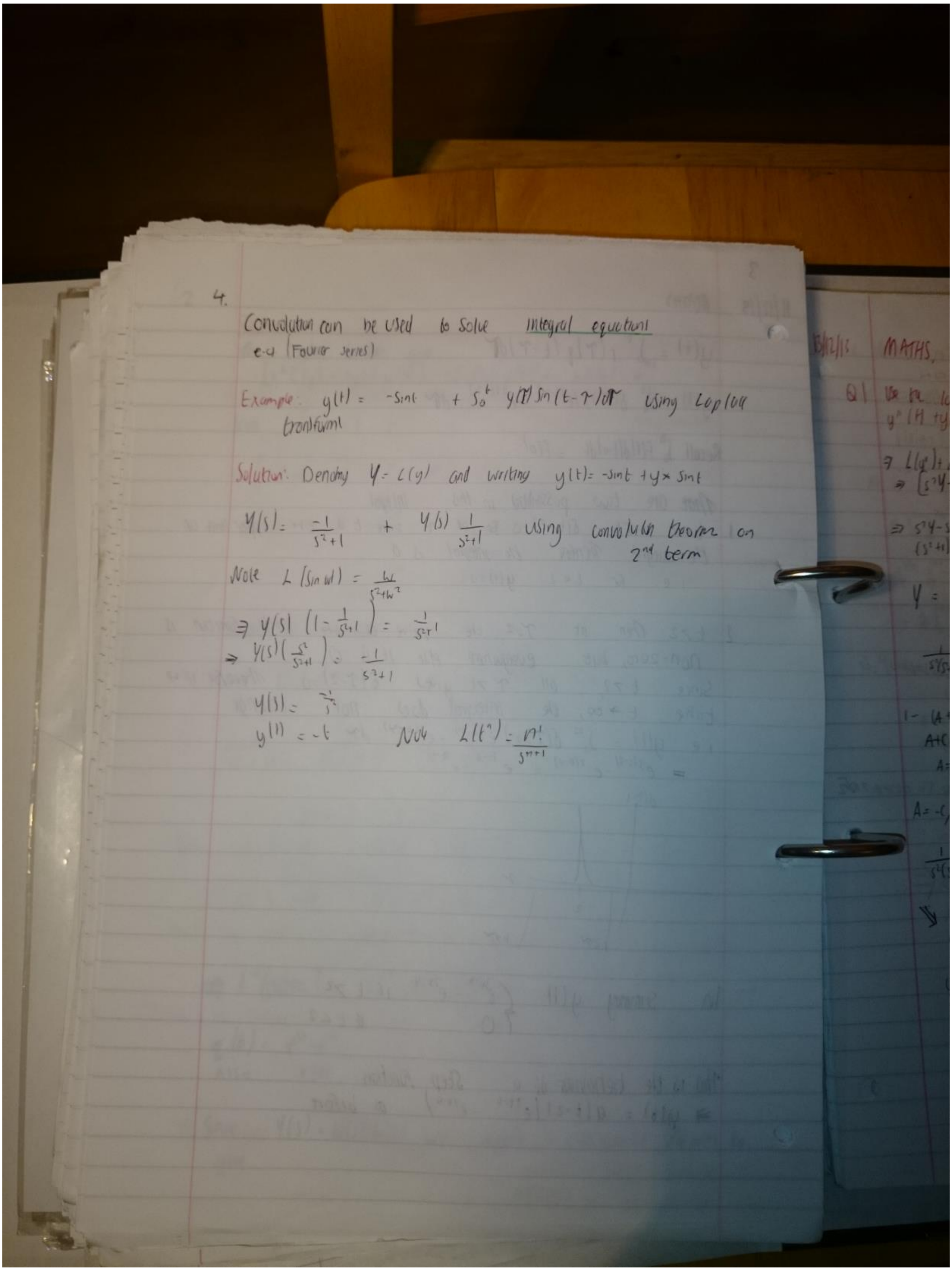


4.

Convolution can be used to solve integral equations
e.g (Fourier series)

Example: $y(t) = -\sin t + \int_0^t y(\tau)\sin(t-\tau)d\tau$ using Laplace transforms

Solution: Denoting $Y = L(y)$ and writing $y(t) = -\sin t + y * \sin t$

$Y(s) = \frac{-1}{s^2+1} + Y(s)\frac{1}{s^2+1}$ using convolution theorem on 2nd term

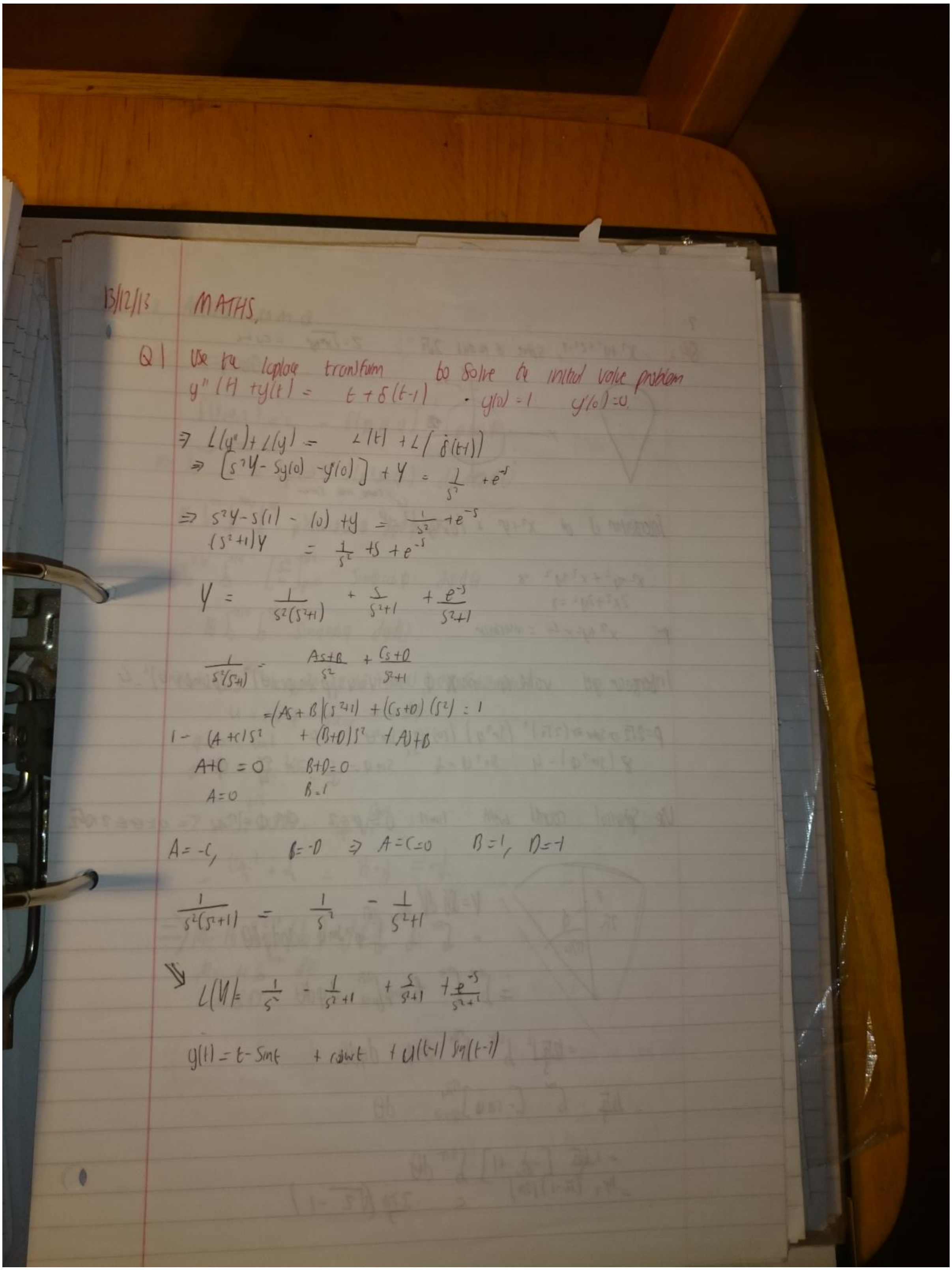
Note $L(\sin wt) = \frac{w}{s^2+w^2}$

$\Rightarrow Y(s)\left(1 - \frac{1}{s^2+1}\right) = \frac{1}{s^2+1}$

$\Rightarrow Y(s)\left(\frac{s^2}{s^2+1}\right) = \frac{-1}{s^2+1}$

$Y(s) = \frac{-1}{s^2}$

$y(t) = -t$ Note $L(t^n) = \frac{n!}{s^{n+1}}$



13/12/13 MATHS.

Q1 Use the laplace transform to solve the initial value problem
$y''(t) + y(t) = t + \delta(t-1)$. $y(0) = 1$ $y'(0) = 0$.

$$\Rightarrow L(y'') + L(y) = L(t) + L(\delta(t-1))$$

$$\Rightarrow [s^2Y - sy(0) - y'(0)] + Y = \frac{1}{s^2} + e^{-s}$$

$$\Rightarrow s^2Y - s(1) - (0) + y = \frac{1}{s^2} + e^{-s}$$

$$(s^2+1)Y = \frac{1}{s^2} + s + e^{-s}$$

$$Y = \frac{1}{s^2(s^2+1)} + \frac{s}{s^2+1} + \frac{e^{-s}}{s^2+1}$$

$$\frac{1}{s^2(s^2+1)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$= (As+B)(s^2+1) + (Cs+D)(s^2) = 1$$

$$1 = (A+C)s^2 + (B+D)s^2 + A)+B$$

$A+C = 0$ $B+D = 0$

$A = 0$ $B = 1$

$A = -C,$ $B = -D$ $\Rightarrow A = C = 0$ $B = 1,$ $D = -1$

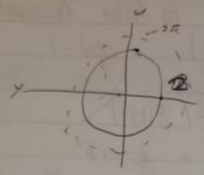
$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$\Rightarrow L(Y) = \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} + \frac{e^{-s}}{s^2+1}$$

$$y(t) = t - \sin t + \cos t + u(t-1)\sin(t-1)$$

?

Q4.A $x^2+y^2+z^2=8$, sphere of radius $2\sqrt{2}$ $z=\sqrt{x^2+y^2}$ = cone



cone inside sphere

Intersection is at $x^2+y^2+(\sqrt{x^2+y^2})^2=8$

$x^2+y^2+x^2+y^2=8$

$2x^2+2y^2=8$

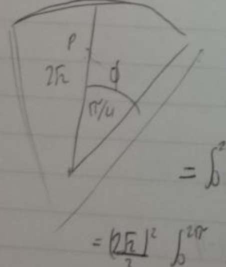
⇒ $x^2+y^2=4$ = intersection

Intersection gives value for max ϕ, $x^2+y^2=4$ $(\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2=4$

$\rho=2\sqrt{2}$ on sphere ⇒ $(2\sqrt{2})^2(\sin^2\phi)\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1}=4$

$8(\sin^2\phi)-4$ $\sin^2\phi=\frac{1}{2}$ $\sin\phi=\frac{1}{\sqrt{2}}=\frac{\pi}{4}=\phi$

Use spherical coords with limits $0\le\rho\le2$ $0\le\phi\le\pi/4$ $0\le\theta\le2\sqrt{2}$



$V=\iiint dV$

$=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sqrt{2}}\rho^2\sin\phi\, d\rho d\phi d\theta$

$=\int_0^{2\pi}\int_0^{\pi/4}\frac{\rho^3}{3}\sin\phi\Big|_{\rho=0}^{2\sqrt{2}}\, d\phi d\theta$

$=\frac{(2\sqrt{2})^2}{3}\int_0^{2\pi}\int_0^{\pi/4}\sin\phi\, d\phi d\theta$

$=\frac{16\sqrt{2}}{3}\int_0^{\pi}\left[-\cos\phi\right]_{\phi=0}^{\pi/4}d\theta$

$=1\frac{16\sqrt{2}}{3}\left[-\frac{1}{\sqrt{2}}+1\right]\int_0^{2\pi}d\theta$

$=\frac{16}{3}(\sqrt{2}-1)(2\pi)$ $=\frac{32\pi}{3}(\sqrt{2}-1)$

/13 2 MATHS EXAM

4c $z=\rho\cos\theta$

$f(x,y,z)=z$

$M=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sqrt{2}}(\rho$

$M=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sqrt{2}}$

$\int_0^{2\pi}\int_0^{\pi/4}[$

$=16\int_0^{2\pi}\int_0^{\pi/4}$

But $\int_0^{\pi/4}$

$u=$

$\phi=0$ u

$\phi=\pi/4$

$\Rightarrow -\int_1^2$

$=$

$\Rightarrow M=16$

$M=$

$=$

3/12/13 3. MATH0 EXAM Q

4C $z = p\cos\theta$

$\delta(x,y,z) = z \quad - \quad \delta(p,\theta,\varphi) = p\cos\phi$

$$M = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sqrt{2}} (p\cos\phi)(p^2\sin\phi)\, dp\, d\varphi\, d\theta$$

$$M = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sqrt{2}} p^3\cos\phi\sin\phi\, dp\, d\varphi\, d\theta$$

$$\int_0^{2\pi}\int_0^{\pi/4}\left[\frac{p^4}{4}\right]_{p=0}^{2\sqrt{2}} \sin\phi\cos\phi\, d\varphi\, d\theta$$

$$= 16\int_0^{2\pi}\int_0^{\pi/4} \sin\varphi\cos\varphi\, d\varphi\, d\theta$$

But $\int_0^{\pi/4} \sin\phi\cos\varphi\, d\phi$ need be Substitution:

$u = \cos\varphi \quad du = -\sin\phi\, d\phi$

$\phi = 0 \quad u = 1$

$\phi = \pi/4 \quad u = 1/\sqrt{2}$

$$\Rightarrow -\int_1^{2\sqrt{2}} u\,du = \frac{-u^2}{2}\Big|_1^{1/\sqrt{2}}$$

$$= \frac{(1/\sqrt{2})}{2} + \frac{1}{2} = \frac{-1}{4} + \frac{1}{2} = \frac{1}{4}$$

$$\Rightarrow M = 16\int_0^{2\pi}\left(\frac{1}{4}\right) d\theta$$

$$M = 4\int_0^{2\pi} d\theta$$

$$= 8\pi$$

4

Q5. $F(xy) = (1+y\overset{f}{e^{xy}})i + (2y+\overset{g}{x}e^{xy})j$

conservative if $\frac{dg}{dx} = \frac{df}{dy}$

$\frac{df}{dy} = e^{xy} + xye^{xy}$ $\frac{dg}{dx} = e^{xy} + xye^{xy}$

$\Rightarrow \frac{df}{dy} = \frac{dg}{dx} \Rightarrow$ conservative

b $f = \frac{d\phi}{dx}$

$\frac{d\phi}{dx} = 1 + ye^{xy}$

$\phi = \int (1+ye^{xy})\,dx$

$\Rightarrow x + y(\frac{1}{y}e^{xy}) + C(y)$

$= x + e^{xy} + C(y)$

$\frac{d\phi}{dy} = xe^{xy} + C'(y)$

$\Downarrow$

$2y + xe^{xy} = xe^{xy} + C'(y)$

$C'(y) = 2y$

$C(y) = = y^2$

$\Rightarrow x + e^{xy} + y^2 + C$ C is any constant!

c Only initial and final point count.

Using fundamental theorem of

$\int_{(0,1)}^{(1,1)} (1+ye^{xy})dx + (2y+xe^{xy})dy$

$= \phi(1,1) - \phi(0,1)$