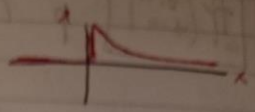
1/4/2014 Applied Probability Question Sheet 2

Q1 a $p(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \text{ (otherwise)} \end{cases}$



$x_1, \ldots, x_n$

$L = p(x_1) \times p(x_2) \ldots \times p(x_n)$

$L = \lambda e^{-\lambda x_1} \lambda e^{-\lambda x_2} \ldots \lambda e^{-\lambda x_n}$

$L = (n\lambda) e^{-n\lambda \sum_{i=1}^{n} x_i}$

Log likelihood $\quad \log (n\lambda) e^{-n\lambda \sum_{i=1}^{n} x_i}$

$\overset{n\log(\lambda)}{\cancel{\log(n\lambda)}} + -n\lambda \sum_{i=1}^{n} x_i \qquad n\log(\lambda) - \lambda \sum x$

$\cancel{\log(n) + \log(\lambda) - n\lambda \sum x_i}$

B $\frac{dL}{d\lambda} = \cancel{\frac{1}{\lambda} - n \sum x_i = 0}$ $\qquad n\log(\lambda) + \lambda \sum x_i$ check is max?

$-\frac{1}{\lambda} = n \sum x_i \qquad \frac{n}{\lambda} + \sum x_i = 0 \qquad \frac{d^2L}{d\lambda^2} = \frac{-n}{\lambda^2} < 0$

at max $= 0$

$\lambda = \qquad \frac{n}{\lambda} = \sum x_i$ $\quad$ negative $\Rightarrow$ maximum

$\lambda = \frac{n}{\sum_{i=1}^{n} x_i}$

2A $p(Y = y; p) = \text{choose} \binom{y-1}{y-k} p^k (1-p)^{y-k} \qquad y = k, k+1, k+2 \ldots$

equivalent $\frac{(y-1)!}{(y-k)!(k-1)!} p^k (1-p)^{y-k}$

p = success $\qquad 1-p$ = failure

y - number of trials $\qquad k$ - number of successes

$y_1, \ldots y_n$ integers - trials

Likelihood $= p(y_1, y_2, \ldots, y_n)$

$= p(y_1)\, p(y_2) \ldots p(y_n)$

$= \binom{y_1-1}{y_1-k} p^k (1-p)^{y_1-k} \binom{y_2-1}{y_2-k} p^k (1-p)^{y_2-k} \ldots \ldots \binom{y_n-1}{y_n-k} p^k (1-p)^{y_n-k}$

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2. $\left[\prod_{i=1}^{n}\binom{y_i-1}{y_i-k}\right] p^{nk}(1-p)^{y_1-k,\ y_2-k, \dots\ y_n-k}$

$= \quad '' \quad '' \quad (1-p)^{\sum_{i=1}^{n} y_i - nk}$

$Log(L) \quad Log\left[\prod\binom{y_i-1}{y_i-k}\right] + Log(p^{nk}) + Log(1-p)^{\sum y_i - nk}$

$Log\left[\prod\binom{y_i-1}{y_i-k}\right] \quad + nk\, Log(p) \quad + (\sum y_i - nk) Log(1-p)$

b $\frac{dL}{dp} \quad 0 \quad + \quad nk/p \quad + \frac{-\sum y_i + nk}{1-p}$

At max $=0$ solve: $\frac{dL}{dp}=0$.

$\frac{nk}{p} = - \frac{\sum y_i - nk}{1-p} = 0$

$nk/p = \sum y_i - nk/1-p$

$nk(1-p) = p(\sum y_i - nk)$

$nk - nkp = p(\sum y_i - nk)$

$nk = p(\sum y_i - \cancel{nk} + \cancel{nk})$

$p = nk / \sum_{i=1}^{n} y_i$

Checking $\frac{dL^2}{dp^2} \quad -\frac{nk}{p^2} \quad - \frac{\left(-(\sum y_i + nk)\right)}{(1-p)^2}$

negative

$\frac{-nk}{p^2} \quad - \quad \frac{(\sum y_i - nk)}{(1-p)^2}$ negative $\Rightarrow$ is a maxmum

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Q3 a.i $P(X=-1) = 1/3$ $P(y \neq 0) = 1/3$

$P(X=0) = 1/3$ $P(y=1) = 2/3$

$P(X=1) = 1/3$

a.ii $P(X+2Y \geq 0)$

x	y
-1	1
0	0
0	1
1	0
1	1

All combination true except for $X=-1, Y=0$ $P(X=-1, Y=0) = 0$

So $P(X+2Y \geq 0) = 1$ (Sum of probability for all possible value ≥ 0 is 1)

a.iii $E(X) = -1(1/3) + 0(1/3) + 1(1/3) = 0$

$E(Y) = 0(1/3) + 1(2/3) = 2/3$

$E(XY) = -1(0)(0) + -1(1)(1/3) + 0(0)(1/3) + 0(1)(0) + 1(0)(0) + 1(1)(1/3) = 0$

for every value of X and Y in table

$P(X=-1, Y=0) = 0 \neq \underset{X=-1}{1/3} \times \underset{Y=0}{1/3}$

$0 \neq 1/9$ $P(X)P(Y)$ Not independent

3b. $P(0 \leq X \leq 0.5,\ 0.5 \leq Y \leq 1.0)$

$\int_{0.5}^{1}\int_{0}^{0.5} \frac{6}{7}(x+y)^2\, dx\, dy$

$\frac{6}{7}\left[\frac{x^3}{3} + x^2y + y^2x\right]\Big|_{x=0}^{0.5}$

$\frac{6}{7}\int_{0.5}^{1}\left[\frac{0.5^3}{3} + 0.25y + 0.5y^2\right] dy$

$\frac{6}{7}\left[\frac{0.5^3}{3}y + \frac{0.25y^2}{2} + \frac{0.5y^3}{3}\Big|_{y=0.5}^{1}\right] = \frac{150}{672} = 25/112$

3 b ii. Marginal density functions of X and Y

$$p(x) = \int_{all\ y} p(x,y)\,dy$$

$$\int_0^1 \frac{6}{7}(x+y)^2\,dy$$

$$\frac{6}{7}\int_0^1 x^2+y^2+2yx\,dy$$

$$\frac{6}{7}x^2y + \frac{6}{21}y^3 + \frac{6}{7}yx^2\Big|_{y=0}^{1}$$

$$\frac{6}{7}x^2 + \frac{6}{21} + \frac{6}{7}x$$

$$\cancel{\frac{6}{7}x^2 + \frac{8}{7}} \qquad \frac{6}{7}x^2 + \frac{2}{7} + \frac{6}{7}x$$

$$p(y) = \int_0^1 p(x,y)\,dx$$

Same $\rightarrow$ $\frac{6}{7}y^2 + \frac{2}{7} + \frac{6}{7}y$

iii. Independent?

Independent $\rightarrow$ $p(x,y) = p(x)p(y)$

but not the case here:

e.g.: $p(1/2, 1/2) = 6/7(\frac{1}{2}+\frac{1}{2})^2 = \frac{6}{7}$

$P(1/2)\,p(1/2) = (6/7\ 1/4 + 6/7\ 1/2 + 2/7)^2$ Not equal!

NOT INDEPENDENT

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Q3. (i Marginal Distribution of X and Y

$\mu_x = -6$ $\sigma_x = 2.5$ $\rho = 0.9$

$\mu_y = 3$ $\sigma_y = 1$

X is $N(-6, 2.5^2)$ $\sim N(\mu_x, \sigma_x^2)$

Y is $\sim N(3, 1^2)$ $\sim N(\mu_y, \sigma_y^2)$

ii Distribution of X given Y=4

Conditional distribution is also normal

$y=4$

mean $= \mu_x + \dfrac{\rho \sigma_x (y - \mu_y)}{\sigma_y}$ $\dfrac{-6 + 0.9(2.5)(4-3)}{1}$

$-6 + 0.9(2.5) = -3.75.$

Variance $= \sigma_x^2 (1-\rho)^2$ $= 2.5^2(1-0.9^2)$

$6.25(1-0.81) = 6.25(0.19)$

$= 1.1875$

$X_{\text{given } Y=4} \sim N(-3.75, 1.1875)$

iii. $P(-4 \leq X \leq 0)$ $(6, 2.5^2)$

$P\left(\dfrac{-4-(-6)}{2.5} \leq \dfrac{X-(-6)}{2.5} \leq \dfrac{0-(-6)}{2.5}\right)$

$P(+0.8 \leq Z \leq 2.4)$

$P(Z \leq 2.4) - P(Z \leq 0.8) \rightarrow 0.992 - 0.788 = 0.204$

$\Rightarrow P(-4 < X \leq 0)$

$P\left(\dfrac{-4-(-3.75)}{\sqrt{1.1875}} \leq \dfrac{X-(-3.75)}{\sqrt{1.1875}} \leq \dfrac{0-(-3.75)}{\sqrt{1.1875}}\right)$

$P(-0.23 \leq Z \leq 3.4)$

$P(Z \leq 3.4) - P(Z \leq -0.23) = 0.591$