21/10/13. Maths ①

Example: Find the surface area of the paraboloid $z=x^2+y$ below the plane $z=1$.

Solution: PREVIOUS SHEET →

$$S = \iint \sqrt{1+4x^2+4y^2}\, dA$$

It's easier if we use polar coordinates $x=\rho\cos\theta$ $y=\rho\sin\theta$

$\Rightarrow z=\rho^2$ $(\rho\cos\theta)^2 = \rho^2$ $\Rightarrow 0 \le \rho \le 1$

$$S = \int_0^{2\pi}\left[\int_0^1 \sqrt{1+4\rho^2}\,\rho\, d\rho\right] d\theta$$

$$= 2\pi \int_0^1 \sqrt{1+4\rho^2}\, d\rho$$

$t=\rho^2 \Rightarrow dt = 2\rho d\rho$

$$S = 2\pi \int_0^1 \sqrt{1+4t}\, \frac{dt}{2}$$

$$= 2\pi\left(\frac{1}{2}\right)(1+4t)^{3/2}\left(\frac{1}{4}\right)\Big|_{t=0}^{1}$$

$$= 2\pi \frac{1}{12}(5^{3/2}-1) = \frac{\pi}{6}(5\sqrt{5}-1) \approx 5.33$$

only works with $z=f(x,y)$

Lamina

A lamina is a region of space with mass m and a variable density $\delta(x,y)$. The mass is given by

$$m = \iint_R \delta(x,y)\, dA$$

The centre of mass/centre of gravity of the lamina is $(\bar{x},\bar{y})$ where $\bar{x} = \frac{1}{m}\iint_R x\,\delta(x,y)\, dA$

$$\bar{y} = \frac{1}{m}\iint_R y\,\delta(x,y)\, dA$$

For a lamina with constant density, the centre of gravity is called the centroid.

This is the same as finding the centre of mass of an object with non-uniform mass distribution.

Example: Compute the mass and centre of mass of the lamina with density $\delta(x,y) = x^2+y^2$ inside the unit circle.

Solution: $M = \iint_R (x^2+y^2)\,dA \overset{\text{polar coord}}{=} \int_0^{2\pi}\int_0^1 p^2 \cdot p\,dp\,d\theta$ ← unit circle

$2\pi \int_0^1 p^3\,dp = 2\pi \frac{p^4}{4}\Big|_{p=0}^{1} = \frac{\pi}{2} = M$

$\bar{x} = \frac{1}{M} \iint_R x(x^2+y^2)\,dA$ $\qquad x = p\cos\theta \quad y = p\sin\theta$

$= \frac{2}{\pi} \int_0^{2\pi}\int_0^1 (p\cos\theta)(p^2)\,p\,dp\,d\theta$

$= \frac{2}{\pi}\int_0^{2\pi}\int_0^1 p^4\cos\theta\,dp\,d\theta$

$= \frac{2}{\pi}\int_0^{2\pi} \frac{p^5}{5}\cos\theta\Big|_{p=0}^{1}\,d\theta$

$= \frac{2}{\pi}\int_0^{2\pi} \frac{1}{5}\cos\theta\,d\theta$

Aside: $\int_0^{2\pi}\cos\theta\,d\theta = \int_0^{2\pi}\sin\theta\,d\theta = 0$ $\qquad \cos(\theta+2\pi) = \cos\theta$, $\sin(\theta+2\pi) = \sin\theta$

$\Rightarrow \frac{2}{\pi}\frac{1}{5}(-\sin\theta)\Big|_{\theta=0}^{2\pi}$

$\frac{2}{5\pi}\big(\underbrace{-\sin 2\pi}_{=0} + \underbrace{\sin 0}_{=0}\big) = 0$

$\bar{y} = \frac{1}{M}\iint \underbrace{y}_{p\sin\theta}(x^2+y^2)\,dA = 0$

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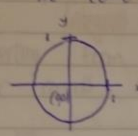
i.e. The centre of graph is at (0,0)



4. TRIPLE INTEGRALS

We are mainly interested in cylindrical or spherical coords, but we will begin in rectangular coords

A Triple Integral is defined as an infinite of the sum

$$\iiint_G f(x,y,z)\,dV = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta V_k$$

In rectangular coordinates we have the same rules as before, except we now integrate over a solid G. and looks like

$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right] dA$$

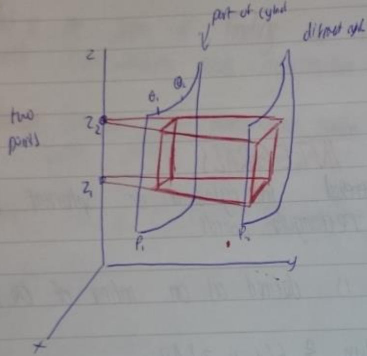
This is also true for cylindrical coordinates but not for spherical coords.

The volumne of ~~a~~ solid G is

$$\text{Volume } G = \iiint_G dV$$

Generally in mechanics fluid dynamics and electrostatics, cylindrical and spherical coords are more useful

Triple Integrals in Cylindrical Coordinates



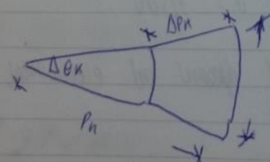
A Small volume element in cylindrical coords is a region between two radii ρ_1 and ρ_2, two angles θ_1 and θ_2 and two heights z_1 and z_2

We integrate over the "cylindrical wedges"

$$\iiint_G f(\rho,\theta, z)\,dV = \lim_{n\to\infty}\sum_{k=1}^{n} f(\rho_k,\theta_k,z_k)\,\Delta V_k$$

where $\Delta V_k = [\text{area of base}].[\text{height}]$

$$= \rho_k\,\Delta\rho_k\,\Delta\theta_k\,\Delta z_k$$



Theorem: Let G be a solid with upper surface $z = g_2(\rho,\theta)$ and ~~upper~~ lower surface $z = g_1(\rho,\theta)$ in cylindrical polar coords. If the projection of G onto the xy-plane is a Simple polar region R and if $f(\rho,\theta,z)$ is continuous on G then:

$$\iiint_G f(\rho,\theta,z)\,dV = \iint_R \left[\int_{g_1(\rho,\theta)}^{g_2(\rho,\theta)} f(\rho,\theta,z)\,dz\right] dA$$

$$= \int_{\theta_1}^{\theta_2}\left[\int_{\rho_1(\theta)}^{\rho_2(\theta)}\left[\int_{g_1(\rho,\theta)}^{g_2(\rho,\theta)} f(\rho,\theta,z)\,dz\right]\rho\,d\rho\right]d\theta$$ further and to colour

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Triple Integrals in Cylindrical Coordinates

Converting from rectangular coords to cylindrical polar coords the triple integral becomes

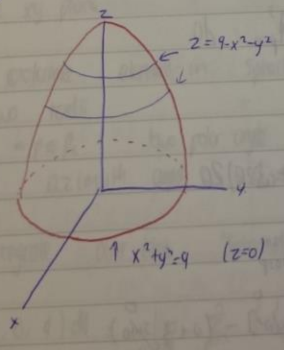
$$\iiint_G f(x,y,z)\,dx\,dy\,dx = \iiint_G f(\rho\cos\theta, \rho\sin\theta, z)\,\rho\,dz\,d\theta$$

Example: Use cylindrical coords to calculate

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2\,dz\,dy\,dx$$

Solution: We see that z has lower limit 0, but the upper limit depends on x and y via $z = 9-x^2-y^2$.

After integrating z, we see that the limits of y depend on x.



with largest value in the xy plane

i.e. $z=0 \Rightarrow x^2+y^2=9$

and so $-\sqrt{9-x^2} \le y \le \sqrt{9-x^2}$

Finally, the x variable when $y=0 \Rightarrow x^2 = 9$

$\Rightarrow -3 \le x \le 3$

Now converting to cylindrical coordinates,

$$9-x^2-y^2 = 9-\rho^2$$

2

2.

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2y^2} x^2 \, dz\,dy\,dz$$

circle area

$$= \iint_R \left[\int_0^{9-p^2} (p\cos\theta)^2 dz \right] dA$$

$x^2+y^2=9$

$p^2=9$ $\quad 0 \leq p \leq 3$

only take 0 to 3 because always positive avoid duplication

area element in polar coord and cylindrical coords

$$= \int_0^{2\pi}\int_0^{3}\int_0^{9-p^2} (p^2\cos^2\theta)dz \; (p\,dp\,d\theta)$$

$$= \int_0^{2\pi}\int_0^{3}\int_0^{9-p^2} p^3\cos^2\theta \; dz\,dp\,d\cancel{\theta}$$

$$= \int_0^{2\pi}\int_0^{3} p^3\cos^2\theta \; z\Big|_{z=0}^{9-p^2} dp\,d\theta$$

$$\int_0^{2\pi}\int_0^{3} (9p^3-p^5)\cos^2\theta \; dp\,d\theta$$

$$\int_0^{2\pi}\int_0^{3} \left(\frac{9}{4}p^4 - \frac{p^6}{6}\right)\Big|_{p=0}^{3} d\theta$$

$$\frac{243}{4}\int_0^{2\pi} \cos^2\theta \, d\theta$$

$$= \frac{243}{4}\int_0^{2\pi} \frac{1}{2}(1+\cos 2\theta) d\theta$$

$$\Rightarrow \frac{243}{4}\left(\frac{1}{2}\right)\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\theta=0}^{2\pi}$$

$$\Rightarrow \frac{243}{4}\cancel{\frac{1}{2}}\left[\cancel{2}\pi + \frac{1}{2}\cancel{\sin 4\pi}^{\,0}\right] - \left(0 + \frac{1}{2}\cancel{\sin 0}^{\,0}\right)$$

$$= \frac{243}{4}\pi$$

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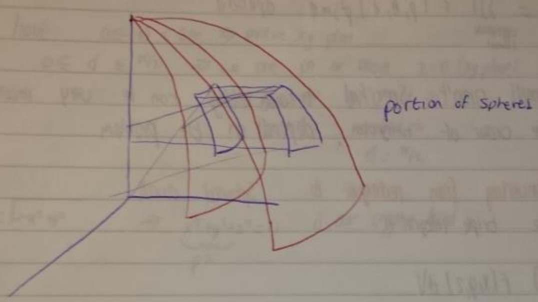
Triple Integrals in Spherical coords



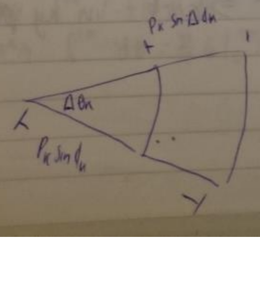
Note $\rho =$ constant gives a Sphere, $\theta =$ constant, produces a half-plane
$\phi =$ constant gives a right circular cone, and $\phi = \pi/2$ gives the xy plane

A small volume element in Spherical coords is the volume between two radii
$\rho_1 \le \rho \le \rho_2$ two polar angles $\theta_1 \le \theta \le \theta_2$
and two azimuth angles $\phi_1 \le \phi \le \phi_2$

We can integrate over these "Spherical wedges"

$$\iiint_G f(\rho, \theta, \phi)\, dV = \lim_{n \to \infty} \sum_{k=1}^{n} f(\rho_k, \theta_k, \phi_k)\, \Delta V_k$$

Where $\Delta V_k = [\text{area of base}] \times [\text{height}] = \rho_k^2 \sin \phi_k \, \Delta\rho_k \, \Delta\theta_k \, \Delta\phi_k$



$\Delta z = \Delta\rho_k \cos\Delta\phi_k \approx \Delta\rho_k$

4.

$$\iiint_G f(\rho, \theta, \phi)\, dV$$

$$= \iiint_{\substack{\text{appropriate} \\ \text{limits}}} f(\rho, \theta, \phi)\, \rho^2 \sin\phi \;\; d\rho\, d\theta\, d\phi$$

limits aren't specified because they can be very involved and the order of integration depends on the problem

Converting from rectangular to spherical coords:
the triple integral is

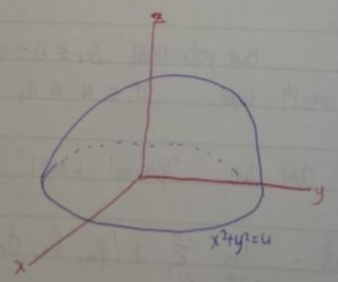
$$\iiint_G f(x, y, z)\, dV$$

$$= \iiint_G f(\rho \sin\phi \cos\theta,\ \rho \sin\phi \cos\theta,\ \rho \cos\phi)\, \rho^2 \sin\phi \;\; d\rho\, d\phi\, d\theta$$

Example: Use spherical coords to evaluate

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2+y^2+z^2} \;\; dz\, dy\, dx$$

Solution:



we have $0 \leq z \leq \sqrt{4-x^2-y^2}$ which depends on x and y, also $-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$ $\xrightarrow{\text{for } z=0}$ $\Rightarrow$ circle in xy plane,
$x^2=4$ for $y=0$ $\quad -2 \leq x \leq 2$

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Jacobians

What does it actually mean to change coordinates? It is ~~even~~ in fact the same as a change of variable when integrating.

In that situation, we change integration variable from x to u via $x = g(u)$ assuming g is differentiable

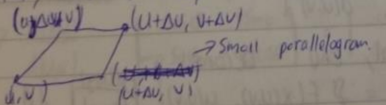
$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} g(u) \frac{dx}{du}\,du$$

$$= \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))\,g'(u)\,du$$

If we have a double integral over x and y, we can change to new coordinates u and v in the same way.

We can represent $\vec{r}$ in terms of the new variables via
$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j}$

We can view integrating over this by finding the region in terms of u, v. If we look at a small area with vertices $\vec{r}(u,v)$, $\vec{r}(u+\Delta u, v) + \vec{r}(u, v+\Delta v) + \vec{r}(u+\Delta u, v+\Delta v)$



We can express the area element dA either in x, y via $dA = dx\,dy$ or using u, v as the area of this parallelogram

For a small region, it has sides $\frac{d\vec{r}}{du}\Delta u$, $\frac{d\vec{r}}{dv}\Delta v$ with area

$$\Delta A = \left\| \frac{d\vec{r}}{du}\Delta u \times \frac{d\vec{r}}{dv}\Delta v \right\|$$

$$= \left\| \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} \right\| \Delta u\,\Delta v$$

$$\frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{dx}{du} & \frac{dy}{du} & 0 \\ \frac{dx}{dv} & \frac{dy}{dv} & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{dx}{du} & \frac{dy}{du} \\ \frac{dx}{dv} & \frac{dy}{dv} \end{vmatrix} \vec{k} = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} \vec{k}$$

This determint is the Jacobian

A Jacobian is produced when we change coordinates, most commonly from the xy-plane a new uv-plane, via the equation $x = x(u,v)$ and $y = y(u,v)$

We will denote it as either $J(u,v)$ or $\frac{d(x,y)}{d(u,v)}$ and is given by

$$J(u,v) = \frac{d(x,y)}{d(u,v)} = \begin{vmatrix} dx/du & dy/du \\ dx/dv & dy/dv \end{vmatrix} = \frac{dx}{du}\frac{dy}{dv} - \frac{dy}{du}\frac{dx}{dv}$$

It appears through the area element

$$\Delta A = \left| \frac{d(x,y)}{d(u,v)} \right| \Delta u \Delta v$$

In a double integral, this becomes

$$\iint_R f(x,y)\, dA_{xy} = \iint_S f(x(u,v), y(u,v)) \left| \frac{d(x,y)}{d(u,v)} \right| dA_{uv}$$

and $dA_{uv} = du\,dv$

S is the region in the uv-plane what corresponds to the region R in the xy-plane

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$ $aei + bfg + cdh - afh - bdi - ceg$

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For triple integral we have Jacobian

$$J(u,v,w) = \frac{d(x,y,z)}{d(u,v,w)} = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} & \frac{dx}{dw} \\ \frac{dy}{du} & \frac{dy}{dv} & \frac{dy}{dw} \\ \frac{dz}{du} & \frac{dz}{dv} & \frac{dz}{dw} \end{vmatrix}$$

which give the change in volume element

$$\Delta V \approx \left|\frac{d(x,y,z)}{d(u,v,w)}\right| \Delta u \Delta v \Delta w$$

and the triple integral becomes

$$\iiint_G f(x,y,z)\, dV_{xyz}$$

$$= \iiint_H f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{d(x,y,z)}{d(u,v,w)}\right| dV_{uvw}$$

where $dV_{uvw} = du\,dv\,dw$ and H the region in uvw space corresponding to G in xyz- space

In polar coordinates $x = \rho\cos\theta$ $y = \rho\sin\theta$

$$\rightarrow \frac{d(x,y)}{d(u,v)} \begin{vmatrix} \cos\theta & +\sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix}$$

$$= \rho\cos^2\theta - (-\rho\sin^2\theta) = \rho(\cos^2\theta + \sin^2\theta) = \rho$$

$$\Rightarrow dA = dx\,dy = \rho\, d\rho\, d\theta$$

Cylindrical coordinates

$x = \rho\cos\theta$ $y = \rho\sin\theta$ $z = z$

$$\frac{d(x,y,z)}{d(\rho,\theta,z)} = \rho \quad \therefore dV = \rho\, d\rho\, d\theta\, dz$$

Spherical coordinates

$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho\cos\phi$

$$\frac{d(x,y,z)}{d(\rho,\theta,\phi)} = \rho^2 \sin\phi$$

$$dV = dx\,dy\,dz = \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi$$

Example Find the Jacobian of the change of variables

$x = \tfrac{1}{2}(u+v) \qquad y = \tfrac{1}{2}(u-v)$

Solution
$$\frac{d(x,y)}{d(u,v)} = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) - \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = -\tfrac{1}{4} - \tfrac{1}{4} = -\tfrac{1}{2}$$

Example Find the jacobian of the change of variable

$x = (au) \qquad y = bv \qquad z = cw$

Solution
$$\frac{d(x,y,z)}{d(u,v,w)} \quad \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} & \frac{dx}{dw} \\ \frac{dy}{du} & \frac{dy}{dv} & \frac{dy}{dw} \\ \frac{dz}{du} & \frac{dz}{dv} & \frac{dz}{dw} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$= abc + 0 + 0 - 0 - 0 - 0 = abc.$$

Mass and centre of gravity of solids.

We previously discussed laminar mass and centre of gravity of lamina with density $\delta(x,y)$. If instead we have a solid G with density $\delta(x,y,z)$, we can generalise to get the mass of G.

$M = \iiint_G \delta(x,y,z)\, dV$ with centre grav $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_x}{M}, \frac{M_y}{M}, \frac{M_z}{M}\right)$

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Mass and centre of gravity of solids

Solid G, with density $\delta(x,y,z)$

Mass $M = \iiint_G \delta(x,y,z)\,dV$

Centre of gravity $(\bar{x},\bar{y},\bar{z})$

$$\bar{x} = \frac{1}{m}\iiint_G x\,\delta(x,y,z)\,dV$$

$$\bar{y} = \frac{1}{m}\iiint_G y\,\delta(x,y,z)\,dV$$

$$\bar{z} = \frac{1}{m}\iiint_G z\,\delta(x,y,z)\,dV$$

Example: Find the mass and centre of gravity of a cube with square base of length in the xy plane centred on (1,1) and a height 5. with density $\delta(x,y,z) = z$

Solution: $M = \iiint_G \delta(x,y,z)\,dV$

$$\int_0^2\int_0^2\int_0^5 z\,dz\,dy\,dx = \int_0^2\int_0^2 \frac{z^2}{2}\Big|_{z=0}^{5}\,dy\,dx$$

$$= \frac{25}{2}\int_0^2\int_0^2 dy\,dx$$

$$= \frac{25}{2}\int_0^2 y\Big|_{y=0}^{2}\,dx$$

$$= 25\int_0^2 dx$$

$$= 25(x|_{x=0}^{2})$$

$$= 50$$

Due to Symmetry, the x and y coordinates of the centre of gravity are the same

We will calculate $\bar{x}$ (check $\bar{y}$ if unsure)

$$\bar{x} = \frac{1}{50} \int_0^2\int_0^2\int_0^5 (x)(z)\,dz\,dy\,dx$$

$$= \frac{1}{50} \int_0^2 \int_0^2 x\left(\frac{25}{2}\right) dy\,dx$$

same as for m

$$= \frac{1}{4} \int_0^2 \int_0^2 x\,dy\,dx$$

$$= \frac{1}{2} \int_0^2 x\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\Big|_{x=0}^{2}\right) = 1$$

By symmetry $\bar{y} = 1$ also. We could also see this by noting that density depends only on z, and so the centre of the square in the xy-plane $(1,1)$ should be the same as $(\bar{x}, \bar{y})$.

$$\bar{z} = \frac{1}{50} \int_0^2\int_0^2\int_0^5 (z)(z)\,dz\,dy\,dx$$

$$= \frac{1}{50} \int_0^2\int_0^2 \frac{z^3}{3}\Big|_{z=0}^{5}\,dy\,dx$$

$$= \frac{1}{50} \int_0^2\int_0^2 \frac{125}{3}\,dy\,dx$$

$$= \frac{5}{6} \int_0^2\int_0^2 dy\,dx$$

$$= \frac{5}{6} \int_0^2 y\Big|_{y=0}^{2}\,dx$$

$$= \frac{5}{6} \int_0^2 2\,dx$$

$$= \frac{5}{3}\, x\Big|_{x=0}^{2}$$

$$= \frac{5}{3}(2) = \frac{10}{3}$$

Centre of mass $(\bar{x}, \bar{y}, \bar{z}) = \left(1, 1, \frac{10}{3}\right)$