LAPLACE TRANSFORMS + ODE

Ordinary Differential Equation (ODE)

ODE - an equation that contains one or several derivatives of an unknown function call $y(x)$ or $y(t)$ where x, t are independent variables

$y' = \cos x \qquad y'' + 9y = e^{-2x} \qquad y'y'' - \frac{3}{2}y^2 = 0$

Partial : two variables : $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$

Explicit form $F(x, y, y') = 0$
Implicit form $y' = f(x, y)$

Function $y = ce^{0.2t}$

Derivate : $y' = 0.2e^{0.2t} = 0.2y$
Hence y is solution of $y' = 0.2y$ form: $y' = ky$
Similarly: $y' = -0.2$ solution $y = ce^{-0.2t}$

INITIAL VALUE PROBLEM

The unique solution of a given problem is obtained from a general solution by an initial condition $y(x_0) = y_0$, with given values x_0 and y_0, which are used to determine value of constant c

Geometrically this condition means that the solution curve should pass through point (x_0, y_0) in xy plane

An ODE together with an initial condition, is called an initial value problem

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Thus if the ODE is explicit $y' = f(x,y)$ the initial value problem is of the form:

$$y' = f(x,y) \qquad y(x_0) = y_0$$

Example:

Solve initial value problem $y' = \frac{dy}{dx} = 3y \qquad y(0) = 5.7$

General solution is $y(x) = ce^{3x}$

From this solution and initial condition, $y(0) = ce^0 = c = 5.7$

Hence the initial value problem has the solution

$y(x) = 5.7e^{3x}$ This is a particular solution

Example:

Solve $y' = -2xy \qquad y(0) = 1.8$

By seperation and integration:

$\frac{dy}{y} = -2x\,dx$ integrate: $\ln y = -x^2 + \tilde{c}$

$y = ce^{-x^2}$ general solution

$y(0) = 1.8 \Rightarrow ce^0 = c = 1.8$

Solution = $y = 1.8e^{-x^2}$

Second Order ODE.

A Second order ODE is called linear if it can be written $y'' + p(x)y' + q(x)y = r(x)$

Homogeneous eq^n if $r(x) = 0$ i.e

$$y'' + p(x)y' + q(x)y = 0$$

LAPLACE TRANSFORM

Initial value problem → Algebraic problem → Solving AP by algebra → Solution of IVP

Initial value problems are solved without first determining a general solution.

Laplace Transform - Linearity - First Shifting Theorem (S-Shifting)

If $f(t)$ is a function defined for all $t \geq 0$, its Laplace transform is the integral of $f(t)$ times e^{-st} from $t=0$ to $t=\infty$. It is a function of s say $F(s)$, and is denoted by $\mathcal{L}(f)$:

(1) $F(s) = \mathcal{L}(s) = \int_0^\infty e^{-st} f(t)\, dt.$

We assume that $f(t)$ is such that an integral exists

$F(s) = \int_0^\infty k(s,t)\, f(t)\, dt$ with kernel $k(s,t) = e^{-st}$.

Laplace transform is called an integral transformation because it transforms a function in one space to a function in another space by a process of integration that involves a kernel. The kernel / kernel function is a function of the variables in the two spaces and defines the integral transform.

$f(t)$ in (1) is called inverse transform of $F(s)$ and denoted by $\mathcal{L}^{-1}(F)$ that is:

$f(t) = \mathcal{L}^{-1}(F)$

$\Rightarrow \mathcal{L}^{-1}(\mathcal{L}(F)) = f$ and $\mathcal{L}(\mathcal{L}^{-1}(F)) = F$

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Example:

$f(t) = 1$ when $t \geq 0$ Find $F(s)$

$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s}$

Laplace Transform $\mathcal{L}(e^{at})$ of exp funtion e^{at}

$f(t) = e^{at}$ when $t \geq 0$ a is constant Find $\mathcal{L}(f)$

$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\,dt = \frac{1}{a-s} e^{-(s-a)t}\Big|_0^\infty$

$\mathcal{L}(e^{at}) = \frac{1}{s-a}$

Laplace transform is a Linear operation:

$\mathcal{L}[\,a f(t) + b g(t)] = a\,\mathcal{L}[f(t)] + b\,\mathcal{L}[g(t)]$

True because integration is a linear operator.

Example:

Find transform of $\cosh(at) = \frac{1}{2}(e^{at} + e^{-at})$

$\mathcal{L}(\cosh(at)) = \frac{1}{2}[\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})]$

$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2-a^2}$

$f(t)$	$\mathcal{L}(f)$	$f(t)$	$\mathcal{L}(f)$
1	$1/s$	$\cos\omega t$	$s/s^2+\omega^2$
t	$1/s^2$	$\sin\omega t$	$\omega/s^2+\omega^2$
t^2	$2!/s^3$	$\cosh(at)$	s/s^2-a^2
t^n $n=0,1.$	$n!/s^{n+1}$	$\sinh(at)$	a/s^2-a^2
t^a a posive	$\Gamma(a+1)/s^{a+1}$	$e^{at}\cos\omega t$	$s-a/(s-a)^2+\omega^2$
e^{at}	$1/s-a$	$e^{at}\sin\omega t$	$\omega/(s-a)^2+\omega^2$

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First Shifting Theorem s-Shifting

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k) then $e^{at}f(t)$ has the transform $F(s-a)$ where $(s-a > k)$.

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

or taking inverse

$$e^{at}f(t) = \mathcal{L}^{-1}(F(s-a))$$

We obtain $F(s-a)$ by replacing s with $s-a$ in integral:

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt = \int_0^\infty e^{-st}[e^{at}f(t)]\,dt$$

$$= \mathcal{L}[e^{at}f(t)]$$

This is how we get eq^n for $e^{at}\cos\omega t$ and $e^{at}\sin\omega t$

Example: $\mathcal{L}(f) = \dfrac{3s-137}{s^2+2s+401}$

Apply the inverse transform, complete square:

$$F = \mathcal{L}^{-1}\left[\frac{3(s+1)-140}{(s+1)^2+400}\right] = 3\mathcal{L}^{-1}\frac{s+1}{(s+1)^2+20^2} - 7\mathcal{L}^{-1}\left[\frac{20}{(s+1)^2+20^2}\right]$$

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t)$$

Laplace Transform of Derivatives

The transforms of the first and second derivatives of $f(t)$ satisfy

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

Proof. $\mathcal{L}(y') = \int_0^\infty e^{-st}y'(t)\,dt = e^{-st}y(t)\Big|_0^\infty - (-s)\int_0^\infty e^{-st}$

$$= 0 - (1)(y)(0) + s\mathcal{L}(y) = s\mathcal{L}(y) - y(0)$$

proved by integration by parts

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Use first result to prove second:

$\mathcal{L}(y'') = \mathcal{L}((y')') = s\mathcal{L}(y') - y'(0)$

$\mathcal{L}(y'') = s\mathcal{L}(y') - y'(0)$
$= s[s\mathcal{L}(y) - y(0)] - y'(0)$
$= s^2\mathcal{L}(y) - sy(0) - y'(0)$

Differential Equations, Initial Value Problems

Consider initial value problem:

$y'' + ay' + by = r(t) \quad y(0) = K_0 \quad y'(0) = K_1$

a, b constant)

r(t) is given input (driving force)

y(t) is output (response to input)

Step 1: Setting up of the subsidiary eq^n

This is an algebraic eq^n for the transform $Y = \mathcal{L}(y)$

$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$

From: $\mathcal{L}(y'') + a\mathcal{L}(y') + b\mathcal{L}(y)$

Collecting the Y-terms, we have subsidiary equation:

$(s^2 + as + b)Y = (s+a)y(0) + y'(0) + R(s)$

Step 2: Solution of the subsidiary eq^n by algebra

We divide by s^2+as+b and use the so-called transfer function

$$Q(s) = \frac{1}{s^2+as+b} = \frac{1}{(s+\frac{1}{2}a)^2 + b - \frac{1}{4}a^2}$$

(7) This gives solution $Y(s) = [(s+a)y(0) + y'(0)]Q(s) + R(s)Q(s)$

If $y(0) = y'(0) = 0$ this is simply $Y = RQ$ hence:

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$

Q depends neither on r(t) nor the initial conditions, only on a and b

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Step 3 Inversion of Y to obtain $y = \mathcal{L}^{-1}(Y)$

We reduce (7) usually by partial fractions to a sum of terms whose inverses can be found from the tables.

Example:

solve. $y'' - y = t \qquad y(0) = 1 \qquad y'(0) = 1$

Solution Step 1: $Y = \mathcal{L}(y)$.

$$s^2 Y - s y(0) - y'(0) - Y = \frac{1}{s^2}$$

thus $(s^2 - 1) Y = s + 1 + \frac{1}{s^2}$

in form of

Step 2: transfer function $Q = 1/(s^2 - 1)$

$$Y = (s+1)Q + \frac{1}{s^2} Q$$

$$Y = \frac{s+1}{s^2 - 1} + \frac{1}{s^2(s^2-1)}$$

Simplification of first fraction and expansion of last fraction gives:

$$Y = \frac{1}{s-1} + \left(\frac{1}{s^2-1} - \frac{1}{s^2}\right)$$

Step 3: We obtain the solution:

$$y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^t + \sinh(t) - t$$

Steps of Laplace Transform method

t-space | s-space

$y'' - y = t$, $y(0) = 1$, $y'(0) = 1$ → Subsidiary eqn: $(s^2-1)Y = s + 1 + 1/s^2$ → Solution of subsidiary eqn: $Y = \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}$ → Solution: $y(t) = e^t + \sinh(t) - t$

Example:

Solve: $y'' + y' + 9y = 0 \quad y(0) = 0.16 \quad y'(0) = 0$

$$s^2Y - 0.16s + sY - 0.16 + 9Y = 0$$

thus $(s^2 + s + 9)Y = 0.16(s+1)$

$$Y = \frac{0.16(s+1)}{s^2+s+9} = \frac{0.16(s+\frac{1}{2}) + 0.08}{(s+\frac{1}{2})^2 + \frac{35}{4}}$$

by first shifting theorem and cos/sin formula

$$y(t) = \mathcal{L}^{-1}(Y) = e^{-t/2}\left(0.16\cos\sqrt{\tfrac{35}{4}}\,t + \frac{0.08}{\frac{1}{2}\sqrt{35}}\sin\sqrt{\tfrac{35}{4}}\,t\right)$$

$$= e^{-0.5t}(0.16\cos 2.96t + 0.027\sin 2.96t)$$

First shifting theorem $(s+\frac{1}{2}) \Leftrightarrow (s)$

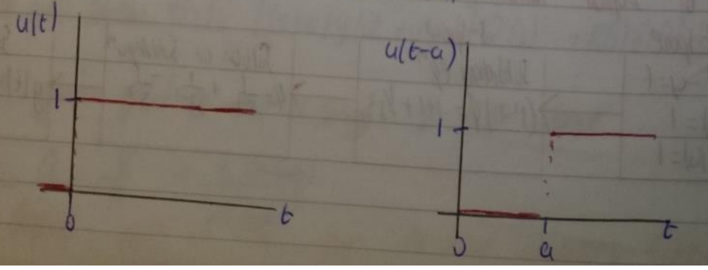
$(s+\frac{1}{2})^2 \Leftrightarrow s^2$ ignore $+\frac{1}{2}$ or constant

UNIT STEP FUNCTION (HEAVISIDE FUNCTION) SECOND SHIFTING THEORM (t-shifting)

Unit step function

The unit step function or Heaviside function $u(t-a)$ is 0 for $t < a$ has a jump of size 1 at $t = a$ (where we can leave it undefined) and is 1 for $t > a$ in a formula:

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \qquad a \geq 0$$



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The transform $u(t-a)$ follows directly from defining integral

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\,dt = \int_a^\infty e^{-st} \cdot 1\,dt$$

$$= \frac{-e^{-st}}{s} \Big|_{t=a}^{\infty}$$

here the integration begins at $t = a\ (\geq 0)$ because $u(t-a)$ is 0 for $t < a$ hence

$$\mathcal{L}[u(t-a)] = \frac{e^{-as}}{s} \qquad s > 0$$

$$\int_0^\infty e^{-st} u(t-a)\,dt = \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt$$

$$= \int_a^\infty e^{-st}\,dt = -\frac{1}{s} e^{-st} \Big|_a^\infty = -\frac{1}{s}(0) + \frac{1}{s}e^{-as}$$

$$= \frac{e^{-as}}{s}$$

Second Shifting Theorem: Time Shifting

If $f(t)$ has the transform $F(s)$, then the "shifted function"

(3)
$$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

has the transform $e^{-as}F(s)$. That is

That is if $\mathcal{L}(f(t)) = F(s)$ then

$$\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$$

or with inverse of both sides

$$f(t-a)u(t-a) = \mathcal{L}^{-1}[e^{-as}F(s)]$$

Example: transform of $5\sin t$ is $5/(s^2+1)$ hence the shifted function $5\sin(t-2)u(t-2)$ has the transform

$e^{-2s}F(s) = 5e^{-2s}/(s^2+1)$

Dirac Delta function

Defined by $\delta(t-a) = \begin{cases} \infty & \text{if } t=a \\ 0 & \text{otherwise} \end{cases}$

and $\int_0^\infty \delta(t-a)\,dt = 1.$

When inserted in an integral it has effect of picking out the function at $t=a$:

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

Laplace transform: $\mathcal{L}[\delta(t-a)] = \int_0^\infty e^{-st}\,\delta(t-a)\,dt = e^{-sa}$

hence $\mathcal{L}^{-1}(e^{-sa}) = \delta(t-a)$

Example: $y'' - 5y' + 6y = \delta(t-2) \quad y'(0) = 0 \quad y(0) = 0$

$[s^2\mathcal{L}(y) - sy(0) - y'(0)] - 5[s\mathcal{L}(y) - y(0)] + 6\mathcal{L}(y) = e^{-2s}$

$[s^2\mathcal{L}(y) - s(0) - (0)] - 5[s\mathcal{L}(y) - y(0)] + 6\mathcal{L}(y) = e^{-2s}$

$(s^2 - 5s + 6)\,Y = e^{-2s}$

$$Y = \frac{e^{-2s}}{s^2 - 5s + 6}$$

Fracln. $\frac{1}{(s-3)(s-2)} = \frac{A}{(s-3)} + \frac{B}{(s-2)}$

$\Rightarrow \quad 1 = A(s-2) + B(s-3)$

$B = -1 \quad A = 1$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2 - 5s + 6}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-3} - \frac{1}{s-2}\right) = e^{3t} - e^{2t}$$

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Convolution

If $F(s)$ and $G(s)$ exist (the transform of f and g) then $H(s)$ exists (transform of h) and $H = FG$

Example: Use convolution to solve

$$y'' - 5y' + 6y = \delta(t-2) \qquad y'(0) = 0 \quad y(0) = 0$$

$$[s^2 \mathcal{L}(y) - sy(0) - y'(0)] - 5[s\mathcal{L}(y) - y(0)] + 6\mathcal{L}(y) = e^{-2s}$$

$Y = \mathcal{L}(y) \Rightarrow$

$$(s^2 - 5s + 6)Y = e^{-2s} \Rightarrow Y = Q(s)R(s)$$

$$Q(s) = \frac{1}{s^2 - 5s + 6}, \quad R(s) = e^{-2s}$$

$$\frac{1}{s^2-5s+6} = \frac{A}{s-3} + \frac{B}{s-2} \qquad B = -1, \; A = 1$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-5s+6}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-3} - \frac{1}{s-2}\right) = e^{3t} - e^{2t}$$

$q(t) = e^{3t} - e^{2t}$ we know $r(t) = \delta(t-2)$

So $Y(s) = Q(s)R(s)$, use convolution theorem

$$y(t) = \int_0^t r(\tau)\, q(t-\tau)\, d\tau$$
$$\int_0^t \delta(\tau-2)\; e^{3(t-\tau)} - e^{2(t-\tau)} \; d\tau$$

Two possibilities, if $t < 2$ $\;$ fn(t) $= 0$. $\;\delta(\tau - 2) = 0$ for all τ in inter

if $t > 2$ $\;$ at some point $\tau - 2 = 0$ in integral

$$y(t) = \int_0^\infty \delta(\tau - 2)\left(e^{3(t-\tau)} - e^{2(t-\tau)}\right) d\tau$$
$$= e^{3(t-2)} - e^{2(t-2)} = e^{3t-6} - e^{2t-4}$$

$$y(t) = \begin{cases} e^{3t-6} - e^{2t-4} & \text{if } t > 2 \\ 0 & \text{if } t < 2 \end{cases}$$

In form of $u(t-a)f(t-a)$ so: $y(t) = u(t-2)\left(e^{3t-6} - e^{2t-4}\right)$

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Example Solve $y(t) = -\sin t + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$

$Y(s) = \mathcal{L}(y) \Rightarrow y(t) = -\sin t + y * \sin(t)$

we get $Y(s) = -\frac{1}{s^2+1} + Y(s)\frac{1}{s^2+1}$

$Y(s)\left(1 - \frac{1}{s^2+1}\right) = \frac{-1}{s^2+1}$

$Y(s) = \frac{-1}{s^2}$ $y(t) = -t.$