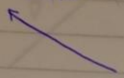
13/1/14 MATHS. 1V

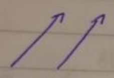
DMITRI ZAITSEV

terminal point B vector



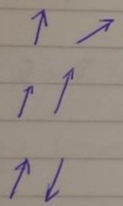
A initial point

Vector $\bar{v} = \vec{v} = AB$

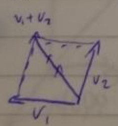


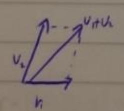
equal (← length + direction the same

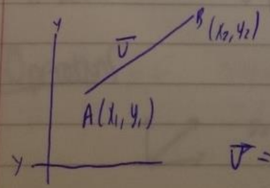
Not equal



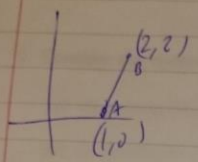
Sum Operations for vectors



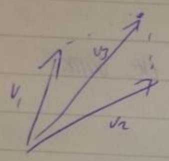




$\bar{v} = \vec{AB} = ((x_2 - x_1), (y_2 - y_1))$



$\bar{v} = (2-1, 2-0) = (1,2)$



$\bar{v}_1\ (u_1, w_1)$

$\bar{v}_2\ (u_2, w_2)$

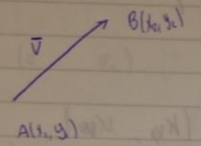
$\bar{v}_3 = (u_1+u_2,\ w_1+w_2)$

$\bar{v}_1\ (1, -5)$

$\bar{v}_2\ (0, 11)$

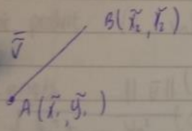
$\bar{v}_1 + \bar{v}_2 = (1, 6)$

15/1/14 small PASS3125 in PASS3125 MATHS



$\bar{v} = (x_2 - x_1,\ y_2 - y_1)$

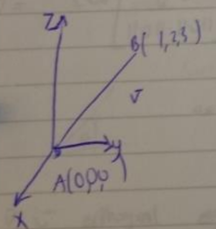
Formula remains the same regardless of coordinates used

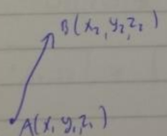


$\tilde{v} = (\tilde{x}_2 - \tilde{x}_1,\ \tilde{y}_2 - \tilde{y}_1)$

$\bar{v} = \tilde{v} \Leftrightarrow x_2 - x_1 = \tilde{x}_2 - \tilde{x}_1$ parallel

$y_2 - y_1 = \tilde{y}_2 - \tilde{y}_1$

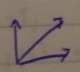




$\bar{v} = (1-0,\ 2-0,\ 3-0)$
$= (1, 2, 3)$
$\bar{v} = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$

n-vector- $\bar{v}(x_1, \ldots, x_n)$

Operations

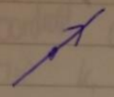


$\bar{v} = v_1 + v_2$

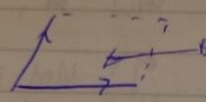
Can't add 3 space and 4 space etc

7/2/14 Maths I

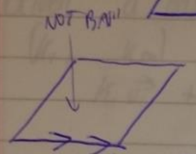
line (one direction)



plane (two directions: linearly independent)



Good



no plane: two vectors in same direction.
BAD.

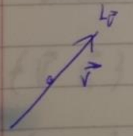
$\{v_1, v_2\}$ linearly dependent if there is a relation

$k_1\vec{v_1} + k_2\vec{v_2} = \vec{0}$

$(k_1, k_2) \neq \vec{0}$ atleast one $k_j \neq 0$

Plane $P_{\vec{v_1} v_2} = \{ k_1\vec{v_1} + k_2\vec{v_2} : k_1, k_2 \in \mathbb{R}\}$

$L_{\vec{v}} = \{ k\vec{v} : k \in \mathbb{R}\}$



$\{\vec{v}\}$ basis if $\vec{v} \neq \vec{0}$
$\{\vec{0}\}$ not basis

Basis of subspace $V \in \mathbb{R}^n$ is collection (set) $\{\vec{v_1}, \dots \vec{v_n}\}$ such that:

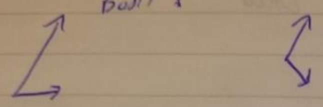
1. It spans V
2. It is linearly independent.

Span $\{\vec{v_1}, \dots, \vec{v_n}\} = \{k_1\vec{v_1} + \dots + k_n\vec{v_n} : k_1, \dots, k_n \in \mathbb{R}\}$

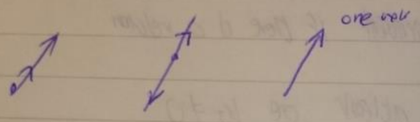
Span $\{\vec{v_1} \dots \vec{v_n}\}$ spans V if $V = \text{Span}\{\dots\}$.

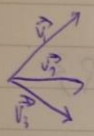
Dimension of V is number of vectors in any basis of V.

Basis:



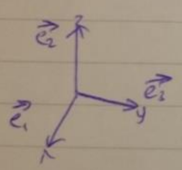
Non-basis:

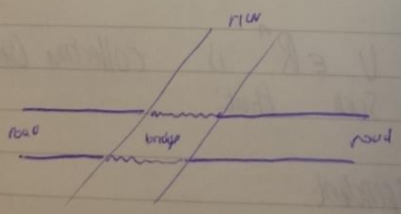


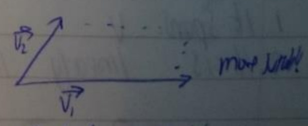


$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span but not basis
but not linearly independent.

Dimension of $\mathbb{R}^3$ is 3 because has standard basis
$\vec{E}_1$ $(1,0,0)$ $\vec{E}_2$ $(0,1,0)$ $\vec{E}_3$ $(0,0,1)$







Non standard coordinates

Multiplication by Scaler (a number)

$\vec{v}$ $\quad 2\vec{v}$

$-\vec{v} = (-1)\vec{v}$

$K\vec{v} = (Kv_1, \ldots, Kv_N)$

i.e. translation km to miles.

Dot product

$\bar{V}_1 \cdot \bar{V}_2 = (v_1w_1 + v_2w_2)$

$(1, 5, 0, 3) \cdot (1, -1, 11, 0)$

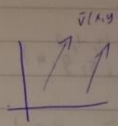
$= (1\cdot 1 + 5(-1), + 0(11), + 3(0))$

$(1 - 5) = 4$

17/1/14 MATHS.

Vectors

$\vec{v} = (v_1 \dots v_n)$



Operations

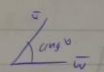
1. $(v_1 \dots v_n) + (w_1 \dots w_n) = (v_1+w_1 \dots v_n+w_n)$
2. $K(v_1 \dots v_n) = (Kv_1 \dots Kv_n)$
3. Dot product $(v_1 \dots v_n)\cdot(w_1 \dots w_n) = v_1 w_1 + \dots + v_n w_n$ (returns a scalar)

Length $\|\vec{v}\|$ or Norm

$\sqrt{v_1^2 + \dots v_n^2}$

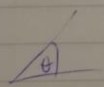
$\vec{v} = (v_1 \dots v_n)$

$\vec{w} = (w_1 \dots w_n)$



Angle between $\vec{v}, \vec{w}$

$$\theta = \cos^{-1}\left[\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|}\right]$$



$0 \le \theta \le \pi$

$\vec{v}, \vec{w}$ orthogonal $\Leftrightarrow$ $\angle$ angle $(\vec{v}, \vec{w}) = \frac{\pi}{2}$

$$\Rightarrow \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|} = \cos\frac{\pi}{2} = 0$$

Q Are $\vec{v}, \vec{w}$ orthogonal?

$\vec{v} = (1, 0, -5)$, $\vec{w} = (5, 0, 1)$

$v \cdot w = 1(5) + 0 \cdot 0 + (-5 \cdot 1) = 0$ = orthogonal

2

2

$\bar{v} = (k, 0, -1, 0)$
$\bar{w} = (k, 5, 9, 4)$

Q. $\bar{v}, \bar{w}$ orthogonal

$\bar{v} \cdot \bar{w} = k^2 + 0 \cdot 5 + (-1)9 + 0$
$= k^2 - 9 = 0$
$K = \pm 3$

$\measuredangle(\bar{v}, \bar{w}) = \cos^{-1}\left(\frac{k^2-9}{\|\bar{v}\| \cdot \|\bar{w}\|}\right)$

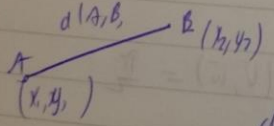
$= \cos^{-1}\left(\frac{k^2-9}{\sqrt{k^2+1}\,\sqrt{2k^2+5^2+9^2}}\right)$

$-1 \leq \frac{\bar{v} \cdot \bar{w}}{\|\bar{v}\| \|\bar{w}\|} \leq 1$

$|\bar{v} \cdot \bar{w}| \leq \|\bar{v}\| \|\bar{w}\|$ Cauchy-Swarz Inequality

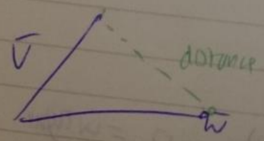
$|v_1w_1 + \dots v_nw_n| \leq \sqrt{v_1^2 + \dots + v_n^2}\,\sqrt{w_1^2 + \dots + w_n^2}$

Distance



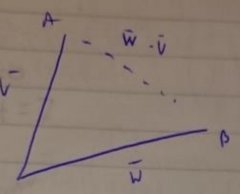
$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \|\bar{v}\|$

$\|\bar{v}\| = \sqrt{v_1^2 + \dots v_n^2}$



17/1/14 3 MATH

$d(\bar{v}, \bar{w}) =$



$d(\bar{v}, \bar{w}) = \|\bar{w} - \bar{v}\|$

$\bar{v} = (1, 0, k)$

$\bar{w} = (-k, 1, k^2)$

$\bar{v} - \bar{w} =$

$\cancel{d = \sqrt{\quad}}$

$\bar{v} - \bar{w} = (1+k, 0-1, k-k^2)$

$d = \|\bar{v} - \bar{w}\|$

$= \sqrt{(1+k)^2 + (-1)^2 + (k-k^2)^2}$

1

20/1/14 MATHS

$x + y^2 = 5$ non linear.

linear equation $x + 5y - 3 = 0$

$5x + 3y + 11t = 7$

System of linear equation

Problem

- Build model
- Algorithm $x(x-2) = 0$ $x = 0$ or $x = \sqrt{2}$
- program (implementation)

$x = 0$ or $x = 1$

Matrix order dont matter

$x \quad y \quad z = 0$

$2x \quad y \quad +10z = 5$

$x \quad +0y \quad -2 = 0$

$2x + y = 5$

$3z = 1$

$x - z = 0$

Write as:

$$\begin{array}{c} \text{matrix } A \quad\quad\quad x \\ \left(\begin{array}{ccc|c|c} 2 & 1 & 0 & x & 5 \\ 0 & 0 & 3 & y & 1 \\ 1 & 0 & -1 & z & 0 \end{array}\right) \end{array}$$

vector $\bar{v}$ (x, y, z)

$\bar{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ ← Same vector can write as row or column

$A \cdot v = \bar{w}$

matrix multiplication (row x colm then carry on row by column etc

Multiplication

$$\begin{pmatrix} 5, & 1, & -1 \\ 1, & 2, & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} 5(0) + 1(1) + (-1)(3) \\ 1(0) + 2(1) + 3(3) \end{pmatrix} = \begin{pmatrix} -2 \\ 11 \end{pmatrix}$$

row x column

$$\begin{pmatrix} 5, 1, 0 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & -2 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 5(1) + 1(0) + 0(5), & 5(3) + 1(-2) + (0,0) \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & & b_{1i} \\ & & \\ b_{m1} & & b_{mi} \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix}$$

any amount

$n \times m$ matrix

same as a row long

m is important

$m \times i$ matrix

$n \times i$

22/1/14 MATHS 1

$x - z = 5$
$y - 2x = 7$ $\Rightarrow$

x	y	z	
x	$+0y$	$-z$	$=5$
$-2x$	$+1$	$+0z$	$=7$

$$\underbrace{\begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{v}} = \underbrace{\begin{pmatrix} 5 \\ 7 \end{pmatrix}}_{\vec{w}}$$

$A\vec{v} = \vec{w}$

Transpose of Matrix

$A \begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$ $A^T \begin{pmatrix} 1 & -2 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$

$A \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_m & & a_{mn} \end{pmatrix}$ $A' = \begin{pmatrix} a_{11} & \dots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{mn} \end{pmatrix}$

Dot Product

$\vec{v} = (v_1 \dots v_n)$
$\vec{w} = (w_1 \dots w_n)$
$\vec{v} \cdot \vec{w} = \vec{v}_1(\vec{w}_1) + \dots \vec{v}_n(\vec{w}_n)$

$$\begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

2

Q Compute dot product $\vec{u}\cdot\vec{v}$

$$\vec{u} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & -5 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

1

$$\begin{pmatrix} 1\cdot 1 - 1\cdot 2 - 1\cdot 3 \\ 2\cdot 1 + 1\cdot 2 + 3\cdot 3 \\ 1\cdot 1 - 1\cdot 2 + 5\cdot 3 \end{pmatrix}$$

$\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ ← dot product switch order $= \vec{v}^T\cdot\vec{u}$ ↙ matrix product

$$= (0,1,0)\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & -5 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= (\cancel{0\cdot 1} + 1\cdot 2 + \cancel{0\cdot 1},\ \cancel{0\cdot 1} + 1\cdot 1 + \cancel{0(-1)},\ \cancel{0(1)} + 1(3) + \cancel{0(5)})\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$(2, 1, 3)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (2\cdot 1 + 1\cdot 2 + 3\cdot 3) = 13$$

Transformation

Input and output

$w_1 = f_1(v_1 \ldots v_n)$

$(v_1 \ldots v_n) \mapsto (w_1 \ldots w_m)$

$w_m = f_m(v_1 \ldots v_n)$

22/1/14 3. Maths

$W_1 = V_1^2 + V_3$
$W_2 = \sin V_2 - 3V_c$

Linear Transformation

$$\begin{cases} W_1 = 2V_1 + 3V_2 \\ W_2 = V_2 - 5V_3 \end{cases}$$

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \overbrace{\begin{pmatrix} 2 & 3 & 0 \\ 0 & 1 & -5 \end{pmatrix}}^{A} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

A matrix of transformation

24/1/14 Maths 1 MATHS

$w = a_1 x_1 \quad \dots \quad a_n x_n$

$\vdots$

$w_m \quad a_{m1} x_1 \quad + \quad a_{mn} x_n$

$$\begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

$$\vec{w} = A\vec{v}$$

Example.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$$

$$\vec{w} = A\vec{v}$$

$$\underbrace{\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}}_{\vec{w}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{\vec{v}}$$

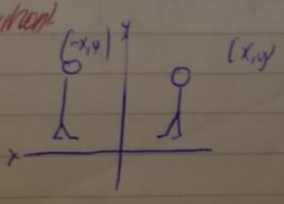
$w_1 = 1(v_1) + 0(v_2) \qquad w_1 = v_1$

$w_2 \quad 0(v_1) + 2(v_2) \qquad w_2 = 2v_2$

$w_3 \quad 1(v_1) + 1(v_2) \qquad w_3 = v_1 + v_2$

Examples of Transformations

reflection operators:



2

$T(x,y) = (-x,y)$

v_1, v_2 w_1, w_2

$w_1 = -v_1$ $x' = -x$

$w_2 = v_2$ $y' = y$

$x' = -1x + 0y$

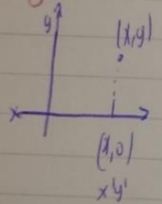
$y' = 0x + 1y$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ reflection matrix (about y axis)

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ reflection matrix about x-axis   Chapter 4.2

Projection



$x' = x$

$y' = 0$

$$\left\{\begin{array}{l} x' = 1x + 0y \\ y' = 0x + 0y \end{array}\right\} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

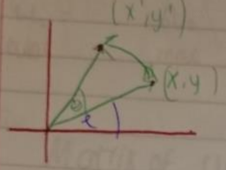
$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ projection matrix to x-axis

Projection to y-axis is $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ projection matrix

Diagonal matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ Diagonal zeros everywhere else

24/1/14 Maths

Rotation Operator



Polar coordinates

$x = r\cos \ell$

$y = r\sin \ell$

$x' = r\cos(\ell+\theta)$

$y' = r\sin(\ell+\theta)$

$x' = r\cos\theta\cos\ell - r\sin\theta\sin\ell$

$y' = r\sin\theta\sin\ell + r\cos\theta\cos\ell$

euler's formula $\quad e^{i\ell} = \cos\ell + i\sin\ell$

$e^{i(\ell+\theta)} = \cos(\ell+\theta) + i\sin(\ell+\theta)$

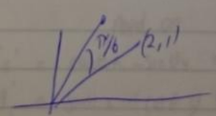
$e^{i\ell}e^{i\theta} = (\cos\ell + i\sin\ell)(\cos\theta + i\sin\theta$

$x' = (\cos\theta)x - (\sin\theta)y$

$y' = (\sin\theta)x + (\cos\theta)y$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & +\cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ Rotation matrix

$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & +\cos\theta \end{pmatrix}$



$\begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\pi/6 & \cos\frac{\pi}{6} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} \frac{\sqrt{3}}{2}\cdot 2 & -\frac{1}{2}\,2 \\ \frac{1}{2}(2) & +\frac{\sqrt{3}}{2}(1) \end{pmatrix}$

11/14 # Maths I

$W_1 = 5v_1 - 1v_3$ $\quad 5v_1 \quad 0v_2 \quad -1v_3$

$W_2 = 6v_2$ $\quad 0v_1 \quad 6v_2 \quad 0v_3$

output input

$$= \begin{pmatrix} 5 & 0 & -1 \\ 0 & 6 & 0 \end{pmatrix}$$

Matrix of your transformation

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} 5 & 0 & -1 \\ 0 & 6 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Examples of reflections

Reflections: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ about x-axis $\quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ about y-axis

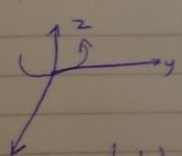
Projection: to x-axis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad$ to y-axis $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Rotations: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ about angle θ $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$x' = \cos\theta x - \sin\theta y$

$y' = \sin\theta x \quad \cos\theta y$

Rotations in 3-dimensional $\quad$ need an axis of rotation



$$\begin{cases} x' = \cos\theta x - \sin\theta y + 0z \\ y' = \sin\theta x + \cos\theta y + 0z \\ z' = 0x + 0y + z \end{cases}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$ rotational matrix around Z-axis

MOUDM

2. MATHS

About x-axis: from y to z

$$\begin{pmatrix} \tilde{y}' \\ \tilde{z} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\,) \begin{pmatrix} \tilde{y} \\ \tilde{z} \\ \tilde{x} \end{pmatrix}$$

new   old

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \longmapsto \begin{pmatrix} \tilde{x}' \\ \tilde{y}' \\ \tilde{z}' \end{pmatrix}$$

$$\begin{pmatrix} \tilde{x}' \\ \tilde{y}' \\ \tilde{z}' \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{1} \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$ About x-axis

28/1/14 Math 1 wednesday

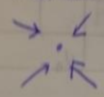
Dilation and Contraction

$(x,y) \mapsto (2x, 2y)$ expand), dilation
$(x,y) \mapsto (kx, ky)$

$k = \frac{1}{2}$ $(x,y) \mapsto \left(\frac{x}{2}, \frac{y}{2}\right)$ contraction

Dilation

Contraction



$x = kx$ $y = ky$

$$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{array}{c} \begin{array}{cc} \hat{} & y \end{array} \\ \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \end{array} \begin{pmatrix} x \\ y \end{pmatrix}$$

$k = -1$ $(x,y) \longmapsto (-x,-y)$

Composition of transformation.

Reflection about x-axis. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

" about y-axis $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Composition $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Dilation / Contraction in any dimension

$$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \rightarrow \begin{pmatrix} kv_1 \\ \vdots \\ kv_n \end{pmatrix} \qquad \begin{cases} w_1 = kv_1 \\ w_n = kv_n \end{cases}$$

$$= k \underbrace{\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}}_{\vec{v}} = k\vec{v}$$

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2

$\vec{w} = k\vec{v}$

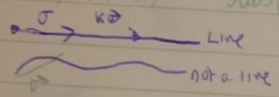
Matrix of dilation/contraction

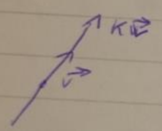
$$\begin{cases} w_1 = kv_1 \\ w_2 = 0v_1 + kv_2 \\ \vdots \\ w_n = 0v_1 + \dots kv_n \end{cases} \qquad \begin{pmatrix} 1 & 0 & \cdot & \\ 0 & 1 & \cdots & \\ \vdots & \ddots & \ddots & \\ 0 & 0 & 0\,0\,0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \underbrace{\begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ \cdots & \cdots & 0 & k \end{pmatrix}}_{A} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

matrix of dilation

Lines, planes, → Subspace





line $k\vec{v}$ $\{k\vec{v},\ k \in \mathbb{R}\}$

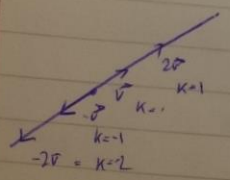
$\vec{v}(1,-2)$ $\Rightarrow$ $L_{\vec{v}} \Rightarrow \{k(1,-2) : k \in \mathbb{R}\}$

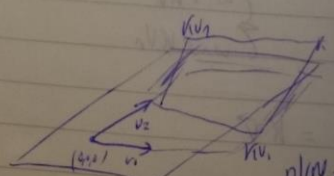
$l_{\vec{v}} \Rightarrow$ $\vec{v} = k\vec{v}$

$\vec{w} = k(1,-2)$

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = k\begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \begin{aligned} w_1 &= k \\ w_2 &= -2k \end{aligned}$$

parametric eqⁿ for line





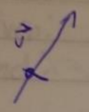
plane needs 2 vectors v_1, v_2 add vectors

plane $= \{kv_1 + kv_2 \quad k_1 k_2 \in \mathbb{R}\}$

linear combination of vectors $v_1\ v_2$ plane generated by $v_1\ v_2$ is set of all linear combinations of $v_1\ v_2$

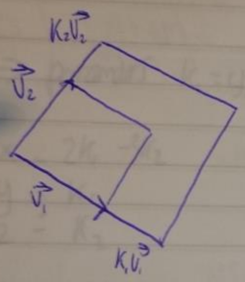
31/1/14 Maths 1

Subspaces



$\vec{v}$: $\{ k\vec{v} : k \in \mathbb{R} \}$

$\vec{v}$ $\begin{pmatrix} u \\ v \end{pmatrix}$ $\quad L\vec{v} = \left\{ k\begin{pmatrix} u \\ v \end{pmatrix} : k \in \mathbb{R} \right\}$



$P\ \vec{v_1}\vec{v_2} = \{ k_1v_1 + k_2v_2 : k_1, k_2 \in \mathbb{R} \}$

$\begin{pmatrix} x \\ y \end{pmatrix} = k\begin{pmatrix} u \\ v \end{pmatrix}$

$\vec{v} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ given $\qquad \begin{pmatrix} x \\ y \end{pmatrix} = k\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$\begin{cases} x = -2k \\ y = 3k \end{cases}$ parametric system

$k = \frac{-x}{2} \qquad y = 3\left(\frac{-x}{2}\right)$

$2x + 3x = 0$

eqn of line generated by $\vec{v}$ $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

2.

Line in $\mathbb{R}$ generated by $\vec{v}$ $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = k \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$$\begin{cases} x = k \\ y = -2k \\ z = 3k \end{cases}$$

Eliminate parameter k

$$\begin{cases} k = x \\ y = -2x \\ z = 3x \end{cases}$$

system of eq^{n}s for line $L\vec{v}$

Plane generated by:

$\vec{v_1}$ $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ $\vec{v_2}$ $\begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + k_2 \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$$

$x = k_1 - k_2$
$y = -2k_1$
$z = 3k_1 + 5k_2$

To get equation ⇒ eliminate parameter

$$\begin{cases} k_1 = \frac{-1}{2}y \\ k_2 = k_1 - x = \frac{1}{2}y - x \end{cases}$$

$z = 3\left(\frac{-1}{2}y\right) + 5\left(\frac{1}{2}y - x\right)$

$2z = -3y + 5y - 10x$

$10x - 7y + 2z = 0$ eq^n for <u>plane</u>

generated by

31/1/14 3

Vile veda

Given: Equation $x - 2y + 3z = 0$
Find: Parametric eqn $x = 2y - 3z$

$\uparrow$ dependent ; y, z free

$\Rightarrow$ parameters $k_1 = y$, $k_2 = z$

$x = 2k_1 - 3k_2$
$y = k_1$
$z = k_2$

$k_1 = y$ $k_2 = z$
parametric system for plane

$$\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{v}} = k_1 \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v_1}} + k_2 \underbrace{\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}_{\vec{v_2}}$$

$v = k_1\vec{v} + k_2\vec{v_2}$
vector form of plane eqn

Subspace:
Parametric description

$y_1 = a_{11} k_1 + \dots a_{1n} k_n$

$\vdots$

$y_m = a_{m1} k_1 + \dots a_{mn} k_n$

System of eqns $\Rightarrow$ $b_{11} y_1 + \dots + b_{1m} y_n = 0$
$b_{n1} y_1 + \dots + b_{nm} y_m = 0$ (implicit eqn)

Line / — not a line

2 conditions
1. $\vec{v}, \vec{w} \in$ line $\Rightarrow \vec{v} + \vec{w} \in$ line
2. $\vec{v} \in$ line $\Rightarrow k\vec{v} \in$ line

3/2/14 Maths 1 week 4

n-space $\mathbb{R}^n$ = Space of n-vector

$\vec{v} = (v_1, \dots, v_n)$

vector Space = set of vectors + operations
(addition, multiplication by scalar
$(\vec{v}, \vec{w})$ $\vec{v}+\vec{w}$ $(k, \vec{v}) \mapsto k\vec{v}$ work equally in all dimensions!

Line. / Subspace of $\mathbb{R}^2$

not line ʃ

Subset $V \in \mathbb{R}^n$ is subspace if 2 rules hold:

1. $\vec{v}, \vec{w} \in V \Rightarrow \vec{v}+\vec{w} \in V$ v+w must be in V. Add
2. $\vec{v} \in V \Rightarrow k\vec{v} \in V$ Scale

Example:
Set of points: $S = \{(a, 2a, 0) : a \in \mathbb{R}\}$
Q. is S a subspace?

1. $\vec{v} = \begin{pmatrix} a \\ 2a \\ 0 \end{pmatrix}$ $\vec{w} = \begin{pmatrix} b \\ 2b \\ 0 \end{pmatrix}$

$\vec{v}+\vec{w} = \begin{pmatrix} a+b \\ 2a+2b \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ 2c \\ 0 \end{pmatrix}$ $c = a+b$

2. $\vec{v} \in S \Rightarrow k\vec{v} \in S$

$\vec{v} \begin{pmatrix} a \\ 2a \\ 0 \end{pmatrix}$ $k\vec{v} \begin{pmatrix} ka \\ 2ka \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ 2c \\ 0 \end{pmatrix}$

works for $c \in ka$

$\Rightarrow k\vec{v} \in S$ A. yes $\Rightarrow$ Subspace

2

Example

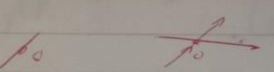
$S = \{ (a, 0, 1, -a) : a \in \mathbb{R} \}$

2nd rule: $\vec{v} \in S \Rightarrow k\vec{v} \in S$ for any k

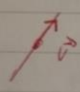
$\vec{v} = \begin{pmatrix} a \\ 0 \\ 1 \\ -a \end{pmatrix} \quad k\vec{v} \begin{pmatrix} ka \\ 0 \\ k \\ -ka \end{pmatrix} \Rightarrow \begin{pmatrix} c \\ 0 \\ 1 \\ -c \end{pmatrix} \quad \begin{matrix} ka = c \\ 0 = 0 \\ k = 1 \\ -ka = -c \end{matrix}$

$k\vec{v} \in S$ does not hold for all k

Not A subspace

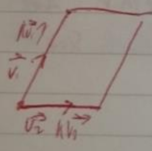


Line is a subspace only if origin $(0,0)$ is on the line



line is spanned or generated by one vector $\vec{v}$

$L\vec{v} : \{ k\vec{v} : k \in \mathbb{R} \}$



$P\vec{v_1}\vec{v_2} \ \{ k_1\vec{v_1} + k_2\vec{v_2} : k_1, k_2 \in \mathbb{R} \}$

Linear Combination of $\vec{v_1}$ and $\vec{v_2}$ or $\vec{v}$

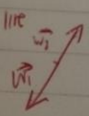
→ n-space $\mathbb{R}^n$

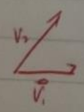
→ m vectors $\vec{v_1} \dots \vec{v_m} \in \mathbb{R}^n$

→ linear combination $k_1\vec{v_1} + \dots + k_m\vec{v_m}$

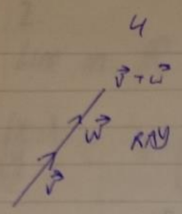
Span $\{ \vec{v_1}, \dots \vec{v_m} \}$ is set of all linear combination

Span line





$\vec{v_1} \quad \vec{v_2}$ span $\mathbb{R}^2$



$$L = \{(t, 2t, 2t^2)\}$$

$$\vec{v}, \vec{w} \in L \Rightarrow \vec{v}+\vec{w} \in L$$

$$\vec{v} = \begin{pmatrix} t \\ 2t \\ t^2 \end{pmatrix} \quad \vec{w} \begin{pmatrix} u \\ 2u \\ u^2 \end{pmatrix} \quad \vec{v}+\vec{w} = \begin{pmatrix} t+u \\ 2t+2u \\ u^2+t^2 \end{pmatrix} = \begin{pmatrix} s \\ 2s \\ s^2 \end{pmatrix}$$

$t+u = s$ $\qquad$ $s = t+u$

$2t+2u = 2s$

$t^2+u^2 = s^2$

$$u^2+t^2 = (t+u)^2 = t^2+u^2+2tu$$

Not Solvable ms,

$\Rightarrow$ <u>Not</u> a line for all t, u.

3/2/14 3 Math 

Q.1. Find equations for Span$\{\vec{v}_1 \dots \vec{v}_n\}$

Q.2. Determine if $\vec{v}_1 \dots \vec{v}_n$ span $\mathbb{R}^n$
(Span$\{\vec{v}_1 \dots \vec{v}_n\} = \mathbb{R}^n$)

$$v_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 2 \\ -2 \\ 0 \end{pmatrix}$$

Span$\{\vec{v}_1, \vec{v}_2\} = ?$

General vector $\vec{v}$ $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$

$$\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2$$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = k_1 \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 2 \\ -2 \\ 0 \end{pmatrix}$$

Method: Eliminate k_1, k_2.

$$\begin{cases} x = 0 \\ y = -k_1 + 2k_2 \\ z = k_1 - 2k_2 \\ t = 0 \end{cases} \qquad \begin{array}{l} x = 0 \\ t = 0 \\ k_1 = 2k_2 - y \quad \text{(sub into) } z \\ z = -y \end{array}$$

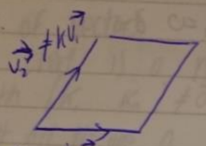
$$\begin{cases} x = 0 \\ t = 0 \\ z + y = 0 \\ k_1 = 2k_2 - y \end{cases}$$

Find all $\vec{v} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ for which system had solution k_1 and k_2 (eliminate)

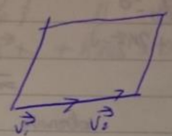
Span given by three eq^{ns}

5/2/14 Maths 1 wednesday

Line.



$\vec{v_1}$ $\vec{v_2}$ not in same line.



$\vec{v_1}$, $\vec{v_2}$ parallel, in same line

Try: $\vec{v_1} = k\vec{v_2}$? Bad $\vec{v_2} = 0$

Zero vectors: $\vec{0} = (0, \dots, 0)$

$\vec{0} = (0,0)$ 2-vector 3-vector etc

$\vec{0}$ $(0,0,0)$ 3 vector

$\vec{v} = (1,2,-3)$

$\vec{v} + \vec{0} = (1,2,-3)$

Try: $\vec{0}$ vector has no direction

Linear Dependence Chapter 5.3

$\vec{v_1}$ $\vec{v_2}$ are linearly dependant if $\vec{v_1}, \vec{v_2}$ are in the same line

$\Leftrightarrow$ $\vec{v_1}$ $\vec{v_2}$ parallel

$\Leftrightarrow$ if there is linear relation $k_1\vec{v_1} + k_2\vec{v_2} = \vec{0}$

with not all $k_j = 0$ either $k_1 \neq 0$ or $k_2 \neq 0$

$\Rightarrow (k_1, k_2) \neq \vec{0}$

Example: $\vec{v_1} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ $\vec{v_2} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$ are they in same line?

$\Leftrightarrow$ linearly dependant?

$k_1\vec{v_1} + k_2\vec{v_2} = 0$ $k_1\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} = \vec{0}$

2.

$$\begin{cases} 0 = 0 \\ k_1 - 2k_2 = 0 \\ -k_1 + 2k_2 = 0 \end{cases} \qquad \begin{cases} k_1 = 2k_2 \\ -2k_2 + 2k_2 = 0 \quad 0=0 \end{cases}$$

$\Rightarrow k_1 = 2k_2$

otherwise $0 = 0$

k_2 free variable $\quad$ k_1 is dependent

Put $k_2 = 1$ $\quad$ but can't be zero $\neq 0$

$k_1 = 2k_2 = 2$

$2(1) = 2$ $\qquad (k_1, k_2) = (1,2) \neq \vec{0}$

$\Longrightarrow$ linearly dependent

3 vectors $\qquad$ Are they in same plane?

$\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad \vec{v_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v_3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Linear dependence of $\vec{v_1}\ \vec{v_2}\ \vec{v_3}$

$\Leftrightarrow$ There is a relation $k_1\vec{v_1} + k_2\vec{v_2} + k_3\vec{v_3} = \vec{0}$

with (not all $k_j = 0$)

$[(k_1, k_2, k_3) \neq \vec{0}]$

$$k_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$k_1\vec{v_1} + k_2\vec{v_2} + k_3\vec{v_3} = 0$$

$$\begin{cases} k_1 + k_3 = 0 \\ k_2 + k_3 = 0 \\ 2k_1 + k_3 = 0 \end{cases} \nearrow \Rightarrow \begin{cases} k_2 = -k_3 \\ k_1 + k_3 = 0 \\ 2k_1 + k_3 = 0 \end{cases} \quad \begin{cases} k_2 = -k_3 \\ k_1 = -k_3 \\ -2k_3 + k_3 = 0 \end{cases} \quad \begin{cases} k_2 = -k_3 \\ k_1 = -k_3 \\ \underline{-k_3 = 0} \end{cases}$$

because $k_3 = 0$

$k_1 = 0$ $\quad \Rightarrow$ no other solution than 0, no linear relation!

$k_2 = 0$ $\qquad$ Linear independence

$k_3 = 0$ $\qquad$ Not in the same plane

5/2/14 3. Math)

Set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly dependent if there is a relation $k_1\vec{v}_1 + \ldots + k_n\vec{v}_n = 0$ with $(k_1 \ldots k_n) \neq \vec{0}$

not all k_i are 0.

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ -1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 3 \end{pmatrix}$$

$$k_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ -1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} + k_3 \begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 3 \end{pmatrix} = 0$$

$k_2 = 0$ ✓

$k_3 = 0$ ✓

$k_1 - 2k_3 = 0$

$3k_1 = 0$ ✓

$-k_1 - k_2 + 3k_3 = 0$

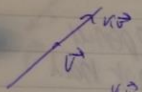
all k_i are zero $\Rightarrow$ Linearly independent

7/2/14 3 Math)

Coordinate of vector $\vec{v}$ relative to basis $\{\vec{v_1}, \vec{v_2}\}$ of are k_1, k_2 where $\vec{v} = k_1\vec{v_1} + k_2\vec{v_2}$

Given a basis $(\vec{v_1}, \dots, \vec{v_m})$ we have correspondence $(k_1, \dots k_m)$ coordinates:

$k_1\vec{v_1} + \dots + k_m\vec{v_m}$



$\Rightarrow k$ is coordinate on line

$k_1\vec{v_1} + k_2\vec{v_2}$

$\Updownarrow$

(k_1, k_2)

EXAMPLE:

Which are bases (plural) of $\mathbb{R}^3$

Dimension $= 3 \Rightarrow$ basis must have 3 vectors.

1. $\{\vec{v_1}\ \vec{v_2}\}$ $\vec{v_1}$ $\begin{pmatrix}1\\0\\3\end{pmatrix}$ $\vec{v_2} = \begin{pmatrix}0\\-1\\5\end{pmatrix}$ not 3 vector $\Rightarrow$ <u>NOT</u> basis.

2. $\{\vec{v_1}\ \vec{v_2}, \vec{v_3}\}$ $\vec{v_3} = \begin{pmatrix}0\\1\\2\end{pmatrix}$

3-vectors in 3-space (number of vectors = dimension)

Fact: if # of vectors = dimension of subspace then it is enough to check linear dependence/independence

$$k_1\begin{pmatrix}1\\0\\3\end{pmatrix} + k_2\begin{pmatrix}0\\-1\\-5\end{pmatrix} + k_3\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases} k_1 = 0 \\ -k_2 + k_3 = 0 \\ 3k_1 - 5k_2 + 2k_3 = 0 \end{cases}$$

from not all $k = 0$

$\forall$

$\begin{cases} k_1 = 0 \\ k_2 = 0 \\ k_3 = 0 \end{cases}$ $\Rightarrow$ no relation $\Rightarrow$ linearly independent

$\Rightarrow$ it is a basis

Hw Friday

4 Math

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\vec{v}_2 = \begin{pmatrix} a \\ b \end{pmatrix}$

$$k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

$$\begin{cases} k_1 + ak_2 = 0 \\ bk_2 = 0 \end{cases} \qquad \begin{cases} k_1 = -ak_2 \\ bk_2 = 0 \end{cases}$$

case 1: for $b \neq 0$

$$\Rightarrow \begin{cases} k_1 = 0 \\ k_2 = 0 \end{cases}$$

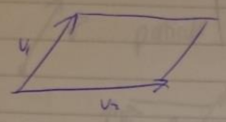
lin independ
$\Rightarrow$ basis

case 2: or $b = 0$

$$\begin{cases} k_1 = -ak_2 \\ 0 = 0 \end{cases}$$

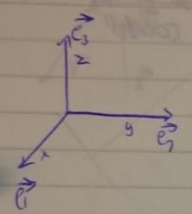
$\Rightarrow$ depend
$\Rightarrow$ not basis

10/2/14 Math 1.



$\{ \vec{v_1}, \vec{v_2} \}$ Basis

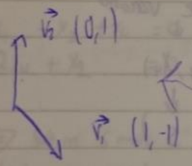
$\vec{v} = k_1\vec{v_1} + k_2\vec{v_2}$ (v_1, v_2) coordinates



$\vec{e_1}$ $(1,0,0)$

$\vec{e_2}$ $(0,1,0)$ relative to basis $\{\vec{v_1}, \vec{v_2}\}$

$\vec{e_3}$ $(0,0,1)$



Question: Find coordinates of $\vec{v} = (1,2)$ relative to the basis $\{\vec{v_1}, \vec{v_2}\}$

$\vec{v} = k_1\vec{v_1} + k_2\vec{v_2}$

$\begin{pmatrix}1\\2\end{pmatrix} = k_1\begin{pmatrix}1\\-1\end{pmatrix} + k_2\begin{pmatrix}0\\1\end{pmatrix}$

$k_1 = 1$

$-k_1 + k_2 = 2 \Rightarrow k_2 = 3$

$(1,3)$ ← non standard coordinates of $\vec{v}$

Find coordinates of $\vec{v}\,(1,2,3)$ relative to standard basis $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$

$\vec{v} = k_1\vec{e_1} + k_2\vec{e_2} + k_3\vec{e_3}$

2

$$\begin{pmatrix}1\\2\\3\end{pmatrix} = k_1\begin{pmatrix}1\\0\\0\end{pmatrix} + k_2\begin{pmatrix}0\\1\\0\end{pmatrix} + k_3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$k_1 = 1$
$k_2 = 2$ ⇒ coordinates relative to standard basis
$k_3 = 3$ are the standard coordinates

LINEAR SYSTEM.

$$\begin{cases} x_1 - x_3 + x_4 = 1 \\ 2x_4 + x_2 = 0 \end{cases} \qquad \begin{cases} x_1 = 1 + x_3 - x_4 \\ x_2 = -2x_4 \end{cases}$$

dependent variables (x_1, x_2)

free variables $(x_3, x_4) \to (t, s)$

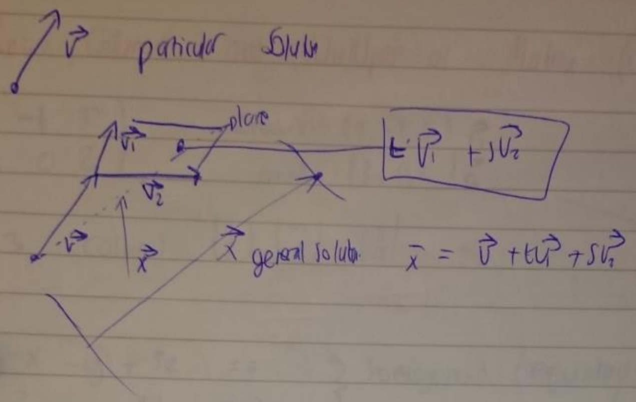
$$\begin{cases} x_1 = 1+t-s \\ x_2 = -2s \\ x_3 = t \\ x_4 = s \end{cases}$$

$$\vec{x} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}1+t-s\\-2s\\t\\s\end{pmatrix} = \underbrace{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}_{\vec{v}} + t\underbrace{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}}_{\vec{v}_1} + s\underbrace{\begin{pmatrix}-1\\-2\\0\\1\end{pmatrix}}_{\vec{v}_2}$$

$\vec{x} = \vec{v} + t\vec{v}_1 + s\vec{v}_2$ ← General Solution

$\vec{v}$: particular solution because we can get when $t = s = 0$

10/2/14 3 Maths



$$\begin{cases} x_1 - x_3 + x_4 = 0 \\ 2x_4 + x_2 = 0 \end{cases}$$ * Associated Homogeneous System $$\begin{cases} x_1 = x_3 - x_4 \\ x_2 = -2x_4 \end{cases}$$

$\Rightarrow$ Sub in t and s.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

$\vec{x} = t\vec{v_1} + s\vec{v_2}$ (no translation compared to above) (same solution but without it.)

General solution is sum of particular solution plus ~~solution of~~ associated homogeneous system solution

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right)$$ $\leftarrow$ augmented matrix (rhs line thing).

$\underbrace{}_{A}$ $\underbrace{}_{\vec{b}}$

$A\vec{x} = \vec{b}$ Matrix eqn for System

$A\vec{x} = \vec{0}$ Associated homogeneous ~~eqn~~ System eqns

12/2/14 Maths 1.

Row, Column and Nullspace of Matrix (Ch. 5.5)

$\begin{pmatrix} 1 & -1 & 5 \\ 3 & 0 & 2 \end{pmatrix}$ 2 rows $(1,-1,5)\ \vec{r_1}$ and $(3,0,2)\ \vec{r_2}$

3 columns $\begin{pmatrix}1\\3\end{pmatrix} \begin{pmatrix}-1\\0\end{pmatrix} \begin{pmatrix}5\\2\end{pmatrix}$

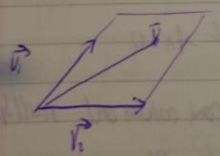
$$\left.\begin{aligned} x - y + 5z &= 0 \\ 3x \quad + 2z &= 0 \end{aligned}\right\}$$ homogenous (equating to zero)

$\cancel{-3y - 35z} + 2z = 0$

$\Rightarrow k_1\vec{v_1} + k_2\vec{v_2}$

$k_1(1,-1,5) + k_2(3,0,2)$

Linear combination of rows $\vec{R_1}$ and $\vec{R_2}$



$\vec{v} = k_1\vec{v_1} + k_2\vec{v_2}$

Plane $P_{\vec{v_1}\vec{v_2}} = \text{Span } \vec{v_1}, \vec{v_2}$

$\Rightarrow$ all linear combinations

$= \{ k_1\vec{v_1} + k_2\vec{v_2} \quad k_1, k_2 \in \mathbb{R}^3$

Row Space of Matrix A is Span of its rows.

Row Span $(A) = \text{Span}(\vec{r_1}, \vec{r_2})$

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ $\quad \vec{r_1} = (a_{11} \ldots a_{1n}) \in \mathbb{R}^n$

$\quad \vec{r_m} = (a_{m1} \ldots a_{mn}) \in \mathbb{R}^n$

2 MATHS

row space = Span $(\vec{r_1}, \vec{r_m}) < \mathbb{R}^n$

$<$: subset

columns.

$\vec{c_1} \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} \quad \vec{c_n} \begin{pmatrix} a_{1n} \\ a_{mn} \end{pmatrix} \in \mathbb{R}^m$

column Space (A) = Span $\{\vec{c_1}, \dots, \vec{c_n}\} < \mathbb{R}^m$

Nullspace of A is the solution of homogeneous system
$A\vec{x} = 0$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$A\vec{x} = b \qquad \vec{x} = \vec{x_n} \cdots + k_1\vec{v_1} + \dots k_n\vec{v_n}$

general — particular solution — solution of $Ax = 0$.

Want way to find basis for row, column and nullspace

$$\begin{cases} 2x_2 - x_4 = 0 \\ x_1 + x_2 + x_3 = 0 \\ 2x_1 + 2x_2 + 2x_3 + x_4 = 0 \end{cases}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\begin{pmatrix} 0 & 2 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 \end{pmatrix}$$

Elementary row operations

1. exchange $\vec{r_1}$ $\vec{r_2}$
2. ~~Add~~ $-2r_1 +$ Replace $\vec{R_3}$ by $r_3 - 2r_1$
3. Replace $\vec{r_2}$ with $\frac{1}{2}\vec{r_2}$

leading 1

$$\begin{pmatrix} \boxed{1} & 1 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 2 & 2 & -1 \end{pmatrix} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

leading 1
row echelon form

12/2/14 3 Math)

<u>Not</u> row echelon form

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 10 \\ 1 & 05 \end{pmatrix}$$

14/2/14. Maths 1

Row column and nullspace

$$A = \begin{pmatrix} 0 & 2 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 \end{pmatrix} \qquad \begin{matrix} \vec{r_1} & (0, 2, 0, -1) \\ \vec{r_2} & (1, 1, 1, 0) \\ \vec{r_3} & (2, 2, 2, 1) \end{matrix}$$

$$\vec{c_1}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \vec{c_2}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \vec{c_3}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \vec{c_4}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Row Space is Span $\{\vec{r_1}, \vec{r_2}, \vec{r_3}\}$

Column Space is Span $\{\vec{c_1}, \vec{c_4}\}$

$A\vec{x} = \vec{b}$

$$\begin{pmatrix} 0 & 2 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

$$x_1\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + x_3\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + x_4\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

$\Leftrightarrow \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ is linear combination of $\vec{c_1}, \vec{c_2}, \vec{c_3}, \vec{c_4}$

$$= x_1\vec{c_1} + x_2\vec{c_2} + x_3\vec{c_3} + x_4\vec{c_4}$$

$A\vec{x} = \vec{b}$ solvable $\Leftrightarrow$ there exist $x_1 \dots x_4$ such that $\vec{b} = x_1\vec{c_1} + \dots + x_4\vec{c_4} \Leftrightarrow \vec{b} \in \text{Span}\{\vec{c_1}, \dots \vec{c_4}\}$

$A\vec{x} = \vec{b}$ solvable $\Leftrightarrow$ $b \in$ column space of A.

Nullspace of A is space of all solution of $A\vec{x} = \vec{0}$

Elementary row operations:

1. exchange row.
2. multiply row by scalar
3. Add row times scalar to row?

Rule for row space: with leading 1's
Row space $(\tilde{A})$ = row spa (A)
Basis of row space consists of rows in the leading 1's
Basis is $\{\vec{r_1}, \vec{r_2}, \vec{r_3}, \vec{r_4}\}$

$\vec{r_1} = (1\ 1\ 1\ 0)$
$\vec{r_2}\ (0\ 1\ 0\ -\tfrac{1}{2})$
$\vec{r_3}\ (0\ 0\ 0\ 1)$

Basis is $\{(1,1,1,0), (0,1,0,-1)\}$

Rule for column space

$A = \begin{pmatrix} 0 & 2 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 \end{pmatrix}$ $\quad \tilde{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -\tfrac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Basis for column space of A consists of columns of A at positions of leading one's in $\tilde{A}$

Basis $= (\vec{c_1}, \vec{c_2}, \vec{c_3}\}$

$\vec{c_1} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ $\quad \vec{c_2} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ $\quad \vec{c_4} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

If $\vec{x} = (\dots)$

Nullspace of A = Nullspace of $\tilde{A}$

$A\vec{x} = \vec{0}$

$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & \tfrac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix} = \vec{0}$

$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2 - \tfrac{1}{2} x_4 = 0 \\ x_4 = 0 \end{cases}$ $\quad x_1 = -x_3$, $\quad x_2 = 0$

$x_1 = -x_3$
$x_2 = x_4 = 0$
dependent, x_3 ind.

$x_3 = t \Rightarrow$ free param

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ t \\ 0 \end{pmatrix} = t \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v_1}}$

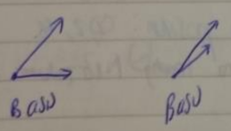
Basis for nullspace of A is $(\vec{v_1})$

17/2/14 MATHS 1

$$A = \begin{pmatrix} 0 & 2 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 \end{pmatrix}$$ matrix $\xrightarrow{\text{Elementry row opeations}}$ $0x + 2y + 0z - 1t$ or $2y - t$

$$\tilde{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Use $\tilde{A}$ to find a basis for
1. Row space of A
2. Column space of A
3. Nullspace of A.



Recall:
Row Space $(\tilde{A})$ = Row Space (A)
Null Space $(\tilde{A})$ = Null Space (A)
NOT FOR COLUMN SPACE

Given a set of vectors $\vec{v_1}\begin{pmatrix}0\\1\\2\\0\end{pmatrix}$ $\vec{v_2}\begin{pmatrix}2\\1\\2\\2\end{pmatrix}$ $\vec{v_3}\begin{pmatrix}0\\1\\2\\0\end{pmatrix}$ $\vec{v_4}\begin{pmatrix}0\\+1\\-1\end{pmatrix}$

Find subset of given set of $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ which is a basis of Span $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$

Method: Construct matrix A with $\vec{v_1}$ $\vec{v_2}$ $\vec{v_3}$ v_4 as columns

2

$A = \begin{pmatrix} 0 & 2 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 \\ 0 & 2 & 0 & -1 \end{pmatrix} \xrightarrow{\text{row operation}} \tilde{A}$

Swap r_1 and r_2

$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 2 & 2 & 1 \\ 0 & 2 & 0 & -1 \end{pmatrix} \underset{\sim}{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \end{pmatrix} \quad r_2 \div 2 \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \end{pmatrix}$

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & & 4 \leftarrow \text{position} \end{array} \\ r_4 - 2r_2 \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

row echelon form

zero row

Span of $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ is column space of $(\tilde{A})$

Take vectors that correspond with position: 1, 2, 4

Answer : $\{\vec{v_1}, \vec{v_2}, \vec{v_4}\}$

no leading 1 in column 3

$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix}$ lot of correlation

$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

zero row $\Rightarrow$ there was a correlation $\Rightarrow$ too many vectors

Rank and Nullity of Matrix

Rank (A) = dimension of row space

= dimension of column space

Nullity of A = dimension of nullspace

~~System~~ $\begin{cases} y + 2z = 0 \\ \cancel{y + 2z = 0} \\ \cancel{-y - 2z = 0} \end{cases}$ two eq^n's redundant rank = 1

rank = 1 = number of independent eq^n's

43+16   51

38   14 $\boxed{51}$ 14 37

17/2/14 Math 3

Find rank
Find Nullity

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Only one leading 1 - rank = number of leading 1's.

Nullity = number of columns - rank.

Rank = 1
Nullity $3 - 2 = 2$

$$\begin{matrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{matrix}$$

2