14 Multiple Integrals

Double Integrals

Finding volume, we break up the region R into infinitesimal areas ΔA_k around the point (x_k^*, y_k^*) and integrate over these areas.

We define the double integral in this region to be given by

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} f(x_k^*, y_k^*)\Delta A_k$$

From this we can calculate the volume between the region R and surface $z=f(x,y) \geq 0$ to be

$$V = \iint_R f(x,y)\,dA$$

If the condition $f(x,y) \geq 0$ does not hold, i.e. $f(x,y)$ does not lie above R for all values considered, we find the difference in volumes above and below the xy plane and we call this the net signed volume

Double integrals over rectangular regions

If R is a rectangle with x-coordinates in the range a to b and y coordinates in the range c to d, then these are the respective limits for the integration.

The area element $dA = dxdy$ for rectangular region and we must understand that we can perform partial definite integration by treating y as a constant when integrating over x and vice versa:

$$\int_a^b f(x,y)\,dx \quad \text{and} \quad \int_c^d f(x,y)\,dy$$

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look at partial integrals of x^3y

$$\int_0^1 x^3y\,dx = y\int_0^1 x^3dx = \frac{x^4y}{4}\Big|_{x=0}^{1} = \frac{y}{4}$$

$$\int_0^1 x^3y\,dx = x^3\int_0^1 y\,dy = \frac{x^3y^2}{2}\Big|_{y=0}^{1} = \frac{x^3}{2}$$

Furthermore a partial definite integral with respect to x can subsequently be integrated with respect to y, or vice versa

We call this process iterated integration.

$$\int_c^d\int_a^b f(x,y)\,dx\,dy = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$$

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx$$

Example: Calculate $\int_1^2\int_0^3(x^2y-2y)\,dx\,dy$ and $\int_0^3\int_1^2(x^2y-2y)\,dy\,dx$

Solution $\int_1^2\int_0^3(x^2y-2y)\,dx\,dy = \int_1^2\left[\int_0^3(x^2y-2y)\,dx\right]dy$

$$= \int_1^2 \frac{x^3y}{3} - 2xy\Big|_{x=0}^{3}\,dy$$

$$= \int_1^2 [9y-6y-(0-0)]\,dy$$

$$= \int_1^2 3y\,dy$$

$$\left[\frac{3}{2}y^2\right]_{y=1}^{2} = \frac{3}{2}(4-1) = \frac{9}{2}$$

Integrating second way will also give same result.

Fubini's theorem:

Let R be the rectangle defined by $a\le x\le b$ $c\le y\le d$

If $f(x,y)$ is continuous on this rectangle then:

$$\iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

Order of integration can be changed without changing the result

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Properties of double integrals:

1. $\iint_R c f(x,y)\, dA = c\iint_R f(x,y)\, dA$ where c is constant

2. $\iint_R (f(x,y) \pm g(x,y))\, dA = \iint_R f(x,y)\, dA \pm \iint_R g(x,y)\, dA$

3. If R_1 and R_2 are regions such that $R = R_1 + R_2$:
$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$.

DOUBLE INTEGRALS OVER NON RECTANGULAR REGIONS

We consider integrating over a region of arbitrary shape and complexity. Two types of regions.

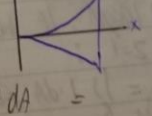
Type 1: Bounded to the left and right by vertical lines $x=a$ and $x=b$, and bounded above and below by curves $y=g_1(x)$ and $y=g_2(x)$

Type 2: bounded below and above by horizontal lines $y=c$ and $y=d$ and bounded to the left and right by curves $x=h_1(y)$ and $x=h_2(y)$

Type 1: $\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$

Type 2: $\iint_R f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$

Example: Compute volume of solid above $z = xy^2$ and below by R, the region between $y=x^3$ above and $y=-x^2$ below $0 \le x \le 1$



Solution: $V = \iint_R xy^2\, dA = \int_0^1 \left[\int_{-x^2}^{x^3} xy^2\, dy \right] dx$

$= \int_0^1 \left[\frac{xy^3}{3} \right]_{y=-x^2}^{x^3} dx$

$= \int_0^1 \left[\frac{x}{3}(x^9 + x^6) \right] dx$

$= \frac{1}{3}\int_0^1 \left[x^{10} + x^7 \right] dx$

$= \frac{1}{3}\left[\frac{x^{11}}{11} + \frac{x^8}{8} \right]_{x=0}^{1}$

$= \frac{1}{3}\left[\frac{1}{11} + \frac{1}{8} \right] = \frac{19}{264}$

Example Find volume of tetrahedron bounded by coordinate planes and the plane $z = 6-2x-3y$

Solution Consider projection onto xy plane region extends from $0 \le x \le 3$ and $g_1(x) \le y \le g_2(x)$. We find this by setting $z=0$, $0 = 6-2x-3y$. Interception with x axis $y=0$ and $x=3$. Therefore limits are $0 \le x \le 3$ in positive x direction and

We consider the positive y direction, so the lower bound for the y integration must be $g_1(x)=0$. Upper bound is given by $0 = 6-2x-3y$, $y = -2x/3 + 2$

$$\text{Volume} = \iint_R (6-2x-3y)\,dA = \int_0^3 \int_{g_1(x)}^{g_2(x)} (6-2x-3y)\,dy\,dx$$

$$\Rightarrow \int_0^3 \int_0^{-\frac{2x}{3}+2} (6-2x-3y)\,dy\,dx$$

$$= \int_0^3 \left[6y - 2xy - \frac{3}{2}y^2\right]_{y=0}^{-\frac{2x}{3}+2} dx$$

$$= \int_0^3 \left[6\left(-\frac{2x}{3}+2\right) - 2x\left(-\frac{2x}{3}+2\right) - \frac{3}{2}\left(-\frac{2x}{3}+2\right)^2\right] dx$$

$$= \int_0^3 \left[6 - 4x + \frac{2x^2}{3}\right] dx$$

$$= \left[6x - 2x^2 + \frac{2x^3}{9}\right]_{x=0}^{3} = 18 - 18 + 6 = 6.$$

Area as a double integral

We can calculate area by double integral. If we are given some surface $z = f(x,y)$ we can set $z=1$:

$$V = A \cdot h = A \cdot 1 = A = \iint_R 1 \cdot dA = \text{area } R.$$

Thus Area of $R = \iint_R dA$

Example: Calculate area between the curves $y = x^3$ and $y = \sqrt{x}$ for $0 \le x < 1$

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Solution: $A = \iint_R dA = \int_0^1 \int_{x^2}^{\sqrt{x}} dy\,dx$

$= \int_0^1 y \Big|_{y=x^2}^{\sqrt{x}} dx$

$= \int_0^1 (\sqrt{x} - x^2)\,dx = \left[\frac{2x^{3/2}}{3} - \frac{x^4}{4}\right]_{x=0}^{1} = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$

Parametric Surfaces

Parametric curve $x = x(t)$, $y = y(t)$, $z = z(t)$

It is also possible to have a parametric surface, which is a surface with two parameters u and v that determine where the point on the surface is. Looks like:

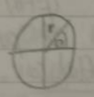
$$x = x(u,v) \quad y = y(u,v) \quad z = z(u,v)$$

Rectangular coordinates:

$x = u$, $y = v$ $z = w$.

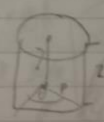
Polar coordinates:

$x = \rho\cos\theta$ $y = \rho\sin\theta$



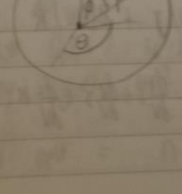
Cylindrical coordinates

$x = \rho\cos\theta$, $y = \rho\sin\theta$ z



Spherical coordinates

$x = \rho\sin\phi\cos\theta$ ~~$y = \rho\sin\theta$~~ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\phi$



$\rho \geq 0$, $0 \leq \theta \leq 2\pi$ $0 \leq \phi \leq \pi$

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Note in polar coordinates double integral is

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \left[\int_{r_1(\theta)}^{r_2(\theta)} f(\rho,\theta)\rho\,d\rho\right]d\theta$$

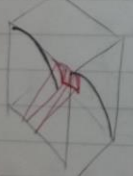
extra ρ is jacobian

Example: Change $z = x^2 + y^2 - 9$ to rectangular and polar coordinates

rectangular $\Rightarrow x = u^2 + v^2 - 9$
In polar $= \rho^2\cos^2\theta + \rho^2\sin^2\theta - 9 = \rho^2 - 9$

Example: change $x^2+y^2+z^2=9$ to Spherical coordinates
$\rho = 3 \Rightarrow x = 3\sin\varphi\cos\theta \quad y = 3\sin\varphi\sin\theta \quad z = 3\cos\varphi$
θ and φ are parameters

Surfaces of Revolution
If we take curve $y=f(x)$ we can rotate about $x$axis to trace out a surface in 3d. The parameter of revolution will be v and we set
$x = u \quad y = f(u)\cos v \quad z = f(u)\sin v$.



Vector valued functions of two variables
Since we have extended curves of one parameter to surfaces of 2 parameters we can likewise extend a vector valued function of one variable to a vector valued function of two variables
$r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$

Partial derivative: $\frac{dr}{du} = \frac{dx}{du}i + \frac{dy}{du}j + \frac{dz}{du}k \qquad \frac{dr}{dv} = \frac{dx}{dv}i + \frac{dy}{dv}j + \frac{dz}{dv}k$

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Tangent plane to parametric surface

Let σ be a parametric surface in three dimensions. A plane is tangent to σ at P_0 if a line through P_0 lies in the plane if and only if it is a tangent line at P_0 of a curve on σ.

Let $r(u,v)$ be a curve on the parametric surface σ and $P(a,b,c)$ a point on σ with $a = x(u_0,v_0)$, $b = y(u_0,v_0)$, $c = z(u_0,v_0)$ then

If $\frac{dr}{du} \neq 0$ it is a tangent to the constant v curve

If $\frac{dr}{dv} \neq 0$ it is a tangent to the constant u curve

From this we see that if $\frac{dr}{du} \times \frac{dr}{dv} \neq 0$ at P_0 then it is orthogonal to both tangent vectors and is therefor normal to the tangent plane at P_0.

We define the principle unit normal vector to the surface $r(u,v)$ at (u_0,v_0) to be $n = \dfrac{\frac{dr}{du} \times \frac{dr}{dv}}{\left\| \frac{dr}{du} \times \frac{dr}{dv} \right\|}$ provided $\frac{dr}{du} \times \frac{dr}{dv} \neq 0$.

Example: find eq^n of tangent plane for the surface

$r(u,v) = ui + 2v^2 j + (u^2+v) k$ at $(2,2,3)$

First we find the normal vector $\frac{dr}{du} \times \frac{dr}{dv}$.

$\frac{dr}{du} = i + 2uk$ $\qquad$ $\frac{dr}{dv} = 4vj + k$

$\frac{dr}{du} \times \frac{dr}{dv} = -8uvi - j + 4vk$

At point $(2,2,3)$ we should have $u=2$, $2v^2=2$ and $u^2+v=3$

Solving gives $u=2$ and $v=-1$

Normal vector $= n = 16i - j - 4k$

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We can rewrite tangent plane using equation as
$16(x-2) - y(-2) - 4(z-3) = 0$
$= 16x - y - 4z = 18$

Note that we can use any normal vector to get the tangent plane

If we used the unit normal vector for example, we would just divide the entire equation by a constant which leaves it unchanged

Denoting the normal vector at $P_0 = (a,b,c)$ by $n = (n_x, n_y, n_z)$ eqn of tangent plane at P_0 becomes
$n_x(x-a) + n_y(y-b) + n_z(z-c) = 0$

Surface Area

In addition to volume of a solid we also might want to know its surface area

Let $r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$ be a smooth parametric surface on a region of the plane R with $\frac{dr}{du} \times \frac{dr}{dv} \neq 0$ on R.

Due to this condition a tangent ~~vector~~ plane exists for every point (u,v) on R

Split area up by gridlines

If we consider the area of the surface defined on a sub region contained in one of the boxes defined by the gridlines in the region R, we find the ~~surface~~ area of the box of size $\Delta u_k \times \Delta v_k$ at a corner point (u_k, v_k) is $\Delta A_k = \Delta u_k \Delta v_k$

Considering the parallelogram on the surface for this region, it will have sides $r(u_k + \Delta u_k, v_k) - r(u_k, v_k) \approx \frac{dr}{du} \Delta u_k$

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and $r(u_k, v_k + \Delta v_k) - r(u_k, v_k) \approx \frac{dr}{dv} \Delta v_k$

For Δu_k and Δv_k small then area of parallelogram is
$\Delta S_k = \| \frac{dr}{du} \Delta u_k \times \frac{dr}{dv} \Delta v_k \|$

$= \| \frac{dr}{du} \times \frac{dr}{dv} \| \Delta A_k$

This is the surface area of parametric surface defined on the box at point (u_k, v_k). Total surface area for this region is the sum of all these boxes

$$S = \sum_{k=1}^{n} \| \frac{dr}{du} \times \frac{dr}{dv} \| \Delta A_k$$

where n is the number of boxes in the region.

If we take the limit, this gives us the surface area of the parametric surface $r(u,v)$ over the region R

$$S = \iint_R dS = \iint_R \| \frac{dr}{du} \times \frac{dr}{dv} \| dA$$

Example: Find the surface area of the sphere of radius 4 that lies in the cylinder above the xy plane with base $x^2+y^2=12$

We need to parameterise surface first to spherical polar coordinates

$r(\theta, \phi) = 4\sin\phi\cos\theta\, i + 4\sin\phi\sin\theta\, j + 4\cos\phi\, k$

Need to work out range of θ and ϕ.

For θ we have a cylinder that goes through an angle of 2π around z axis $\quad 0 \leq \theta \leq 2\pi$

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ϕ is the rotation away from the z-axis, so the limits of ϕ will come from the limits of z on the cylinder

We therefore substitute the values of x and y when the surface intersects the cylinder i.e. when $x^2+y^2=12$

$$x^2+y^2+z^2=16$$
$$12+z^2=16$$
$$z^2=4$$
$$z \pm 2$$

Then because we are in spherical polar coordinates

$$4\cos\phi = \pm 2$$
$$\cos\phi = \pm \tfrac{1}{2}$$
$$\phi = \pi/3$$

We only take $z=+2$ because we only want region above xy plane

limits are then $0 \le \phi \le \pi/3$ since the angle runs from the z-axis ($\phi=0$) to the intersection of the surfaces $\phi = \pi/3$.

Derivatives of r are $r_\theta = -4\sin\phi\sin\theta\, i + 4\sin\phi\cos\theta\, j$

$r_\phi = 4\cos\phi\cos\theta\, i + 4\cos\phi\sin\theta\, j - 4\sin\phi\, k$

Cross product is
$$\begin{vmatrix} i & j & k \\ -4\sin\phi\sin\theta & 4\sin\phi\cos\theta & 0 \\ 4\cos\phi\cos\theta & 4\cos\phi\sin\theta & -4\sin\phi \end{vmatrix}$$

$$= -16\sin^2\phi\cos\theta\, i - 16\sin^2\phi\sin\theta\, j + (-16\sin\phi\cos\phi\sin^2\theta - 16\sin\phi\cos\phi\cos^2\theta)k$$
$$= -16\sin^2\phi\cos\theta\, i - 16\sin^2\phi\sin\theta\, j - 16\sin\phi\cos\phi\, k$$

The magnitude $\|r_\theta \times r_\phi\|$ is

$$\sqrt{(-16\sin^2\phi\cos\theta)^2 + (-16\sin^2\phi\sin\theta)^2 + (-16\sin\phi\cos\phi)^2}$$
$$16\sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi}$$
$$16\sqrt{\sin^2\phi(\sin^2\phi + \cos^2\phi}$$
$$16\sqrt{\sin^2\phi}$$

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$= 16\,|\sin\phi|$

For the region we are interested in $\sin\phi$ is always positive, so we drop the absolute value

$$S = \iint_R 16\sin\phi \, dA$$

$$= \int_0^{2\pi} \left[\int_0^{\pi/3} 16\sin\phi \, d\phi \right] d\theta$$

$$= \int_0^{2\pi} \left[-16\cos\phi \right]_0^{\pi/3} d\theta$$

$$\int_0^{2\pi} \left[-16\left(\cos\tfrac{\pi}{3} - \cos 0\right) \right] d\theta$$

$$\int_0^{2\pi} 8 \, d\theta$$

$$= 8\theta \,\Big|_0^{2\pi} = 16\pi$$

Surface areas of surfaces of the form $z = f(x,y)$

If our surface is written in form of $r(x,y) = x(u,v)i + y(u,v)j + z(u,v)k$ where $z = f(x,y)$, then we can look to parameterise using $x=u$ and $y=v$ instead to give

$$r(u,v) = ui + vj + f(u,v)k$$

The derivatives are then $r_u = i + \frac{dz}{du}k$

$$r_v = j + \frac{dz}{dv}k$$

From which we find $r_u \times r_v = k - \frac{dz}{du}i - \frac{dz}{dv}j$

We therefore have $\|r_u \times r_v\| = \sqrt{1 + \left(\frac{dz}{du}\right)^2 + \left(\frac{dz}{dv}\right)^2}$

Since $x=u$ and $y=v$ we recover $S = \iint_R \sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2} \, dA$

Example: Find surface area of the paraboloid $z=x^2+y^2$ below the plane $z=1$.

Solution: note that $z \geq 0$ because it is a sum of squares

Therefore we want surface area between xy plane and circle of radius 1 at $z=1$.

Using the eqn $\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$ we get.

$$S=\iint_R \sqrt{1+4x^2+4y^2}\, dA$$

Easier to calculate using polar coordinates $x=\rho\cos\theta$ $y=\rho\sin\theta$ which gives $z=\rho^2$ and $\sqrt{1+4x^2+4y^2} = \sqrt{1+4\rho^2}$

$$\Rightarrow S=\int_0^{2\pi}\left[\int_0^1 \sqrt{1+4\rho^2}\,\rho\, d\rho\right] d\theta$$

$$= 2\pi \int_0^1 \sqrt{1+4t}\, \frac{dt}{2} \qquad (t=\rho^2 \Rightarrow dt = 2\rho\, d\rho)$$

$$= 2\pi \cdot \tfrac{1}{2}\cdot\tfrac{2}{3}\cdot\tfrac{1}{4}(1+4t)^{3/2}\Big|_0^1$$

$$2\pi\left(\tfrac{1}{12}\right)\left(5^{3/2}-1\right)$$

$$= \frac{\pi}{6}\left(5\sqrt{5}-1\right) = 5.33041$$

Lamina

A lamina is a region of space that has a M and a variable density given by the density function $\delta(x,y)$

Mass is given by $M = \iint_R \delta(x,y)\,dA$

Centre of mass or centre of gravity of lamina region is:

$$(\bar{x},\bar{y}) = \left(\frac{M_x}{M}, \frac{M_y}{M}\right)$$

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Where $M_x = \iint_R x\,\delta(x,y)\,dA$ $\quad M_y = \iint_R y\,\delta(x,y)\,dA$

For a lamina with a constant density, the centre of gravity is also called the centroid. Note that this is essentially the same as finding the centre of mass of an object with non uniform mass distribution.

Example: Compute mass and centre of gravity of lamina inside unit circle with density $\delta(x,y) = x^2+y^2$

Solution: Mass is given by $M = \iint_R (x^2+y^2)\,dA$

$$= \int_0^{2\pi}\int_0^1 \rho^2 \cdot \rho\,d\rho\,d\theta$$

$$= 2\pi \int_0^1 \rho^3\,d\rho$$

$$= 2\pi \left.\frac{\rho^4}{4}\right|_0^1 = \frac{\pi}{2}$$

x-component is given by

$$\bar{x} = \frac{1}{m}\iint x(x^2+y^2)\,dA = \frac{2}{\pi}\int_0^{2\pi}\int_0^1 \rho\cos\theta \cdot \rho^2 \cdot \rho\,d\rho\,d\theta$$

$$= \frac{2}{\pi}\int_0^{2\pi}\int_0^1 \rho^4\cos\theta\,d\rho\,d\theta$$

$$= \frac{2}{\pi}\int_0^{2\pi} \frac{1}{5}\cos\theta\,d\theta$$

$$= \frac{2}{5\pi}\left(-\sin\theta\right)_0^{2\pi} = 0$$

Similarly $\bar{y} = 0$ and therefore the centre of gravity is a $(0,0)$.
Note that we could swap order of integration because the limits are independent of the other coordinate.

TRIPLE INTEGRALS

Triple integrals in either cylindrical or Spherical coordinate

Triple Integral defined as a infinite limit of a Sum by

$$\iiint_G f(x,y,z)\,dV = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k,y_k,z_k)\Delta V_k$$

The Same rules that apply to double integrals apply here except we now integrate over a Solid G.

In general, triple integrals in rectangular coordinates will look like:
$$\iiint_G f(x,y,z)\,dx\,dy\,dz = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right] dA$$

Where we treat x and y coordinates as we did for double integrals.

This is certainly the case for cylindrical coord but Spherical coord require different treatment.

Using triple integrals volume of a Solid G is
$$\text{Volume of } G = \iiint_G dV$$

Triple Integrals in Cylindrical Coordinates

Consider a small volume element in cylindrical coordinates, is a region between two radii $\rho_1 \le \rho \le \rho_2$, two angles $\theta_1 \le \theta \le \theta_2$ and two height $z_1 \le z \le z_2$

If we want to find the volume of Some Solid G, we can integrate over these "cylindrical wedges".

$$\iiint_G f(\rho,\theta,z)\,dV = \lim_{n\to\infty} f(\rho_k,\theta_k,z_k)\Delta V_k$$

where ΔV_k = [area of base] $\cdot$ [height] $= \Delta\rho_k\, \rho_k\, \Delta\theta_k\, \Delta z_k$

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Theorem: Let G be a solid, with upper surface $z = g_2(\rho,\theta)$ and lower surface $z = g_1(\rho,\theta)$ in cylindrical polar coordinates. If the projection of G onto the xy-plane is a plane is a simple polar region R, and if $f(\rho,\theta,z)$ is continuous on G then.

$$\iiint_G f(\rho,\theta,z)\,dV = \iint_R \left[\int_{g_1(\rho,\theta)}^{g_2(\rho,\theta)} f(\rho,\theta,z)\right] dA$$

$$= \int_{\theta_1}^{\theta_2} \left[\int_{\rho_1(\theta)}^{\rho_2(\theta)} \left[\int_{g_1(\rho,\theta)}^{g_2(\rho,\theta)} f(\rho,\theta,z)\rho\,dz \right] d\rho \right] d\theta$$

Converting from rectangular to cylindrical polar coordinates, triple integral becomes.

$$\iiint_G f(x,y,z)\,dx\,dy\,dz = \iiint_G f(\rho\cos\theta, \rho\sin\theta, z)\rho\,dz\,d\rho\,d\theta$$

Example: Use cylindrical coordinates to calculate

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} x^2\,dz\,dy\,dx$$

Volume between $x^2+y^2=9$ for $z=0$ and $z = 9-x^2-y^2$.

Solution: We see that z has lower limit 0, but the upper limit depends on y via $z = 9-x^2-y^2$.

Then after integrating z, we see that the largest values y can take are at the base of the solid, on the circle $x^2+y^2=9$, hence $-\sqrt{9-x^2} \le y \le \sqrt{9-x^2}$

Finally the x variable will have limits when $y=0$, giving $x^2=9$ $-3 \le x \le 3$

Converting to cylindrical we get

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} x^2\,dz\,dy\,dx = \iint_R \left[\int_0^{9-\rho^2} \rho^2\cos^2\theta\,dz\right] dA$$

$$\int_0^{2\pi}\int_0^3\int_0^{9-\rho^2} \rho^2\cos^2\theta\,\rho\,dz\,d\rho\,d\theta$$

$$=\int_0^{2\pi}\int_0^3 \left[z\rho^3\cos^2\theta\right]_{z=0}^{9-\rho^2} d\rho\,d\theta$$

$$=\int_0^{2\pi}\int_0^3 \left[(9\rho^3-\rho^5)\cos^2\theta\right] d\rho\,d\theta = \int_0^{2\pi}\left[\left(\frac{9\rho^4}{4}-\frac{\rho^6}{6}\right)\cos^2\theta\right]_{\rho=0}^{3} d\theta$$

$$=\frac{243}{4}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{243}{4}\int_0^{2\pi}(1+\cos 2\theta)\,d\theta$$

$$=\frac{243}{4}\left(\frac{1}{2}\right)\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = \frac{243}{4}\pi$$

Triple integrals in Spherical Coordinates

Note that ρ = constant gives a sphere, θ = constant gives a half plane and ϕ = constant gives a right circular cone and $\phi = \pi/2$ gives the xy plane

Similarly to cylindrical coordinates, a small volume element in spherical coordinates is a region between two radii $\rho_1 \le \rho \le \rho_2$, two ~~POLAR~~ polar angles $\theta_1 \le \theta \le \theta_2$ and two azimuth angles $\phi_1 \le \phi \le \phi_2$.

We integrate over "spherical wedges" to find volume

$$\iiint_G f(\rho,\theta,\phi)\,dV = \lim_{n\to\infty}\sum_{k=1}^{n} f(\rho_k,\theta_k,\phi_k)\Delta V_k$$

Where ΔV_k = [area of base]·[height] $= \rho_k^2 \sin\phi_k\,\Delta\rho_k\,\Delta\theta_k\,\Delta\phi_k$

Volume given by:

$$\iiint_G f(\rho,\theta,\phi)\,dV = \underset{\text{app limits}}{\iiint} f(\rho,\theta,\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

The limits are intentionally left unspecified because they can be quite involved and the order of integration may depend on how a problem is presented

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Converting from rectangular to spherical coordinates, the triple integral becomes

$$\iiint_G f(x,y,z)\,dx\,dy\,dz = \iiint_G f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\,\rho^2\sin\varphi\ d\rho\,d\theta\,d\varphi$$

Example: Use spherical coordinates to calculate

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\ dz\,dy\,dx$$

Volume between $x^2+y^2=4$ for $z=0$ and $z=\sqrt{4-x^2-y^2}$

Solution: z has lower limit 0 but upper limit depends on x and y via $z=\sqrt{4-x^2-y^2}$. The largest value can take are at the base of the solid, on the circle $x^2+y^2=4$, hence $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$. x variable will have have limits when $y=0$ giving $x^2=4$ or $-2\le x\le 2$.

Converting to spherical coordinates, we integrate over entire plane and so $0\le\theta\le 2\pi$, the surface $z=\sqrt{4-x^2-y^2}$ can be written as $x^2+y^2+z^2=4 \Rightarrow \rho^2=4,\ 0\le\rho\le 2$

We are above the xy plane as $z\geq 0$ and so $0\le\varphi\le\pi/2$ when $\varphi=\pi/2$ gives 0, i.e. the value of φ ~~gives~~ in xy plane

$$\int_{-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\ dz\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2}(\rho\cos\varphi)^2\sqrt{\rho^2}\,(\rho^2\sin\varphi)\ d\rho\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2}\rho^5\cos^2\varphi\sin\varphi\ d\rho\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\frac{\rho^6}{6}\cos^2\varphi\sin\varphi\Big|_{\rho=0}^{2}\ d\varphi\,d\theta$$

$$= \frac{32}{3}\int_0^{2\pi}\int_0^{\pi/2}\cos^2\varphi\sin\varphi\ d\varphi\,d\theta$$

$$= \frac{32}{3}\int_0^{2\pi}\int_0^{\pi/2}(-\cos^2\varphi)\,d(\cos\varphi)$$

$$\frac{32}{3}\int_0^{2\pi}\left[-\frac{1}{3}(\cos)^3\theta\right]_{\varphi=0}^{\pi/2} d\theta = \frac{32}{9}\int_0^{2\pi} d\theta = \frac{64\pi}{9}$$

In the φ integral we used the substitution $u = \cos\varphi$

Jacobian!

What does it mean to change from rectangular to cylindrical coords?

In short we can change the integration variable from x to u via $x = g(u)$ assuming g is a differentiable function giving

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))\frac{dx}{du}du = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)\,du$$

If we now have a double integral over x and y, we can change to new coordinates u and v in the same way

We can imagine that we have a vector-valued function that represents the original variables and may be expressed through the new variables

$$r(u,v) = x(u,v)i + y(u,v)j$$

Then looking at the small area with vertices $r(u_0, v_0)$, $r(u_0+\Delta u, v_0)$ $r(u_0, v_0+\Delta v_0)$ and $r(u_0+\Delta u, v_0+\Delta v)$ we see that we can express the area either by $\Delta A = \Delta x\,\Delta y$, the area as expressed through x and y, or alternatively, we see the parallelogram can be parametrised through u and v in which case for a small region the sides are $\frac{dr}{du}\Delta u$ and $\frac{dr}{dv}\Delta v$.

The area element for a small area is then:

$$\Delta A \approx \left\| \frac{dr}{du}\Delta u \times \frac{dr}{dv}\Delta v \right\| = \left\| \frac{dr}{du} \times \frac{dr}{dv} \right\| \Delta u \Delta v$$

The cross product is:

$$\frac{dr}{du} \times \frac{dr}{dv} = \begin{vmatrix} i & j & k \\ \frac{dx}{du} & \frac{dy}{du} & 0 \\ \frac{dx}{dv} & \frac{dy}{dv} & 0 \end{vmatrix} = \begin{vmatrix} \frac{dx}{du} & \frac{dy}{du} \\ \frac{dx}{dv} & \frac{dy}{dv} \end{vmatrix} k$$

This determinant is what we know as the Jacobian.

A Jacobian is produced when we change from the xy-plane to the uv-plane via equation $x = x(u,v)$ $y = y(u,v)$.

We denote this by either $J(u,v)$ or $\delta(x,y)/\delta(u,v)$ given by:

$$J(u,v) = \frac{\delta(x,y)}{\delta(u,v)} = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \frac{dx}{du}\frac{dy}{dv} - \frac{dy}{du}\frac{dx}{dv}$$

It appears in the area element: $\Delta A \approx \left| \frac{\delta(x,y)}{\delta(u,v)} \right| \Delta u \Delta v$

In a double Integral this gives us:

$$\iint_R f(x,y)\,dA_{xy} = \iint_S f(x(u,v), y(u,v)) \left| \frac{\delta(x,y)}{\delta(u,v)} \right| dA_{uv}$$

where S is the region in the uv plane corresponding to the region R in the xy plane.

In a similar way for triple Integrals we have Jacobian $\Rightarrow$

$$J(u,v,w) = \frac{\delta(x,y,z)}{\delta(u,v,w)} = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} & \frac{dx}{dw} \\ \frac{dy}{du} & \frac{dy}{dv} & \frac{dy}{dw} \\ \frac{dz}{du} & \frac{dz}{dv} & \frac{dz}{dw} \end{vmatrix}$$

20. Triple Integrals → Jacobian

Which gives the change in the volume element: $\Delta V = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \Delta u \Delta v \Delta w$

and the triple integral becomes:

$$\iiint_G f(x,y,z)\,dV_{xyz} = \iiint_H f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV_{uvw}$$

Where H is the region in the uvw space corresponding to the region G in the xyz space

Polar Coordinates: $x = \rho\cos\theta \quad y = \rho\sin\theta$

$$\frac{\partial(x,y)}{\partial(\rho,\theta)} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho(\cos^2\theta + \sin^2\theta) = \rho$$

$$\Rightarrow dx\,dy = \rho\,d\rho\,d\theta$$

Cylindrical Coordinates: $x = \rho\cos\theta \quad y = \rho\sin\theta \quad z$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,z)} = \rho \Rightarrow dx\,dy\,dz = \rho\,d\rho\,d\theta\,dz$$

Spherical Coordinates: $x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,z)} = \rho^2\sin\phi \Rightarrow dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

Example: Find Jacobian of $x = \frac{1}{2}(u+v) \quad y = \frac{1}{2}(u-v)$

Solution: $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

Example: Find Jacobian of $x = au, \quad y = bv \quad z = cw$

Solution: $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} & \frac{dx}{dw} \\ \frac{dy}{du} & \frac{dy}{dv} & \frac{dy}{dw} \\ \frac{dz}{du} & \frac{dz}{dv} & \frac{dz}{dw} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

Mass and centre of Gravity of Solids

Already found mass and centre of gravity for lamina with density function $\delta(x,y,z)$. Can generalise to get mass of G:

$$M = \iiint_G \delta(x,y,z)\,dV$$

and centre of gravity: $(\bar{x},\bar{y},\bar{z}) = \left(\frac{M_x}{M}, \frac{M_y}{M}, \frac{M_z}{M}\right)$

Where:

$$M_x = \iiint_G x\,\delta(x,y,z)\,dV$$

$$M_y = \iiint_G y\,\delta(x,y,z)\,dV$$

$$M_z = \iiint_G z\,\delta(x,y,z)\,dV$$

Example: Find mass and centre of gravity of a cube with square base of length two in xy plane centred on (1,1) and height of 5 with density function $\delta(x,y,z) = z$

Solution: $M = \iiint_G \delta(x,y,z)\,dV = \int_0^2\int_0^2\int_0^5 z\,dz\,dy\,dx$

$$= \int_0^2\int_0^2 \tfrac{1}{2}(z^2)\Big|_0^5\,dy\,dx$$

$$= \frac{25}{2}\int_0^2\int_0^2 dy\,dx$$

$$= 25\int_0^2 dx = 50$$

The x and y coordinates for the centre of gravity will clearly give the same result due to the symmetry so we only calculate one:

$$\bar{x} = \frac{1}{50}\int_0^2\int_0^2\int_0^5 xz\;dz\,dy\,dx$$

$$= \frac{1}{50}\int_0^2\int_0^2 x\frac{25}{2}\,dy\,dx$$

$$= \frac{1}{4}\int_0^2 xy\Big|_{y=0}^{2}\,dx$$

$$= \frac{1}{2}\int_0^2 x\,dx = \frac{x^2}{4}\Big|_0^2\,dx = 1 \qquad \text{Similarly } \bar{y} = 1$$

Could of seen this by fact that density doesn't vary in xy plane and so the centre of the square would coincide with $\bar{x}\,\bar{y}$.

$$\bar{z} = \frac{1}{50}\int_0^2\int_0^2\int_0^5 z^2\,dz\,dy\,dx$$

$$= \frac{1}{50}\int_0^2\int_0^2 \frac{z^3}{3}\Big|_{z=0}^{5}\,dy\,dx$$

$$= \frac{1}{50}\int_0^2\int_0^2 \frac{125}{3}\,dy\,dx$$

$$= \frac{5}{6}\int_0^2\int_0^2 dy\,dz = \frac{5}{6}(4) = \frac{10}{3}$$

$\Rightarrow$ Centre of gravity is $\left(1, 1, \frac{10}{3}\right)$