13. Partial Derivatives

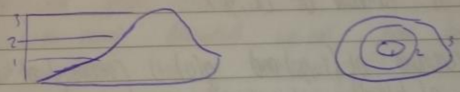
$z = f(x,y)$ means z is a function of x and y in the sense that a unique value of the dependent variable z is determined by specifying values for the independent variables x and y.

$f(x,y,z) = x^2 + xy + y^2 - \sqrt{z}$ at point $(1,3,4)$ =

$1^2 + (1)(3) + 3^2 - \sqrt{4} = 11$

Level curves:

like taking contour map of curve



If the surface $z = f(x,y)$ is cut by the horizontal plane $z = k$, then at all points on the intersection we have $f(x,y) = k$.

The projection of this intersection onto the xy plane is called the level curve of height k.

Level surface.

$z = f(x,y)$ in 3d gives curve. If k is a constant, graph of equation in 4d is $f(x,y,z) = k$ will be a surface in 3 space called i.e. level surface with constants.

13.2 Limits and Continuity

In analogy to functions of a single variable, we define the limit of a function $f(x,y,z)$ along a smooth curve C as (x,y,z) approaches $(x_0,y_0,z_0) \equiv (x(t_0), y(t_0), z(t_0))$

to be: $\lim_{(x,y,z) \to (x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0)$

$f(x(t), y(t), z(t))$

2

A function $f(x,y,z)$ is continuous at (x_0,y_0,z_0) if $f(x_0,y_0,z_0)$ is defined and

$$\lim_{t \to t_0} f(x,y,z) = f(x_0,y_0,z_0)$$

In two variables, if $f(x,y)$ is continuous at every point in a region D, then it is continuous on D, and if it is continuous on the entire xy-plane we say that $f(x,y)$ is continuous everywhere

1. If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x,y) = g(x).h(y)$ is continuous at (x_0,y_0)

2. If $h(x,y)$ is continuous at (x_0,y_0) and $g(u)$ is continuous at $u_0 = h(x_0,y_0)$ then $f(x,y) = g(h(x,y))$ is continuous at (x_0,y_0) ie. composition of continuous functions is continuous

3. Sums, differences and products of continuous functions are continuous.

4. Quotients of 2 differentiable ~~functi~~ functions are continuous unless the denominator is zero

13.3 Partial Derivatives of functions of two variables

To get a derivative we have to fix one of the variables to see how the others change. This is idea behind partial derivatives

Take $z = f(x,y)$. We can fix y at some value say $y = y_0$ then the derivative of $f(x,y_0)$ in x is

$$\frac{d}{dx} f(x,y_0)$$

3.

In other words, treat y as a constant. We can observe for x

Example $z = x^2 \sin y$ find $\frac{dz}{dx}\big|_{\pi,\pi}$ $\frac{dz}{dy}\big|_{(\pi,\pi)}$

$$\frac{dz}{dx} = 2x\sin y \,\big|_{\pi,\pi} = 2\pi \sin\pi = 0$$

$$\frac{dz}{dy} = x^2\cos y = \pi^2\cos\pi = -\pi^2$$

Higher order partial derivatives

As with normal derivatives we can have higher order derivatives but now there can be mixed partial

$$f_{xy}(x,y) = \frac{d^2f}{dy\,dx} = \frac{d}{dy}\left(\frac{df}{dx}\right)$$

$$f_{xx}(x,y) = \frac{d^2f}{dx^2}$$

Example Find $f_{xy}(x,y)$ for $f(x,y) = x^2(y^2-y)$

$$f_{xy}(x,y) = \frac{d}{dy}\left(\frac{d(x^2(y^2-y))}{dx}\right)$$

$$= \frac{d}{dy}\left(2x(y^2-y)\right)$$

$$= 2x(2y-1)$$

let f be a function of 2 variables. If f_{xy} and f_{yx} are continuous on some open disk then $f_{xy} = f_{yx}$ on that disk

13.4

One dimension wave equation

If a string is oscillating in one direction (up and down), the position of any point on the string depends on both a x coordinate and time t and can be described by a function $u(x,t)$. Then it can be shown wave equation is:

$$\frac{d^2u}{dt^2} = c^2 \frac{d^2u}{dx^2}$$

The constant c^2 depends on properties of string

Local linear approximation

If a function $f(x,y,z)$ is differentiable at a point, it can be approximated by a linear function. Consider the function at a point (x_0, y_0, z_0) and consider shifting to a nearby point ($x = x_0 + \Delta x$, $y = y_0 + \Delta y$, $z = z_0 + \Delta z$)

we can approximate:

$$f(x,y,z) \approx f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)\Delta x + f_y(x_0,y_0,z_0)\Delta y + f_z(x_0,y_0,z_0)\Delta z$$

and since $\Delta x = x - x_0$, $\Delta y = y - y_0$, $\Delta z = z - z_0$

local linear approximation: $L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0)$
$+ f_z(x_0,y_0,z_0)(z-z_0)$

Example find LLA of $f(x,y) = x^a y^b + \frac{y^a}{x^b}$ at $(1,1)$

we need $L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$f(1,1) = 2$ $\quad f_x(x,y) = ax^{a-1}y^b - b\frac{y^a}{x^{b+1}} \Rightarrow f_x(1,1) = a-b$

$f_y(x,y) = bx^a y^{b-1} + a\frac{y^{a-1}}{x^b} \Rightarrow f_y(1,1) = a+b$

$L(x,y) = 2 + (a-b)(x-1) + (a+b)(y-1)$

5.

The chain rule

Generally a function $f(x,y,z)$ depends on a parameter t via $f(x(t), y(t), z(t))$. Vary t will change each $x, y,$ and z.

Recalling that the chain rule for a function $v(u(t))$ gives $\frac{dv}{dt} = \frac{dv}{du}\frac{du}{dt}$ we define chain rule for derivatives a

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt} + \frac{df}{dz}\frac{dz}{dt}$$

If $z = f(x,y)$ has variables that depend on two parameters u and v i.e. $x(u,v)$ and $y(u,v)$ we have chain rule for partial derivatives

$$\frac{df}{du} = \frac{df}{dx}\frac{dx}{du} + \frac{df}{dy}\frac{dy}{du}$$

$$\frac{df}{dv} = \frac{df}{dx}\frac{dx}{dv} + \frac{df}{dy}\frac{dy}{dv}$$

Example: Use chain rule to find dz/du and dz/dv for

$z = x^2y, \quad x = 2u + v \quad y = u - v^2$

$$\frac{dz}{du} = \frac{dz}{dx}\frac{dx}{du} + \frac{dz}{dy}\frac{dy}{du}$$

$= 2xy(2) + x^2(1\cancel{-v^2})$

$4xy + x^2$

$= 4(2u+v)(u-v^2) + (2u+v)^2$

$$\frac{dz}{dv} = \frac{dz}{dx}\frac{dx}{dv} + \frac{dz}{dy}\frac{dy}{dv} = 2xy(1) + x^2(-2v)$$

$2xy + -2vx^2$

$2(2u+v)(u-v^2) - 2v(2u+v)^2$

6

Directional derivative and gradient

If we consider a function at a given point $f(x,y,z)$ there are many different directions in which we could move away from the initial point. In general, any linear combination which is a unit vector $(a^2+b^2+c^2=1)$

$$u = ai + bj + ck.$$

if we fix the origin to be (x_0, y_0, z_0)

In terms of arc length parameter s, then we express subsequent motion away from (x_0, y_0, z_0) through the equations

$$x = x_0 + as, \quad y = y_0 + bs, \quad z = z_0 + cs$$

When we take $s=0$ we get our initial point. Differentiation with respect to s will give the slope in the direction of u when $s=0$.

In other words we use s to tell how a small change affects the function f at (x_0, y_0, z_0). If we didn't set $s=0$ at the end we would not find the derivative at x_0, y_0, z_0, but at a point an arc length s away in the relevant direction.

We define the directional derivative of f in the direction of u to be

$$D_u f(x_0, y_0, z_0) = \frac{d}{ds}\left[f(x_0 + as, y_0 + bs, z_0 + cs) \right]\Big|_{s=0}$$

$$= f_x(x_0,y_0,z_0)a + f_y(x_0,y_0,z_0)b + f_z(x_0,y_0,z_0)c$$

This can be regarded as the slope of surface ~~wrt~~ $w = f(x,y,z)$ in the direction ~~of~~ u.

a

In other words $\nabla F(x_0, y_0, z_0)$ is normal to the tangent line of the curve C at P. We therefore define the tangent plane to be the plane with normal vector

$$n = \nabla F(x_0, y_0, z_0) = (F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0) \; F_z(x_0, y_0, z_0))$$

and the tangent plane is given by:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \quad ①$$

Since it is a plane that touches the surface $F(x,y,z)$ at the point (x_0, y_0, z_0) in analogy to the tangent line

The normal line is the line that is parallel to the normal vector and has parametric form $r(t) = r_0 + nt$.

$$x = x_0 + F_x(x_0, y_0, z_0)t \quad y = y_0 + F_y(x_0, y_0, z_0)t \quad z = z_0 + F_z(x_0, y_0, z_0)t$$

A more useful form of ① comes from considering $z = f(x,y)$ at the point $(x_0, y_0, f(x_0, y_0))$ and gives tangent plane as

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

In this form normal vector is:

$$n = (-f_x(x_0, y_0), \; -f_y(x_0, y_0), 1)$$

Since we have $F(x,y,z) = z - f(x,y)$, the normal line can be written as

$$r(t) = r_0 + t\,(-f_x(x_0, y_0)i \; - f_y(x_0, y_0)j + k)$$

These are the forms of the tangent plane and normal line that we will use for calculations. Notice it is identical to the LLA given earlier for surface $z = f(x,y)$ which reads

LL

$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

This means that the graph of the local linear approximation $z = L(x,y)$ is the tangent plane to surface $z = f(x,y)$ at point (x_0, y_0)

Example: Find tangent plane and normal line of surface $z = -(x^2+y^2)$ at point $(1,1,-2)$

We notice point is $(x_0, y_0, f(x_0,y_0))$ since $f(x_0,y_0) = -2$.

Find the derivatives $\frac{df}{dx}\Big|_{(1,1)} = -2 \qquad \frac{df}{dy}\Big|_{(1,1)} = -2$

We use these to get eqⁿ of tangent plane.

$$z = -2 + (-2)(x-1) + (-2)(y-1)$$
$$= z = 2(1-x-y)$$

Normal vector given by $n = (-f_x(x_0,y_0), -f_y(x_0,y_0), 1)$

so $n = (2,2,1)$ and vector normal line is

$$r = (1,1,-2) + t(2,2,1)$$
$$= (1+2t, 1+2t, t-2)$$

We could of wrote $F = z + x^2 + y^2$

normal vector given by $\nabla F = (F_x(1,1,-2), F_y(1,1,-2), F_z(1,1,-2)) = (2,2,1)$

Tangent plane using $(F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0), F_z(x_0,y_0,z_0)(z-z_0))$

$$2(x-1) + 2(y-1) + 1(z-(-2)) = 0$$
$$2x + 2y + z - 2 = 0$$

rewrite as $\quad z = 2(1-x-y)$.

~~which~~ which is same as before

normal line eqⁿ = $\quad x = 1+2t, \quad y = 1+2t \quad z = -2+t$

A critical point is any point which is either a stationary point or where one or more of the derivatives doesn't exist.

The Second Partial Derivative Test.
Let $f(x,y)$ be a function with continuous second order partial derivatives in a disk centred around a critical point (x_0,y_0)

$$D = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2.$$

If $D>0$ and $f_{xx}(x_0,y_0)>0$ then $f(x,y)$ has a relative min. at (x_0,y_0)

If $D>0$ and $f_{xx}(x_0,y_0)<0$ then $f(x,y)$ has a relative max at (x_0,y_0).

If $D<0$ then $f(x,y)$ has a saddle point at (x_0,y_0)

If $D=0$ no conclusion can be drawn

Saddle point is a stationary point that is not a relative ~~max~~ or absolute extremum

Example: Find the critical points of $f(x,y) = xy - x^3 - y^2$ and determine whether they are maxima, minima or saddle points.

Solution: To find critical point we set $f_x(x,y)=0$ and $f_y(x,y)=0$
Gives us $y-3x^2=0$, $x-2y=0$.

therefore from second equation we can rewrite first eq^n as:
$\frac{x}{2} - 3x^2 = 0 \Rightarrow x(x-\frac{1}{6}) = 0$ $x=0$ or $x=\frac{1}{6}$

corresponding y values are then $x=0 \Rightarrow y=0$ and $x=\frac{1}{6}$ $y=\frac{1}{12}$
critical points. $(0,0)$ $(\frac{1}{6}, \frac{1}{12})$

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Second order Partial Derivatives:

$f_{xx}(x,y) = -6x$ $\quad f_{yy}(x,y) = -2$ $\quad f_{xy}(x,y) = 1$

$$D = (-6x)(-2) - (1)^2 = 12x - 1$$

At point $(0,0)$ $D = -1 < 0$, therefor $(0,0)$ is a saddle point.

At point $(\frac{1}{6}, \frac{1}{12})$ $D = 12(\frac{1}{6}) - 1 = 1 > 0$ and therefor either a min or max

Check $f_{xx}(\frac{1}{6}, \frac{1}{12}) = -6(\frac{1}{6}) = -1 < 0$ therefor $(\frac{1}{6}, \frac{1}{12})$ is a global maximum

2.

The gradient

Calculating directional derivatives is made easier using the gradient denoted by ∇ which is "nabla" but generally read as 'del' and is given by

$$\nabla f(x,y,z) = f_x(x,y,z)i + f_y(x,y,z)j + f_z(x,y,z)k$$

Using this we see that we can use it to express directional derivatives

$$D_u f(x,y,z) = \nabla f(x,y,z) \cdot u$$

This is why it is called the gradient, because it can give the slope in any direction if the dot product with a unit vector is taken.

Properties of gradient are:

1. $z = f(x,y,z)$ has its maximum slope in the direction of the gradient, and the maximum slope is $\|\nabla f(x,y,z)\|$
2. $z = f(x,y,z)$ has its minimum slope in the direction opposite to that of the gradient and minimum slope is $-\|\nabla f(x,y,z)\|$
3. If $\nabla f = 0$ at a point, all directional derivatives are zero at that point.
4. Since level curves are curves of equal $z = f(x,y)$ then the gradient is normal to the level curves. Therefore on level curves $\nabla f \cdot T = 0$

Example: Find the unit vector in the direction which $f(x,y) = 10 - 2x^2 - y^2$ increases most quickly at $p = (1,1)$ and compute the rate of change

Direction in which f increases most is: $\nabla f|_{(1,1)} = \frac{df}{dx} i + \frac{df}{dy} j \Big|_{(1,1)}$

$$= -4x i - 2y j \Big|_{(1,1)}$$

$$= -4i - 2j.$$

8.
this has magnitude $\|\nabla F\| = \sqrt{4^2+2^2} = 2\sqrt{2}$

unit vector $= u = \frac{-2}{\sqrt{5}} i - \frac{1}{\sqrt{5}} j$

rate of change is $+\|\nabla F\| = +2\sqrt{5}$

Tangent planes and normal vectors

We want to consider how to find the tangent plane to a surface. A tangent plane is the surface that contains all possible tangent lines of all curves at a point P_0

~~At a point $P_0 = (x_0, y_0, z_0)$ the surface is continuous at P_0 and that its partial derivatives are also continuous~~

At a point $P_0 = (x_0, y_0, z_0)$ the surface $F(x(t), y(t), z(t))$ has value $c = F(x_0, y_0, z_0)$. We assume that the surface is continuous at P_0 and that its partial derivatives are also continuous

Then at point P_0 we have

~~$0 = F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0) + F_z(x_0, y_0, z_0)$~~

$$0 = F_x(x_0, y_0, z_0)\, x'(t_0) + F_y(x_0, y_0, z_0)\, y'(t_0) + F_z(x_0, y_0, z_0)\, z'(t_0)$$

We now consider a curve C parametrized by $r(t) = (x(t), y(t), z(t))$ and we note that the tangent line C runs parallel to $r'(t) = (x'(t), y(t), z'(t))$.

Above equation can be rewritten as

$$0 = \big(F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0)\big) \cdot \big(x'(t_0), y'(t_0), z'(t_0)\big)$$

which can be written as $0 = \nabla F(x_0, y_0, z_0) \cdot r'(t_0)$