4/12/13 Mang Science 1

Types of Integer Programming

- LP where all variables are restricted to be integers. - All integer P ILP

- If only a subset of variables are restricted to be integers - Mixed integer L

- Binary variables 1 or 0. Binary Integer L.P.

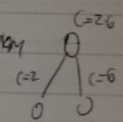
- LP that results from dropping the integer requirements is called LP relaxation of the ILP.

GRAPHICAL

Pick integer solution nearest to the relaxed optimum

Branch and Bound

- Divides the set of feasible solutions to an integer problem into smaller subsets (branching)

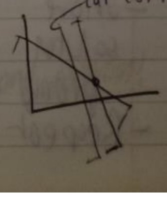


- From the subsets we then identify the subsets that are most likely to contain the optimal solution and the subsets that need not be explored further because they could not possibly contain the optimum solution

- Branch on largest fraction with variable.

Cutting Plane

- Solve problem using Simplex procedure disregarding the integer condition
- Taking the basic variable with the largest fraction from the final simplex table and from a cut constraint



- Constraint added to table
- Dual Simplex is used to restore feasibility
- Iteratively added until I.S. is obtained

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BALAS
- Transform problem
- Convert max to min x (-1)
- Replace variable negative with $x_j = (1 - x_j)$
- Constraints of $\geq$ converted to $\leq$ by -1
- No (=) constraints
- Change back $x_j = (1 - x_j)$ at end

Solution To Balas.
N = Set of decision variables = 0
T = Set of decision variables = 1
Z = value of objective function
S = level of infeasibility in each constraint.

Goal Programming
Decision maker needs to consider multiple criteria in arriving at overall best decision.

Formulating
Identify "hard" constraints, then the goals and any constraint on achieving the goals, then the priority of the constraints, the Decision Vars and lastly the O.F.

- A "hard" constraint is one that must be satisfied

Solving
- Identify the feasible solution
- Identify feasible space that will achieve highest priority goal. If no such solutions select the closest point to the achievement
- Move to next lower priority level and determine best solution possible without sacrificing any achievement of higher priority goals
- Repeat step 3 until all [illegible]

1. Integer linear Programming

Problems that are modelled as linear programs with the additional requirement that one or more of the variables must be integer - integer linear programming

- If all variables must be integer we have all integer linear program
- The LP that results from dropping the integer requirement is called LP relaxation
- If only a subset of variables is restricted to be integers, mixed integer linear programming
- Binary variables are 0 or 1. if restricted to binary → binary integer linear program

Graphical Solution for all integer linear program:

T = number townhouses

A = number of apartment buildings

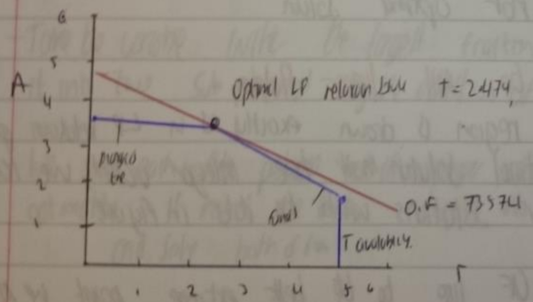
Objective function is Max $10T + 15A$.

ST: $282T + 400A \le 2000$ (1000's) Funds available

$4T + 40A \le 140$ Manager's time (hours)

$T \le 5$ Townhouses available

$T, A \ge 0$ and integer



Graphical Solution

If we drop integer requirement, $T = 2.474$, $A = 3.252$ and profit = 73.574.
This is LP relaxation of problem

Rounding to obtain an integer solution

Whenever rounding has a minimal impact on the O.F. and constraints, most managers find it acceptable.

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- When the decision variables take on small values that have a major impact on the value of the objective function or feasibility, an optimal integer solution is required

- Suppose we rounded solution to LP relaxation to obtain T=2 and A=3 with an objective function of 10(2) + 15(3) = 65.
- Annual cash flow of 6500 is far less than 73,574 provided by LP relaxation

- Integer solution T=3 and A=3 is infeasible because it exceeds funds constraint
- Rounded T=2 and A=4 is also infeasible

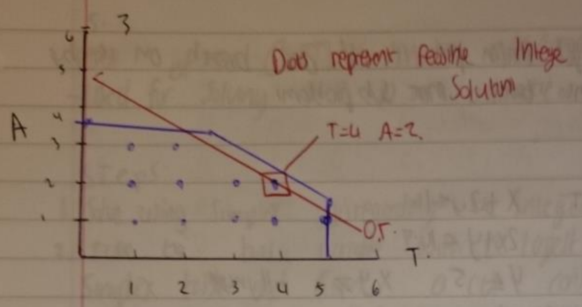
- Rounding to an integer solution is a trial and error approach. Each rounded solution must be checked for optimality and feasibility
- Even ~~in cases~~ ~~where a rounded solution must~~ be evaluated for feasibility ~~as well as for its impact on the value~~ of the ~~objective function~~
- Even in cases where a rounded solution is feasible, we do not guarantee that we have found the optimal ~~...~~ integer solution
- T=3, A=2 is not optimal solution

Graphical Solution to All Linear Problem

- Graph of feasible region is drawn exactly as in LP relaxation problem
- Then because the optimal solution must have integer values, we identify the feasible integer solutions with the dots in figure

- Instead of moving OF line to the best extreme point, we move it in an improving direction as far as possible until reaching a dot.

- We find optimal solution T=4 and A=2, value of 10(4) + 15(2) = 7000 better than T=2 A=3.



Branch and Bound Algorithm

- Divide the set of feasible solution to an integer programming problem into smaller subsets (branching)

- From subsets we identify subsets which are most likely to contain the optimal Solution. Subsets need to be explored further because they might contain O.S.

- A tight upper/lower bound on the value of the best solution in each subset is obtained by solving a relaxed linear Program

- Take the variable with the largest fraction part and sub divide it into two sub problems by the addition of 2 extra constraints

- We solve each sub problem and if integer solution is found we check for optimality. If not: we further divide into 2 or more sub problems and solve both of them

- We repeat repeat until all solutions are found or are infeasible

Example: Max $z = x+y$

ST: $x+7y \le 16$ $\quad x, y, \geq 0$ all integer

$2x+y \le 14.9$

Solution: $x = 4.6$, $y = 5.7$ $z = 10.3$

- Take variable with largest fraction part i.e. y and branch on it by partitioning it into two more sub problems

Sub problem 1:

Max $z = x + y$ ST: $x + 2y \leq 16$

$2x + y \leq 14.9$

$y \leq 5$ $\quad x, y \geq 0$ all integer

Solution $x = 4.95$, $y = 5$, $z = 9.95$.

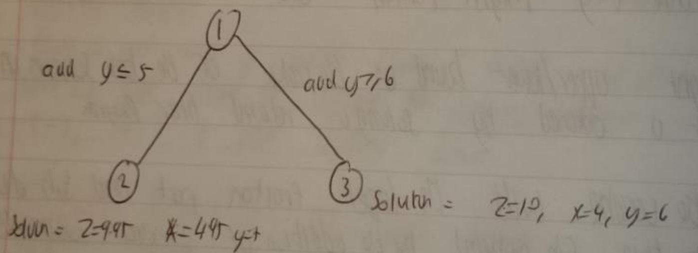
Sub problem 2

Max $z = x + y$ ST: $x + 2y \leq 16$

$2x + y \leq 14.9$

$y \geq 6$ $\quad x, y \geq 0$ all integer value

Solution: $x = 4$, $y = 6$ $\quad z = 10$



5.

~~Use~~ ~~of the~~ CUTTING PLANE ALGORITHM

- Used for solving L.P as an alternative to Branch and bound

Steps:

1. Solve using simplex disregarding the integer conditions
2. Take the basic variable with the largest fraction from the final simplex table and form a cut constraint
3. The constraint is added to the table
4. Dual simplex is ~~applied~~ used to restore feasibility
5. Cut constraints are iteratively added until an integer solution is found

Example: Max $z = 7a + 9b$ ST: $-a + 3b \le 6$
$7a + b \le 35$

$a, b \geq 0$ and all integers

Final simplex table.

Basic	z	a	b	S_1	S_2	RHS
z	1	0	0	28/11	15/11	63
b	0	0	1	7/22	1/22	7/2
a	0	1	0	-1/22	3/22	9/2

Solution (i)

- Basic variable with largest fraction solution is now selected (b) which gives

$b + \frac{7}{22}S_1 + \frac{1}{22}S_2 = \frac{7}{2}$

$b = \frac{7}{2} + \frac{7}{22}(-S_1) + \frac{1}{22}(-S_2)$

- Express each coefficient as an integer with a positive fraction

$b = [3 + \frac{1}{2}] + [0 + \frac{7}{22}](-S_1) + [0 + \frac{1}{22}](-S_2)$

- Group the integer and fractional parts:

$b = [3 + 0(-S_1) + 0(-S_2)] + [\frac{1}{2} - \frac{7}{22}S_1 - \frac{1}{22}S_2]$

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- Theory of cut constraints is that the fractional part must be ≤ 0.

$[1/2 - 7/22 s_1 - 1/22 s_2] \le 0$

$-7/22 s_1 - 1/22 s_2 \le -1/2$

- Convert this constraint into an equality by adding a slack variable and add to the final table:

$-7/22 s_1 - 1/22 s_2 + s_3 = -1/2$

Basic	z	a	b	s_1	s_2	s_3	RHS
z	1	0	0	28/11	15/11	0	63
b	0	0	1	7/22	1/22	0	7/2
a	0	1	0	-1/22	3/22	0	9/2
s_3	0	0	0	-7/22	-1/22	1	-1/2

Table is now valid, optimum but infeasible

- Use dual simplex to remove infeasibility while maintaining optimality
- We need to find leaving and entering variable

- ⟹ leaving variable s_3, entering is s_1

Solution 2.

Basic	z	a	b	s_1	s_2	s_3	RHS
z	1	0	0	0	1	8	59
b	0	0	1	0	0	1	3
a	0	1	0	0	1/7	-1/7	32/7
s_1	0	0	0	1	1/7	-22/7	11/7

Still not in integer form. Construct a new cut constraint

Express each coefficient as an integer with a positive fraction:

$a = [4 + 4/7] + [0 + 1/7](-s_2) + [-1 + 6/7](-s_3)$

Gives us cut constraint: $1/7(-s_2) + 6/7(-s_3) + s_4 = -4/7$

ADD to table:

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Basic	Z	a	b	s_1	s_2	s_3	s_4	RHS
Z	1	0	0	0	1	8	0	54
b	0	0	1	0	0	1	0	3
a	0	1	0	0	1/7	-1/7	0	32/7
s_1	0	0	0	1	1/7	-22/7	0	11/7
s_4	0	0	0	0	-1/7	-6/7	1	-4/7

STILL NOT FEASIBLE : REPEAT

$\Rightarrow$

Basic	Z	a	b	s_1	s_2	s_3	s_4	RHS
Z	1	0	0	0	0	2	7	55
b	0	0	1	0	0	1	0	3
a	0	1	0	0	0	-1	1	4
s_1	0	0	0	1	0	-4	1	1
s_2	0	0	0	0	1	6	-7	4

Optimum and integer with: $Z=55$, $a=4$ and $b=3$

BALAS

Used for solving $\{0,1\}$ integer problems

Program must be in the following form:

1. The objective function must be minimised
2. All the coefs in objective function must be positive
3. All constraints must be $\leq$ type

Transformation

1. Convert a max to a min by multiplying by -1
2. Replace variables x_i that have negative signs by $1-x_i$
3. Constraints of $\geq$ are converted to $\leq$ by multiplying both sides by -1

Example: Max $Z = 3A + 2B - 5C - 2D + 3E$

ST: $A + B + C + 2D + E \leq 4$

$7A + 3C - 4D + 3E \leq 8$

$11A - 6B + 3D - 3E \geq 3$ $\quad A, B, C, D, E \in \{0,1\}$

1 Transform O.F by multiplying by -1 to minimise

Min $Z = -3A - 2B + 5C + 2D - 3E$

2 Replace negative variables in O.F. by (1-var) A, B and E

Min $Z = -3(1-A) - 2(1-B) + 5C + 2D - 3(1-E)$

$\Rightarrow Z = 3A + 2B + 5C + 2D + 3E - 8.$

We can omit the constant (8)

3 Change the corresponding variables in constraints A, B and E.

Constraint 1: $(1-A) + (1-B) + C + 2D(1-E) \leq 4$

$-A - B + C + 2D - E \leq 1$

Constraint 2: $7(1-A) + 3C - 4D + 3(1-E) \leq 8$

$-7A + 3C - 4D - 3E \leq -2.$

Constraint 3: ~~-7A+3C-4D~~ $11(1-A) - 6(1-B) + 3D - 3(1-E) \geq 1$

$-11A + 6B + 3D + 3E \geq 1.$

4 Multiply all $\geq$ constraints by -1

Constraint 3 becomes: $11A - 6B - 3D - 3E \leq -1$

Transformed Problem:

Min $Z = 3A + 2B + 5C + 2D + 3E - 8.$

Subject to: $-A - B + C + 2D - E \leq 1$

$-7A + 0 + 3C - 4D - 3E \leq -2$

$11A - 6B + 0 - 3D - 3E \leq -1$ $\quad A, B, C, D, E \in \{0,1\}$

Method for solution:

- After initial transformation we set all variable to zero as we are minimising a

We then look at the problem for infeasibility using the following notation

N = Set of decision variables $= 0$
T = set of decision variables $= 1$
Z = value of O.F.
S = Level of infeasibility in each constraint

Setting up table of coefs.

- Labeled across the first row are the decision variables.
- Second row contains O.F. coefs, all positive
- Remaining rows contain the coefficients of the constraints.

Table of coefficients

Variable	A	B	C	D	E	RHS
Coef Z	3	2	5	2	3	0
Coef C1	-1	-1	1	2	-1	1
Coef C2	-7	0	3	-4	-3	-2
Coef C3	1	-6	0	-3	-3	-1

Initial values:

$N = \{A, B, C, D, E\}$ variables with value 0.
$T = \{0\}$ variables with value 1
$Z = 0$ objective function value
$S = \{0, -2, -1\}$ the amount of infeasibility

Procedure:

- If problem is infeasible, each of the variables is given its alternative value of 1 in turn starting with variable with the smallest coef in O.F.
- A variable will only contribute to an improvement in feasibility if that variable has a negative coef in a constraint.
- Set up solution as a tree similar to Branch and Bound and check all nodes until they have a valid solution or fathomed

10.

First Iteration:

Choose B:

$N = \{A, C, D, E\}$ variable value 0

$T = \{B\}$ variable value 1

$Z = 2$ O.F. value

$S = \{0, -2, 0\}$ amount of infeasib

Second Iteration

Choose D

$N = (A, B, C, E)$ variable value 0

$T = \{D\}$ variable value 1

$Z = 2$ O.F. value

$S = \{-1, 0, 0\}$ amount infeasibility

Third iteration

Choose A.

$N = (B, C, D, E)$ variable values 0

$T = (A)$ variable value 1

$Z = 3$ O.F. value

$S = \{0, 0, -2\}$ amount infeasibility

Fourth iteration:

Choose E:

$N = (A, B, C, D)$ variable value $= 0$

$T = (E)$ value $= 1$

$Z = 3$ O.F value

$S = \{0, 0, 0\}$ infeasibility.

Optimum solution here

Solution

- Remember to replace the variables that were changed in original problem from x_i to $(1 - x_i)$ again

  $A = B = C = D = 0$ $E = 1$ at optim

- Replace A, B, E by $(1 - var)$

  $A = B = 1$ $C = D = E = 0$

- $\Rightarrow$ $Z = 2$.

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Special 0-1 Constraints:
When x_i and x_j represent binary variables denoting whether projects i and j have been completed, the following special constraints may be formulated:

- At most k out of n projects will be completed
$$\sum_j x_j \le k$$
- Project j is conditional on project i:
$$x_j - x_i \le 0$$
- Project i is corequisite for project j:
$$x_j - x_i = 0$$
- Projects i and j are mutually exclusive
$$x_i + x_j \le 1$$

Example
Metro Microwave planning to expand. Seven new product lines available. Formulate a {0,1} linear program.

Product line	Initial Invest	Floor Space	rate of return
1 TV/VCR	6000	125	8.1%
2 Color TV	12 000	150	9.0
3 Projection TV	20 000	200	11.0
4 VCRs	14 000	40	10.2
5 DVD Player	15 000	40	10.5
6 Video Game	2000	20	14.1
7 Home Comp	32000	100	13.2

- Decided should not stock projection TV unless they stock either TV/VCR or color TVs
- Will not stock both VCRs and DVD players
- Stock video game if stock color TV
- Introduce at most 3 new product lines
- 45000 to invest, 420 floor space, maximise expected return

12.

$x_j = 1$ if product line introduced 0 otherwise

$$\text{Max } Z = 0.081(6000)x_1 + 0.09(12000)x_2 + 0.11(20000)x_3 + 0.102(14000)x_4 + 0.105(15000)x_5 + 0.141(2000)x_6 + 0.132(32000)x_7$$

Constraints:

1. Money: $6x_1 + 12x_2 + 20x_3 + 14x_4 + 15x_5 + 2x_6 + 32x_7 \leq 45$.
2. Space: $125x_1 + 150x_2 + 200x_3 + 40x_4 + 40x_5 + 20x_6 + 100x_7 \leq 420$
3. Stock projection TVs only if stock TV/VCR or color TVs

   $x_1 + x_2 \geq x_3 \Rightarrow$ or $x_1 + x_2 - x_3 \geq 0$
4. Do not stock both VCR and DVD

   $x_4 + x_5 \leq 1$
5. Stock video game if stock color TVs

   $x_2 - x_6 \geq 0$
6. Introduce at least 3 new lines

   $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \geq 3$
7. Variables are 0 or 1:

   $x_j = 0$ or 1 for $j = 1, 2, \ldots 7$

Solution:

Introduce: TV/VCRs, projection TVs, and DVD players

Do not Introduce: color TVs, VCRs, Video games, home computers

Total expected return = £4261