Maths 2

Q1

Limits

Type 1 : $\lim_{x \to a} f(x) = b$ – Defined

Type 2 : $\lim_{x \to a} f(x) = \frac{b}{0}$ – Undefined

Type 3 : $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ – Apply l'Hôpital's Rule

Use $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ instead of $\lim_{x \to a} \frac{f(x)}{g(x)}$

Note: May have to apply multiple times

Keep applying until you get a number b or $\frac{b}{0}$

Type 4: $\lim_{x \to a} f(x)g(x) = 0 \cdot \infty$ – Change to $\frac{f(x)}{1/g(x)}$ and Apply l'Hôpital's Rule

Note: Sometimes may have to go $\frac{f(x)}{1/g(x)} \to f(x)\,g(x)$

Type 5: $\lim_{x \to a} \left(\frac{1}{f(x)} + \frac{1}{g(x)}\right) = \infty - \infty$ – Combine the fractions and Apply l'Hôpital's Rule

Type 6: $\lim_{x \to a} f(x)^{g(x)} = 0^0, \infty^0, 1^\infty$ – take the Log of the expression

$f(x)^{g(x)} \to \ln\left(f(x)^{g(x)}\right) \to g(x)\ln f(x) \to \lim_{x \to a} g(x)\ln f(x)$

Note: Type 6 may go straight to Type 3 or to Type 4

Useful expressions

(i) $1/\infty = 0$ (ii) $1/0 = \infty$ (iii) $e^{\infty} = \infty$

(iv) $e^{-\infty} = 0$ (v) $\ln(0) = -\infty$ (vi) $\ln(\infty) = \infty$

Extra notes on limits

(i) $\lim_{x \to \infty} \sin(x)$ or $\cos(x)$ undefined

(ii) $\lim_{x \to \infty}$ of polynomials (a) Higher Power on top $\frac{x^3 \ldots}{x^2 \ldots} = \infty$

(b) Higher Power on bottom $\frac{x^2 \ldots}{x^6 \ldots} = 0$

(c) Equal Powers (Highest) $\frac{3x^3 \ldots}{7x^3 \ldots} = \frac{3}{7}$

Q 2

Integration

Substitution

Replace some part of the integral with u, and then integrate with respect to u.

Note: need to choose a u such that du (the differential of u) is equal to the rest of the integral OR some multiple of it. i.e. $\int \frac{\ln(x)}{x} dx \Rightarrow u = \ln(x) \quad du = \frac{1}{x} dx \Rightarrow \int u\,du$

General order for substitution: LIATE

Logarithms $\ln(x)$, $\ln(x^2)$

Inverse functions $\sin^{-1}(x)$, $\cos^{-1}(x)$

Algebraic functions x^3, x^2, $\sqrt{x}$

Trigonometric functions $\sin(x)$, $\cos(x)$, $\tan(x)$

Exponential functions e^x, e^{x^2}

Note: limits must be changed if evaluating with u, otherwise sub back in for x first.

$$\int_b^a x\,dx = \int_d^c u\,du = [u]_d^c = [x]_b^a$$

Integration by parts

$$\int u\,dv = uv - \int v\,du \quad \text{(In tables)}$$

Separate the integral you are given into two parts, u and dv. Differentiate u to get du, integrate dv to get v. Sub all four values into right hand side of equation.

Note: May have to do integration by parts a number of times if $\int v\,du$ is not easily integratable.

Choose u by same order as substitution above: LIATE

Note: In integration by parts the limits are the same as those of the original integral $\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$

Special Case: $\int e^x \sin(x)\,dx$ In notes!

Difficult Example $\int \cos(\sqrt{x})\,dx$ In notes!

Maths 2

Q2

Partial Fractions (Still integration)

If an integral is made up of a polynomial over a polynomial i.e. $f(x) = \frac{p(x)}{q(x)}$ split into a sum of simpler fractions and integrate those.

How do you split it?

(i) $\dfrac{(\cdots\cdots)}{(x+5)(\cdots\cdots)} = \dfrac{A}{(x+5)} + \cdots\cdots$

(ii) $\dfrac{(\cdots\cdots)}{(x+5)^3(\cdots\cdots)} = \dfrac{A}{(x+5)} + \dfrac{B}{(x+5)^2} + \dfrac{C}{(x+5)^3} + \cdots\cdots$

(iii) $\dfrac{(\cdots\cdots)}{(x^2+2x+5)\ (\cdots\cdots)} = \dfrac{Ax+B}{(x^2+2x+5)} + \cdots\cdots$

(iv) $\dfrac{(\cdots\cdots)}{(x^2+2x+5)^2\ (\cdots\cdots)} = \dfrac{Ax+B}{(x^2+2x+5)} + \dfrac{Cx+D}{(x^2+2x+5)^2} + \cdots\cdots$

To find the constants, join the fractions and equate the coefficients (i.e. $Ax^2 = 3x^2 \therefore A = 3$ etc)

Note: Do Not try to find the constants unless you have time at the end of the exam, minimal marks go for it.

Note: See if you can simplify the quadratic factors before splitting the fraction.

The General Integrals:

(i) $\int \frac{A}{x \pm b}\, dx = A \ln |x \pm b|$

(ii) $\int \frac{1}{x^2+x+1}\, dx = (\cdots\cdots) \tan^{-1}(\cdots\cdots)$

(iii) $\int \frac{1}{ax^2+1}\, dx = \frac{\tan^{-1}(\sqrt{a}\, x)}{\sqrt{a}}$

* Will add to this, need to e-mail McManus to ask him a question

Note: Remember to add $+C$ when integrating without limits

Q2

Differential Equations (First order only)

Example

Basic form: $\frac{dy}{dr} + p(r)y = f(r)$

Method: $\frac{dy}{dr} + 3r^2 y = e^{-r^2} r$

(i) Integrate the function in front of y, $p(r)$

$p(r) = 3r^2$

$\int p(r)\,dr = \int 3r^2\,dr = r^3$

Integrating factor $I(r) = r^3$

(ii) Multiply across by $e^{I(r)}$ (e^{r^3})

$e^{r^3}\frac{dy}{dr} + 3r^2 e^{r^3} y = r$

These two terms combine to form one term

(iii) $\frac{d}{dr}(e^{r^3}y) = r$

(iv) Integrate both sides

$\int \frac{d}{dr}(e^{r^3}y)\,dr = \int r\,dr$

$e^{r^3}y = \frac{r^2}{2} + C$

(v) Only y on left hand side

$y = \frac{r^2}{2e^{r^3}} + \frac{C}{e^{r^3}}$

Note: can be a function of x instead of r

Note: remember the $+C$

Maths 2

Q3

Sequences and Series

Geometric Series (General)

$$\sum_{n=1}^{\infty} ar^n = ar + ar^2 + ar^3 + \dots$$

1. This sum has some value S

$$S = ar + ar^2 + \dots$$

2. Factor out r

$$S = r(a + ar^1 + ar^2 + \dots)$$

3. Everything after the first term is a copy of S

$$S = r(a + S)$$

4. Solve for S

$$S = ra + rS \Rightarrow S - rS = ra \Rightarrow S(1-r) = ra$$

$$S = \frac{ra}{(1-r)}$$

Note: r must be less than 1

General Series: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{\pi^4}{90}$

Two methods to test for convergence: Ratio Test and Integral Test

Ratio Test

Method:
1. $\sum_{n=1}^{\infty} a_n$, some series
2. Work out $\left|\frac{a_{n+1}}{a_n}\right|$
3. Find $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$
4. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$
   - >1 Diverges
   - $=1$ Could be either
   - <1 Converges

Factorials

$$\frac{n!}{(n+1)!} = \frac{1}{(n+1)}$$

$$\frac{n!}{(n+2)!} = \frac{1}{(n+2)(n+1)}$$

$$\frac{(n+1)!}{(n+2)!} = \frac{1}{(n+2)}$$

Q3

Integral Test

Method: 1. Convert a_n to a function $f(x)$ by replacing n with x

2. Integrate $\int_1^\infty f(x)\,dx$

3. Find a_1

4. $\sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^\infty f(x)\,dx$

Note: Have to use ratio test when:

(i) A factorial appears $(n+1)!$

(ii) Powers of negative numbers $(-1)^n$, $(-2)^n$ etc

Maths 2

Q4

Matrices

To multiply matrices you multiply Row × Column

ie.

$$\underset{A}{\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}} \quad \underset{B}{\begin{pmatrix} C_1 & C_2 \end{pmatrix}} = \underset{AB}{\begin{pmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{pmatrix}}$$

If $R_1 = (3 \quad 2 \quad 1)$ and $C_1 = (2 \quad 4 \quad -1) \to \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$

Then $R_1C_1 = (3\times 2) + (2\times 4) + (1 \times -1) = 13$

Note $AB \neq BA$

A simultaneous equation

$$2x + 3y = 11$$
$$4x - 2y = 3$$

Can be written as $\begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \end{pmatrix}$

Or as an augmented matrix $\left(\begin{array}{cc|c} 2 & 3 & 11 \\ 4 & -2 & 3 \end{array}\right)$

Gaussian Elimination

Convert Matrix into: $\left(\begin{array}{cc|c} 1 & \cdot & \cdot \\ 0 & 1 & \cdot \end{array}\right)$ Some number (Row Echelon Form)

Multiply, divide and subtract rows to achieve this.

Gauss - Jordan Elimination

Convert matrix into $\left(\begin{array}{cc|c} 1 & 0 & \cdot \\ 0 & 1 & \cdot \end{array}\right)$ (Reduced Row Echelon Form)

As above, multiply, divide and subtract rows to achieve this.

Note: can subtract multiples of rows (handy when you have 1's)

For a 3 × 3 matrix, order is:

$$\left(\begin{array}{ccc|c} \overset{\to 1}{1} & \overset{\to 0}{9} & \overset{\to 0}{8} & \cdot \\ \overset{\to 0}{2} & \overset{\to 1}{4} & \overset{\to 0}{7} & \cdot \\ \overset{\to 0}{3} & \overset{\to 0}{5} & \overset{\to 1}{6} & \cdot \end{array}\right)$$

Q4

Special case

If one equation is a multiple of another, infinite number of solutions.

Matrix will become: $\left(\begin{array}{ccc|c} \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet \end{array}\right)$

Solving using inverse matrix

The inverse of a matrix is another matrix A^{-1} such that

$$A^{-1}A = \mathbb{1} \qquad (AA^{-1} = \mathbb{1}) \qquad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

identity matrix

Using Gauss-Jorden steps convert $A \Rightarrow \mathbb{1}$

i.e. Gauss-Jorden steps $= A^{-1}$

$$A^{-1}A = \mathbb{1} \qquad A^{-1}\mathbb{1} = A^{-1}$$

Method: 1. Convert simultaneous equations into matrix

2. Include the identity matrix

$$\left(\begin{array}{ccc|ccc} \bullet & \bullet & \bullet & 1 & 0 & 0 \\ \bullet & \bullet & \bullet & 0 & 1 & 0 \\ \bullet & \bullet & \bullet & 0 & 0 & 1 \end{array}\right)$$

3. Use Gauss-Jordan to convert left-hand side into identity

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 1 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 1 & \bullet & \bullet & \bullet \end{array}\right) \leftarrow A^{-1}$$

4. Act on the answers $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with A^{-1}

$$\begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \bullet \\ \bullet \\ \bullet \end{pmatrix} = \left.\begin{matrix} x \\ y \\ z \end{matrix}\right\} \text{answers}$$

Note $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ are from original equations, i.e.

$$\begin{aligned} \bullet x + \bullet y + \bullet z &= a \\ \bullet x + \bullet y + \bullet z &= b \\ \bullet x + \bullet y + \bullet z &= c \end{aligned} \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Maths 2

Q4

Fundamental Theorem of Linear Algebra * On Exam

Conditions equivalent to the existence of A^{-1} (making it possible to solve a set of equations)

(i) $\det(A) \neq 0$

(ii) $Ax = 0$ only has the solution $x = 0$.

(iii) $Ax = b$ only has one solution

(iv) The rows of A are linearly independant

(v) The columns of A are linearly independant

(Rows aren't multiples of each other)

(vi) A is a product of elementary matrices.