Maths Semester 1 2022 JF

CHAPTER 5

APPLICATIONS OF THE DEFINATE INTEGRAL

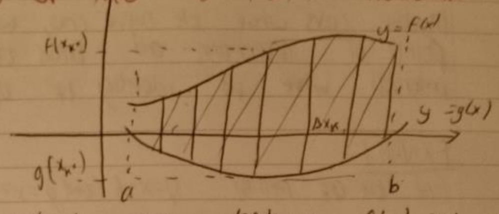
Week 11 Chapter 6. Chapter 6

3/12/12 6 Application of the Definite Integral in Geometry

APPLICATIONS OF THE DEFINATE INTEGRAL

Area between Curve

Split area into subintervals



We wish to compute the area between $f(x)$ and $g(x)$ enclosed by $x=a$ and $x=b$ where $f(x) > g(x)$ and $x \in [a,b]$

The area of each segment $\underbrace{\Delta x_k}_{\text{width}} \underbrace{(f(x_k) - g(x_k))}_{\text{height}}$ where $x_k^* \in [x_{k-1}, x_k]$

Then the area is approximated by the Riemann Sum

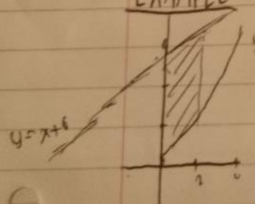
$$A = \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)] \Delta x_k$$

Taking the limit of the width of the subintervals shrinking to zero gives us the exact area

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)] \Delta x_k$$

$$= \int_a^b (f(x) - g(x))\,dx \quad \text{Where} \quad f(x) \geq g(x)$$

EXAMPLE: Find area bounded by $y=x+6$ and $y=x^2$ over interval $[0,2]$



$$A = \int_0^2 ((x+6) - x^2)\,dx$$

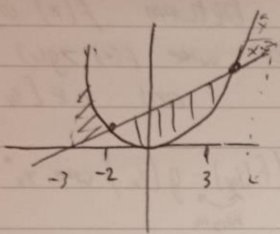
$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2$$

$$2 \Rightarrow \left[\frac{4}{2} + 12 - \frac{8}{3}\right] = \frac{34}{3}$$

In the cases where the curves cross, we must identify the points of intersection and break the integral up into intervals where we identify the upper and lower curve

EXAMPLE:

Find the area between $y=x+6$ and $y=x^2$ over $[-3,4]$



The intersection points are obtained by solving the simultaneous equations

$y=x+6$

$y=x^2$

$x+6=x^2$

$x^2-x-6=0$

$(x+2)(x-3)$

$x=-2 \quad x=3$

$$A=\int_{-3}^{-2} x^2-(x+6)\,.dx + \int_{-2}^{3}(x+6)-x^2\,.dx + \int_{3}^{4}x^2-(x+6)\,.dx$$

$$=\left[\frac{x^3}{3}-\frac{x^2}{2}+6x\right]_{-3}^{-2}+\left[\frac{x^2}{2}+6x-\frac{x^3}{3}\right]_{-2}^{3}+\left[\frac{x^3}{3}-\frac{x^2}{2}-6x\right]_{3}^{4}$$

$\frac{17}{6}+\frac{125}{6}+\frac{17}{6}=\frac{[illegible]}{6}$ units^2

31/12/15

3. MoM weekl

$$= \frac{u^6}{12}\Big|_0^{\pi/8} = \frac{1}{96}$$

Tutorial 10: MA1E01

Applications of the Definite Integral in Geometry

1. Find the area of the region enclosed between $y = 2x^2 - 1$ and $y = 2x + 3$ and on the sides by $x = -2$ and $x = 3$.
2. A solid is generated by revolving the region enclosed by $x = (y-2)^2$ and $x = 4$ about the x-axis, find the volume.
3. The region between $y = x^2$ and $y = x^3$ over the interval $[0, 1]$ is revolved about the y-axis. Find the volume.
4. Find the arclength of the curve $x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$ from $y = 1$ to $y = 4$.
5. Find the area of the surface generated by revolving the curve $y = \sqrt{4 - x^2}$ over the interval $[-1, 1]$ about the x-axis.

1. $2x^2 - 1 = 2x + 3$

$2x^2 - 2x - 4 = 0$

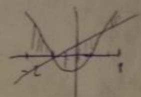
$(x-2)(x+1) = 0$

$x = 2,\ x = -1$

$-2,\ 3$

$[-2,-1]\ [-1,2]\ [2,3]$

$[-2,-1]$ $y = 2x^2 - 1$
-2 $2(4) - 1 = 7$
0 -1
3 17



$A = \int_{-2}^{-1} [f(x) - g(x)]\,dx + \int_{-1}^{2} [g(x) - f(x)]\,dx + \int_{2}^{3} [f(x) - g(x)]\,dx$

$\int_{-2}^{-1} [2x^2 - 2x - 4]\,dx + \int_{-1}^{2} [-2x^2 + 2x + 4]\,dx + \int_{2}^{3} [2x^2 - 2x - 4]\,dx$

$\frac{2x^3}{3} - x^2 - 4x \Big|_{-2}^{-1} + \frac{-2x^3}{3} + x^2 + 4x \Big|_{-1}^{2} + \frac{2x^3}{3} - x^2 - 4x \Big|_{2}^{3}$

$= \frac{49}{3}$

2. $\int y\,dx$

$x \in [0, 4]$

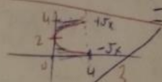
$x = (y-2)^2$

$\pm\sqrt{x} = y - 2$

$y = 2 \pm \sqrt{x}$

$f(x)$ $g(x)$
$y = 2 + \sqrt{x}$ & $y = 2 - \sqrt{x}$
seperate regions

~~$A = \int_0^4 (f(x) - g(x))$~~



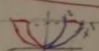
$V = \int \pi (f^2 x - g^2(x))$

$V = \pi \int_0^4 (4 + 4\sqrt{x} + x - (4 - 4\sqrt{x} + x))\,dx$

$8\pi \int_0^4 x^{\frac{1}{2}}\,dx$

$= \frac{128\pi}{3}$

3.



$\int f(y) - g(y)\,dy$

	$y = x^2$	$y = x^3$
$x = 0$	0	0
$x = 1$	1	1
$x = \frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$x = \frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$

$y = x^2$ $y = x^3$

$x = y^{1/2}$ $x = y^{1/3}$

x goes $[0, 1]$

sub in points: y goes $[0, 1]$

$\int_0^1 \pi (f^2 y - g^2 y)\,dy$

$\pi \int_0^1 y^{2/3} - y$

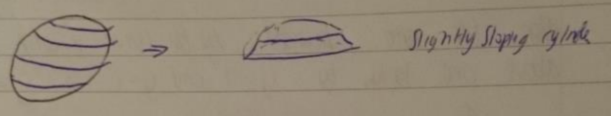
$= \frac{\pi}{10}$

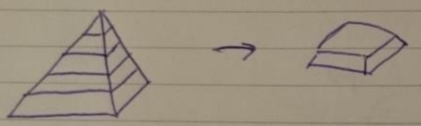
2. MSISS

Area $= \int\!\!\int \int_{-1}^{2} (y-2-y^2) = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2} = \frac{9}{2}$

Volume by slicing
Riemann sums - area under a curve by subdividing into strips and summing, taking an appropriate limit.

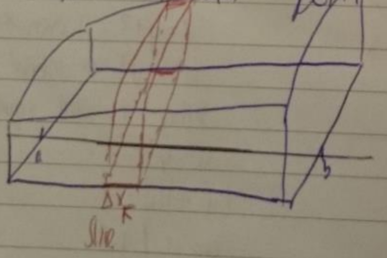
Volumes by slicing - divide solid into slices, add values of each slice to approximate the volume by a Riemann sum, then take a limit to obtain the exact volume.





Let S be a solid that extends along the x-axis and is bounded on the left and right by planes perpendicular to the x-axis at $x=a$ and $x=b$ respectively.

We can find the volume of the solid assuming the cross sectional area $A(x)$ is known at each point $x \in [a, b]$



3 marks

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If $x_k^* \in [x_{k-1}, x_k]$ the the volume of the k^{th} slab is approximately $A(x_k^*)\Delta x_k$ and an
height width

approximation to the total volume by the Riemann Sum. $V \approx \sum_{k=1}^{n} A(x_k^*)\Delta x_k$

Taking the limit as $\Delta x_k \to 0$ gives the exact volume.

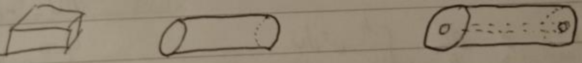
$$V = \lim_{\max(\Delta x_k)\to 0} \sum_{k=1}^{n} A(x_k^*)\Delta x_k$$

$$= \int_a^b A(x)\,dx$$

Similarly $V = \int_c^d A(y)\,dy$ for a solid extending in the y-axis

Example: Derive the formula for the volume of a right cylinder. (a solid generated when a plane region is translated along an x-axis perpendicular to the plane)

e.g.

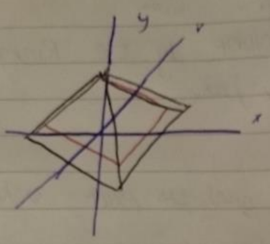


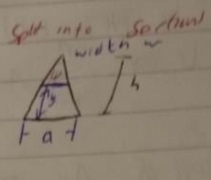
$V = \int_a^b A(x)\,dx = A\int_a^b dx = A(b-a) = A_0(h)$

eg cylinder $V = \pi r^2 h$
area $= \pi r^2$ $\qquad \pi r^2 h$ $\qquad$ for area constant

For a triangular prism $\frac{1}{2}b \times L h = \frac{1}{2}b.L\,(h)$

Example: Derive the formula for the volume of a pyramid when base is square and whose height is h.





Similar triangles $\frac{a}{w} = \frac{h}{h-y}$ $\quad W = \frac{a}{h}(h-y)$

Area of each slice which is a square is w^2

$$= \frac{a^2}{h^2}(h-y)^2$$

$$V = \int_0^h \overset{\text{area}}{\frac{a^2}{h^2}(h-y)^2}\,dy$$

$$= \frac{a^2}{h^2}\int_0^h (h-y)^2\,dy$$

$u = h-y$

$du = -dy$

$y=0,\ u=h$

$y=h\ \ u=0$

$$= \frac{a^2}{h^2}\int_0^h u^2\,du$$

$$= \frac{a^2}{h^2}\left[\frac{u^3}{3}\right]_h^0$$

$$= \frac{a^2}{h^2}\cdot\frac{h^3}{3}$$

$$= \frac{a^2 h}{3}$$ = volume of a pyramid

6/12/12 Week 11 Math

Volumes by Slicing

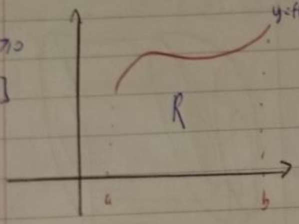
$V = \int_a^b A(x)\,dx$

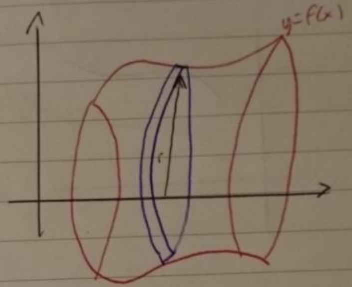
or

$V = \int_c^d A(y)\,dy$

Solids of Revolution

$f(x) \geq 0$ on $[a,b]$





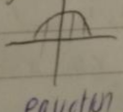
like a lath.

We divide the volume up into discs, whose area at the point $x \in [a,b]$ is πr^2 or $\pi [f(x)]^2$

Hence the volume is $V = \int_a^b \pi [f(x)]^2\,dx$

Called <u>Method of Discs</u>.

Example: Derive the formula for the volume of a sphere obtained by revolving the upper half semi circle about the x-axis



The equation of a circle is $x^2+y^2=r^2$
$\Rightarrow y^2 = r^2 - x^2$

The upper half semi circle is $y = \sqrt{r^2-x^2}$

Volume is given by $V = \int_{-r}^{r} \pi \left[\sqrt{r^2-x^2}\right]^2 dx = \int_{-r}^{r} \pi (r^2-x^2)\,dx$

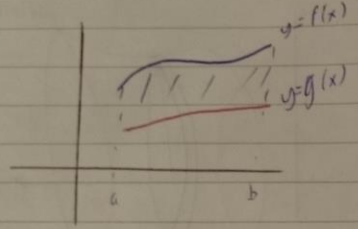
$= \left[\pi r^2 x - \frac{\pi x^3}{3}\right]_{-r}^{r}$

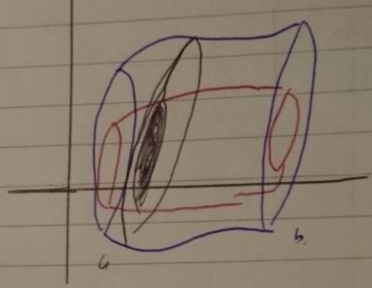
$= \pi r^3 - \frac{\pi r^3}{3} + \pi r^3 - \frac{\pi r^3}{3}$

$= 2\pi r^3 - \frac{2\pi r^3}{3}$

$= \frac{4\pi r^3}{3}$

We can also consider solids of revolution with hollow intervals





The area of each slice in this case is the area of the large circle minus the area of the small circle

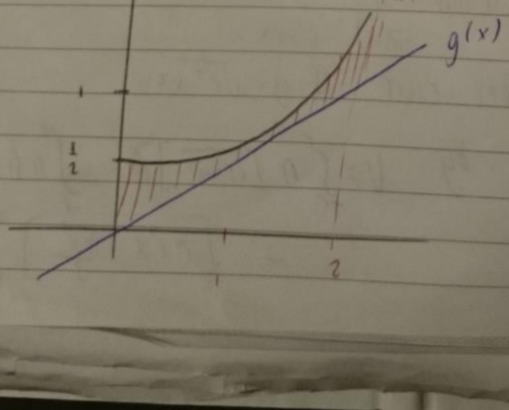
$A(x) = \pi[f^2(x) - g^2(x)]$

$\Rightarrow V = \int_a^b \pi[f^2(x) - g^2(x)]\,dx$

Method of Washers

*

Example: Find the volume of the solid generated when the region between graphs of the equation $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved around the x-axis



6/12/12. Maths week 11

$$V = \int_0^2 \pi \left(\left(\tfrac{1}{2}+x^2\right)^2 - x^2 \right) dx$$

$$= \int_0^2 \pi \left(\tfrac{1}{4} + x^2 + x^4 - x^2 \right) dx$$

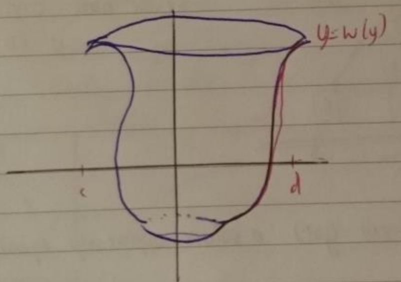
$$= \int_0^2 \pi \left(\tfrac{1}{4} + x^4 \right) dx$$

$$= \pi \left[\tfrac{x}{4} + \tfrac{x^5}{5} \right]_0^2$$

$$= \pi \left[\tfrac{2}{4} + \tfrac{2^5}{5} \right]$$

$$= \frac{69\pi}{10} \ \text{m}^3$$

Revolution about the y-axis:



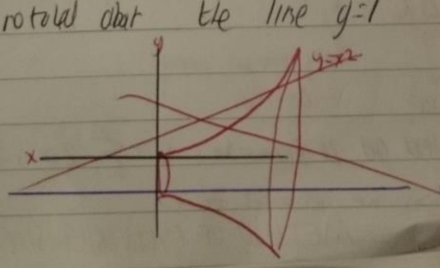
$$V = \int_c^d \pi [w(y)]^2 dy$$

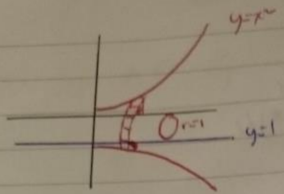
Similarly for the method of washers $V = \int_c^d \pi [w^2(y) - u^2(y)] dy$

where $w(y) \geq u(y)$

Other axis of Revolution

Example: Find the volume of the solid generated when the region under the curve $y = x^2$ on the interval $[0, 2]$ is ~~retated~~ rotated about the line $y = 1$





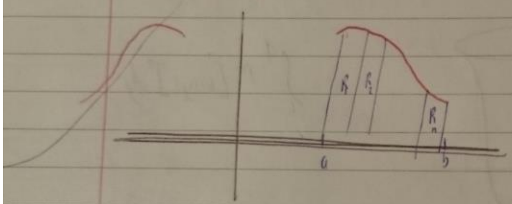
Area of a slice is $A(x) = \pi(x^2+1)^2 - \pi(1)^2$

$$\Rightarrow V = \int_0^2 \pi\left[(x^2+1)^2 - 1\right]dx$$

$$= \int_0^2 \pi\left[x^4 + 2x^2 + 1 - 1\right]dx$$

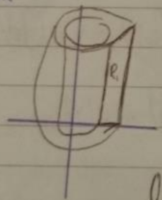
$$= \pi\left[\frac{x^5}{5} + \frac{2x^3}{3}\right]_0^2 = \frac{176\pi}{15}$$

Volume by Cylinderical Shells



revolve about the y-axis
$y = f(x)$

Revolving region R_k about y-axis gives a volume approximatly equal to cylindrical shell



The volume of a cylindrical shell with inner radius r_1 and outer r_2 $V = \pi r_2^2 h - \pi r_1^2 h$ (big-small)
$= \pi h(r_2^2 - r_1^2)$

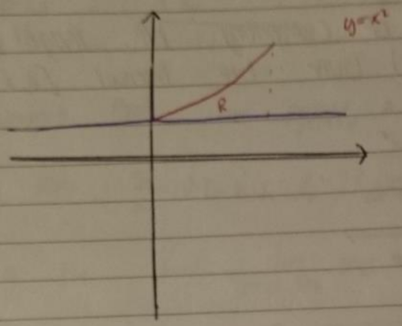
Or written another way $V = 2\pi h \underbrace{(r_2 - r_1)}_{\text{thickness of shell}} \overbrace{\tfrac{1}{2}(r_2 + r_1)}^{\text{average radius of 2 cylinders}}$

The volume of the k^{th} cylindrical shell can be approximated as
$V \approx 2\pi f(x_k^*)\Delta x_k$, x_k^*

$f(x_k^*)$ ↑ height, Δx_k ↑ shell thickness, x_k^* ↑ any point on interval average

Adding each of these volumes gets the Riemann Sum $V \approx \sum_{k=1}^{n} 2\pi f(x_k^*) x_k^* \Delta x_k$
In limit max $|\Delta x_k| \to 0$, we get exact value
$\int_a^b 2\pi x\, f(x)\,dx$ METHOD OF CYLINDRICAL SHELLS

10/12/12. Maths Week 12

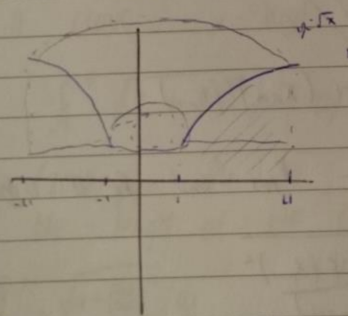


revolving region R enclosed between $y=x^2$ and the x-axis produce a solid with a hollow interior.

Volumes by cylindrical shells:

$$V = \int_a^b 2\pi x\, f(x)dx$$

Example: Use cylindrical shells to find the volume of the solid generated when the region between $y=\sqrt{x}$, $x=1$ and $x=4$ and x-axis is revolved about the y axis.



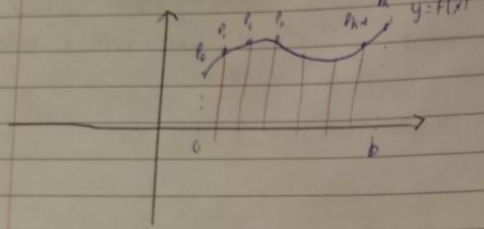
$$V = \int_1^4 2\pi x(\sqrt{x})\,dx = \int_1^4 2\pi x^{3/2}dx = \frac{2\pi x^{5/2}}{5/2}\Big|_1^4 = \frac{4\pi}{5}\left[x^{5/2}\right]_1^4$$

$$= \frac{4\pi}{5}\left[4^{5/2}-1\right] = \frac{124\pi}{5}$$

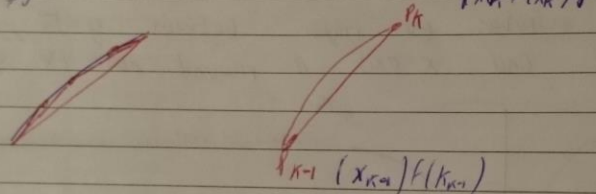
2

Lenght of a plane curve

We consider the problem of computing the lenght of a smooth curve, $y = f(x)$ over the interval $[a, b]$



To compute the lenght of the curve over $[a, b]$ (arc length) we obtain a Riemann sum approximation by dividing the curve into a number of straight line segments and computing the length of each segment



The lenght between the point P_{k-1} and P_k is approximately

$$L_k = \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}$$
$$= \sqrt{\Delta x_k^2 + (f(x_k) - f(x_{k-1}))}$$

By the mean value theorem there is a point in $[x_k, x_{k-1}]$ x_k^* say, such that the slope at x_k^* is equal to slope of line segment

i.e. $\dfrac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*)$

$\Rightarrow f(x_k) - f(x_{k-1}) = f'(x_k^*)\,\Delta x_k$

In limit max $|\Delta x_k| \to 0$, we get...
$\int_0^b 2\pi x f(x)\,dx$ METHOD OF CYLINDRICAL SHELLS

10/12/12 3.

$$\Rightarrow L_k = \sqrt{\Delta x_k^2 + f'(x_k)^2 \Delta x_k^2}$$
$$= \sqrt{1 + f(x_k)^2} \; \Delta x_k$$

Summing each line segment

$$L \approx \sum_{k=1}^{n} \sqrt{1 + f'(x_k^*)^2} \, \Delta x_k$$

In the limit $\lim_{\max \Delta x_k} \to 0$ we get the exact lenght.

$$L = \lim_{\max(\Delta x_k) \to 0} \sum_{k=1}^{n} \sqrt{1 + f'(x_k^*)^2} \; \Delta x_k$$

$$= \int_a^b \sqrt{1 + f'(x)^2} \, dx$$

or

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

For a curve $x = v(y)$ over $[c, d]$

$$L = \int_c^d \sqrt{1 + \left(\frac{dv}{dy}\right)^2} \, dy$$

Example: (2009 paper Q4c)

Find the lenght of the curve $y = x^{3/2}$ from $x=0$ to $x=1$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$y = x^{3/2}$

$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$

$$\int_0^1 \sqrt{1 + \tfrac{9}{4} x} \; dx$$

let $u = 9/4 x \Rightarrow du = \frac{9}{4} dx \quad dx = \frac{4}{9} du$

$x=0 \quad u=1$,

$x=1, \quad u = 13/4$

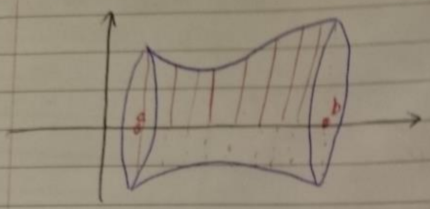
$$\int_1^{13/4} \frac{4}{9} \sqrt{u} \, du = \frac{4}{9} \left[\frac{2u^{3/2}}{3}\right]_1^{3}$$

$$= \frac{8}{27} \left[\left(\tfrac{13}{4}\right)^{3/2} - 1\right] = \frac{(\sqrt{13})^3 - 8}{27}$$

2

4.

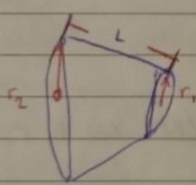
Area of a Surface of revolution



f is smooth, non negative function on $[a, b]$
A Surface of revolution is generated by revolving the portion of the curve $y = f(x)$ between $x = a$, $x = b$ about the x-axis
We require a formula for the area S of the surface

Divide the area up into n parts with the partition, $x_0 = a$, x_1, x_2, x_{n-1}, $x_n = b$

The shape generated by revolving a particular straight line segment between two points of the partition is a frustum



$A = \pi(r_1 + r_2)L$

The lenght L here is just the arclenght between the points of the partition given by

$$L_k = \sqrt{\Delta x_k^2 + [f(x_k) - f(x_{k-1})]^2}$$

- While the radii r_1 and r_2 are simply $f(x_k)$ and $f(x_{k-1})$ respectively
- The area of the k^{th} frustum is

$$S_k = \pi \, (f(x_k) + f(x_{k-1})) \sqrt{\Delta x_k^2 + (f(x_k) + f(x_{k-1}))^2}$$

10/12/12 maths

Adding these gives:

$$S \approx \sum_{k=1}^{n} \pi (f(x_k) + f(x_{k-1})) \sqrt{\Delta x_k^2 + (f(x_k) - f(x_{k-1}))^2}$$

By the mean value theorem

$$f(x_k) - f(x_{k-1}) = f'(x_k^*) \Delta x_k$$

By the intermediate value theorem the function f(x) must take on every value on $[x_{k-1}, x_k]$ and since the average lies in this range there must exist x_k^{**} such that

$$f(x_k^{**}) = \frac{1}{2}(f(x_k) - f(x_{k-1}))$$

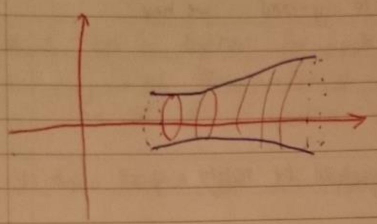
$$S \approx \sum_{k=1}^{n} 2\pi f(x_k^{**}) \sqrt{1 + f'(x_k^*)^2} \Delta x_k$$

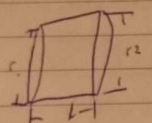
The limit $\max |\Delta x_k| \to 0$ doesn't depend on x_k^{**} choose $x_k^{**} = x_k^*$

$$S = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 2\pi (f(x_k^*) \sqrt{1 + f'(x_k^*))^2} \Delta x_k$$

$$= \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} dx$$

11/12/12 Maths Week 12.





$S = \pi(r_1 + r_2) \cdot L$

$S_k = \pi\left(f(x_{k-1}) + f(x_k)\right)\sqrt{\Delta x_k^2 + \left(f(x_k) - f(x_{k-1})\right)^2}$

Mean value theorm $\Rightarrow$ $f(x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x_k$

Intermediate-value theorm
$\Rightarrow$ $f(x_k^{**}) = \frac{1}{2}\left(f(x_{k-1}) + f(x_2)\right)$

$S_k = 2\pi f(x_k^{**})\sqrt{1 + f'(x_k^*)}\,\Delta x_k$

The limit $\max \Delta x_k \to 0$ doesn't depend on choice of x_k^{**} or x_k^* $\quad x_k^* = x_k^{**}$

which gives the Riemann sum $S = \sum_{k=1}^{n} 2\pi f(x_k^*)\sqrt{1 + f'(x_k^*)}\,\Delta x_k$

$$\Rightarrow S = \lim_{\max(\Delta x_k) \to 0} \sum_{k=1}^{n} 2\pi f(x_k^*)\sqrt{1 + f'(x_k^*)}\,\Delta x_k$$

$$= \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\,dx$$

or

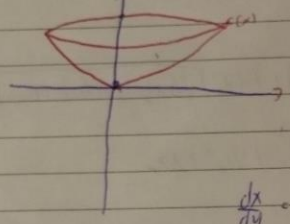
$$\int_0^1 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

For revolution $x = w(y)$ about the y-axis we have

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example 2009 pap Q4d

Find the area of the surface generated by rotating $y = x^2$ about the y-axis for $0 \leq x \leq 2$



when $x = 0$ $y = 0$

when $x = 2$ $y = 4$

$x = \sqrt{y}$

positive square.

$$\frac{dx}{dy} = \frac{1}{2} y^{-\frac{1}{2}}$$

$$S = \int_0^4 2\pi\sqrt{y} \sqrt{1 + \left(\tfrac{1}{2}y^{-\frac{1}{2}}\right)^2}\, dy$$

$$= \int_0^4 2\pi\sqrt{y}\sqrt{1 + \frac{1}{4y}}\, dy$$

$$= \int_0^4 2\pi\sqrt{y + \tfrac{1}{4}}\, dy$$

$u = y + \frac{1}{4}$ $du = dy$

$y = 0$ $u = \frac{1}{4}$

$y = 4$ $u = \frac{17}{4}$

$$S = \int_{1/4}^{17/4} 2\pi\sqrt{u}\, du = \frac{4\pi}{3}\left(u^{\frac{3}{2}}\right)\Big|_{1/4}^{17/4}$$

$$\frac{4\pi}{3}\left[\left(\frac{17}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right]$$

$$\frac{\pi}{6}\left[(17)^{3/2} - 1\right]$$

Exam

Q1 Functions + limits - Def^n of a function, finding domain + range, finding inverse, def^n of limit epsilon prove limit

Q2 Continuity / Differentiation def^n of continuity, piecewise functions derivatives, domain, related rates, linear approximations

Q3 Derivative in graphing and application - Sketch a polynomial/rational function max/min

Q4 Integration and application in geometry - Def^n Riemann Sum, fundamental theorem calculus, integrate by substitution, area, rotation, arc length, S.A