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Q1 $y''(t) + y(t) = t + \delta(t-1) \qquad y(0)=1 \quad y'(0)=0$

$s^2 Y(s) - sy(0) - y'(0) + 1Y = \frac{1}{s^2} + e^{-s}$

$Y(s)(s^2+1) = \frac{1}{s^2} + e^{-s} + 1$

$Y = \frac{1}{s^2(s^2+1)} + \frac{e^{-s}}{(s^2+1)} + \frac{1}{(s^2+1)}$

$\frac{1}{s^2} - \frac{1}{s^2+1} + e^{-s}\frac{(1)}{s^2+1} + \frac{s}{s^2+1^2}$

$t - \sin t + \cancel{\delta(t-1)\sin t} + \cos t$

$t - \sin t + u(t-1)\sin(t-1)$

Q2 $f(x,y) = \sin\left(\frac{x^3y^2+\pi}{xe^y+1}\right)$

a $\frac{df}{dx} = \cos\left(\frac{x^3y^2+\pi}{xe^y+1}\right)\left(\frac{\cancel{(xe^y+1)(3x^2y^2)} - (x^3y^2+\pi)(e^y)}{(xe^y+1)^2}\right)$ at $(2,0)$

$\cos\left(\frac{\pi}{3}\right)\ \frac{(-\pi)}{9}$

$\frac{1}{2}\frac{(-\pi)}{9} = -\frac{\pi}{18}$

$\frac{df}{dy} = \cos\frac{(x^3y^2+\pi)}{(xe^y+1)}\left(\frac{(xe^y+1)(2x^3y) - (x^3y^2+\pi)(xe^y)}{(xe^y+1)^2}\right)$ at $(2,0)$

$\cos\left(\frac{\pi}{3}\right)\ \left(\frac{-2\pi}{9}\right) \quad \frac{1}{2}\left(\frac{-2\pi}{9}\right) = -\frac{\pi}{9}$

$\left(-\frac{\pi}{18}\, i,\ -\frac{\pi}{9}\, j\right)$

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b Rate of change = magnitude $\left(\frac{-\pi}{18} i, \frac{-\pi}{9} j\right)$

$$= \sqrt{\left(\frac{-\pi}{18}\right)^2 + \left(\frac{-\pi}{9}\right)^2} = \sqrt{\frac{\pi^2}{324} + \frac{\pi^2}{81}} = \frac{\sqrt{5\pi^2}}{18} = \text{rate of change}$$

c Unit vector in direction of decreasing most rapidly at P is unit vector divided by negative of magnitude

$$V = \frac{-\vec{\nabla} f}{\|\vec{\nabla} f\|}$$

$$\left(\frac{-\pi/18}{-\pi\sqrt{5}/18} i, \frac{-\pi/9}{-\pi\sqrt{5}/18}\right)$$

$$\frac{1}{\sqrt{5}} \vec{i}, \quad \frac{2}{\sqrt{5}} \vec{j}$$

D Eqⁿ of tangent plane

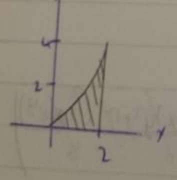
$f(x_0, y_0) \quad F_x(2,0)(x - x_0) + F_y(2,0)(y - y_0)$

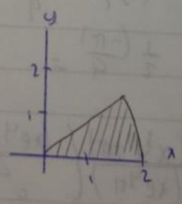
$\sin\left(\frac{\pi}{3}\right) \quad -\frac{\pi}{18}(x-2) + \frac{-\pi}{9}(y) = \cancel{2}$ = eqⁿ of plane

$$\frac{\sqrt{3}}{2} - \frac{\pi}{18} x + \frac{\pi}{9} + \frac{\pi}{9} y$$

$$18z = -\pi x + 2\pi y + 9\sqrt{3} + 2\pi$$

Q 3





b $\int_{\sqrt{2}}^{2} \int_{0}^{4} (x^3+1)^{1/2} \, dy\, dx$

$$\int_{\sqrt{y}}^{2} y\sqrt{x^3+1} \Big|_0^4$$

$$= 4\int_{\sqrt{y}}^{2} \sqrt{x^3+1} \, dx$$

$$= 4\left[\frac{3x^2}{2\sqrt{x^3+1}}\right]\Big|_{\sqrt{y}}^{2}$$

$$= 8 - \frac{6y}{\sqrt{y\sqrt{y}+1}}$$

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Q3c $x = r\cos\theta \quad y = r\sin\theta \quad (x^2+y^2) = r^2$

Limits $0 \le \theta \le \pi/4 \qquad 0 \le r \le 2$

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} x^2 x^2+y^2 \, dx\,dy$$

$$= \int_0^2 \int_0^{\pi/4} r^5 \cos^2\theta \, d\theta \, dr$$

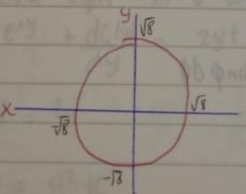
$$\int_0^2 r^5 \int_0^{\pi/4} \cos^2\theta \, d\theta \, dr$$

$$\left[\frac{r^6}{6}\right]_2^0 \left[\tfrac{1}{2}\left(x + \tfrac{1}{2}\sin 2x\right]_0^{\pi/4} \right]$$

$$= \frac{32}{3}\left[\tfrac{1}{2}\left(\pi/4 + \tfrac{1}{2}\sin \pi/2\right)\right) - 0 \Big]$$

$$= \frac{32}{3}\left(\pi/8 + 1/4 \right)$$

Q4



b Spherical $\rho, \theta, \phi \qquad x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi \qquad$ Jacob is $\rho^2\sin\phi$

$0 \le \rho \le 2\sqrt{2}$.

bounded at bottom by $z = \sqrt{x^2+y^2}$

$\rho\cos\phi = \sqrt{\rho^2\sin^2\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$

$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi}$

$\rho\cos\phi = \rho\sin\theta$

$\cos\phi = \sin\theta$ or $\phi = \pi/4$

$0 \le \phi \le \pi/4 \qquad 0 \le \theta \le 2\pi \qquad 0 \le \rho \le 2\sqrt{2}$

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$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sqrt{2}} p^2 \sin\phi \; dp\,d\phi\,d\theta$$

$$\int_0^{2\pi}\int_0^{\pi/4} \sin\phi \left.\frac{p^3}{3}\right|_0^{2\sqrt{2}} d\phi\,d\theta$$

$$\frac{16\sqrt{2}}{3}\int_0^{2\pi} (-\cos)\Big|_{\phi=0}^{\phi=\pi/4} d\theta$$

$$\frac{16\sqrt{2}}{3}\left(\frac{2-\sqrt{2}}{2}\right)(-2\pi)$$

$$= \left(\frac{-32+32\sqrt{2}}{3}\right)\pi = \text{volume}$$

C $\delta(x,y,z) = z$ change to sperical, $z = p\cos\phi$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sqrt{2}} p\cos\phi\, p^2\sin\phi \; dp\,d\phi\,d\theta$$

$$\left.\frac{p^4}{4}\right|_0^{2\sqrt{2}}$$

$$16\int_0^{2\pi}\int_0^{\pi/4} \cos\phi\sin\phi \; d\phi\,d\theta$$

$u = \cos\phi \quad du = -\sin\phi\, d\phi$

$\phi = \pi/4 \quad u = \sqrt{2}/2$

$\phi = 0 \quad \Rightarrow u = 1$

$$\int_1^{\sqrt{2}/2} u\,du$$

$$\left.\frac{u^2}{2}\right|_1^{\sqrt{2}/2}$$

$$\left.\frac{\cos^2(\phi)}{2}\right|_1^{\sqrt{2}/2} = \frac{1}{4}$$

$$\frac{16}{1}\left(\frac{1}{4}\right)\int_0^{2\pi} d\theta$$

$= 8\pi = \text{mass}$

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Q5 A If conservative, ⇒ path independent

$\frac{df}{dy} \stackrel{?}{=} \frac{dg}{dx}$ $\qquad$ $\frac{dg}{dx} = x(xe^{xy}) + e^{xy}$

$e^{xy} + y$ $\qquad$ $0 + e^{xy} + xye^{xy}$

$\cancel{e^{xy} 2 + yxe^{xy}}$

$0 + e^{xy} + xye^{xy} \quad = \Rightarrow$ conservative

B Potential function exists if it is conservative

$\frac{d\phi}{dx} = 1 + ye^{xy}$ $\qquad$ $\frac{d\phi}{dy} = 2y + xe^{xy}$

integrate wrt x

$\int \frac{d\phi}{dx} = x + e^{xy} + G(y) = \phi$

- where $G(y)$ depends on y only
- Now differentiate expression for ϕ wrt y

$\frac{d\phi}{dy} = xe^{xy} + \frac{dG(y)}{dy} = 2y + xe^{xy}$

$\frac{dG(y)}{dy} = 2y$

$G(y) = y^2 + C$

$\Phi = x + e^{xy} + y^2 + C$ where C is any constant

C When $F(x,y) = \nabla \Phi(x,y)$

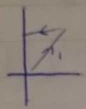
$(1,1) - (0,1)$

$(1 + e^{0} + 1^2) - (0 + e^{0} + 1^2$

$1 + 1 + \quad - (1 + 1)$

$= 1$

D Green's theorem $\int_C f(x,y)dx + g(x,y)dx = \iint_R \left(\frac{dg}{dx} - \frac{df}{dy}\right)dA$



$0 \le x \le 1$

$x \le y \le 1$

$$\int xy\,dx + x^2y^3\,dy = \int_0^1 \int_x^1 (2xy^3 - x)\,dy\,dx$$

$$= \int_0^1 \left[\frac{xy^4}{2} - xy\right]_x^1 dx$$

$$= \int_0^1 \left(\frac{x}{2} - x - \frac{x^5}{2} + x^2\right)dx$$

$$= \frac{x^2}{4} - \frac{x^2}{2} - \frac{x^6}{12} + \frac{x^3}{3}\Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{2} - \frac{1}{2} + \frac{1}{3} = 0$$