CHAPTER 15 VECTOR CALCULUS + LINE INTEGRAL

Vector field:

In 2d or 3d we define a vector field to be the function that defines a unique vector $F(p)$ for every point p.

This vector field can be written as:

$$F(x,y,z) = f(x,y,z)i + g(x,y,z)j + f(x,y,z)k$$

Acting on a vector field: Gradient, Divergence and Curl

We defined the gradient when dealing functions of several variables. The gradient defines a vector at every point in space, so in other words it is a vector field.

We will also see: $\nabla\varphi = \frac{d\varphi}{dx}i + \frac{d\varphi}{dy}j + \frac{d\varphi}{dz}k$ which is the gradient field of φ.

It is the same as before except we will use φ here rather than f so we can continue to use F in F.

The gradient field by the definition of the gradient, will point in the direction in which φ increases most. Vector fields that can be defined in this way are special.

A vector field F is conservative if there exists a φ such that $F = \nabla\varphi$ where φ is called the potential function for F.
Not all vector fields can be defined in this way.

Example: Check if the potential function $\varphi(x,y,z) = x^2 - 3y^2 + 4z^3$ gives rise to vector field $F(x,y,z) = 2xi - 6yj + 12z^2k$

Solution: We have $F(x,y,z) = \nabla(x^2 - 3y^2 + 4z^3)$

$$= \frac{d}{dx}(x^2 - 3y^2 + 4z^3)i + \frac{d}{dy}(x^2 - 3y^2 + 4z^3)j + \frac{d}{dz}(x^2 - 3y^2 + 4z^3)k$$

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$= 2xi - 6yj + 12z^2k$ as required. F is a conservative vector field.

The Divergence of a vector field $F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$ is

$$\text{div } F = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}$$

Note that the result is a scalar (a number) NOT a vector.

Curl of a vector field $F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$ is given by

$$\text{curl } F = \left(\frac{dh}{dy} - \frac{dg}{dz}\right)i + \left(\frac{df}{dz} - \frac{dh}{dx}\right)j + \left(\frac{dg}{dx} - \frac{df}{dy}\right)k$$

The curl itself is a vector field. Note that the value of both that depends on the point (through the explicit value) of f, g, h) and they should give the result in any coord system

The curl can be represented by: $\text{Curl } F = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ f & g & h \end{vmatrix}$

where we interpret $(\frac{d}{dx})(g)$ as $\frac{dg}{dx}$

Example: Find the divergence and curl of the vector field $F = xyzi + y^2zj + zk$

Solution: $\text{Div } F = \frac{d(xyz)}{dx} + \frac{d(y^2z)}{dy} + \frac{d(z)}{dz}$

$= yz + 2yz + 1 = 3yz + 1$

Curl is $\begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ xyz & y^2z & z \end{vmatrix}$

$= \left[\frac{dz}{dy} - \frac{d(y^2z)}{dz}\right]i + \left[\frac{d(xyz)}{dz} - \frac{dz}{dx}\right]j + \left[\frac{d(y^2z)}{dx} - \frac{d(xyz)}{dy}\right]k$

$= [0 - y^2]i + [xy - 0]j + [0 - xz]k$

$$= -y^2 i + xy j - xz k$$

Up until now, we have treated the gradient operator ∇ as an abstract object. However, in the terminology of this section, the natural interpretation of it is

$$\nabla = \frac{d}{dx} i + \frac{d}{dy} j + \frac{d}{dz} k$$

Hence applying it to any scalar $\phi(x,y,z)$ will give us gradient:

$$\nabla \phi = \frac{d\phi}{dx} i + \frac{d\phi}{dy} j + \frac{d\phi}{dz} k$$

It also allows us to write the divergence and curl in a convenient way: $\operatorname{div} F = \nabla \cdot F = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}$

and $\operatorname{curl} F = \nabla \times F = \left(\frac{dh}{dy} - \frac{dg}{dz}\right) i + \left(\frac{df}{dz} - \frac{dh}{dx}\right) j + \left(\frac{dg}{dx} - \frac{df}{dy}\right) k$

Geometrically, the divergence represents the tendency of the vector field to "flow" at a point. It measures how much "stuff" moves past a point.

The curl measures the rotation of the vector field at the point

In addition, we can consider forming a new operator from the del operator by taking the dot product with itself.

$$\nabla^2 = \nabla \cdot \nabla = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$$

This called the Laplace Operator because when applied to $\phi(x,y,z)$ it generates Laplace's equation which we recall is:

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0$$

and which now can be expressed in the convenient notation:

$$\nabla^2 \phi = 0$$

This appears in fluid flow and heat flow

Example: Prove that $\varphi = xz^2 - \frac{z^3}{3}$ satisfies Laplace's equation

Solution: $\frac{d\varphi}{dx} = 2xz \qquad \frac{d^2\varphi}{dx^2} = 2z$

$\frac{d\varphi}{dy} = 0 \qquad \frac{d^2\varphi}{dy^2} = 0$

$\frac{d\varphi}{dz} = x^2 - z^2 \qquad \frac{d^2\varphi}{dz^2} = -2z$

and therefore $\nabla^2\varphi = 2z + 0 + (-2z) = 0$ as required

LINE INTEGRALS:

By a line integral, we mean integrating a function along a curve C. We denote this by $\int_C f(x,y,z)\,ds$

where s is the arc length parameter along the curve

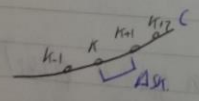
If we break the curve up into segments such that k^{th} segment has length Δs_k then the integral can be written in terms of the limit of a sum:

$$\int_C f(x,y,z)\,ds = \lim_{\max \Delta s_k \to 0} \sum_{k=0}^{n} f(x_k, y_k, z_k)\,\Delta s_k$$

However, a section of arc length is given by:

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \|r'(t)\|\,dt = \|r'(t_k)\|\,\Delta t_k$$

by mean value theorem for integral $\Delta t_k = t_k - t_{k-1}$.



The point of this is that in the limit in which all these intervals approach zero, that is transformed into a continuous variable and we arrive at the following definition:

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If C is smoothly parameterized by:

$$r(t) = xi + yj + zk$$

then the line integral of $f(x,y,z)$ along C is given by

$$\int_C f(x,y,z)\,ds = \int_a^b f(x,y,z)\,\|r'(t)\|\,dt$$

Note that if we want to integrate with respect to the x-direction we would simply replace r' with x' and ds with dx

Example: Evaluate $\int_C (1+xy^2)\,ds$ along the curve $r(t) = ti + 2t^2 j$ $(0 \le t \le 1)$

$r_1 = r_0 + t(r_1 - r_0)$

$0 + t(1)$

Solution: We find $r' = i + 2j$ and hence $\|r'\| = \sqrt{4}$

$ti + 2t^2 j$

$t(2t)^2$

Therefore line integral becomes $\int_C (1+xy^2)\,ds$

$$= \int_0^1 [1 + t(2t)^2]\sqrt{5}\,dt = \int_0^1 [1+4t^3]\sqrt{5}\,dt$$

$$= \sqrt{5}[t + t^4]_0^1 = 2\sqrt{5}$$

A more general and more useful approach to this is which we used for vector fields. Consider vector field:

$$F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$$

for which we can find the line integral for each component in the same way as before

However if we have a small length in each direction, this can be represented by $dr = dxi + dyj + dxk$

Then the line integral of F along C is denoted by:

$$\int_C F \cdot dr = \int_C (f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz)$$

When we calculate by an analogous method to the simple case as

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\,dt$$

Example: Evaluate $F = \cos t\, i + \sin t\, j$ along the curve $r = t i + t^2 j$ in the interval $-\pi/2 \le t \le \pi$

Solution: Firstly we have $r' = i + 2t j$ and

$$\int_C F \cdot dr = \int_{-\pi/2}^{\pi} (\cos t\, i + \sin t\, j) \cdot (i + 2t j)\, dt$$

$$= \int_{-\pi/2}^{\pi} (\cos t + 2t \sin t)\, dt$$

$$= \left[\sin t + (-2t\cos t + 2\sin t)\right]_{-\pi/2}^{\pi}$$

$$= 2\pi \cos \pi + 3\sin \pi - (-\pi \cos \pi/2 + 3\sin \pi/2)$$

$$= 2\pi + 3$$

Note that for the second term we used integration by parts

If we parameterize the curve using the arc length parameter s, then $T = r'(s)$ is the unit tangent vector along C and

$$\int_C F \cdot T\, ds$$

where $F \cdot T = \|F\| \|T\| \cos\theta = \|F\| \cos\theta$ where θ is the angle between the vectors and $\|T\| = 1$ (unit vector)

Line Integrals along Smooth Piecewise curves

Consider a line integral taken along a series of smooth curves joined end to end. At the joins there are not smooth, but each section is. i.e. it is a piecewise smooth function

If we have a smooth curves $C_1, C_2, C_3 \dots C_n$, then C represents the curve made by joining these together and the line integral is given by:

$$\int_C F \cdot dr = \int_{C_1} F \cdot r + \int_{C_2} F \cdot r + \dots + \int_{C_n} F \cdot r$$

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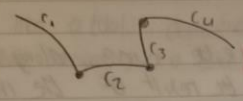
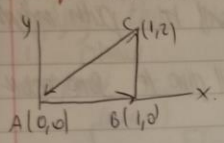


piecewise smooth function

Example: Evaluate $\int_C x^2y\,dx + x\,dy$ along the triangular path



Solution: Recall that the line segment connecting r_0 to r_1 is given by $r = (1-t)r_0 + tr_1$ for $0 \le t \le 1$. Therefore we parameterize the line as follows:

C_1: $r(t) = (1-t)(0,0) + (1,0) = (t,0)$

C_2: $r(t) = (1-t)(1,0) + (1,2) = (1,2t)$

C_3: $r(t) = (1-t)(1,2) + (0,0) = (1-t, 2-2t)$

We now integrate along each curve from 0 to 1, noting that each integral is broken up into x and y components. On C_1, $y=0$ and $y'(t)=0$ so we immediately see

$$\int_{C_1} (x^2y\,dx + x\,dy) = 0$$

Second one: $x'(t)=0$ $y'(t)=2$ so first term vanishes and

$$\int_{C_2} (x^2y\,dx + x\,dy) = \int_{C_2} x\,dy = \int_0^1 (1)\frac{d}{dt}(2t)\,dt = \int_0^1 2\,dt = 2$$

For third: $x'(t)=-1$ $y'(t)=-2$ and we have

$$\int_{C_3} (x^2y\,dx + x\,dy) = \int_0^1 (1-t)^2(2-2t)\frac{d}{dt}(1-t)\,dt + \int_0^1 (1-t)\frac{d}{dt}(2-2t)\,dt$$

$$= 2\int_0^1 (t-1)^3\,dt + 2\int_0^1 (t-1)\,dt$$

$$= \left[2\frac{(t-1)^4}{4} + 2\frac{(t-1)^2}{2}\right]_0^1$$

$$= \frac{1}{2}(0) + 0 - \left(\frac{1}{2}(1) + (1)\right) = -\frac{3}{2}$$

Therefore $\int_C x^2y\,dx + x\,dy = 0 + 2 - \frac{3}{2} = \frac{1}{2}$

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Path Independence

The path of integration, i.e. the curve along which the integral is taken can affect the result of the integral.

If a vector field is conservative, it will be path independent

In other words, any path chosen will give the same result which is determined by the end points

Conservative fields are those for which F is defined as the gradient of a function ϕ i.e. $F = \nabla\phi(x,y,z)$.

The Fundamental Theorem of Line Integrals:

Suppose that the vector field $F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$ is conservative on an open region D containing (x_0, y_0, z_0) and that the components of f, g, h are continuous on this region

If we define ϕ such that $F(x,y,z) = \nabla\phi(x,y,z)$ then for the curve C from (x_0, y_0, z_0) to (x_1, y_1, z_1)

$$\int_C F(x,y,z)\cdot dr = \int_C \nabla\phi\, dr = \phi(x_0,y_0,z_0) - \phi(x_1,y_1,z_1)$$

Proof: $\int_C F\, dr = \int_C \frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy + \frac{d\phi}{dz}dz$

$$\int_a^b \left(\frac{d\phi}{dx}\frac{dx}{dt} + \frac{d\phi}{dy}\frac{dy}{dt} + \frac{d\phi}{dz}\frac{dz}{dt}\right)dt$$

$$= \int_a^b \frac{d}{dt}\left[\phi(x(t), y(t), z(t))\right] dt$$

$$= \phi(x(b), y(b), z(b)) - \phi(x(a), y(a), z(a))$$

$$= \phi(x_1, y_1, z_1) - \phi(x_0, y_0, z_0)$$

In the 1st line we broke the integral into integrals along each direction and treated the eqn in x, y, z separately

We also used chain rule to write to combine the partial

derivative into a total derivative, before we integrated, where we recall that integrating a total derivative gives us the function differential

Applying this theorem to the idea of path independence, it is obvious that a conservative vector field is path independent since

$$\int_{(x_0,y_0,z_0)}^{(x_1,y_1,z_1)} F(x,y,z)\cdot dr = \int_{(x_0,y_0,z_0)}^{(x_1,y_1,z_1)} \nabla\phi \cdot dr = \phi(x_0,y_0,z_0) - \phi(x_1,y_1,z_1)$$

Example: Consider vector field $F = yi + xj$ with paths $y=x$ and $x=x^3$ with the same endpoints evaluated along $y=x$ and $y=x^3$ from $(0,0)$ to $(1,1)$. Prove it is independent of the given path!

Solution: We must parameterize curves using $x=t$ in both cases, which gives $y=t$ and $y=t^3$. Note also, $dr = dxi + dyj$. Then for first curve we have:

$$\int_{C_1} F\cdot dr = \int_{C_1} (yi + xj)\cdot(dxi + dyj)$$
$$= \int_{C_1} y\,dx + x\,dy$$
$$= \int_0^1 \left(y(t)\frac{dx}{dt} + x(t)\frac{dy}{dt}\right) dt$$
$$= \int_0^1 (t)(1) + (t)(1) = \int_0^1 2t\,dt = t\Big|_0^1 = 1$$

Similarly for second curve $\int_{C_2} F\cdot dr = \int_{C_2} (yi + xj)\cdot(dxi + dyj)$

$$= \int_{C_2} (y\,dx + x\,dy)$$
$$= \int_0^1 \left(y(t)\frac{dx}{dt} + x(t)\frac{dy}{dt}\right) dt$$
$$= \int_0^1 \left((t^3)(1) + (t)(3t^2)\right) dt$$
$$= \int_0^1 (4t^3)\, dt$$
$$t^4\Big|_0^1 = 1$$

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Both of these have the same result. Hence the line integral of F is independent with respect to the path

Moreover one can check that $\phi = xy$ generate F since

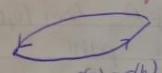
$$\nabla \phi = \frac{d}{dx}(xy)i + \frac{d}{dy}(xy)j = yi + xj = F$$

and therefore F is conservative and

$$\int_C F \cdot dr = \int_{(0,0)}^{(1,1)} F \cdot dr = \phi(1,1) - \phi(0,0) = (1)(1) - (0)(0) = 1$$

is the result for all path from $(0,0)$ to $(1,1)$

Line integral along closed path



$r(a) = r(b)$ = closed path

If we have a closed path such that $(x_1, y_1, z_1) = (x_0, y_0, z_0)$ or $r(a) = r(b)$ in vector notation, then a conservative vector field, $F = \nabla \phi$ will have the integral:

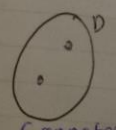
$$\int_C F \cdot dr = \phi(r(b)) - \phi(r(a)) = \phi(x_1, y_1, z_1) - \phi(x_0, y_0, z_0) = 0$$

for all closed path

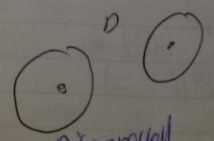
The converse is also true, any vector field for which all closed paths result with line integral 0 are conservative

In order to make this more general, we must define what it means for a region to be connected.

A region D is connected if we can join any 2 point in the region by a smooth curve otherwise it is disconnected



Connected



Disconnected

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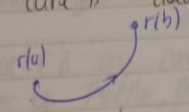
Theorem: If $f(x,y)$ and $g(x,y)$ are continuous on some open connected region D, the the following statements are equivalent:

- $F(x,y) = f(x,y)i + g(x,y)j$ is a conservative vector field on the region D
- $\int_C F \cdot dr = 0$ for every piecewise smooth closed curve on C
- $\int_C F \cdot dr$ is independent of the path from any point P in D to any point Q in D for every piecewise smooth curve C in D

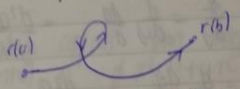
Testing for a conservative vector field

A curve is simple if it does not intersect itself

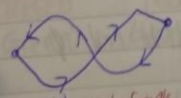
A curve is closed if its endpoints are the same.



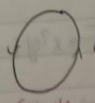
Simple not closed



not simple not closed



$r(a)=r(b)$ not simple, closed

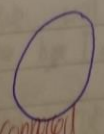


$r(a)=r(b)$

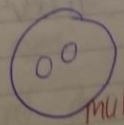
simple, closed

A **simply connected region** is a region such that any closed curve in a region D does not enclose any point not in D i.e. has no holes

If a closed curve in a connected region does enclose region outside D, it is **multiply connected** i.e has holes



Simply connected



multiply connected region

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Conservative Field test

If $f(x,y)$ and $g(x,y)$ are continuous and have continuous first order partial derivatives on some open region D, and if the vector field $F(x,y) = f(x,y)i + g(x,y)j$ is conservative on D then

$$\frac{df}{dy} = \frac{dg}{dx} \quad \text{at each point in } D$$

Conversely if D is simply connected and the condition holds at each point in D, then $F(x,y) = f(x,y)i + g(x,y)j$ is conservative

This is not a proof but note that $F = \nabla\phi$ then

$$f = \frac{d\phi}{dx} \qquad g = \frac{d\phi}{dy}$$

and so $\frac{df}{dy} = \frac{d}{dy}\frac{d\phi}{dx} = \frac{d^2\phi}{dy\,dx}$

$\frac{dg}{dx} = \frac{d}{dx}\frac{d\phi}{dy} = \frac{d^2\phi}{dx\,dy}$ which are equal if these derivatives are continuous

Example: Is $F = x^2y\,i + y\,j$ conservative? If so find ϕ.

Solution: We have $f = x^2y$ and $g = x$ and so

$\frac{df}{dy} = x^2$ and $\frac{dg}{dx} = 1$ so F is not conservative

Example: Is $F = 2xy^3\,i + (1+3x^2y^2)\,j$ conservative? If so find ϕ.

Solution: $\frac{df}{dy} = 6xy^2$ $\quad \frac{dg}{dx} = 6xy^2$ so F is conservative

so we have eqn: $\frac{d\phi}{dx} = 2xy^3$

$$\frac{d\phi}{dy} = 1+3x^2y^2$$

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To make progress we must integrate one of these equations. If we integrate the first with respect to x we have

$$\int \frac{d\phi}{dx} = \int 2xy^3 dx$$

$$\Rightarrow \phi = x^2y^3 + C_1(y)$$

where $C_1(y)$ depends on y only and is therefore a constant with respect to integration of x.

We then differentiate this expression for ϕ with respect to y and equate it to our 2nd eqn.

$$\frac{d\phi}{dy} = \frac{d}{dy}(x^2y^3 + C_1(y)) = 3x^2y^2 + \frac{dC_1}{dy}$$

and so $3x^2y^2 + \frac{dC_1}{dy} = 1 + 3x^2y^2$

which gives us $\frac{dC_1}{dy} = 1 \Rightarrow C_1(y) = y + C$

where C is any constant, which appears from integrating the eqn over y. Therefore, the potential is $\phi\ x^2y^3 + y + C$

Greens Theorem

Green's theorem allows us to rewrite an area integral as a line integral

Greens theorem: Let R be a simply connected plane region whose boundary is a simple, closed piecewise smooth curve C oriented anti-clockwise

If f(x,y) and g(x,y) are continuous and have continuous first partial derivatives on some open set containing R, then

$$\int_C f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{dg}{dx} - \frac{df}{dy}\right) dA$$

Proof: We will prove this for regions which can be viewed as either type I or II region.

In this case it comes down to proving:

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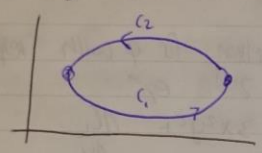
$$\int_C f(x,y)\,dx = -\iint_R \frac{\partial f}{\partial y}\,dA$$

$$\int_C g(x,y)\,dy = \iint_R \frac{\partial g}{\partial x}\,dA.$$

We will prove the first of these; the second is proved using identical logic. Denote lower part of curve by C_1 and upper by C_2.



We then break up the line integral into these upper and over parts $\int_C f(x,y)\,dx = \int_{C_1} f(x,y)\,dx + \int_{C_2} f(x,y)\,dx = \int_{C_1} f(x,y)\,dx - \int_{-C_2} f(x,y)\,dx$ where both integrals are now expressed via curves that go from left to right.

We now represent the curves parametrically:

C_1: $x=t$, $y=g_1(t)$ $(a \le t \le b)$

C_2: $x=t$ $y=g_2(t)$ $(a \le t \le b)$

Which allows us to write the line integral in terms of t:

$$\int_C f(x,y)\,dx = \int_a^b f(t,g_1(t))\,x'(t)\,dt - \int_a^b f(t,g_2(t))\,x'(t)\,dt$$

$$= \int_a^b f(t,g_1(t))\,dt - \int_a^b f(t,g_2(t))\,dt$$

$$= -\int_a^b \left[f(t,g_2(t)) - f(t,g_1(t))\right]dt = -\int_a^b \left[f(t,y)\right]_{y=g_1(t)}^{g_2(t)}$$

$$= -\int_a^b \left[\int_{g_1(t)}^{g_2(t)} \frac{\partial f}{\partial y}\,dy\right]d = -\int_a^b \left[\int_{g_1(t)}^{g_2(t)} \frac{\partial f}{\partial y}\,dy\right]\frac{1}{x'(t)}\,dx$$

$$= \int_a^b \int_{g_1(t)}^{g_2(t)} \frac{\partial f}{\partial y}\,dy\,dx = \iint_R \frac{\partial f}{\partial y}\,dA \quad \text{as required}$$

Similar calculation for the other eq^n finishes the proof.

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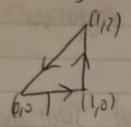
Example: Evaluate $\int_C x^2y\,dx + x\,dy$ around triangle path using Green's theorem



Solution: We have $f(x,y) = x^2y$ and $g(x,y) = x$.

Note that line from $(0,0)$ to $(1,2)$ has eq^n $y = 2x$ which will be the upper limit for the y integration.

Using theorem: $\int_C x^2y\,dx + x\,dy = \iint \left[\frac{d}{dx}(x) - \frac{d}{dy}(x^2y)\right] dA$

$$= \int_0^1 \int_0^{2x} (1 - x^2)\,dy\,dx$$

$$= \int_0^1 \left[y - x^2y\right]_{y=0}^{2x} = \int_0^1 (2x - 2x^3)\,dx$$

$$= \left[x^2 - \frac{x^4}{2}\right]_{x=0}^{1} = \frac{1}{2}$$

Green's theorem allows us to simplify problems using either an area or a line integral depending on situation

If a line integral is taken around a closed curve, we use notation

$$\oint f(x,y)\,dx$$

Green's theorem: $\oint f(x,y)\,dx + g(x,y)\,dy = \iint \left(\frac{dg}{dx} - \frac{df}{dy}\right) dA$

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Finding Area Using Green's Theorem

Can find area of region via the line integral around its boundary

If we choose $f(x,y)$ and $g(x,y)$ such that $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1$ we will get an area integral on the rhs.

Useful choices are for $f(x,y) = x$, $g(x,y) = 0$ which gives

$$A = \iint_R dA = \oint x\,dy$$

and for $f(x,y) = 0$ $g(x,y) = -y$ which gives

$$A = -\iint_R dA = \oint y\,dx$$

Often combining both is useful

$$A = \frac{1}{2}\oint -y\,dx + x\,dy$$

Example: Find the area of the circle $x^2 + y^2 = a^2$

Solution: The circle can be parameterized $\Rightarrow$

$x = a\cos t$, $y = a\sin t$ $\quad 0 \le t \le 2\pi$

which has an anti-clockwise orientation

$$A = \frac{1}{2}\oint -y\,dx + x\,dy$$

$$= \frac{1}{2}\int_0^{2\pi} \left[-(a\sin t)(-a\sin t) + (a\cos t)(a\cos t) \right] dt$$

$$= \frac{a^2}{2}\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$\frac{a^2}{2}\int_0^{2\pi} dt = \pi a^2$$

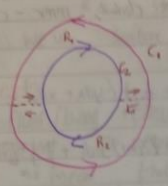
Note that $dx = -a\sin t\,(dt)$ and $dy = a\cos t\,(dt)$

Green's Theorem for multiply connected regions:

If a region has holes in it, Green's theorem doesn't appear to hold

However we can break up the region into several connected regions

Example:



Now that we can consider boundaries on the inside as well as the outside, we need to generalize our notions about orientation

A region R is positively oriented if when traversing the boundary, the region lies to the left of the boundary's orientation

Apply Green's theorem to each region:

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) = \iint_{R_1} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA + \iint_{R_2} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

$$= \oint_{R_1} f(x,y)dx + g(x,y)dx + \oint_{R_2} f(x,y)dx + g(x,y)dy$$

It turns out the contributions from the common boundary (ie) where we cut the region in 2) cancel each other out as we traverse them twice in opposite directions

Hence, the only line integrals that contribute are those over C_1 and C_2 giving:

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \oint_{C_1} f(x,y)dx + g(x,y)dy + \oint_{C_2} f(x,y)dx + g(x,y)dy$$

Example Consider the region with outer boundary a circle of radius 2 and inner boundary a circle of radius 1. Use Green's theorem to find area:

Solution Easy to work out, we know answer is 3π.
Outer circle oriented anti clockwise, inner - clockwise.
integrate from 2π to 0

$$A = \frac{1}{2}\oint_{C_1}(-y\,dx + x\,dy) + \frac{1}{2}\oint_{C_2}(-y\,dx + x\,dy)$$

$$\frac{1}{2}\int_0^{2\pi}\left[-(2\sin t)(-2\sin t) + (2\cos t)(2\cos t)\right]dt + \frac{1}{2}\int_{2\pi}^{0}\left[-\sin t(\sin t) + \cos t(\cos t)\right]dt$$

$$= \frac{1}{2}(4)\int_0^{2\pi}\sin^2 t + \cos^2 t\,dt - \frac{1}{2}\int_0^{2\pi}\sin^2 t + \cos^2 t\,dt$$

$$= 2\int_0^{2\pi}dt - \frac{1}{2}\int_0^{2\pi}dt$$

$$= 2(2\pi) - \frac{1}{2}(2\pi) = (4-1)\pi = 3\pi.$$

Note $\int_{2\pi}^{0} = -\int_0^{2\pi}$ reversing order of integration produces a minus sign.

CHAPTER 6 IN THIS NOTES

Surface Integral

The idea is to integrate a function $f(x,y,z)$ over a surface σ. Integration follows normal form.

If $r(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$ the surface integral of $f(x,y,z)$ over σ is:

$$\iint_\sigma f(x,y,z)\,d\sigma = \iint_R f(x(u,v), y(u,v), z(u,v))\left\|\frac{dr}{du}\times\frac{dr}{dv}\right\|du\,dv$$

Note that $f(x,y,z) = 1$ the surface area comes from letting:

$$S = \iint_R \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dA$$

For a surface $z = f(x,y)$ this reduces to the simpler form:

$$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,f(x,y)) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

in section 3: we use surface integral for flux, and flux integral

Flux Integrall

Consider a fluid flowing through 3d.
We can represent the velocity of the fluid as a vector field $F(r)$, which mean that every point in the fluid has a velocity that depends on that point.

We have a flow represented by a vector field.
In this context the vector field is called a <u>flow field</u>

Oriented Surfaces

Consider surface of sphere. We have an inside and outside to the surface
This is what we mean by orientable.

In case of ball, if we start on the inside surface we can never reach the outside surface by moving on the surface and vice versa
This is a normal surface with 2 side

With Möbius strip we start at inside and end at "outside". Only 1 side

A two sided surface is said to orientable and a one-sided surface is non orientable

How to distinguish between sides?

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- Recall that a unit normal vector n is perpendicular to the surface which we call σ.
- Since $-n$ is parallel to n, but points in opposite direction, it too is perpendicular to the surface but points in other direction, so these 2 vectors distinguish the sides on the surface

If σ is a smooth orientable surface, it is always possible to choose the direction of $n = n(x,y,z)$ so that it varies continuously over surface

These unit vectors form an orientation of the surface. A smooth orientable surface has exactly 2 possible orientations.

Orientation of a smooth parametric surface

Recall that for a parametric surface σ traced out the graph of $r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$

the principle unit normal vector is: $n = \dfrac{\frac{dr}{du} \times \frac{dr}{dv}}{\left\|\frac{dr}{du} \times \frac{dr}{dv}\right\|}$

In fact this vector defines the positive orientation of the surface σ with n pointing in the positive direction

$-n$ defines the negative orientation, with $-n$ pointing in the negative direction

Example: Find the orientation of the cylinder given by $r = \cos u\, i + v j - \sin u\, k$

Solution: To find normal vector we need derivative $\frac{dr}{du} = -\sin u\, i - \cos u\, k$, $\frac{dr}{dv} = j$

then $\frac{dr}{du} \times \frac{dr}{dv} = \cos u\, i - \sin u\, k$

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and $\left\| \frac{dr}{du} \times \frac{dr}{dv} \right\| = 1$

and $n = \frac{dr}{du} \times \frac{dr}{dv} = \cos u\, i - \sin u\, k$

This points in the outward direction radially
$-n$ points inwards.

Flux

Fluid is either a gas which is compressible or a liquid which is incompressible. We will only be concerned with incompressible fluids which are in a steady state

This means that the velocity of a given point in the liquid is fixed i.e. it does not vary with time

What happens when we insert a permeable, oriented surface σ into a flowing incompressible fluid in a steady state? What is the net volume Φ that passes through surface per unit of time?

By net volume we mean that if there is fluid flow in the positive direction minus the volume that flows in the negative direction

If we have an incompressible fluid with velocity
$F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$
flowing through a surface σ with normal vector n pointing in the positive orientation.

The projection of F onto n is $(n \cdot F)n$ which gives the flow of the fluid through the surface

The remaining contribution is the flow along the surface

The sign of $F \cdot n$ determines the direction of flow, a $\oplus$ sign means the flow is in the direction of n "out of the surface" a $\ominus$ sign means the flow in the direction of $-n$ ie "into surface"

We now break up the surface into patches $\sigma_1, \sigma_2, \sigma_n$ with areas $\Delta S_1, \Delta S_2 \ldots \Delta S_n$. For a point (x_k, y_k, z_k) in the kth patch, the velocity through the surface σ_k is then

$$F(x_k, y_k, z_k) \cdot n(x_k, y_k, z_k)$$

The the volume of fluid that passes through the surface σ_k in unit time is the velocity (times the time which is 1) multiplied by the surface area ΔS_k ie.

$$F(x_k, y_k, z_k) \cdot n(x_k, y_k, z_k) \Delta S_k$$

We obtain the volume that passes through the entire surface by summing all the contributions

$$\sum_{i=1}^{n} F(x_k, y_k, z_k) \cdot n(x_k, y_k, z_k) \Delta S_k$$

Finally we define the total volume passing through σ to be the limit of the sum @ the areas $\Delta S_k \to 0$ i.e. when $n \to \infty$

$$\Phi = \lim_{n \to \infty} \sum_{k=1}^{n} F(x_k, y_k, z_k) \cdot n(x_k, y_k, z_k) \Delta S_k$$

which becomes the integral:

$$\Phi = \iint_{\sigma} F(x,y,z) \cdot n(x,y,z)\, dS$$

known as the flux of F across σ

Evaluating Flux Integrals

We can now evaluate flux integrals

$$\iint_{\sigma} F \cdot n\, dS = \iint_{R} F \cdot n \left\| \frac{dr}{du} \times \frac{dr}{dv} \right\| dA$$

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$$= \iint_R F \cdot \frac{\frac{dr}{du} \times \frac{dr}{dv}}{\left\| \frac{dr}{du} \times \frac{dr}{dv} \right\|} \left\| \frac{dr}{du} \times \frac{dr}{dv} \right\| dA.$$

$$= \iint_R F \cdot \left(\frac{dr}{du} \times \frac{dr}{dv} \right) dA$$

Example: Find the flux of vector field $F = zk$ across the sphere $x^2 + y^2 + z^2 = a^2$ oriented outwards

Solution A sphere with outward orientation is represented as a vector valued function in spherical coordinates by:

$$r(\theta, \phi) = a \sin\phi \cos\theta i + a \sin\phi \sin\theta + a \cos\phi$$

We in fact used this when discussing surface area and similarly we get: $r_\phi = a\cos\phi\cos\theta i + a\sin\phi\sin\theta j - a\sin\phi k$

$$r_\theta = -a\sin\phi\sin\theta i + a\sin\phi\cos\theta j.$$

The cross product:

$$r_\theta \times r_\phi = \begin{vmatrix} i & j & k \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

We now substitute this into our eqn

$$\Phi = \iint_R F \cdot \left(\frac{dr}{d\phi} \times \frac{dr}{d\theta} \right) dA$$

We shall also rewrite $F = a\cos\phi k$

$$F \cdot \left(\frac{dr}{d\phi} \times \frac{dr}{d\theta} \right) = a^3 \sin\phi \cos^2\phi$$

Subbing into the eqn for the flux we get

$$\Phi \int_0^{2\pi} \int_0^{\pi} a^3 \sin\phi \cos^2\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} a^3 [-\cos^2\phi] \, d(\cos\phi) \, d\theta$$

2u

$$= \int_0^{2\pi} a^3 \left[\frac{-\cos^3 u}{3} \right]_{\phi=0}^{\pi} d\theta$$

$$= \int_0^{2\pi} a^3 \left[1 - \frac{-1}{3} \right] d\theta$$

$$\frac{2a^3}{3} \int_0^{2\pi} d\theta = \frac{4\pi a^3}{3}$$

We made use of fact that $d(\cos\phi) = -\sin\phi \, d\phi$

Orientation and Flux of non-parametric Surfaces

Let σ be a non parametric surface of the form:

$z = g(x,y)$ or $y = g(z,x)$ or $x = g(y,z)$

which can be represented by vector valued functions

$r = ui + vj + g(u,v)k$ or $r = vi + g(u,v)j + uk$ $\quad r = g(u,v)i + uj + vk$

We can then write these surfaces as

$G(x,y,z) = z - g(x,y) = 0 \quad G(x,y,z) = y - g(z,x) \quad G(x,y,z) = z - g(y,z) = 0$

Since $G(x,y,z)$ in all cases is just another way to represent the surface σ, the Gradient ∇G is normal to the surface and the principle unit normal vector is given by:

$$n = \frac{\nabla G}{\|\nabla G\|}$$

In other words: $\nabla G = \frac{dr}{du} \times \frac{dr}{dv}$

and so, flux may be written as:

$$\Phi = \iint_\sigma F \cdot n \, dS = \iint_R F \cdot \nabla G \, dA$$

If we take the first of the three cases above $z = g(x,y)$ then the gradient of $G(x,y,z) = z - g(x,y)$ is

$$\nabla G = -\frac{dg}{dx} i - \frac{dg}{dy} j + k = -\frac{dz}{dx} i - \frac{dz}{dy} + k$$

and we get:

$$\iint_\sigma F\cdot n\,dS = \iint_R F\cdot\left(-\frac{dz}{dx}i - \frac{dz}{dy}j + k\right)dA \quad \text{for } \sigma \text{ oriented up}$$

$$\iint_\sigma F\cdot n\,dS = \iint_R F\cdot\left(\frac{dz}{dx}i + \frac{dz}{dy}j - k\right)dA \quad \text{for } \sigma \text{ oriented down}$$

Example Let σ be the portion of the surface $z = 1-x^2-y^2$ that lies above the xy plane, with σ oriented up. Find the flux of the vector field $F(x,y,z) = xi + yj + zk$ and σ.

Solution: We can use eqn for σ oriented up:

$$\Phi = \iint_R F\cdot\left(-\frac{dz}{dx}i - \frac{dz}{dy}j + k\right)dA$$

$$= \iint_R (xi + yj + zk)\cdot(2xi + 2yj + k)\,dA$$

$$= \iint_R (2x^2 + 2y^2 + z)\,dA$$

$$= \iint_R (2x^2 + 2y^2 + (1-x^2-y^2))\,dA$$

$$= \iint_R (x^2 + y^2 + 1)\,dA$$

$$\int_0^{2\pi}\int_0^1 (r^2+1)(r)\,dr\,d\theta \quad \text{CHANGING TO POLAR COORD}$$

$$= \int_0^{2\pi}\left[\frac{r^4}{4} + \frac{r^2}{2}\right]_{r=0}^{1} d\theta$$

$$= \int_0^{2\pi}\frac{3}{4}\,d\theta = \frac{3\pi}{2}$$

The Divergence Theorem.

We focus on a closed surface, which is a surface which contains a finite solid (cube, sphere etc)

A sphere is a smooth surface as it has one face and no corners or vertex

A box is not smooth and has 6 faces

$$\iint_{\sigma} F\cdot n\,d\sigma = \iint_R F\cdot\left(-\frac{\partial z}{\partial x} i - \frac{\partial z}{\partial y} j + k\right) dA \quad \text{for } \sigma \text{ oriented up}$$

$$\iint_{\sigma} F\cdot n\,d\sigma = \iint_R F\cdot\left(\frac{\partial z}{\partial x} i + \frac{\partial z}{\partial y} j - k\right) dA \quad \text{for } \sigma \text{ oriented down}$$

Example Let σ be the portion of the surface $z = 1 - x^2 - y^2$ that lies above the xy plane, with σ oriented up. Find the flux of the vector field $F(x,y,z) = xi + yj + zk$ across σ

Solution: We can use eqn for σ oriented up:

$$\Phi = \iint_R F\cdot\left(-\frac{\partial z}{\partial x} i - \frac{\partial z}{\partial y} j + k\right) dA$$

$$= \iint_R (xi + yj + zk)\cdot(2xi + 2yj + k)\,dA$$

$$= \iint_R (2x^2 + 2y^2 + z)\,dA$$

$$= \iint_R \left(2x^2 + 2y^2 + (1 - x^2 - y^2)\right) dA$$

$$= \iint_R (x^2 + y^2 + 1)\,dA$$

$$\int_0^{2\pi}\int_0^1 (r^2+1)(r)\,dr\,d\theta \quad \text{CHANGING TO POLAR COORD}$$

$$= \int_0^{2\pi} \left[\frac{r^4}{4} + \frac{r^2}{2}\right]_{r=0}^{1} d\theta$$

$$= \int_0^{2\pi} \frac{3}{4}\,d\theta = \frac{3\pi}{2}$$

The Divergence Theorem.

We focus on a closed surface, which is a surface which contains a finite [illegible] space in it.

[illegible]

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We only look at cases where surfaces are orientable and have either inward orientation or outward orientation

Divergence = $F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$

is $\text{div } F = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}$

The physical understanding is stated in the divergence theorem where we consider fluid flowing through a surface.

The Divergence Theorem

Let G be a solid whose surface σ is oriented outward if

$F(x,y,z) = f(x,y,z)i + g(x,y,z)j + f(x,y,z)k$

is a vector field where f, g, h have continuous first partial derivatives on some open set containing G, and if n is the outward unit normal vector on σ then

$$\iint_{\sigma} F \cdot n \, dS = \iiint_{G} \text{div} F \, dV$$

In other words, the flux through the surface is equal to the divergence of F integrated over the solid. This means that the divergence gives the amount of fluid that flows per unit volume

Divergence is a scalar not a vector, so taking a triple integral can be easier than taking a surface integral

Flux from the Divergence Theorem

Flux through a surface is given by $\Phi = \iint_{\sigma} F \cdot n \, dS$

and therefore we can apply divergence theorem to calculate it.

Example

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Example: Find the outward flux of vector field $F(x,y,z) = zk$ across the sphere $x^2+y^2+z^2=a^2$

Solution: To apply divergence theorem we need the divergence,

$$\operatorname{div} F = \frac{\partial(z)}{\partial z} = 1$$

and hence $\Phi = \oiint_S F\cdot n\,dS = \iiint_\sigma (1)\,dV = \iiint_\sigma dV = \frac{4\pi a^3}{3}$

Since the volume of a sphere with radius a is $4\pi a^3/3$, and the triple integral with 1 as the argument is just the volume

- Hard because we don't have to calculate a unit vector for each section of piecewise smooth curve with divergence theory
- Only requirement for surface is that it is outward oriented

Example: Find flux of vector field $F(x,y,z) = (2x^3+xy-2zy)i + (xy^2-x^2z)j + (4xz-x^2y)k$ across the unit cube $\square$ $L=1$

Solution: Surface has 6 faces, use divergence theory.

$$\operatorname{div} F = \frac{d(2x^3+xy-2zy)}{dx} + \frac{d(xy^2-x^2z)}{dy} + \frac{d(4xz-x^2y)}{dz}$$

$$= (6x^2+y) + (2xy) + (4x) = 6x^2+4x+2xy+y$$

Since we have a unit cube it makes sense to integrate using cartesian coord

$0 \le x,y,z \le 1$

$$\Phi = \oiint F\cdot n\,dS = \iiint (6x^2+4x+2xy+y)\,dV$$

$$= \int_0^1\int_0^1\int_0^1 (6x^2+4x+2xy)\,dz\,dy\,dz$$

$$= \int_0^1\int_0^1 (6x^2+4x+2xy)z\Big|_{z=0}^{1}\,dy\,dz$$

$$\int_0^1\int_0^1 (6x^2+4x+2xy+y)\,dy\,dx$$

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$\int_0^1 (6x^2y+4xy+xy^2+\frac{y^2}{2})\Big|_{y=0}^{1} dx$

$= \int_0^1 (6x^2+4x+x+\frac{1}{2})dx = \int_0^1 (6x^2+5x+\frac{1}{2})dx$

$= (2x^3+\frac{5x^2}{2}+\frac{x}{2})\Big|_{x=0}^{1} = 2+\frac{5}{2}+\frac{1}{2} = 5$

Example Find the outward flux of the vector field.

$F(x,y,z) = \frac{x^3}{3}i + \frac{y^3}{3}j + \frac{z^3}{3}k$

across the region enclosed by the hemisphere $z=\sqrt{a^2-x^2-y^2}$ and plane $z=0$.

Solution: The Surface is piecewise Smooth, Divergence Theory works best:

$\operatorname{div} F = \frac{d(x^3/3)}{dx} + \frac{d(y^3/3)}{dy} + \frac{d(z^3/3)}{dz} = x^2+y^2+z^2$

Spherical coordinates are best, since we have a hemisphere with $z \geq 0$ and radius a, the limits are $0 \leq \rho \leq a$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2$ Hence the integral becomes:

$\Phi = \oiint F \cdot dS = \iiint_G (x^2+y^2+z^2)dv = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (\rho^2)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$

$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a \rho^4 \sin\phi \, d\rho\, d\phi\, d\theta$

$\int_0^{2\pi}\int_0^{\pi/2} \frac{\rho^5}{5}\sin\phi \Big|_{\rho=0}^{a} d\phi\, d\theta$

$= \int_0^{2\pi}\int_0^{\pi/2} \frac{a^5}{5}\sin\phi \, d\phi\, d\theta = \int_0^{2\pi} \frac{a^5}{5}\left[-\cos\phi\right]_{\phi=0}^{\pi/2} d\theta$

$= \int_0^{2\pi} \frac{a^5}{5}[0-(-1)]d\theta = \int_0^{2\pi} \frac{a^5}{5} d\theta = \frac{2\pi a^5}{5}$

An Interpretation of the Divergence

Consider small spherical region G, region has volume $vol(G)$ and surface denoted $\sigma(G)$

If region is sufficiently small, div F will not vary much for any vector field F on the region, and therefore we can take it to be constant, which we take to be the divergence at the centre of the region, P_0, and so $\operatorname{div} F \approx \operatorname{div} F(P_0)$.

The flux is then approximately given by:

$$\Phi(G) = \iint_{\partial(G)} F \cdot n \, dS = \iiint_G \operatorname{div} F \, dV \approx \operatorname{div} F(P_0) \iiint_G dV = \operatorname{div} F(P_0) \operatorname{vol}(G)$$

Hence we get $\operatorname{div} F(P_0) \approx \dfrac{\Phi(G)}{\operatorname{vol}(G)}$

This expression is called the outward flux density of F across G.
In the limit that the volume of G approaches 0, ie $\operatorname{vol}(G) \to 0$ this expression becomes exact and can be expressed as:

$$\operatorname{div} F(P_0) = \lim_{\operatorname{vol}(G) \to 0} \frac{\Phi(G)}{\operatorname{vol}(G)} \iint_{\partial(G)} F \cdot n \, dS$$

This is the outward flux density of F at P_0. In other words, the divergence measures how much "stuff" moves through a region of unit volume.

Sources and Sinks.

If we have no sources or sinks in an incompressible fluid, there can be no net flow out of or into a surface and $\operatorname{div} F(P) = 0$ for every point P.

If divergence is positive ($\operatorname{div} F(P) > 0$) then P is a source.
If divergence is negative ($\operatorname{div} F(P) < 0$) then P is a sink

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STOKES THEORM

If a curve C bounds a surface σ, then if you walk along the curve with your head in same direction as the orientation of σ, walking around the curve so that σ is on your left induces the positive orientation of C and walking around the curve so that σ is on your right induces the negative orientation of C

Stokes theorm: Let σ be a piecewise smooth oriented surface that is bounded by a simple, closed, piecewise smooth curve C with positive orientation. If the vector field

$F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$ and has continuous components f, g, h with continuous first partial derivatives on some open set containing σ, and if T is the unit tangent vector to C, then:

$$\oint_C F \cdot T \, dl = \iint_\sigma (\text{curl } F) \cdot n \, dS$$

In terms of arc length parameter, s, the unit tangent vector T is given by $T = dr(s)/ds$ Hence $T\,ds = dr$, theorm becomes:

$$\oint_C F \cdot dr = \iint_\sigma (\text{curl } F) \cdot n \, dS$$

Stokes theorm is a generalization of Green's theorm

Example: Prove that $F(x,y,z) = 2z\,i + 3y\,j + 5y\,k$ defined on the surface σ which is the portion of the paraboloid $z = 4 - x^2 - y^2$ for $z \geq 0$ with upward orientation satisfies Stoke's theorm if σ is bounded by the positively oriented circle $x^2 + y^2 = 4$ in the xy plane

Solution: We need to evaluate both sides of equation. The curve is a circle of radius 2 parametrized in the

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anticlockwise direction

$x = 2\cos t, \quad y = 2\sin t, \quad z = 0 \qquad (0 \le t \le 2\pi)$

The line integral is: $\oint_C F \cdot dr = \oint_C 2z\,dz + 3x\,dy + 5y\,dz$

$= \int_0^{2\pi} [2(0)(-2\sin t) + 3(2\cos t)(2\cos t) + 5(2\sin t)(0)]\,dt$

$= 12\int_0^{2\pi} \cos^2 t\,dt = 6\int_0^{2\pi} [1+\cos 2t]\,dt = 3\left[t + \frac{1}{2}\sin 2t\right]_0^{2\pi} = 12\pi$

For the other side of the equation we need curl F:

$$\text{curl } F = \nabla \times F \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 2z & 3x & 5y \end{vmatrix} = 5i + 2j + 3k$$

Since σ is represented by $z = f(x,y)$ and is oriented upwards,

$n = -\frac{dz}{dx} i - \frac{dz}{dy} j + k$ and so:

$\iint_\sigma (\text{curl } F) \cdot n\, d\sigma = \iint_R (5i + 2j + 3k) \cdot \left(\frac{-dz}{dx} i - \frac{dz}{dy} j + k\right) dA$

$= \iint_R (5i + 2j + 3k) \cdot (2x i + 2y j + k)\, dA$

$= \iint_R (10x + 4y + 3)\, dA$

We need to decide which coords to use: Polar

$0 \le r \le 2 \qquad 0 \le \theta \le 2\pi$

$\iint_\sigma (\text{curl } F) \cdot n\, d\sigma = \int_0^{2\pi} \int_0^2 (10r\cos\theta + 4r\sin\theta + 3)\, r\, dr\, d\theta$

$= \int_0^{2\pi} \int_0^2 (10r^2\cos\theta + 4r^2\sin\theta + 3r)\, dr\, d\theta$

$\int_0^{2\pi} \left[\frac{10r^3\cos\theta}{3} + \frac{4r^3\sin\theta}{3} + \frac{3r^2}{2}\right]_0^2 d\theta$

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$$\int_0^{2\pi} \left[\frac{80\cos\theta}{3} + \frac{32\sin\theta}{3} + 0\right] d\theta$$

$$= \left[\frac{80\sin\theta}{3} - \frac{32\cos\theta}{3} + 6\theta\right]_{\theta=0}^{2\pi}$$

$$\left[\frac{80(0)}{3} - \frac{32(1)}{3} + 6(2\pi)\right] - \left[\frac{80(0)}{3} - \frac{32(1)}{3} + 6(0)\right] = 12\pi$$

Both results agree, so Stokes Theorem is satisfied

Finally, let's mention that we can use Stokes Theorem to evaluate work.

$$W = \oint_C F\,dr = \iint_S (\text{curl } F)\cdot n\,dS$$

Laplace Transform

Used to solve linear ordinary differential equations (ODEs). They look like:
$y' + 2y = 0$
or $y'' - 3y' + 4y = 0$

where the ' prime means derivative

We used this earlier in the course when solving for the potential function ϕ of a vector field F

When we integrated formulae such as $\frac{d(\phi(y))}{dy} = y$
to find $\phi(y)$ we are actually solving an ODE.

Defining the Laplace Transform

A Laplace transform is a type of integral transform where we integrate the original function $f(t)$ with some kernel $K(s,t)$ to produce a new function which we call $F(s)$. Generally:

$$F(s) = \int_0^\infty K(s,t) f(t)\,dt$$

is an integral transform.

If $f(t)$ is defined for all $t \geq 0$ then the Laplace transform of f is $F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$

For a Laplace transform the kernel is e^{-st}. We also have the inverse transform $f(t) = \mathcal{L}^{-1}(F)$

We will always use f and t for the original function and F and s for the Laplace transform

Example: Find Laplace transform of $f(t) = 1$

Solution: $\mathcal{L}(1) = \int_0^\infty e^{-st}(1)\,dt = \int_0^\infty e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_0^\infty$

$= 0 - \left[-\frac{1}{s}(1)\right] = \frac{1}{s}$ since $e^{-\infty} = 0$

Example: Find Laplace transform of $f(t) = e^{at}$.

Solution: $\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$

$= -\frac{1}{s-a} e^{-(s-a)t}\Big|_0^\infty = 0 - \left(-\frac{1}{s-a}(1)\right) = \frac{1}{s-a}$

Since $e^{-\infty} = 0$ again

Laplace transforms satisfy the linearity property
$\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f) + b\mathcal{L}(g) = aF(s) + bG(s)$.

where $\mathcal{L}(g) = G(s)$ is the Laplace transform of $g(t)$. It follows from the properties of integration.

One result of table is the result of the first shifting theorem. Allows us to find $\mathcal{L}(e^{at}f(t))$ directly from $\mathcal{L}(f(t))$

$$\mathcal{L}[e^{at} f(t)] = F(s-a).$$

Conversely: $e^{at}f(t) = \mathcal{L}^{-1}[F(s-a)]$

To see this, consider:
$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt = \int_0^\infty e^{-st}[e^{at}f(t)] = \mathcal{L}[e^{at}f(t)]$$

In other words, if $F(s) = \mathcal{L}(f(t))$ is known, we can easily find $\mathcal{L}[e^{at}f(t)]$ which is $F(s-a)$.

Example: Find Laplace transform of $y(t) = 2e^{t} + 3e^{2t}$

Solution: We make use of above properties

$$L(y) = L(2e^{t} + 3e^{2t}) = 2L(e^{t}) + 3L(e^{2t})$$

$$= 2\frac{1}{s-1} + 3\frac{1}{s-2} = \frac{5s-7}{(s-1)(s-2)}$$

Example: Find Laplace Transform of $y(t) = e^{at}t^{n}$.

Solution: We make use of above property

$$L(y) = L[e^{at}t^{n}] = F(s-a)$$

where $F(s-a) = L(t^{n}) = \frac{n!}{s^{n+1}}$

Hence $L(y) = F(s-a) = \frac{n!}{(s-a)^{n+1}}$

Example: Find $L^{-1}\left(\frac{1}{s^2-4s+5}\right)$

Solution: We can rewrite the fraction in the following way

$$\frac{1}{s^2-4s+5} = \frac{1}{(s-2)^2-4+5} = \frac{1}{(s-2)^2+1}$$

Therefore using table :- $L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$

If we call $F(s) = \frac{1}{s^2+1}$ and $f(t) = \sin t$, then we see that shifting a by 2 will give us

$$L^{-1}\left(\frac{1}{s^2-4s+5}\right) = L^{-1}\left(\frac{1}{(s-2)^2+1}\right) = e^{2t}\sin t$$

i.e. $F(s-2) = \frac{1}{s^2-4s+5} \Rightarrow L^{-1}[F(s-2)] = e^{2t}F(s)$

4.

Ordinary Differential Equations

Transforms of derivatives

The laplace transforms of derivatives of y and its derivatives (assuming y and y' are continuous) are

$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$$
$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0)$$

To prove these, we substitute into eqn's defining laplace transform and then integrate by parts

$$\mathcal{L}(y') = \int_0^\infty e^{-st}(y'(t))dt = e^{-st}y(t)\Big|_0^\infty - (-s)\int_0^\infty e^{-st}(y(t))dt$$
$$= (0) - (1)(y)(0) + s\mathcal{L}(y) = s\mathcal{L}(y) - y(0)$$

We can now use the first result to help prove second

$$\mathcal{L}(y'') = \mathcal{L}((y')') = s\mathcal{L}(y') - y'(0)$$

Which gives us

$$\mathcal{L}(y'') = s\mathcal{L}(y') - y'(0) = s[s\mathcal{L}(y) - y(0)] - y'(0) = s^2\mathcal{L}(y) - sy(0) - y'(0)$$

First and Second order differential eqn's:

The most basic type are first order differential eqn's which means that only first derivatives appear. We assume that we have y, which is a function of t ie. $y(t)$

The eqn $y' + ay = 0$ is a general form of a homogeneous first order differential eqn

It contains the first derivative y' and the ~~original~~ original function y. It is called homogeneous because the right hand side is zero.

An inhomogeneous f ODEs is something like

$$y' + ay = r(t)$$

where

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Similarly we can define second order differential which also include y''. They can be homogeneous

$$y'' + ay' + by = 0$$

or inhomogeneous:

$$y'' + ay' + by = r(t)$$

Usually when solving ODEs we solve the homogeneous eq^n to give a general solution and then the inhomogeneous eq^n to give the particular solution and add these to get the complete solution

We can also define n^{th} order differential eq^n including the nth derivative.

In order to solve odes we also need initial values. For a first order differential equation we need a condition $y(0) = k_0$ and for a second order differential eq^n we need 2 conditions $y(0) = k_0$ and $y'(0) = k_1$

An ODE with necessary initial condition is called an initial value problem. We use Laplace transforms to solve this

Solving ordinary differential eqns

Write in terms of $L(y)$ and initial conditions. Therefore we can reduce differential eq^{n}s to polynomials.

First order ODEs become linear and second order become quadratic eqns

An inhomogeneous eq^n can be solved directly without need to solve for both a general and particular solution

The initial conditions are accounted for without needing to use

Here to eliminate constants as happens in standard methods of solving ODEs

First order ODEs

Example Solve the first order eqn $y'(t) = y(t)$ with initial condition $y(0) = y_0$.

Solution We take Laplace transform of both sides
$$\mathcal{L}(y') = \mathcal{L}(y)$$

Using eqn we get
$$s\mathcal{L}(y) - y(0) = \mathcal{L}(y) \Rightarrow s\mathcal{L}(y) - y_0 = \mathcal{L}(y).$$

We now solve for $\mathcal{L}(y)$
$$(s-1)\mathcal{L}(y) = y_0 \Rightarrow \mathcal{L}(y) = \frac{y_0}{s-1}$$

In order to solve the problem we need to use the inverse transform to return to y.
$$y(t) = \mathcal{L}^{-1}\left(\frac{y_0}{s-1}\right) = y_0\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = y_0 e^t.$$

Thus is the solution to eqn $y'(t) = y(t)$ with initial condition $y(0) = y_0$. Notice that $y'(t) = y_0 e^t$ and hence $y'(t) = y(t)$

You should always check that the solution satisfies the original eqn. Sometimes we need to make use of partial fraction decomposition as the next example shows

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Example Solve the eqn $y'(t) - 2y(t) = 4$ subject to the condition $y(0) = 1$ using Laplace transform

Solution

This gives us $\mathcal{L}(y') - 2\mathcal{L}(y) = \mathcal{L}(4)$

$\Rightarrow s\mathcal{L}(y) - y(0) - 2\mathcal{L}(y) = 4/s$

Using the first eqn in II, we simplify to get

$$(s-2)\mathcal{L}(y) = y(0) + \frac{4}{s} = 1 + \frac{4}{s}$$

We now isolate $\mathcal{L}(y)$

$$\mathcal{L}(y) = \frac{1}{s-2} + \frac{4}{s(s-2)}$$

and we now can make use of partial fractions to write

$$\frac{A}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

Simplify we get $B = -A$, and $A = -2$

$$\frac{4}{s(s-2)} = -\frac{2}{s} + \frac{2}{s-2}$$

Returning to our eqn for $\mathcal{L}(y)$

$$\mathcal{L}(y) = \frac{1}{s-2} + \frac{4}{s(s-2)} = \frac{1}{s-2} - \frac{2}{s} + \frac{2}{s-2} = \frac{3}{s-2} - \frac{2}{s}$$

Taking the inverse transform and consulting our table ①

we get

$$y(t) = \mathcal{L}^{-1}\left(\frac{3}{s-2}\right) - \mathcal{L}^{-1}\left(\frac{2}{s}\right) = 3e^{2t} - 2$$

In the original eqn $y'(t) - 2y(t) = 4$

$$6e^{2t} - 2(3e^{2t} - 2) = 4$$

$$6e^{2t} - 6e^{2t} + 4 = 4 \quad 4 = 4 \checkmark$$

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Second order ODEs

For second order ODES the method is similar but more complicated

General second order eq^n $y'' + ay' + by = r(t)$ and take Laplace transform of it

We can view $r(t)$ as the input to the system, i.e. the driving force, and $y(t)$ as the output, i.e. the response to the input.

Using notation $Y = L(y)$, the Laplace transform gives

$$L(y'') + a L(y') + b L(y) = L(r)$$
$$[s^2 Y - s y(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

where we used both eq^{n}s in II and that $R = L(r)$. This is called the subsidiary eq^n. We want to solve for Y, so collecting the terms:

$$Y(s^2 + as + b) = (s+a)y(0) + y'(0) + R(s)$$

We now introduce a transfer function

$$Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{(s + \frac{a}{2})^2 + b - \frac{1}{4}a^2}$$

with which we can rewrite

$$Y = [(s+a)y(0) + y'(0)]Q(s) + R(s)Q(s)$$

Final step is to take inverse transform to find $y = L^{-1}(Y)$ to give us the solution

It is not necessary to calculate $Q(s)$ explicitly if you don't want to

a

As long as you isolate Y you can solve the ODE.

Note that all the information about the homogeneous eqn appears in the initial conditions and driving force appear in addition to Q

When solving them it is often necessary to use partial fraction decomposition in order to take the inverse transform.

Example Solve the initial value problem

$$y''(t) + 4y(t) = 2e^t \qquad y(0) = 1 \quad y'(0) = 0$$

Solution Laplace transform of eqn is

$$\mathcal{L}(y'') + 4\mathcal{L}(y) = 2\mathcal{L}(e^t)$$

Substituting the initial conditions, we solve for $\mathcal{L}(y)$

$$(s^2+4)\mathcal{L}(y) = s + \frac{2}{s-1}$$

$$\Rightarrow \mathcal{L}(y) = \frac{s}{s^2+4} + \frac{2}{(s-1)(s^2+4)}$$

Now use partial fraction decomposition on the latter term.

$$\frac{2}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}$$

Worked out $A = 2/5 \quad B = -2/5 \quad C = -2/5$

Replacing into eqn we get

$$\mathcal{L}(y) = \frac{s}{s^2+4} + \frac{2/5}{s-1} + \frac{(-2/5)s + (-2/5)}{s^2+4}$$

$$= \frac{s}{s^2+4} + \frac{2}{5}\frac{1}{s-1} - \frac{2}{5}\frac{s}{s^2+4} - \frac{2}{5}\frac{1}{s^2+4}$$

Finally taking the inverse transform and using table 1 for reference we get

$$y(t) = \cos(2t) + \frac{2}{5}e^{t} - \frac{2}{5}\cos(2t) - \frac{1}{5}\sin 2t$$

$$= \frac{3}{5}\cos 2t + \frac{2}{5}e^{t} - \frac{1}{5}\sin 2t$$

In our original eqn $\quad y''(t) + 4y(t) = 2e^{t}$

$$\left(-\frac{12}{5}\cos(2t) + \frac{2}{5}e^{t} + \frac{12}{5}\sin(2t)\right) + 4\left(\frac{3}{5}\cos(2t) + \frac{2}{5}e^{t} - \frac{3}{5}\sin(2t)\right) = 2e^{t}$$

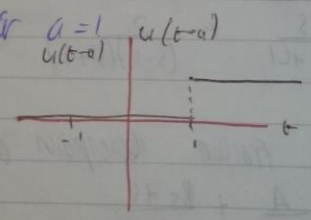
$$\frac{2}{5}e^{t} + \frac{8}{5}e^{t} = 2e^{t} \qquad 2e^{t} = 2e^{t} \checkmark$$

Heaviside function / unit step function

The Heaviside function or unit step function which we will denote $u(t-a)$ is

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

This is shown for $a = 1$



$$L(u(t-a)) = \int_0^{\infty} e^{-st}u(t-a)\,dt = \int_0^{a} e^{-st}(0)\,dt + \int_a^{\infty} e^{-st}(1)\,dt$$

$$= \int_a^{\infty} e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_a^{\infty} = -\frac{1}{s}(0) + \frac{1}{s}e^{-as} = \frac{e^{-as}}{s}$$

The reason we are interested in the step function is because it might appear along with a function

The second shifting theorem says that if $L[f(t)] = F(s)$ then

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$$\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$$

and going the other way $f(t-a)u(t-a) = \mathcal{L}^{-1}[e^{-as}F(s)]$

Note that $f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t<a \\ f(t-a) & \text{if } t>a \end{cases}$

Any function given in this form can be written using an appropriate Step function

The main use of the Step function is when we have something like $\frac{e^{-s}}{s^2}$ in a solution to an ODE and we need to use an inverse transform on it. How would we do this?

If we consider it to be of the form $e^{-s}F(s)$ where $F(s) = \frac{1}{s^2}$ then we can make use of eqn to give

$$\mathcal{L}^{-1}[e^{-s}F(s)] = f(t-1)u(t-1)$$

and so if $F(s) = \frac{1}{s^2}$ then $f(t) = t$ and hence

$$\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2}\right] = (t-1)u(t-1)$$

Dirac Delta function

The dirac delta function is defined by the equation

$$\delta(t-a) = \begin{cases} \infty & \text{if } t=a \\ 0 & \text{otherwise} \end{cases} \qquad 29.$$

and $\int_0^\infty \delta(t-a)\,dt = 1$

Even more importantly because of property 29, when integrated in an integral it has effect of picking out the function at $t=a$ i.e.

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

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The Laplace transform is

$$\mathcal{L}[\delta(t-a)] = \int_0^\infty e^{-st}\,\delta(t-a)\,dt = e^{-sa}$$

and hence $\mathcal{L}^{-1}(e^{-sa}) = \delta(t-a)$