# 1 Axiomatic Bargaining Theory

## 1.1 Basic definitions

What we have seen from all these examples, is that we take a bargaining situation and we can describe the utilities possibility set that arises from it. In general, we get a compact convex set that contains the utility point corresponding to the disagreement outcome.

Nash, instead of starting with a real life bargaining situation, chose to take the set of feasible utilities and the disagreement utilities as a primitive of the model. In other words, he chooses to abstract from the details of the bargaining situation and consider only the set of attainable utilities. This is what is behind the following definition:

**Definition 1** A *bargaining problem* is a pair $\langle S; d\rangle$, where $S \subseteq \mathbb{R}^2$ is a compact convex set, $d \in S$, and there exists $s \in S$ such that $s_i > d_i$ for $i = 1, 2$.

We must understand a bargaining problem as coming from some real life bargaining situation. On the other hand, we do not know the bargaining situation. There might be several bargaining situations that give rise to the same bargaining problems. Two bargaining situations that induce the same pair $\langle S; d\rangle$ are treated identically.

The assumption that the set of feasible utility pairs is bounded means that the utilities obtainable as an outcome of bargaining are limited. Behind the convexity assumption on $S$ lies the idea that the set of agreements is a set of lotteries over some set of physical outcomes and that individuals' preferences can be represented by a von Neumann-Morgenstern utility function. Players can agree to disagree $d \in S$ (the disagreement is a possibility), and there is some agreement preferred by both to the disagreement outcome. This ensures that the agents have mutual interest in reaching an agreement, although there is a conflict of interest over the particular agreement to be reached.

The set of all bargaining problems is denoted by $\mathcal{B}$.

**Definition 2** A *bargaining solution* is a function $f : \mathcal{B} \to \mathbb{R}^2$ that assigns to each bargaining problem $\langle S; d\rangle \in \mathcal{B}$ a unique element of $S$.

Examples of Solutions:

(i) The disagreement solution. It assigns to each bargaining problem $\langle S; d\rangle \in \mathcal{B}$, the disagreement point $d$. Since $d \in S$, it is a well-defined solution.

(ii) The player 1 dictatorial solution. It assigns to each bargaining problem $\langle S; d\rangle$ the strongly efficient and individually rational point that maximizes player 1's utility function.

(iii) The egalitarian solution. It assigns to each bargaining problem, the greatest feasible point $(s_1^*, s_2^*)$ that satisfies $s_1^* - d_1 = s_2^* - d_2$.

(iv) The Kalai-Smorodinsky solution:

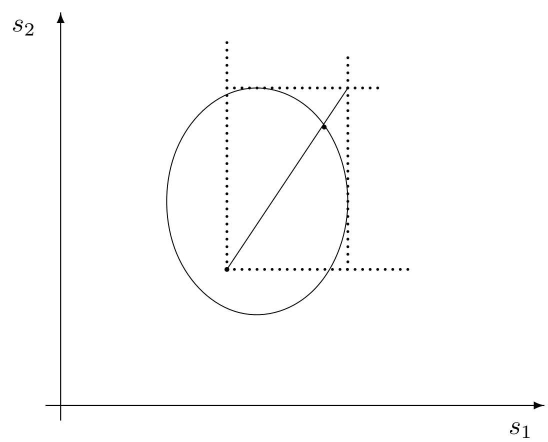



Figure 1: Kalai-Smorodinsky.

(v) The Nash bargaining solution: It selects the unique solution to the following maximization problem.

$$\max_{(s_1,s_2)\in S} (s_1 - d_1) \cdot (s_2 - d_2)$$
$$\text{s.t. } s_1 \geq d_1$$
$$s_2 \geq d_2$$

The Nash bargaining solution will be denoted by $f^N$.

## 1.2 Nash's Axioms

**Definition 3** The bargaining problem $\langle S', d' \rangle$ is obtained from the bargaining problem $\langle S, d \rangle$ by the transformations $s_i \to \alpha_i s_i + \beta_i$, for $i = 1, 2$ if

$$d'_i = \alpha_i d_i + \beta_i \qquad \text{for} \qquad i = 1, 2$$

and

$$S' = \{(\alpha_1 s_1 + \beta_1;\ \alpha_2 s_2 + \beta_2) \in \mathbb{R}^2 :\ (s_1, s_2) \in S\}.$$

Example: Consider $\langle S; d \rangle$ where

$$S = \{(s_1, s_2) \in \mathbb{R}^2 : \; s_1 + s_2 \leq 1 \; s_1, s_2 \geq 0\}$$

$$d = (0; 0)$$

Consider the following transformations:

$$\begin{array}{rcl} s_1 & \rightarrow & 2s_1 + 1 \\ s_2 & \rightarrow & s_2 + 2 \end{array}$$

When we apply these transformations to $\langle S; d \rangle$ we get

$$S' = \{(2s_1 + 1; \; s_2 + 2) : \; s_1 + s_2 \leq 1 \; s_1 + s_2 \geq 0\}$$

$$\begin{array}{rcl} d_1' & = & 2d_1 + 1 = 1 \\ d_2' & = & d_2 + 2 = 2. \end{array}$$

The first property that we may want a bargaining solution to satisfy is

**Invariance to equivalent utility representations (INV)**: Suppose that the bargaining problem $\langle S', d' \rangle$ is obtained from $\langle S; d \rangle$ by the transformations $s_i \rightarrow \alpha_i s_i + \beta_i$ where $\alpha_i > 0$ for $i = 1, 2$. Then $f_i(S', d') = \alpha_i f_i(S; d) + \beta_i$ for $i = 1, 2$.

This axiom requires that the utility outcome co-vary with the representation of preferences, so that any physical outcome that corresponds to the solution of the problem $\langle S; d \rangle$ also corresponds to the solution of $\langle S', d' \rangle$.

In order to present the next axiom, we need another technical definition.

**Definition 4** A bargaining problem is *symmetric* if $d_1 = d_2$ and $(s_1, s_2) \in S$ if and only if $(s_2, s_1) \in S$.

We can now state the symmetry axiom.

**Symmetry (SYM)**: If the bargaining problem is symmetric, then $f_1(S; d) = f_2(S; d)$.

The next axiom is more problematic.

**Independence of irrelevant alternatives** (IIA): If $\langle S; d \rangle$ and $\langle T; d \rangle$ are bargaining problems with $S \subset T$ and $f(T; d) \in S$, then $f(S; d) = f(T; d)$.

The axiom relates to the (unmodeled) bargaining process. If the negotiators gradually eliminate outcomes as unacceptable, until just one remains, then it may be appropriate to assume IIA. On the other hand, there are procedures in which the fact that a certain agreement is available influences the outcome, even if it is not the one that is reached.

**Pareto efficiency** (PAR): Suppose $\langle S; d \rangle$ is a bargaining problem, $s \in S$, $t \in S$ and $t_i > s_i$ for $i = 1, 2$. Then $f(S; d) \neq s$.

This axiom implies that the players never disagree (since we have assumed that there is an agreement on which the utility of each player $i$, exceeds $d_i$).

Note that the axioms SYM and PAR restrict the behavior of the solution on single bargaining problems, while INV and IIA require the solution to exhibit some consistency across bargaining problems.

**Theorem 1** There is a unique bargaining solution $f : \mathcal{B} \to \mathbb{R}^2$ satisfying the axioms INV, SYM, IIA and PAR. It is the Nash bargaining solution.

**Proof** : We proceed in a number of steps

a) $f^N$ is well-defined.

b) We check that $f^N$ satisfies the axioms.

Invariance: Suppose that $\langle S'; d' \rangle$ and $\langle S; d \rangle$ are like in the statement of the axiom. Then

$$S' = \{(\alpha_1 s_1 + \beta_1;\ \alpha_2 s_2 + \beta_2) \in \mathbb{R}^2 :\ (s_1, s_2) \in S\}$$

and $d'_i = \alpha_i s_i + \beta_i \quad i = 1, 2$.

In other words, $s' \in S'$ if and only if there exists $s \in S$ such that $s'_i = \alpha_i s_i + \beta_i$ for $i = 1, 2$.

Therefore, if $(s'_1, s'_2) \in S'$, we have

$$\begin{array}{rcl} (s'_1 - d'_1)(s'_2 - d'_2) & = & (\alpha_1 s_1 + \beta_1 - \alpha_1 d_1 - \beta_1)(\alpha_2 S_2 + \beta_2 - \alpha_2 d_2 - \beta_2) \\ & = & (\alpha_1 s_1 - \alpha_1 d_1)(\alpha_2 s_2 - \alpha_2 d_2) \\ & = & \alpha_1 \alpha_2 (s_1 - d_1)(s_2 - d_2) \end{array}$$

for some $(s_1, s_2) \in S$.
Now, $(s_1^*, s_2^*)$ maximizes $(s_1 - d_1)(s_2 - d_2)$ over $S$, if and only if

$$(s_1^* - d_1)(s_2^* - d_2) \geq (s_1 - d_1)(s_2 - d_2) \qquad \forall (s_1, s_2) \in S$$

if and only if

$$\alpha_1 \alpha_2 (s_1^* - d_1)(s_2^* - d_2) \geq \alpha_1 \alpha_2 (s_1 - d_1)(s_2 - d_2) \qquad \forall (s_1, s_2) \in S$$

if and only if

$$({s'}_1^* - d'_1)({s'}_2^* - d'_2) \geq (s'_1 - d'_1)(s'_2 - d'_2) \qquad \forall (s'_1, s'_2) \in S'$$

where ${s'}_i^* = \alpha_i s_i^* + \beta_i$.
Namely $(\alpha_1 s_1^* + \beta_i, \alpha_2 s_2^* + \beta_2)$ maximizes $(s'_1 - d'_1)(s'_2 - d'_2)$ over $S'$.

Symmetry: Let $H(s_1, s_2) = (s_1 - d_1)(s_2 - d_2)$. Let $(S; d)$ be a symmetric bargaining problem. Assume that $(s_1^*, s_2^*) \in S$ maximizes $H$ over $S$, namely

$$(s_1^* - d_1)(s_2^* - d_2) \geq H(s_1 s_2) \qquad \forall (s_1, s_2) \in S.$$

Since $(S;d)$ is symmetric, $d_1 = d_2$. Therefore,

$$(s_2^* - d_1)(s_1^* - d_2) \geq H(s_1, s_2) \quad \forall (s_1, s_2) \in S. \tag{1}$$

Since $S$ is symmetric $(s_2^*, s_1^*) \in S$. Thus (1) means that $(s_2^*, s_1^*)$ also maximizes $H$ over $S$. But since the maximizer is unique, it must be that $(s_1^*, s_2^*) = (s_2^*, s_1^*)$ which implies $s_1^* = s_2^*$.

<u>IIA</u>: Assume $S \subset T$ and that $(s_1^*, s_2^*) \in S$ maximizes $H$ over $T$. Namely

$$(s_1^* - d_1)(s_2^* - d_2) \geq (s_1 - d_1)(s_2 - d_2) \qquad \forall (s_1, s_2) \in T.$$

In particular,

$$(s_1^* - d_1)(s_2^* - d_2) \geq H(s_1, s_2) \qquad \forall (s_1, s_2) \in S.$$

Since $s^* \in S$, the result follows.

<u>PAR</u>: Since $H(s_1, s_2)$ is increasing both in $s_1$ and $s_2$ —in the sense that if $s_1' > s_1$ and $s_2' > s_2$ then $H(s') > H(s)$— $(s_1, s_2)$ cannot maximize $H$ if there exist $(t_1, t_2) \in S$ with $t_1 > s_1$ and $t_2 > t_2$.

c) Finally, we show that $f^N$ is the only bargaining solution that satisfies all four axioms.

Suppose that $f$ is a bargaining solution that satisfies the four axioms. We shall show that $f = f^N$. Let $\langle S, d \rangle$ be an arbitrary bargaining problem. We need to show $f(S, d) = f^N(S, d)$.

<u>Step 1</u>. Let $f^N(S, d) = (z_1, z_2)$. Since by the definition of bargaining problem there exists $(s_1, s_2) \in S$ with $s_1 > d_1$ and $s_2 > d_2$, we have $z_i > d_i$ for $i = 1, 2$.

Let $\langle S'; d' \rangle$ be the bargaining problem that is obtained from $\langle S; d \rangle$ by the transformations $s_i \to \alpha_i s_i + \beta_i$, for $i = 1, 2$, which move the disagreement point to the origin and the solution $z = f^N(S; d)$ to the point $(\frac{1}{2}, \frac{1}{2})$.

That is

$$\left\{ \begin{array}{lcl} \alpha_i d_i + \beta_i & = & 0 \\ \\ \alpha_i z_i + \beta_i & = & \frac{1}{2} \end{array} \right. \qquad \text{for } i = 1, 2$$

which yields

$$\alpha_i = \frac{1}{2(z_i - d_i)}; \qquad \beta_i = -\frac{d_i}{2(z_i - d_i)} \quad \text{for } i = 1, 2.$$

Since both $f$ and $f^N$ satisfy INV we have

$$\begin{array}{lcl} f_i(S', 0) & = & \alpha_i f_i(S, d) + \beta_i \ \text{ and} \\ f_i^N(S', 0) & = & \alpha_i f_i^N(S, d) + \beta_i = \frac{1}{2}. \end{array} \qquad \text{for } i = 1, 2.$$

Hence, $f(S,d) = f^N(S,d)$ if and only if $f(S',0) = f^N(S',0)$. Since $f^N(S',0) = (\frac{1}{2};\frac{1}{2})$ it remains to show that $f(S',0) = (\frac{1}{2};\frac{1}{2})$.

Step 2: We claim that $S'$ contains no point $(s_1, s_2)$ for which $s_1+s_2 > 1$, $s_1 \geq 0$ $s_2 \geq 0$.

If it does let $\Delta = s_1 + s_2 - 1 > 0$ and let

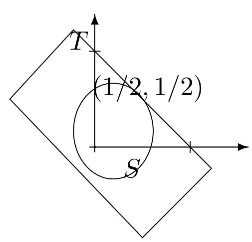
$$(t_1,t_2) = ((1-\epsilon)\frac{1}{2} + \epsilon s_1; (1-\epsilon)\frac{1}{2} + \epsilon s_2) = (1-\epsilon)(\frac{1}{2},\frac{1}{2}) + \epsilon(s_1,s_2).$$

Since $S'$ is convex, $(t_1,t_2) \in S'$. Now

$$\begin{array}{rcl} t_1 \cdot t_2 & = & (1-\epsilon)^2\frac{1}{4} + (1-\epsilon)\epsilon\frac{1}{2}s_1 + \epsilon(1-\epsilon)\frac{1}{2}s_2 + \epsilon^2 s_1 s_2 \\ & = & (1-\epsilon)^2\frac{1}{4} + \frac{(1-\epsilon)\epsilon}{2}[s_1+s_2] + \epsilon^2 s_1 s_2 \\ & \geq & (1-\epsilon)^2\frac{1}{4} + \frac{(1-\epsilon)\epsilon}{2}(s_1+s_2) + \epsilon^2 \cdot 0 \\ & \geq & (1-\epsilon)^2\frac{1}{4} + \frac{(1-\epsilon)\epsilon}{2}(1+\Delta) \\ & \geq & \frac{1}{4}[(1-\epsilon)^2 + 2(1-\epsilon)\epsilon(1+\Delta)] \\ & \geq & \frac{1}{4}[1 + 2\Delta\epsilon - (2\Delta+1)\epsilon^2] \\ & \geq & \frac{1}{4} + \frac{1}{4}\left[2\Delta - (2\Delta+1)\epsilon\right]\epsilon \end{array}$$

So $t_1 t_2 > \frac{1}{4}$ for $\epsilon < 2\frac{\Delta}{2\Delta+1}$, contradicting the fact that $\frac{1}{2}\cdot\frac{1}{2} \geq t_1 t_2$ (namely that $f^N(S',0) = (\frac{1}{2},\frac{1}{2})$).

Step 3: Since $S'$ is bounded, step 2 ensures that we can find a rectangle $T$ that is symmetric about the $45^o$ line and that contains $S'$, on the boundary of which there is $(\frac{1}{2},\frac{1}{2})$.




Step 4: By PAR and SYM of $f$ we have $f(T,0) = (\frac{1}{2},\frac{1}{2})$.

Step 5: By IIA we have $f(S',0) = f(T,0)$ so that $f(S',0) = (\frac{1}{2},\frac{1}{2})$ completing the proof. $\square$

## 1.3 Is any axiom superfluous?

INV: The egalitarian solution satisfies PAR SYM and IIA. It does not satisfy INV.

SYM: The bargaining solution defined by

$$\arg\max(s_1-d_1)^\alpha(s_2-d_2)^{1-\alpha}$$

$$(d_1, d_2) \leq (s_1 s_2) \in S$$

where $\alpha \neq 1/2$ and $0 < \alpha < 1$, satisfies INV, PAR and IIA but it is not symmetric.

IIA: The Kalai-Smorodinsky bargaining solution satisfies INV, PAR and SYM but does not satisfy IIA.

PAR: The disagreement point solution satisfies all the axioms except for PAR.