19/11/12

week 9

Management Science Queues

2 broad types of model:

- Single server queue
- Multiple Server queue

Defined by 4 characteristics (factors called operating)

- Queue discipline - (first in first out) or (last in first out)
- calling population - size - customers where they are coming from
- Arrival rate — average people per unit ~~time~~ λ λ
- Service rate — average time spent to serve customer.

Single queue

server

□ ○ ○ ○ ← ○ queue

Arrival rate:

- Actual number of customers to arrive per unit time is a random quantity whose expected value is λ
- Formula for the probability of number of customers that shown to work in many situations

$$P(n \text{ arrivals in 1 unit of time}) = \frac{\lambda^n}{n!} e^{-\lambda},$$

for $n = 0, 1, 2, 3, 4, \ldots$

Formula called the Poisson distribution

- Expected value of this distribution is λ

2.

Supose $\lambda = 5$ per hour Then: $P(n) = \frac{\lambda^n}{n!} e^{-\lambda}$ $0! = 1$ $1! = 1$

- $P(0 \text{ customers in } 1 \text{ hour}) = \frac{5^0}{0!} e^{-5} = \frac{1}{1} e^{-5} = 0.00674$

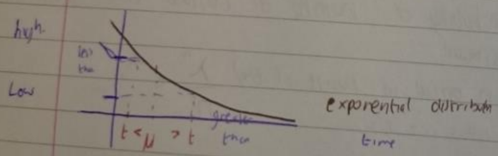
- $P(1 \text{ customer in } 1 \text{ hour}) \frac{5^1}{1!} e^{-5} = 0.0337$

- $P(5 \text{ customer in } 1 \text{ hour}) = \frac{5^5}{5!} e^{-5} = 0.175$

Service rate
- average number of customers that can be served per unit time per server.
- denoted by μ.
- Rate is an average, actual figure is actually random

$P(\text{Service time} > t) = \exp(-\mu t) = e^{-\mu t}$ need to know μ

particular prob units of time



Single Service

Most common single server model has
- Infinite calling population
- FIFO que discipline
- Poisson arrival at λ
- ~~Service rate~~ exponential service times with rate μ

(4).

Multi-server ques

- Complicated situation when more than 1 server but 1 queue
- number of servers = c
- assum:
  - FIFO que disciple
  - infinite calling population
  - Poisson arrival (with rate λ)
  - Exponential service time. Each server serves at a rate μ

$P(\text{service time} > t) = e^{-\mu t}$ average num of customers served per unit time for server.

$P(n) \quad \frac{\lambda^n}{n!} e^{-\lambda}$ - average num of customers to arrive per unit time

server

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- Collectively all c servers will serve at a rate $c\mu$. This is called the mean efficient service rate
- assume that $c\mu > \lambda$
- If $c\mu$ is $>= \lambda$ que is unstable

SCHAUM outline

3. WEEK 9.

9/11/12 Mang Sciency

Example.

cars arrive 15 per hour

average car served in 2 mins

① λ = 15 on average per hour.

μ = 30 cars are average per hour

② station empty $= P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{15}{30} = 0.5$ hours or 30 minutes

③ 2 customers $P_n = P_2 = \left(\frac{\lambda}{\mu}\right)^n P_0 = \left(\frac{15}{30}\right)^2 \left(\frac{1}{2}\right) = 0.125$

④ Average in Station $= L = \frac{\lambda}{\mu - \lambda} = \frac{15}{30-15} = 1$

⑤ Avg waiting to be served $= L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{15^2}{30(15)} \approx 0.5$

⑥ Avg waiting or being served $= W = \frac{1}{\mu - \lambda} = \frac{L}{\lambda} = \frac{1}{15}$ hours or 4 mins

⑦ Avg waiting $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{15}{30 \times 15} = \frac{1}{30}$

8 Utilisation factor $= U = 1 - P_0 = 1 - 0.5 = 0.5$ hours or 30 mins

/12

20/11/12. Mangement Science

3 cashiers = c $\quad c = 3$

serving rate 20/hr $\quad \mu = 20$

customers rate 50/hr $\quad \lambda = 50$

$c\mu > \lambda$ queueing system stable

1. proportion time bank empty = $P_0 = 1 \Big/ \left[\left(\sum_{n=0}^{c-1} \frac{1}{n!} \left(\frac{50}{20}\right)^n \right) + \frac{1}{3!} \cdot \left(\frac{50}{20}\right)^3 \left(\frac{60}{60-50}\right) \right]$

$1 \Big/ \left[\left(1 + \frac{5}{2} + \frac{1}{2} \times \left(\frac{5}{2}\right)^2 \right) + \frac{1}{6} \left(\frac{5}{2}\right)^3 \times 6 \right] = \frac{1}{22.25} = 0.0449$ of an hour

2. Probability 5 customers in bank = P_5 = as $n > c \Rightarrow \left[\frac{1}{3!\,3^{5-3}} \left(\frac{50}{20}\right)^5 P_0 \right]$

$= 0.289 \times P_0$

≈ 0.013

3. Average number customers in queue = $L = \frac{50(20)\left(\frac{50}{20}\right)^3}{(2!)(60-50)^2} \cdot P_0 + \frac{50}{20}$

$= 6.00$ in queue

4. Length of Queue = $L_Q = 35$

5. average wait $\quad W = 0.12$ hour $= 7.2$ min

6. average wait time in queue $= 0.047$ of hour $= 4.2$ mins

7. Probability has to wait (someone in the service bay) $= 0.701$

5 week 9

19/11/12

5. Find absolute extrema:

$$\frac{dA}{dr} = 6 - 4r - \pi r = 0$$

$$\Rightarrow r = \frac{6}{4+\pi}$$

$A\left(\frac{6}{4+\pi}\right) \approx 2.52$ absolute extreme

$A\left(\frac{6}{2+\pi}\right) \approx 2.14$

Radius for which area is maximised is $r = \frac{6}{4+\pi}$

6/11/12 Management Science Queue

Another multi-server queue

□ ○ 000 different queuing arrangement original

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Assume:

- each server has same arrival rate, but may be that a customer joins a que behind a busy server when other servers are idle!

- If customers arrive at rate λ then they arrive to each server at rate λ/c

- now view each server independently (single server) arrival λ/c service μ

One queue vs c queues

Question: Suppose customers arrive at rate λ. If you have c servers is the average time a customer waits different if there is one queue for all or a seperate queue for each server?

With c queues average arrival rate to a server is λ/c so average waiting time at any server will be

$$w_q = \frac{\lambda/c}{\mu(c\mu - \lambda)}$$

$$W_q = \frac{L}{\lambda} - \cancel{\frac{1}{\mu}}$$

$$c = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^c}{(c-1)!(c\mu-\lambda)^2} P_0 + \cancel{\frac{\lambda}{\mu}}$$

$$W_q = \frac{\mu\left(\frac{\lambda}{\mu}\right)^c}{(c-1)!(c\mu-\lambda)^2} P_0$$

c servers or c servers
1 queue c queues.

	case	multiple servers Single queue $\frac{\mu}{(c\mu-\lambda)^2[\Sigma(\ )]}$ ...	Multiple servers c queue $W_q = \frac{\lambda}{\mu(c\mu-\lambda)}$
1.	$\lambda=10, \mu=6, c=3$	$W_q = \frac{6}{(18-10)^2[2![(\frac{10}{6})^{-3}+2!(\frac{10}{6})^{-2}+(\frac{10}{6})^{-1}]+6\times8}$ $= 0.0375$	$W_q = \frac{10}{6\times8} = 0.208$
2	$\lambda=5, \mu=6, c=3$	$W_q = \frac{6}{13^2[2!(\frac{5}{6})^{-3}+2!(\frac{5}{6})^{-2}+(\frac{5}{6})^{-1}]+6\times}$ $= 0.00444$	$W_q = \frac{5}{6\times13} = 0.064$
3.	$\lambda=10\ \mu=6\ c=2$	$W_q = \frac{6}{2^2[(\frac{10}{6})^{-2}+(\frac{10}{6})^{-1}]+(6\times2)}$ 0.379	$W_q = \frac{10}{6\times2} = 0.853$
4	$\lambda\ 10\ \mu\ 4\ c=3$	$W_q =$	$W_q\ \frac{10}{4\times2} = 1.25$

Single queue
multiple server is faster

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$c = 3 \qquad \lambda = 15 \qquad \mu = 6 \qquad c\mu > \lambda$

2 a $P_0 = \dfrac{1}{\left[\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\left(\frac{c\mu}{c\mu-\lambda}\right)} = \cancel{\dfrac{1}{(1)(1)(15)+(\frac{1}{2})(}}$

$\dfrac{1}{(1)\left(\frac{15}{6}\right)^1 + \left(\frac{1}{2}\right)\left(\frac{15}{6}\right)^2 + \frac{1}{6}\left(\frac{15}{6}\right)^3\left(\frac{18}{18-15}\right)} = \cancel{\frac{85}{4}} - \cancel{21+25} \; \frac{4}{85} = 0.0470588$ o/artkav.

b $P_5 = \dfrac{1}{c!\,c^{n-c}}\left(\dfrac{\lambda}{\mu}\right)^n P_0 \quad \dfrac{1}{6 \cdot 6^2}\left(\dfrac{15}{6}\right)^5\left(\dfrac{4}{85}\right) = 0.02127587146$

c $L = \dfrac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^c}{(c-1)!\,(c\mu-\lambda)^2} P_0 + \dfrac{\lambda}{\mu}$

$\dfrac{15(6)\left(\frac{15}{6}\right)^3}{2 \times 9} \cdot \dfrac{4}{85} + \dfrac{15}{6} = 6.176470588$

d $L_q = L - \dfrac{\lambda}{\mu} \quad \dfrac{105}{17} - \dfrac{15}{6} = \dfrac{125}{34} = 3.676470588$

e $W = \dfrac{L}{\lambda} = \dfrac{\frac{105}{17}}{15} = \dfrac{7}{17} = 0.411764$

f. $W_q = W - \dfrac{1}{\mu} = \dfrac{L_q}{\lambda} \quad \dfrac{\frac{125}{34}}{15} = \dfrac{25}{102} = 0.2450980$

g $P_W = \dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c \dfrac{c\mu}{c\mu-\lambda} P_0$

$\dfrac{1}{6} \cdot \left(\dfrac{15}{6}\right)^3 \cdot \dfrac{18}{3} \cdot \dfrac{4}{85} = \dfrac{25}{34} = 0.7352441176$

20/11/12 Theory of Queues Tutorial

①
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1. a $\lambda = 5$

$\mu = 6$

b. $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{5}{6} = \frac{1}{6}$ of an hour $= 10$ minutes

c. $P_3 = \left(\frac{\lambda}{\mu}\right)^n P_0 = \left(\frac{5}{6}\right)^3 \left(\frac{1}{6}\right) = \frac{125}{1296} = 0.09645$

d $L = \frac{\lambda}{\mu - \lambda}$ $\frac{5}{6-5} = \frac{5}{1} = 5$

e. $L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$ $\frac{5^2}{6(6-5)} = \frac{25}{6} = 4.1666$

f $W = \frac{1}{\mu - \lambda} = \frac{L}{\lambda}$ $\frac{5}{5} = 1$

g $W_q = \frac{\lambda}{\mu(\mu - \lambda)}$ $\frac{5}{6(6-5)} = \frac{5}{6}$

h $U = 1 - P_0 = \frac{\lambda}{\mu} = \frac{5}{6}$

26/11/12 Management Science Tutorial Queues

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$c = 3$ $\lambda = 15$ $\mu = 6.$

- A Single queue works best.
- It works because it eliminates problem of being stuck behind someone who is taking much longer than average
- With one queue people are distributed to the next available server avoiding the above problem

Q2 1 queue

$$= \frac{\mu}{(c\mu - \lambda)^2 \sum_{n=0}^{c-1}}$$

$$\frac{6}{3^2 \left[\frac{(3-1)!}{1!} \left(\frac{15}{6}\right)^{-2} + \frac{(2!)}{2!} \left(\frac{15}{6}\right)^{-1} + 6(3)\right]}$$

$$\frac{6}{9\left[\frac{8}{25} + \frac{2}{5}\right] + 18} = 0.24504$$

multiple ques

$$= \frac{\lambda}{\mu(c\mu - \lambda)} \quad \frac{15}{15(3)} = 0.3333$$

1 queue Quicker than multiple queues

Q1