ST3451: Problem set 4

December, 2015

Problem 1 is due at class on Monday 14th December 5pm class.

1. Suppose an analyst assumes the model

$$\mathrm{E}\{Y_i\} = \beta_0 + \beta_1 X_i, \; i = 1, \ldots, n$$

when the true model is

$$\mathrm{E}\{Y_i\} = \beta_0 + \beta_1 X_i + \beta_2 X_i^2.$$

If we use observations of Y at $X_1 = -1, X_2 = 0, X_3 = 1$ to estimate β_0 and β_1 in the assumed model, what biases will be introduced?

2. A one-way classification model assumes that J observations are taken from each of I normal populations, that is

$$Y_{ij} = \mu_i + \varepsilon_{ij} \qquad (i = 1, 2, \ldots, I; j = 1, 2, \ldots, J)$$

where the ε_{ij} are iid $\mathrm{N}(0, \sigma^2)$.

(a) Find the least squares estimates of $\mu_1, \ldots, \mu_I$.

(b) Suggest an estimator for σ^2 giving reason for your answer.

(c) Write down the design matrix for this model.

(d) Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_I)^T$. Write the test of hypothesis

$$H_0 : \mu_1 = \mu_2 = \ldots = \mu_I, \qquad H_A : H_0 \text{ not true}$$

in the form $H_0 : \mathbf{L}\boldsymbol{\mu} = \mathbf{0}$, identifying $\mathbf{L}$ explicitly.

(e) Describe how you would carry out the test in part (d).

3. Consdier the model from Q2 but now instead of J observations from each population, we have J_i observations from population $i = 1, \ldots, I$. How do your answers to Q2 (a) change?

4. Suppose that $\mathrm{E}\{Y_t\} = \beta_0 + \beta_1 \cos(2\pi k_1 t/n) + \beta_2 \sin(2\pi k_2 t/n)$, where $t = 1, \ldots, n$ and k_1 and k_2 are positive integers. Find the least squares estimates of β_0, β_1 and β_2.

5. What is meant by a studentized residual, and what is the motivation for using studentized residuals? Outline the ways in which a plot of studentized residuals versus fitted values can be used as a diagnostic tool to detect violations of assumptions in fitting a linear regression model.

6. (*) Let $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon}$ has mean $\mathbf{0}$, variance-covariance $\sigma^2\mathbf{I}$ and the columns of $\mathbf{X}$ are linearly independent. If $\mathbf{X}$ and $\boldsymbol{\beta}$ are partitioned in the form

$$\mathbf{X}\boldsymbol{\beta} = [\mathbf{X}_1 \, \mathbf{X}_2] \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix}$$

prove that the least squares estimate $\widehat{\boldsymbol{\beta}}_2$ of $\boldsymbol{\beta}_2$ is given by

$$\widehat{\boldsymbol{\beta}}_2 = \left[\mathbf{X}_2^T\mathbf{X}_2 - \mathbf{X}_2^T\mathbf{X}_1(\mathbf{X}_1^T\mathbf{X}_1)^{-1}\mathbf{X}_1^T\mathbf{X}_2\right]^{-1}\left[\mathbf{X}_2^T\mathbf{Y} - \mathbf{X}_2^T\mathbf{X}_1(\mathbf{X}_1^T\mathbf{X}_1)^{-1}\mathbf{X}_1^T\mathbf{Y}\right]$$

and find $\mathrm{var}\{\widehat{\boldsymbol{\beta}}_2\}$.

16/12/15 ALSM 1

PROBLEM SHEET 4

Q1 $\mathbb{E}[y_i] = \beta_0 + \beta_1 x_i \qquad i = 1, \dots, N$

but the model is polynomial $\mathbb{E}[y_i] = \beta_0 + \beta_1 x_i + \beta_2 x_i^2$

From PS 3 Q6, Q7

Sampling properties of $\hat{\beta}$

$\hat{\beta} \sim (\beta + (X^T X)^{-1} X^T Z \gamma, \; \sigma^2 (X^T X)^{-1})$

In Q6 PS 3 $\mathbb{E}[y] = X\beta$ vs $\mathbb{E}[y] = X\beta + Z\gamma$

In this case, Z holds the squared terms

Here $\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \qquad \gamma = [\beta_2]$

From Q1 PS3 $(X^T X)^{-1} = \frac{1}{n S_{xx}} \begin{bmatrix} \Sigma x_i^2 & -\Sigma x_i \\ -\Sigma x_i & n \end{bmatrix}$

$x_1 = -1, \; x_2 = 0, \; x_3 = 0 \implies \Sigma x_i = 0 \quad \Sigma x_i^2 = 2, \; n = 3, \; S_{xx} = 2$

$Z = \begin{bmatrix} (-1)^2 \\ 0^2 \\ 1^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad X^T Z = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\text{Bias} = (X^T X)^{-1} X^T Z \gamma = \frac{1}{3(2)} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \beta_2$

$= \frac{\beta_2}{6} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \, \beta_2 \\ 0 \end{bmatrix} =$ bias introduced into β_0

Q7 PS 3 states that if $\hat{\sigma}^2 = \frac{y'(I-H)y}{n-p}$ then

$\mathbb{E}[\hat{\sigma}^2] = \sigma^2 + \frac{\gamma Z^T (I-H) Z \gamma}{n-p}$

bias for σ^2 is $\frac{\beta_2 [1 0 1] (I-H) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{3-1}$

From Q1 PS 3 the (k,l) entry of the hat matrix is $\frac{1}{n} + \frac{(x_k - \bar{x})(x_l - \bar{x})}{S_{xx}}$

$I - H = \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix} \qquad \text{Bias}[\sigma^2] = \frac{\beta_2^2}{2} [1 0 1] \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix} = \frac{\beta_2^2}{3}$

10/04/16

Q2 $y_{ij} = \mu_i + \varepsilon_{ij}$ $\quad i = 1, \ldots, I$ $\quad$ I different populations

$\quad j = 1, \ldots, J$ $\quad$ equal sample size → called a balanced design

LS criterion: $Q(\hat{\mu}_1, \ldots, \hat{\mu}_I) = \sum_{i=1}^{I} \sum_{j=1}^{J} (y_{ij} - \hat{\mu}_i)^2$

$= \sum_i \sum_j (y_{ij}^2 - 2\hat{\mu}_i y_{ij} + \hat{\mu}_i^2)$

$= \sum \sum y_{ij}^2 - 2 \sum \hat{\mu}_i \sum y_{ij} + \sum_i J \hat{\mu}_i^2$

$dQ/d\hat{\mu}_i = -2 \sum_j y_{ij} + 2J\hat{\mu}_i = 0$

$\hat{\mu}_i = \sum_j y_{ij} / J = y_{i\cdot}$ sample mean for j^{th} population

dot notation - summing over j

B Estimate I parameters $SSE = \sum_i \sum_j (y_{ij} - \bar{y}_{i\cdot})^2$ has associated $df = n - I = IJ - 1 = I(J-1)$

$\hat{\sigma}^2 = MSE = SSE / I(J-1)$

C Design Matrix $\underline{y} = X\mu + \varepsilon$

$$\begin{array}{c} J \text{ rows} \{ \\ \\ J \text{ rows} \{ \\ \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & & 0 \\ \vdots & & & \\ 1 & 0 & & \\ 0 & \vdots & & \\ & \vdots & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix} \begin{array}{c} \\ \\ \\ \\ \\ \} J \text{ rows} \\ \\ \end{array} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_I \end{bmatrix}$$

$IJ \times I$ matrix

D $H_0: L\mu = \underline{0}$ $\quad$ (I different intercepts, one for each population)

vH, $\mu_1 = \mu_2 = \ldots = \mu_I$

Implies that $\mu_i - \mu_j = 0$ for $i \neq j$

Write L as $\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

L is $\binom{I}{2} \times I$ matrix