$ $2x - 3 = 0$
$x = \frac{3}{2}$ or $x = 1$
$(-\infty, 1)$ $(1, \frac{3}{2})$ $(\frac{3}{2}, \infty)$
check if positive or negative for interval

2 prove $f(x)$ greater than 0
$\frac{\sin x}{\cos x} - x$
dy $\sin x \cos x^{-1} - x$
$\sin x(-\sin x^{-2}) + \cos x^{-1}(\cos^{-1}) - 1$
$\sec^2 x - 1 = 0$
$\sec^2 x = 1$
$\sec x = \sqrt{1} = 1$
$\sec x = 1$: max or min

$f'(x) = \sec^2 x - 1 = \frac{1}{\cos^2 x} - 1$

on interval $\cos x$ is on $[0, 1]$
$\cos^2 x$ is between $[0, 1]$
$\frac{1}{\cos^2 x}$ invert ∞ — 1
$1 \leq \frac{1}{\cos x^2} \leq \infty$ (-1 from it)

$0 \leq \frac{1}{\cos^2 x} \leq \infty$
$f'(x)$ $[0, \infty)$ increasing $\forall x \in [0, \frac{\pi}{2}]$
$f(x) > f(0)$ $\forall x \in [0, \frac{\pi}{2}]$
$\tan x - x > 0 \Rightarrow \tan x - x > 0 \Rightarrow \tan x > x \Rightarrow x < \tan x$

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3. critical points stationary point

a. $f(x) = 4x^4 - 16x^2 + 17$

$f'(x) = 16x^3 - 32x = 0$

$16x(x^2 - 2x) = 0$

$x = 0 \qquad x = \pm\sqrt{2} \qquad$ = critical points

S.P $\quad x = 0 \qquad x = \pm\sqrt{2}$

b. $f(x) = \dfrac{x^2}{x^3+8} \qquad x^2(x^3+8)^{-1}$

$x^2(-1)(3x^2) + (x^3+8)^{-1}(2x)$

$-3x^4 \quad \dfrac{+2x}{x^3+8} \quad = 0$

$f'(x) = \dfrac{2x^4 + 16x - 3x^4}{(x^3+8)^2}$

$= \dfrac{16x - x^4}{(x^3+8)^2} \qquad 16x - x^4 = 0$

$x = 0$ or $\quad 16x - x^4 = 0$

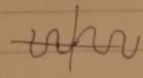
$x = \sqrt[3]{16}$

$x =$

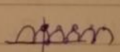
$f'(0)$ is defined $\qquad f'(\sqrt[3]{16}) \Rightarrow \dfrac{(\sqrt[3]{16})^2}{(\sqrt[3]{16})^3 + 8} = 0$ ✓

$\frac{0}{8} = 0$ ✓ $\qquad$ it is a stationary point

c. $f(x) = |\sin x|$



$|\sin x| =$



critical point

$= \begin{cases} \sin x & \sin x > 0 \\ -\sin x & \sin x < 0 \end{cases}$

$f'(x) = \begin{cases} \cos x & > 0 \\ -\cos x & < 0 \end{cases} \qquad \cos x \quad [-\pi, \pi]$

$= \begin{cases} \cos x & 0 < x < \pi \\ -\cos x & -\pi < x < 0 \end{cases}$

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$f'(x) = 0$ where $\cos x = 0$ $= (2n+1)(\frac{\pi}{2})$ odd multiple of $\frac{\pi}{2}$ 1, 3, 5, 7 …

other critical point $x = 0, \pm n\pi$ $n \in \mathbb{N}$

C.P $\pm(2n+1)(\frac{\pi}{2})$, $+n\pi$

S.P $x = \pm(2n+1)(\frac{\pi}{2})$

4 a. $x(x^2-1)^2$

y intercept $x = 0 \Rightarrow 1$

x intercept $y = 0$

$x(x^2-1)^2 = 0$

$x = 0$ $(x^2-1)^2 = 0$

$(x^2-1)(x^2-1) = 0$

$(x+1)(x-1)(x+1)(x-1) = 0$

cut y axis $(0,1)$ $x = 1$ $x = -1$ $x = 0$

cuts x axis $(1,0)$ $(-1,0)$ $(0,0)$

b. $(1,0)$, $(-1,0)$ $(0,1)$ not $(0,0)$ $p'(x) = \cancel{4x} \; 5x^4 - 6x^2 + 1$

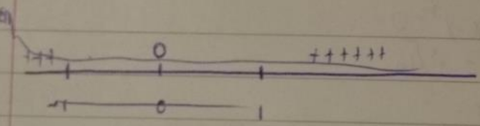
$u = x2$

$5u^2 - 6u + 1$

$(\cancel{5u}$ $(5u-1)(u-1)$

$p'(x) = 0$ $u = \frac{1}{5}$ or $u = 1$

$x = \pm\sqrt{5}$ or $x = \pm 1$



$(-\infty, -1)$ $(-1, -\frac{1}{\sqrt{5}})$ $(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}})$, $(\frac{1}{\sqrt{5}}, 1)$ $(1, \infty)$

$p'(x) = 5x^4 - 6x^2 + 1$

$p''(x) = 20x^3 - 12$

$= 4x(5x^2 - 3)$

$x = 0$ or $x = \pm\sqrt{\frac{3}{5}}$ pof inflexion

$(-\infty, -\frac{\sqrt{3}}{5})$ $(\frac{-\sqrt{3}}{5}, 0)$ $(0, \frac{\sqrt{3}}{5})$ $(\frac{\sqrt{3}}{5}, \infty)$