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CHAPTER 4

INTEGRATION

26/11/12 Maths Week 10 Chapter 4

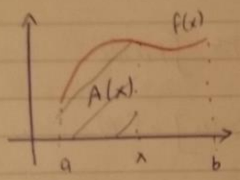
~~Application of Definite Integral~~ INTEGRATION

~~Anti-Derivative and the indefinite integrals~~

It turns out that if a continuous function $f(x)$ over $[a,b]$ and $A(x)$ is the area under $f(x)$ between $[a,x]$ where $x \in [a,b]$ then

$$A'(x) = f(x)$$

i.e. The area under $f(x)$ between $[a,x]$ is a function whose derivative gives us $f(x)$.



Example: Find the area under the graph $y = x^2$, over the interval $[3,6]$

We require $A(x)$ such that $A'(x) = x^2$

$A(x) = \frac{1}{3}x^3 + C$ gives us the area under $f(x)$ over the interval $[a,x]$

To find the constant C we use the fact that

$A(3) = 0 \Rightarrow \frac{1}{3}(3)^3 + c = 0 \Rightarrow C = -9$

$A(x)$ $\frac{1}{3}x^3 - 9$ gives area under $f(x)$ over interval $[3,x]$

~~The~~ area over the interval $[3,6]$ given by

$A(6)$ $\frac{1}{3}(6)^3 - 9 = 63$

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Anti derivatives: A function $F(x)$ is called an anti-derivative of a function $f(x)$ on an interval I if

$$F'(x) = f(x) \quad \text{for all } x \in I$$

Given an anti-derivative $F(x)$ of $f(x)$, all other anti-derivatives may be expressed as $F(x) + C$, where C is a constant

Indefinite Integral: Finding an anti-derivative is known as anti-differentiation or Integration

$$\frac{dF(x)}{dx} = f(x) \Leftrightarrow F(x)+c = \int f(x)dx$$

Differentiating both side $\frac{d}{dx}\left[\int f(x)dx\right] = F'(x) = f(x)$

i.e. differentiating the indefinite integral gives us back the function

Important Integrating formulas:

$$\int dx = x + c \qquad \int \sec^2 x\,dx = \tan x + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \qquad \int \operatorname{cosec}^2 x\,dx = -\cot x + C$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \sin x\,dx = -\cos x + c$$

Properties of the indefinite Integral

Let $F(x)$, $G(x)$ be anti derivative of $f(x)$ and $g(x)$ respectively

i.e. $\int f(x)dx = F(x) + c$

$\int g(x)dx = G(x) + c$

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then: $\int c f(x)dx = cF(x) + C$

$\int f(x) \pm g(x) = F(x) \pm G(x) + C$

Example

Evaluate $\int(4x^3+2x+9\cos x)dx$

$= \frac{4x^4}{4} + \frac{2x^2}{2} + 9\sin x + c$

$x^4 + x^2 + 9\sin x + c$

Solving differential equations (initial value problem)

Given some differential equation of the form:

$\frac{dy}{dx} = f(x)$ (1)

Solutions are anti-derivatives of $f(x)$ $y = \int f(x)dx + c$

To determine C we require some initial data $y(x_0) = y_0$ (2)

Equation (1) and (2) give a unique solution. This is an initial value problem

exam

Example: $\frac{dy}{dx} = x^2 - 4$, $y(0) = -1$

$y(x) = \int(x^2-4)dx + c$

$\frac{1}{3}x^3 - 4x + c$

$y(0) = \frac{1}{3}(0)^3 - 4(0) + c = -1$

$\Rightarrow c = -1$

$\frac{1}{3}x^3 - 4x - 1$

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Integration by Substitution:

Suppose $F(x)$ is an antiderivative of $f(x)$.

i.e. $F'(x) = f(x) \Leftrightarrow \int f(x)dx = F(x) + c$

Consider the composition $F(g(x))$ $\quad \frac{dF(g(x))}{dx} = \frac{dF(x)}{dy} \cdot \frac{dy}{dx} = F'(g(x)) \cdot g'(x)$

$\Leftrightarrow \int F'(g(x))g'(x)dx = F(g(x)) + C$ $\quad$ F antiderivative of f

$\Leftrightarrow \int f(g(x)) \cdot g'(x) = F(g(x)) + c$

To compute the integral, we take $u = g(x) \Rightarrow du = g'(x)dx$

$\Rightarrow \int f(u)du = F(u) + c$

Examples: $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$ $\quad$ we take $u = \sqrt{x} \Rightarrow du = \frac{1}{2}x^{-\frac{1}{2}}dx$

$\Rightarrow 2du = \frac{dx}{\sqrt{x}}$

$\int \sin(u) \cdot (2du)$

$2\int \sin u \, du = -2\cos u + c$

$= -2\cos\sqrt{x} + c$

Example: $\int (2x+7)(x^2+7x+3)^{4/5}dx$ $\quad$ $u = x^2+7x+3 \Rightarrow du = (2x+7)dx$

$\int (2x+7)(x^2+7x+3)^{4/5}dx$

$\int u^{4/5}du$

$\frac{5u^{9/5}}{9} + C \Rightarrow \frac{5}{9}(x^2+7x+3)^{9/5} + c$

Example: $\int \cos^3\theta \, d\theta \Rightarrow \int \cos^2\theta \cdot \cos\theta \, d\theta$ $\quad$ $(1-\sin^2\theta)$

$\int (1-\sin^2\theta)\cos\theta \, d\theta$

$= \int \cos\theta \, d\theta - \int \sin^2\theta\cos\theta \, d\theta$

$u = \sin\theta \rightarrow du = \cos\theta \, d\theta$

$\sin\theta - \int u^2 du$ $\quad$ ~~$= \sin\theta$~~ $\frac{1}{3}$

$\sin\theta - \frac{1}{3}\sin^3\theta + c$

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Area as a Limit:

We adopt the 'Sigma' notation

recursion $\rightarrow \sum_{k=m}^{n} f(k)$ finish value, summand, start value

e.g. $\sum_{n=4}^{8} k^3 = 4^3+5^3+6^3+7^3+8^3$

Properties of sums:

1. $\sum_{n=1}^{n} c.a_k = c\sum_{n=1}^{n} a_k$ $\Rightarrow$ c doesn't depend on k

2. $\sum_{n=k}^{n} (a_k \pm b_k) = \sum_{n=1}^{n} a_k \pm \sum_{n=1}^{n} b_k$

we can also change the starting value of sum

e.g. $\sum_{n=m}^{n} a_n = \sum_{k=0}^{n-m} a_{k_m}$

$\sum_{-m}^{n} a_k = \sum_{m=0}^{n+m} a_{k-m}$

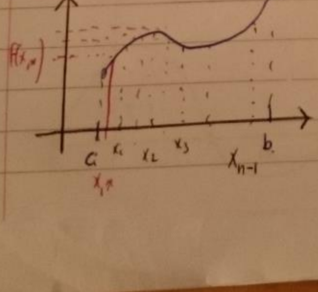
Summation formulae:

1. $\sum_{k=1}^{n} k = \underbrace{1+2+3+\ldots+n}_{\text{open form}} = \underbrace{\frac{n(n+1)}{2}}_{\text{closed form}}$

2. $\sum_{k=1}^{n} k^2 = 1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\sum_{n=1}^{n} k^3 = 1^3+2^3+3^3\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Area as a limit:



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Width of each interval $\Delta_x = \frac{b-a}{n}$ $\frac{\text{width}}{\text{Subinterval}}$ = lenght of Interval [a, b] divided by number of Sub intervals

$x_k = a + k\Delta x$, $k = 0, 1, \ldots n$, $(x = a, x_n = b)$

Area of rectangle = $f(x_1*)\Delta x$ where $x_1* \in [x_0, x_1]$
" Second " = $f(x_2*)\Delta x$ x_2* $[x_1, x_2]$
area k^{th} " = $f(x_k*)\Delta x$ x_k* $[x_{k-1}, x_k*]$

Approximation of the area is

$$A \approx \sum_{k=1}^{n} f(\cancel{x_k*}\, x_n*)\Delta x$$

as $n \to \infty$ we get the exact area

Area under curve: If f is continuous on $[a, b]$ and $f(x) \geq 0$ $\forall x \in [a, b]$ then the area under $y = f(x)$ over $[a, b]$ is $\lim_{n \to \infty} \sum_{k=1}^{n} (f(x_n*)\Delta x)$

Natural choices for x_n^* area:
- left end point: $x_k^* = x_{k-1} = a + (k-1)\Delta x$
- right end point $x_k^* = x_k = a + (k)\Delta x$
- midpoint $x_k^* = \frac{1}{2}(x_k' + x_{k-1}^*) = a + (k-\frac{1}{2})\Delta x$

A is independent of our choice of x_k^*

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Example We revisit the example of the area under $f(x)=x^2$ over the interval $[0,1]$ taking x_k^* to be right end points.

$$\Delta x = \frac{1}{n}$$

$$x_k^* = x_k = \overset{a}{0} + k\Delta x = \frac{k}{n}$$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\frac{(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{1}{6n^3}(2n^3+3n^2+n)$$

$$= \lim_{n\to\infty} \left(\frac{1}{3} + \cancel{\frac{1}{2n}}^{0} + \cancel{\frac{1}{6n^2}}^{0}\right)$$

$$= \frac{1}{3}$$

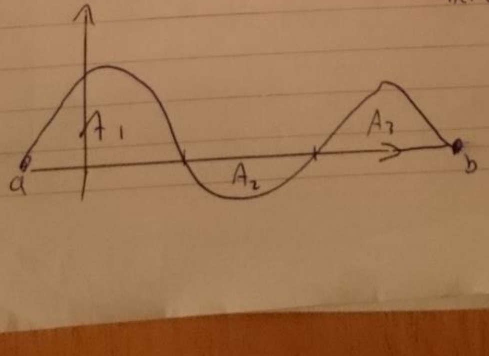
Net Signed area

When f is continuous over $[a,b]$ and f can take on both positive and negative values on $[a,b]$ we say that

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

is the net signed area

net signed area $= (a_1+a_3) - A_2$



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The definite Integral and Riemann Sums:

We now consider the case where the subinterval are not neccessarily of equal lenght

A partition of the interval $[a,b]$ is a collection of point $a = x_0 < x_1 < x_2 \quad < x_n = b$ Such that

$\Delta x_k := x_k - x_{k-1}$

The partition is regular if the subintervals have equal lenght i.e. $\Delta x_k = \Delta x = \frac{b-a}{n}$

Δx_k need not approach 0 as $n \to \infty$ for a partition that is not regular, instead let $\max(\Delta x_k) \to 0$

max all partitions

the largest of Δx_k if it goes all go

$\| \max(\Delta x_k) \| =$ max size

then the net signed area is given by $\boxed{A = \lim_{\max(\Delta x_k) \to 0} \sum_{k=1}^{n} f(x_n^*)(\Delta x_n)}$

Riemann Sum

Def^n: A function f is said to be integrable on $[a,b]$ if the limit $\lim_{\max(\Delta x_k) \to 0} \sum_{k=1}^{n} f(x_k^*)(\Delta x_k)$ exists and does not depend on the choice of partition or on the choice of x_k^*.

We denote this limit by $\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)(\Delta x_k)$

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Definite (Riemann Integral)

$$\int_a^b f(x)(dx) = \lim_{\max(\Delta x_n) \to 0} \sum_{k=1}^{n} f(x_n^*)\Delta x_n$$

where $x_n^* = [x_{n-1}, x_n]$
and $\Delta x_n = x_n - x_{n-1}$
$\max(\Delta x_n)$ = mesh size

If f is continuous on $[a,b]$, then the net Signed area is $\int_a^b f(x)dx$

Properties of the definite Integral:

1. $\int_a^a f(x)\,dx = 0$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ c is a constant

4. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)\,dx$

5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
regardless of how the points a, b, c are ordered

Theorm:

If f is integrable on a closed interval $[a,b]$ and $f(x) \geq 0$ on $[a,b]$ then $\int_a^b f(x)dx \geq 0$

If f, g are integrable on $[a,b]$ and $f(x) \geq g(x)$ on $[a,b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

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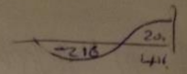

Integrability:

Def^n: A function is bounded on an interval I if there exists a positive M such that $-M \le f(x) \le M$ $\forall_x \in I$

Theorem: let f be a function that is defined on $[a, b]$

(a). If f has finite many discontinuities and $f(x)$ is bounded on $[a, b]$ then f is integrable

(b) If f is not bounded on $[a, b]$ then f is not integrable on $[a, b]$

Fundamental Theorem of Calculus (part 1)

If f is continuous on $[a, b]$ and F is <u>any</u> anti derivative, then $\int_a^b f(x)dx = F(b) - F(a)$

for indefinite integrals, we can ignore the integration const.

$$\int_a^b f(x)dx = \left[\int f(x)dx\right]_a^b$$
$$\left[F(x) + c\right]_a^b$$
$$F(b) + c - (F(a) + c)$$
$$= F(b) - F(a)$$

Example: Return to the area under $f(x) = x^2$ over $[0, 1]$

$$A = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1$$
$$= \frac{1}{3} - \frac{0}{3} = \frac{1}{3}$$

example: $f(x) = \begin{cases} x & x \le 1 \\ x^2 & x > 1 \end{cases}$ $\qquad \int_0^2 f(x)dx$

$$= \int_0^1 f(x)dx + \int_1^2 f(x)dx$$
$$\int_0^1 x\,dx + \int_1^2 x^2 dx$$
$$\left[\frac{x^2}{2}\right]_0^1 + \left[\frac{x^3}{3}\right]_1^2$$
$$F\left(\frac{1^2}{2}\right) - F\left(\frac{0^2}{2}\right) + F\left(\frac{1^3}{3}\right) + \left(\frac{2^3}{3}\right) = \frac{17}{6}$$

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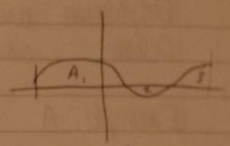
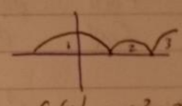
Total area between $f(x)$ and $[a, b]$
The net signed area is $A = \int_a^b f(x)\,dx$



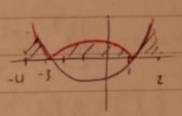
Then the total area is $= \int_a^b |f(x)|\,dx$

$A_1 - A_2 + A_3$



Example: Find the total area between $f(x) = x^2 + 2x - 3$ and $[-4, 2]$ total area $= \int_{-4}^{2} |x^2 + 2x - 3|\,dx$

$x^2 + 2x - 3 = 0$
$(x + 3)(x - 1)$
$x = -3 \quad x = 1$ roots



$$|f(x)| = \begin{cases} x < -3 & x^2 + 2x - 3 \\ -3 \leq x < 1 & -x^2 - 2x + 3 \\ x \geq 1 & x^2 + 2x - 3 \end{cases}$$

$$\int |f(x)|\,dx$$
$$= \int_{-4}^{-3} (x^2 + 2x - 3)\,dx + \int_{-3}^{1} (-x^2 - 2x + 3)\,dx + \int_{1}^{2} (x^2 + 2x - 3)\,dx$$
$$= \left[\frac{x^3}{3} + x^2 - 3x\right]_{-4}^{-3} + \left[\frac{-x^3}{3} - x^2 + 3x\right]_{-3}^{1} + \left[\frac{x^3}{3} + x^2 - 3x\right]_{1}^{2}$$
$$= \frac{46}{3}$$

In the context of linear motion we had displacement $s(t)$ velocity $= v(t) = s'(t)$
acceleration $= a(t) = v't = s''(t)$

Hence $s(t) = \int v(t)\,dt$
$v(t) = \int a(t)\,dt$

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Displacement over $[t_0, t_1]$ is $\int_{t_0}^{t_1} v(t)dt$ net signed area
Distance over $[t_0, t_1]$ total area $\int_{t_0}^{t_1} |v(t)| dt$

Example: A particle moves along a coordinate line with a velocity $v(t) = t^2 - 2t$ m/s
Find displacement over $0 \leq t \leq 3$
Find distance " "

Displacement $= \int_0^3 v(t)dt = \int_0^3 (t^2-2t)dt$
$= \left[\frac{t^3}{3} - t^2\right]_0^3$

$\left[\frac{3^3}{3} - 3^2\right] - \left[\frac{0^3}{3} - 0\right] = 0$

Distance $= \int_0^3 |t^2-2t| dt = \begin{cases} 0 \leq x < 2 & t^2-2t \\ 2 \leq 3 < 3 & -t^2+2t \end{cases}$

$t=0, t=2$

$\left[\frac{t^3}{3} - t^2\right]_0^2 + \left[-\frac{t^3}{3} + t^2\right]_2^3$
$= \frac{8}{3}$ m

Note: Definite integral $\int_a^b f(x)dx$ is a number, and hence doesn't depend on choice of variable
$\int_a^b f(x)\,dx = \int_a^b (f(u))\,du$

Indefinite integral is a function
eg. $\int x^2 dx = \frac{x^3}{3} + c = F(x)$