MA1E01: Chapter 6 Summary

Applications of the Definite Integral in Geometry

- **Area Between Curves**: Consider two functions f and g which are continuous on $[a, b]$ and which satisfy

$$f(x) \geq g(x) \qquad \forall\, x \in [a, b],$$

then the area bounded above by $f(x)$, below by $g(x)$ and on the sides by $x = a$ and $x = b$ is

$$A = \int_a^b \big[f(x) - g(x)\big]\, dx.$$

If we now allow the curves to cross, then the area enclosed between the curves over the interval $[a, b]$ is given by

$$A = \int_a^b \big|f(x) - g(x)\big|\, dx.$$

In order to perform this integral, we must calculate the intersection points (which are the x-values at which the curves cross) and break the integral up into intervals where we identify the upper and lower curves.

- **Reversing the Role of x and y**: Analogous results to those above hold for continuous functions of y. For example, if w and v are continuous functions of y over the y interval $[c, d]$ and satisfying

$$w(y) \geq v(y) \qquad \forall\, y \in [c, d],$$

(i.e., the curve $w(y)$ is everywhere to the right of $v(y)$), then the area enclosed between these curves and above and below by $y = d$ and $y = c$ is

$$A = \int_c^d \big[w(y) - v(y)\big]\, dy.$$

- **Volumes by Slicing**: Let S be a solid that extends along the x-axis and is bounded on the left and right by planes perpendicular to the x-axis at $x = a$ and $x = b$, respectively. Assuming the cross-sectional area $A(x)$ at each point $x \in [a, b]$ is known, then the volume of the solid is given by

$$V = \int_a^b A(x)\, dx.$$

Analogously, the volume of a solid extending along the y-axis bounded by $y = c$ and $y = d$ is

$$V = \int_c^d A(y)\, dy.$$

- **Solids of Revolution: Disks**: Let f be continuous and non-negative on $[a, b]$ and let R be the region bounded above by $y = f(x)$, below by the x-axis and on the sides by $x = a$ and $x = b$. The volume of the solid of revolution that is generated by revolving the region R about the x-axis is

  $$V = \int_a^b \pi \big[f(x)\big]^2 dx.$$

  Analogously, revolving the curve $x = w(y)$ about the y-axis generates a solid of revolution whose volume is

  $$V = \int_c^d \pi \big[w(y)\big]^2 dy.$$

- **Solids of Revolution: Washers**: If we now consider two continuous non-negative functions f and g on $[a, b]$ satisfying

  $$f(x) \geq g(x) \qquad \forall\, x \in [a, b],$$

  and let R be the region enclosed between f and g and on the sides by $x = a$ and $x = b$. Revolving the region R about the x-axis generates a solid of revolution with a hollow interior. the volume of this solid is

  $$V = \int_a^b \pi \Big\{ \big[f(x)\big]^2 - \big[g(x)\big]^2 \Big\}\, dx.$$

  Analogously, for continuous functions of y, $w(y)$ and $v(y)$ such that $w(y) \geq v(y)$ for all $y \in [c, d]$, then revolving the region enclosed by these curves about the y-axis generates a solid of revolution with a hollow interior whose volume is

  $$V = \int_c^d \pi \Big\{ \big[w(y)\big]^2 - \big[v(y)\big]^2 \Big\}\, dy.$$

- **Solids of Revolution: Cylindrical Shells**: Let f be continuous and non-negative on $[a, b]$ and let R be the region that is bounded by $y = f(x)$, below by the x-axis and on the sides by $x = a$ and $x = b$. The volume of the solid of revolution that is generated by revolving the region R about the y-axis is

  $$V = \int_a^b 2\pi\, x\, f(x)\, dx.$$

  Analogously, for the region enclosed between $x = w(y)$, the y-axis and above and below by $y = d$ and $y = c$ revolved about the x-axis generates a volume given by

  $$V = \int_c^d 2\pi\, y\, w(y)\, dy.$$

- **Arclength**: The arclength of a smooth curve $y = f(x)$ over the interval $[a, b]$ is

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

  Analogously, the arclength of the smooth curve $x = w(y)$ from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + \left[w'(y)\right]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

- **Surfaces of Revolution**: Let f be a smooth non-negative function on $[a, b]$. A surface of revolution is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis. The area S of the surface of revolution is

$$S = \int_a^b 2\pi\, f(x) \sqrt{1 + \left[f'(x)\right]^2}\, dx = \int_a^b 2\pi\, y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

  Analogously, the area of the surface of revolution generated by revolving the portion of the curve $x = w(y)$ between $y = c$ and $y = d$ about the y-axis is

$$S = \int_c^d 2\pi\, w(y) \sqrt{1 + \left[w'(y)\right]^2}\, dy = \int_c^d 2\pi\, x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$