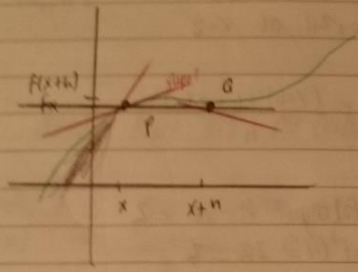
Maths Sender 1 2012 JF

CHAPTER 2

THE DERIVATIVE

22/10/12 Derivatives ① Chapter 2

Derivatel = Slope of tangent lines or rate of change



$$M_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x+h) - f(x)}{x+h-x}$$

$$= \frac{f(x+h) - f(x)}{h}$$

$$M_{tangent} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example Compute the slope of the tangent line of $f(x) = x^2$ at $x = 1$

$$M_{tangent} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \quad \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$\lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x + h = 2x$$

at $x = +1$

Slope $= 2(1) = 2$

Def^n The function $f'(x)$ defined by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

is called the derivative of $f(x)$ with respect to x

The domain of $f'(x)$ consists of all $x \in D(f)$ for which the limit above exists

Example:

Compute the derivative of $f(x)=x^2+1$ and find the equation of the tangent line to $y=x^2+1$ at $x=2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2+1-x^2-1}{h}$$

$$\lim_{h\to 0} = 2x \quad \text{slope of tangent line at } x=2$$
$$= f'(2) = 2(2) = 4$$

need a point on the line $(2, f(2)) = (2,5)$ $m=4$

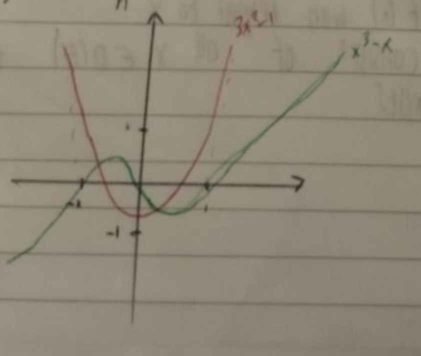
$$y-5 = 4(x-2)$$
$$y-5 = 4x-8$$
$$\cancel{4x-y-3=0} \quad y = 4x-3$$

In general, the eq^n of the tangent line to $y=f(x)$ at $x=x_0$ is $\boxed{y=f(x_0)+f'(x_0)(x-x_0)}$

Example: Compute the derivative of $f(x)=x^3-x$ and plot $f(x)$ and $f'(x)$ on the same graph

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3+(x+h)-x^3+x}{h} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x-h-x^3+x}{h}$$

$$= \lim_{h\to 0} \frac{3x^2+3xh+h^2-1}{h} = 3x^2-1$$



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Derivatives Chapter 2

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Differentiability

The limit that defines $f'(x)$ does not always exist

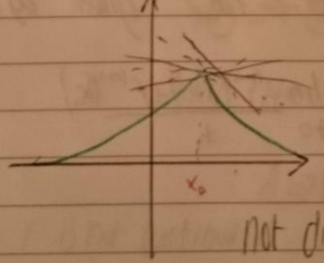
Def^n: A function f is said to be differentiable at x_0 if the limit $f'(x_0) = \lim_{h\to 0} \frac{f(x+h)-f(x_0)}{h}$ exists. If f is differentiable at each point of (a,b), then we say f is differentiable on (a,b). Similarly for (a,∞), $(-\infty,b)$ etc.

For closed interval, $[a,b]$ say then if f is differentiable on (a,b) and if f is right differentiable at $x=a$ and left differentiable at $x=b$, we say f is differentiable on $[a,b]$

Left-Sided derivative $f'_-(x_0) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$

Right-Sided derivative $f'_+(x_0)$ $\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$

Intuitively, a function f is differentiable at x_0 if the graph of f has a tangent line at x_0 i.e. Not differentiable at corner point and points of vertical tangency



not differentiable

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Derivatives (Chapter 3)

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Differentiability

The limit that defines $f'(x)$ may not always exist

Def: A function f is said to be differentiable at x_0 if the limit $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$ exists. If f is differentiable

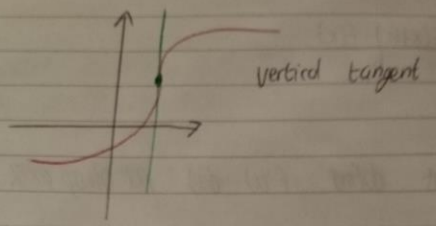
at each point of (a, b), then we say f is differentiable on (a, b).
Similarly for (a, ∞), $(-\infty, a)$ etc.

For closed interval $[a, b]$ say then if f is differentiable on (a, b) and if f is right differentiable at $x = a$ and left differentiable at $x = b$ we say f is differentiable on $[a, b]$

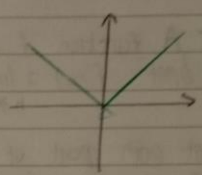
Left-sided derivative $f'_-(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$

Right-sided derivative $f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$

Geometrically: a function f is differentiable at x_0 if its graph has a tangent line at x_0 i.e. not different[illegible]
point of vertical [illegible]



Example: Prove that $f(x)=|x|$ is not differentiable at $x=0$



$$f'(x) = \lim_{h\to 0} \frac{|x+h|-|x|}{h}$$

$$f'(0) = \lim_{h\to 0} \frac{|h|}{h} \qquad \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

Hence two sided limit does not exist and so $f(x)$ is NOT differentiable at $x=0$

Higher Derivatives:

If $f'(x)$ is itself differentiable at x_0, then we can take the 'second derivative'. $f''(x_0) = \left.\frac{d^2f}{dx^2}\right|_{x=x_0} = \lim_{h\to 0} \frac{f'(x_0+h)-f'(x_0)}{h}$

Generally, the n^{th} derivative of $f(x)$ is given by:

$$f^{(n)}(x_0) = \left.\frac{d^nf}{dx^n}\right|_{x=x_0} = \lim_{h\to 0} \frac{f^{(n-1)}(x+h)+f^{(n-1)}(x_0)}{h}$$

evaluated at

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If all of these derivatives exist for $n \in \mathbb{N}$ at x_0, we say $f(x)$ is infinitely differentiable at x_0

e.g. $f(x) = x^n$ $n > 0$ infinitely differentiable

Differentiability and Continuity

Theorem: If a function f is differentiable at x_0, then f is continuous at x_0

Proof: We know that f is differentiable at x_0 $\Rightarrow$ $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists

We wish to show $\lim_{x \to x_0} f(x) = f(x_0)$ (A)

$\Leftrightarrow \lim_{x \to x_0} [f(x) - f(x_0)] = 0$

$\Leftrightarrow \lim_{h \to 0} [f(x_0+h) - f(x)] = 0$ Setting $h = x - x_0$ (B)

To prove this statement we take $\lim_{h \to 0} [f(x_0+h) - f(x_0)]$

$$= \lim_{h \to 0} \left[\frac{f(x_0+h) - f(x_0)}{h} \cdot h \right]$$

$$\lim_{h \to 0} \left[\frac{f(x_0+h) - f(x_0)}{h} \right] \lim_{h \to 0} h$$

$$= f'(x_0) \cdot 0 = 0$$

$\therefore$ (B) is true

(A) $\Leftrightarrow$ (B)

(A) is true hence $f(x)$ is continuous at x_0 Q.E.D

Remarks

(1) $A \Rightarrow B$ same as Not(B) $\Rightarrow$ Not(A)

If f is not continuous at x_0 then f is not differentiable at x_0.

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2 The inverse is not necessarily true
i.e. continuity does not imply differentiability

Techniques of Differentiations:

- Constants $\frac{d(c)}{dx} = 0$
- Constants times a function $\frac{d(cf(x))}{dx} = c\frac{d(f(x))}{dx}$
- Power rule $f(x) = x^{\alpha}$ $\alpha \in \mathbb{R}$

$$\frac{df}{dx} = \alpha x^{\alpha - 1} \quad \text{eg } f(x) = x^{-7/2} \quad f'(x) = -\frac{7}{2}x^{-9/2}$$

- Sums and differences $\frac{d(f(x) \pm g(x))}{dx} = \frac{df}{dx} \pm \frac{dg}{dx}$

Example: At what point does $y = x^3 - 48x + 12$ have a horizontal tangent line

$f'(x) = 3x^2 - 48 = 0 \Rightarrow 3x^2 = 48 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

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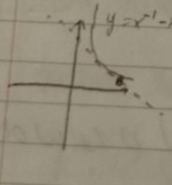
Example: Find the area of the triangle formed by the coordinate axis and the tangent line to $y = x^{-1} - x$ at $(1, 0)$



Need equation of the tangent line at $x=1$

$y'(x) = \frac{dy}{dx} = -x^{-2} - 1 \Rightarrow$ slope at $x=1 \Rightarrow y'(1) = -2$

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eqⁿ tangent line $= y - 0 = -2(x-1)$

$y = -2x + 2$

This line intersects the y-axis when $x = 0$, i.e. at $y = 2$.

Area $= \frac{1}{2}(\text{base} \times \text{height}) = \frac{1}{2}(1)(2) = 1$ square unit.

Product Rule:

If f and g are differentiable at x, then so is the product f.g

$$\frac{d[f(x)g(x)]}{dx} = \frac{df}{dx} \cdot g + \frac{dg}{dx} \cdot f = f'g + g'f \quad \text{where } f'(x) = \frac{df}{dx} \text{ etc}$$

example: $y = (4x^2 + (6x^3 + x^2 - 3))(x^2 + 2x - 4)$

$$\frac{dy}{dx} = (8x^6 + 18x + 2x)(x^2 + 2x - 4) + (4x^2 + 6x^3 + x^2 - 3)(2x + 2)$$

Quotient Rule

If f and g are differentiable at x and if $g(x) \neq 0$, then f/g is differentiable at x

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \Rightarrow \frac{\frac{df}{dx} \cdot g(x) - \frac{dg}{dx} \cdot f(x)}{g(x)^2} \Rightarrow \frac{f'g - g'f}{g^2}$$

example: $y = \frac{x^3 + 2x^2 - 1}{x + 5}$ $\quad \frac{dy}{dx} = \frac{(3x^2 + 4x)(x+5) - (1)(x^3 + 2x^2 - 1)}{(x+5)^2}$

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Derivates of Trigonometric Functions:

For proving derivates of trig functions the limits:

$\lim_{h\to 0} \frac{\sin h}{h} = 1$ $\quad\quad$ $\lim_{h\to 0} \frac{(1-\cos(h))}{h} = 0$

aswell as the addition formulae $\sin(x+y) = \sin x \cos y + \cos x \sin y$

example:

$f(x) = \sin x$ $\quad \frac{d\sin x}{dx} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$

$= \lim_{h\to 0} \frac{\sin x \cos(h) + \cos x \sin(h) - \sin(x)}{h}$

$= \lim_{h\to 0} \frac{\sin x(\cos(h) - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h}$

take sin outside cosh=0

$= \quad 0 \quad + \cos x \lim \frac{\sin h}{h}$ ← =(1) rule above

$= \cos x$

$\frac{d \sin(x)}{dx} = \cos x$ valid when x measured in radians

Similarly we can show that $\frac{d \cos(x)}{dx} = -\sin(x)$

All other trigonometric derivatives can be derived using the quotient rule.

e.g. $f(x) = \tan x = \frac{\sin x}{\cos x}$

$\frac{d(\tan x)}{dx} = \frac{\cos x . \cos x - (-\sin x)(\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$= \sec^2(x)$

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Similarly we can show $\frac{d(\sec(x))}{dx} = \sec(x)\tan(x)$

$$\frac{d(\cot x)}{dx} = \operatorname{cosec}^2(x)$$

$$\frac{d(\operatorname{cosec}(x))}{dx} = -\operatorname{cosec}(x)\cot(x)$$

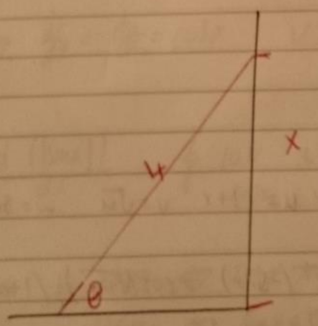
$$\sec(x) = \frac{1}{\cos(x)}$$

$$\operatorname{cosec}(x) = \frac{1}{\sin(x)}$$ 1/over third letter.

$$\cot(x) = \frac{1}{\tan(x)}$$

Example:

A 4m ladder leans against a wall at an angle θ with the horizontal. The top of the ladder is x-meters above the ground. If the bottom of the ladder is pushed towards the wall, find the rate of change at which x changes with respect to the angle (θ) when $\theta = 60°$. Answer in meters/degrees



$$\sin\theta = \frac{x}{4}$$

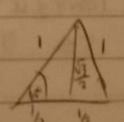
$$\Rightarrow x = 4\sin\theta$$

$$\Rightarrow \frac{dx}{d\theta} = 4\cos\theta \qquad \theta = 60° = \frac{\pi}{3}$$

$$\Rightarrow \frac{dx}{d\theta} \text{ at } \frac{\pi}{3} = 4\cos\frac{\pi}{3}$$

$$= 4\left(\tfrac{1}{2}\right) = 2 \text{ meter/degree}$$

$$= 2\frac{\pi}{180} \text{ meters/degree}$$



$= \frac{1}{2}$

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$= 0.035$ meters/degree

Chain Rule: Derivative of Compositions:

If g is differentiable at some point x, and f is differentiable at $g(x)$, then the composition of $f \circ g$ is differentiable at x.

If $y = f(g(x))$ set $u = g(x)$

$\Rightarrow y = f(u)$

and $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{df}{du} \cdot \frac{dg}{dx}$

Example: $y = \cos(x^5)$

Let $u = x^5$ $y = \cos(u)$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin(u) \cdot 5x^4 = 5x^4 \sin(x^5)$

$-\sin(x^5) \cdot 5x^4$

Example: $y = \sqrt{4x^3 + x}$ $u = 4x^3 + x$ $y = \sqrt{u}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2} u^{-\frac{1}{2}} \cdot (12x^2 + 1)$ $u^{-\frac{1}{2}} = \frac{1}{u^{\frac{1}{2}}} = \frac{1}{\sqrt{u}}$

$= \frac{12x^2 + 1}{2\sqrt{4x^3 + x}}$

Multiple Chain Rule Applications:

e.g. $\frac{d}{dx}(\sin(\sqrt{1+\cos x}))$ $u = 1 + x$ $v = \sqrt{u}$ $w = \sin(v)$

$\frac{dw}{dx} = \frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = \cos v \cdot \frac{1}{2} u^{-\frac{1}{2}} (-\sin x) = \cos(\sqrt{1+x}) \frac{1}{2} (1+\cos x)^{-\frac{1}{2}} (-\sin x)$

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Example: $y = \sqrt{\cos^2((x^2+7)^2)}$

$u = x^2+7$

$v = u^2$

$w = \cos v$

$z = w^2$

$t = z^{1/2}$

$$\frac{dy}{dx} = \frac{dt}{dz} \cdot \frac{dz}{dw} \cdot \frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

$$\frac{1}{2} z^{-\frac{1}{2}} \cdot 2w(-\sin v) \cdot 2u \cdot 2x$$

$$= \frac{1}{2}\left(\cos^2((x^2+7)^2)\right)^{-\frac{1}{2}} \cdot 2\cos((x^2+7)^2) \cdot -\sin(x^2+7)^2 \cdot 2(x^2+7) \cdot 2x$$

$y = \sqrt{\cos^2((x^2+7)^2)}$

$$= \frac{1}{2}\left(\cos^2(x^2+7)^2\right)^{-\frac{1}{2}} \cdot 2\cos(x^2+7)^2 - \sin(x^2+7)^2 \cdot 2(x^2+7) \cdot 2x$$ chain rule

Example: $\frac{d(|x|)}{dx} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$

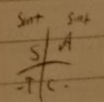
use this result and the chain rule to find $\frac{d(\sin|x|)}{dx}$ on $-\pi < x < \pi$

$u = \sin x \quad f = |u| \qquad \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$

if $u > 0$ then $\frac{df}{du} = 1$ | if $u < 0$ then $\frac{df}{du} = -1$

$\Rightarrow \frac{df}{dx} = \frac{du}{dx} = \cos x \quad u > 0$ | $\Rightarrow \frac{df}{dx} = \frac{du}{dx} = -\cos x \quad u < 0$

$$\frac{d(|\sin x|)}{dx} \begin{cases} \cos x & u > 0 \\ -\cos x & u < 0 \end{cases}$$



But $u > 0 \Leftrightarrow \sin x > 0 \Leftrightarrow x \in (0, \pi)$

$u < 0 \Leftrightarrow \sin x < 0 \Leftrightarrow x \in (-\pi, 0)$

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$$\frac{d(|\sin x|)}{dx} = \begin{cases} \cos x & 0 < x < \pi \\ -\cos x & -\pi < x < 0 \end{cases}$$ Not differentiable at $x=0$

Implicit Differentiation

In some cases, it is not possible to write y as an explicit function of x. For example $x^3+y^3=3xy$, defines y implicitly in terms of x.

In such cases, it is still possible to compute the derivative in terms of y and x.

Example: $5y^2 + \sin y = x^2$

Find $\frac{dy}{dx}$

$$\Rightarrow 10y\frac{dy}{dx} + \cos y\frac{dy}{dx} = 2x$$

$$\Rightarrow \frac{dy}{dx}(10y + \cos y) = 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x}{(10y+\cos y)}$$

Example: $4x^2 - 2y^2 = 9$ Find $\frac{d^2y}{dx^2}$

$$8x - 4y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x}{y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2(y) - 2x\left(\frac{dy}{dx}\right)}{y^2}$$

$$\frac{2y - 2x\left(\frac{2x}{y}\right)}{y^2} \Rightarrow \frac{2y^2-4x^2}{y^3} \Rightarrow \frac{-9}{y^3}$$ (Using the fact that $4x^2-2y^2=0$)

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Example: Find the equation of the tangent line to the curve $y^2 - x + 1 = 0$ at the point $(2,-1)$

$y = \sqrt{x-1}$

$\Rightarrow 2y \cdot \frac{dy}{dx} - 1 = 0$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2y}$

Slope at $(2,-1)$ is $\frac{dy}{dx}\Big|_{(2,-1)} = \frac{-1}{2}$

$y - y_1 = m(x - x_1)$

$y - (-1) = \frac{-1}{2}(x-2)$

$y = \frac{-1}{2}x$

Related Rates:

$V = \frac{\pi}{3} r^2 h$ rate of change of volume with respect to time is dV/dt

which depends on $dr/dt \cdot dh/dt$

$\frac{dV}{dt} = \frac{\pi}{3}\left\{ 2r\frac{dr}{dt} \cdot h + r^2 \frac{dh}{dt} \right\}$

Need to know $r, h, \frac{dr}{dt}, \frac{dh}{dt}$

Example: Let V be the volume of a cylinder having height h and radius r.

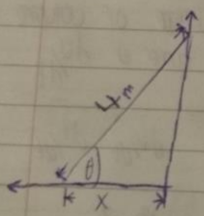
At a certain instant h = 6cm and increasing at 1cm/s, r = 10cm and decreasing at 1cm/s. How fast is the volume changing at this instant?

$$V = \pi r^2 h$$

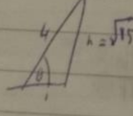
$$\frac{dV}{dt} = 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$$

$$\left.\frac{dV}{dt}\right|_{t=t_0} = 2\pi (10)(6)(-1) + \pi r^2 (10)^2 (+1)$$
$$= -120\pi + 100\pi$$
$$= -20\pi \text{ cm}^3/s$$ - volumne is decreasing

Example: A 4m plank is leaning against a wall, if at a certain instant the back of the plank is 1m from the wall and is pulled towards the wall at a rate of 0.2m/s. How fast is acute angle that the plank makes with the ground increasing?



$$\cos\theta = \frac{x}{4}$$
$$\Rightarrow -\sin\theta . \frac{d\theta}{dt} = \frac{1}{4}\frac{dx}{dt}$$
$$\Rightarrow \left.\frac{d\theta}{dt}\right|_{t=t_0} = \frac{-1}{4\sin\theta} . \left.\frac{dx}{dt}\right|_{t=t_0}$$



$\sin\theta = \frac{\sqrt{15}}{4} \Rightarrow \frac{-1}{4\left(\frac{\sqrt{15}}{4}\right)} . -0.2 \Rightarrow \frac{1}{5\sqrt{15}}$ radians

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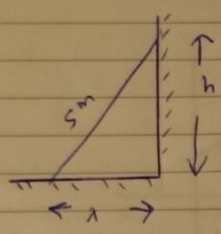
Example:

A 5m ladder is leaning against a wall. If the top of the ladder slips down the wall at a rate of 0.5m/s how fast will the front be moving away from the wall, when the top is 3m from the ground



$\frac{dx}{dt}$ $\quad 5^2 = h^2 + x^2$

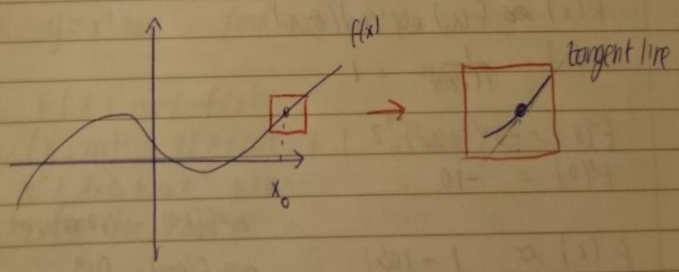
$0 = 2h\frac{dh}{dt} + 2x\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{-h}{x}\frac{dh}{dt}$

When $h = 3$ $\quad x = \sqrt{5^2 - 3^2} = \sqrt{16} = 4$

$\frac{dx}{dt} = \frac{-3}{4}(-0.5) = +\frac{3}{8}$

Local Linear Approximations



- Zoom in - curve locally looks linear (looks like straight line)
- A reasonable approximation is the tangent line to the curve at $x = x_0$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

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(linear approximation to $f(x)$ about $x=x_0$)

Example:
Approximate $\sqrt{80.9}$ using a local linear approximation and without using a calculator

Take $f(x)=\sqrt{x}$ with $x_0=81$

$f(80.9) \approx f(81) + f'(81)(-0.1)$
$f(81)=\sqrt{81}=9$
$f'(81)=\frac{1}{2\sqrt{x}}\Big|_{x=81} = \frac{1}{18}$

$9+\left(\frac{1}{18}\right)(-0.1)$
$= 9 - \frac{1}{180}$
$= 8.99444$

$\sqrt{80.9} = 8.99444272.9$

Example:
obtain a linear approximation to $f(x)=\frac{1}{(1+2x)^5}$ at $x_0=0$

$f(x) \approx f(x_0)+f'(x_0)(x-x_0)$
$f(0)=\frac{1}{(1+2(0))^5}=1$

$f'(x)=-5(1+2x)^{-6}\cdot 2$
$f'(0) = -10$

$f(x) \approx 1-10(x)$

eg $f(0.01)= 0.905730809$
app $f(0.01)= 0.9$

It is called the linear approximation because we are approximating the curve $y=f(x)$ by the tangent line $x=x_0$. For x close to x_0 i.e. small (h) this should be a reasonable approximation

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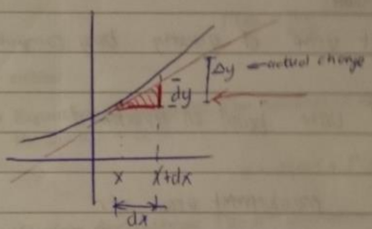
Differentials

$f'(x) = \frac{dy}{dx} \Rightarrow dy = f'(x)d(x)$

"dy", "dx" are differentials.



Δy - change in $f(x)$ as x runs from x to $x + \Delta x$ (dx)

dy - change in tangent line of $f(x)$ at x as x runs from x to $x + dx$

Example: $y = x^2 - 2x + 1$ find dy and Δy

$\Delta y = f(x + \Delta x) - f(x)$

$\Rightarrow (x + \Delta x)^2 - 2(x + \Delta x) + 1 - x^2 + 2x - 1$

$\Rightarrow x^2 + 2x\Delta + \Delta x^2 + 2\Delta x$

$= 2(x-1)\Delta x + \Delta x^2$

$dy = f'(x)dx$

$= 2(x-1)dx$

1-11-12

Local linear approximation revisited

dy is a good approximation to Δy when $dx = \Delta x$ is small (i.e. close to 0)

$\Delta y \approx dy = f'(x)dx$

error propagation due to measurement.

x_0 exact value of quantity

x measured value

$y_0 = f(x_0)$ exact value of quantity being computed

$y = f(x)$ computed value based on measurement

$dx = \Delta x = x - x_0$ measurement error of x

$\Delta y = f(x) - f(x_0)$ propagated error

Approximate the propagated error by $\Delta y \approx dy$

$\frac{\Delta y}{y} \approx \frac{dy}{y}$ relative error / percentage error

Example:

Radius of a sphere is measured with a percentage error of $\pm 0.04\%$. Estimate the percentage error in the calculated volume of the sphere

$V = \frac{4\pi}{3} r^3 \Rightarrow dV = 4\pi r^2 dr \Rightarrow \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = \frac{3dr}{r}$ = percentage error

Relative error is $\pm 0.04\%$ $\Rightarrow -0.0004 < \frac{dr}{r} \leq 0.0004$

$\Rightarrow -0.0012 \leq \frac{3dr}{r} \leq 0.0012$

$-0.0012 \leq \frac{dV}{V} \leq 0.0012$

percentage error in volume is $\pm 0.12\%$