20/11/13 Maths 6 Surface Integrals

Integrate $f(x,y,z)$ over a surface σ, parameterised by $\vec{r}(u,v)$

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

The surface integral is:

$$\iint_{\sigma} f(x,y,z)\,dS = \iint_{R} f(x(u,v), y(u,v), z(u,v)) \times \left\| \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} \right\| dA$$

If $f(x,y,z) = 1$, this gives surface area

$$S = \iint_{R} \left\| \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} \right\| dA$$

If $z = g(x,y)$, then $\iint f(x,y,z)\,dS$

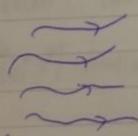
$$= \iint_{R} f(x,y,g(x,y)) \sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2}\, dA$$

Flux Integrals:

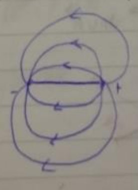
Consider a fluid flowing in 3d.

Velocity of fluid, $\vec{F}(\vec{r})$ ← vector field

This represents a flow of the fluid ⇒ $\vec{F}(\vec{r})$ is the flow field



A fluid flowing.



Electric field for a dipole

Oriented Surfaces:

Consider a ball. It has an inside and an outside. If we start inside we can never get outside by following the surface

For a Mobius strip there is only one side to the surface

A two sided surface is orientable

A one-sided surface is non-orientable.

For a 2-sided surface, $\vec{n}$ and ~~an~~ $-\vec{n}$ point away from the opposite surfaces.

If σ is smooth and orientable, we can choose ~~$\vec{n}$ ... to vary~~ the direction of $\vec{n} = \vec{n}(x,y,z)$ to vary continuously over σ.

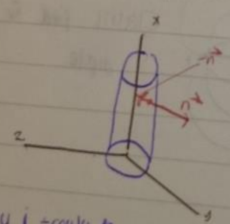
The unit vector form an orientation of the surface.

Orientation of a Smooth Parametric Surface

Surface σ, parameterised by $\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$ has principle unit vector $\vec{n} = \dfrac{\frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv}}{\left\| \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} \right\|}$

This defines the positive orientation of σ. $-\vec{n}$ defines the negative orientation.

Example: Find the orientation of the cylinder
$\vec{r} = \cos u\,\vec{i} + v\,\vec{j} - \sin u\,\vec{k}$



$\frac{d\vec{r}}{du} = -\sin u\, i - \cos u\, k$

$\frac{d\vec{r}}{dv} = \vec{j}$

$$\Rightarrow \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin u & 0 & -\cos u \\ 0 & 1 & 0 \end{vmatrix} = \cos u\,\vec{i} - \sin u\,\vec{k}$$

$$\Rightarrow \left\| \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} \right\| = \sqrt{(\cos u)^2 + (-\sin u)^2} = 1$$

$$\vec{n} = \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} = \cos u\,\vec{i} - \sin u\,\vec{k}$$

maths

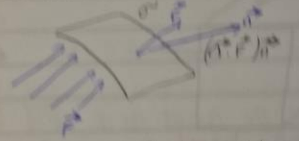
Flux

A liquid (generally) is incompressible and a gas is compressible

We are only interested in incompressible fluid in a steady state this means that the velocity at a point is fixed.

We want to know the net volume Q of fluid that passes through a surface S per unit of time.
(i.e. +ve flow −ve flow)

Consider an incompressible fluid with velocity
$\vec{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\vec{k}$
flowing through S with unit vector $\vec{n}$ giving a positive orientation



$\vec{n}\cdot\vec{F}$ is the magnitude of flow parallel to $\vec{n}$, which in vector form is $(\vec{n}\cdot\vec{F})\vec{n}$

$(\vec{n}\cdot\vec{F})$ flows out of the surface

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$$\iint_{\sigma} \vec{F}\cdot\vec{n}\,dS = \iiint_{G} \operatorname{div}\vec{F}\,dV$$

Divergence gives the amount of fluid flow per unit time

Flux from the Divergence theorem:

Flux through a surface $\Phi = \iint_{\sigma} \vec{F}\cdot\vec{n}\,dS = \iiint_{G} \operatorname{div}\vec{F}\,dV$ by the divergence theorem

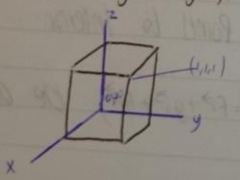
Example: Find the outward flux of $\vec{F}(x,y,z) = z\vec{k}$ accross the sphere $x^2+y^2+z^2=a^2$

Solution: $\operatorname{div}\vec{F} = \frac{d(z)}{dz} = 1$

$\Phi = \iint_{\sigma} \vec{F}\cdot\vec{n} = \iiint (1)\,dV$

$= \iiint_{G} dV = \frac{4}{3}\pi a^3$ Since G is a ball

Example: Find the outward flux of $\vec{F}(x,y,z) = (2x^3 + xy - 2zy)\vec{i} + (xy^2 - x^2z)\vec{j} + (4xz - x^3y)\vec{k}$ across the unit cube



$0 \le x, y, z \le 1$

Solution: $\operatorname{div}\vec{F} = \frac{d(2x^3-xy-2zy)}{dx} + \frac{d(xy^2-x^2z)}{dy} + \frac{d(4xz-x^3y)}{dz}$

$= (6x^2 - y) + (2xy) + (4x)$

$= 6x^2 + 4x + 2xy + y$

$\Phi = \iint_{\sigma} \vec{F}\cdot\vec{n}\,dS = \iiint \operatorname{div}\vec{F}\,dV = \iiint_{G} (6x^2+4x+2xy+y)\,dz\,dy\,dx$

$= \int_0^1\int_0^1\int_0^1 (6x^2+4x+2xy+y)\,dz\,dy\,dx$

$= \int_0^1\int_0^1 \left[(6x^2+4x+2xy+y)(z)\right]_{z=0}^{1}\,dy\,dx$

$$\iint_{\sigma} \vec{F}\cdot\vec{n}\,dS = \iiint_{G} \operatorname{div}\vec{F}\,dV$$

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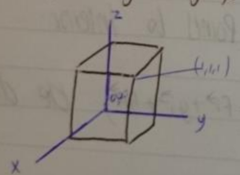
$\Phi = \iint_{\sigma} \vec{F}\cdot\vec{n} = \iiint (1)\,dV$

$= \iiint_{G} dV = \frac{4}{3}\pi a^3$ since G is a ball

Example: Find the outward flux of ~~$\vec{F}$~~ $\vec{F}(x,y,z) = (2x^3+xy-2zy)\vec{i} + (xy^2-x^2z)\vec{j} + (4xz-x^3y)\vec{k}$ across the unit cube



$0 \le x,y,z \le 1$

Solution: $\operatorname{div}\vec{F} = \frac{d(2x^3-xy-2zy)}{dx} + \frac{d(xy^2-x^2z)}{dy} + \frac{d(4xz-x^3y)}{dz}$

$= (6x^2-y) + (2xy) + (4x)$

$= 6x^2 + 4x + 2xy + y$

$\Phi = \iint_{\sigma} \vec{F}\,\vec{n}\,dS = \iiint_{G} ((6x^2+4x+2xy+y))\,dz\,dy\,dx$ $\quad = \iiint \operatorname{div}\vec{F}\,dV$

$= \int_0^1\int_0^1\int_0^1 (6x^2+4x+2xy+y)\,dz\,dy\,dx$

$= \int_0^1\int_0^1 \left[(6x^2+4x+2xy+y)(z)\right]_{z=0}^{1}\,dy\,dx$

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$$= \int_0^1 \int_0^1 (6x^2 + 4x + 2xy + y)\, dy\, dx$$

$$= \int_0^1 \left[6x^2 y + 4xy + xy^2 + \frac{y^2}{2} \right]_{y=0}^{1} dx$$

$$= \int_0^1 6x^2 + 4x + x + \tfrac{1}{2}\, dx$$

$$= \left[2x^3 + 2x^2 + \frac{x^2}{2} + \frac{x}{2} \right]_{x=0}^{1}$$

$$2 + 2 + \tfrac{1}{2} + \tfrac{1}{2} = 5$$

Example: Find the flux of $\vec{F}(x,y,z) = \frac{x^3}{3}\vec{i} + \frac{y^3}{3}\vec{j} + \frac{z^3}{3}\vec{k}$ across the region enclosed by the hemisphere $z = \sqrt{a^2 + x^2 - y^2}$ and plane $z = 0$

Solution: $\operatorname{div} \vec{F} = \frac{d}{dx}\left(\frac{x^3}{3}\right) + \frac{d}{dy}\left(\frac{y^3}{3}\right) + \frac{d}{dz}\left(\frac{z^3}{3}\right)$

$$= x^2 + y^2 + z^2$$

Spherical coords with $0 \le \rho \le a$, $0 \le \theta \le 2\pi$ $0 \le \phi \le \pi/2$

$$\Phi = \iint_{\sigma} \vec{F} \cdot \vec{n}\, d\sigma = \iiint_G (x^2 + y^2 + z^2)\, dV$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \underbrace{(\rho^2)}_{\operatorname{div}\vec{F}} \underbrace{(\rho^2 \sin\phi)}_{\text{Jacobian}}\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \frac{\rho^5}{5} \Big|_{\rho=0}^{a} \sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \frac{a^5}{5} \sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \frac{a^5}{5} \left[-\cos\phi \right]_{\phi=0}^{\pi/2}$$

$$= \int_0^{2\pi} \frac{a^5}{5} \left[-\cos\tfrac{\pi}{2} - \underbrace{(-\cos(0))}_{=+1} \right] d\theta$$

$$= \int_0^{2\pi} \frac{a^5}{5}\, d\theta = \frac{2\pi a^5}{5}$$

Note $\int_0^{\pi/2} \sin\phi\, d\phi = 1$

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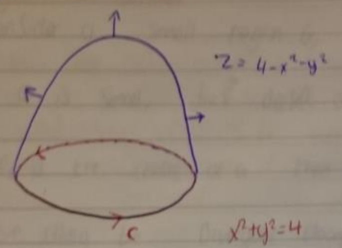


Solution: The curve is a circle parameterised in the anti-clockwise direction $x = 2\cos t$ $y = 2\sin t$ $0 \le t \le 2\pi$ $z = 0$.

$\Rightarrow \oint_C \vec{F} \, d\vec{r} = \oint_C 2z\,dx + 3x\,dy + 5y\,dz$

$dx = -2\sin t\,dt$ $dy = 2\cos t\,dt$ $dz = 0$

$\Rightarrow \oint_C \vec{F} \cdot d\vec{r}$

$= \int_0^{2\pi} [2(0)(-2\sin t) + 3(2\cos t)(2\cos t) + 5(2\sin t)(0)]\,dt$

$= 12\int_0^{2\pi} \cos^2 t\,dt = 6\int_0^{2\pi} [1 + \cos 2t]\,dt$

$= 6\left[t + \frac{1}{2}\sin 2t\right]_{t=0}^{2\pi}$

$= 6\left[2\pi + \cancel{\frac{1}{2}\sin 4\pi}^{\,0} - 0 - \cancel{\frac{1}{2}\sin 0}^{\,0}\right]$

$\cos^2 t = \frac{1}{2}[1 + \cos 2t]$

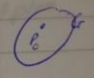
$= 12\pi$

27/11/13 Divergence theorem Continued. Week 10.

Consider a small region G, with volume $vol(G)$

If G is small, $\operatorname{div}\vec{F}$ doesn't change much on G

If P_0 is the centre of G then $\operatorname{div}\vec{F} \approx \operatorname{div}\vec{F}(P_0)$ on G



Therefore, using the Divergence theorem, the flux is

$$\Phi(G) = \iint \vec{F}\cdot\vec{n}\, dS = \iiint \operatorname{div}\vec{F}\, dV$$

$$\approx \operatorname{div}\vec{F}(P_0) \iiint_G dV = \operatorname{div}\vec{F}(P_0)\, vol(G)$$

$$\Rightarrow \operatorname{div}\vec{F}(P_0) \approx \frac{\Phi(G)}{vol(G)}$$

Outward flux density of $\vec{F}$ around G.

Take the limit $vol(G) \to 0$

$$\Rightarrow \operatorname{div}\vec{F}(P_0) = \lim_{vol(G)\to 0} \frac{1}{vol(G)} \iint_{S(G)} \vec{F}\cdot\vec{n}\, dS$$

This is the outward flux density of $\vec{F}$ at P_0

For an incompressible fluid there should be no net flow unless we can add and remove fluid. Anywhere that $\operatorname{div}\vec{F}(P_0) > 0$ is called a Source and anywhere that $\operatorname{div}\vec{F}(P_0) < 0$ is called a Sink

Fluid enters the flow at source and leaves at sink

For an incompressible fluid with no sources or sinks has $\operatorname{div}\vec{F}(P_0) = 0$ for all P.

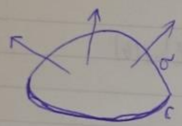
This is the continuity equation for incompressible fluid

2 Maths

STOKES THEORM

If a curve C bounds a surface σ, then with your head in the direction of the orientation of σ, walking along C so that σ is on your left gives C a <u>positive orientation</u>

If σ is on your right, this gives a <u>negative orientation</u>



$\rightarrow$ positive orientation of C

Stokes Theorem: Let σ be a piecewise smooth oriented surface that is bounded by a simple, closed, piecewise smooth curve C with positive orientation.

If the vector field $\vec{F}(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k}$ has continuous components f, g, h with continuous first partial derivatives on some open set containing σ and if $\vec{T}$ is the unit tangent vector on C, then

$$\oint_C \vec{F}\cdot\vec{T}\, ds = \iint_\sigma (\text{curl}(\vec{F})\cdot\vec{n}\, dS$$

$$\vec{T} = \frac{d\vec{r}(s)}{ds} \quad (\text{Claim!})$$

$$\oint \vec{F}\cdot d\vec{r} = \iint_\sigma (\text{curl}\,\vec{F})\cdot\vec{n}\, dS \qquad \text{since } \vec{T}\, ds = d\vec{r}$$

Example: Prove that $\vec{F}(x,y,z) = 2z\,\vec{i} + 3x\,\vec{j} + 5y\,\vec{k}$ defined on σ where σ is the portion of the paraboloid $z = 4 - x^2 - y^2$ for $z \geq 0$ with upward orientation satisfies Stokes Theorem if σ is bounded by the positively oriented circle $x^2+y^2=4$ in the xy-plane

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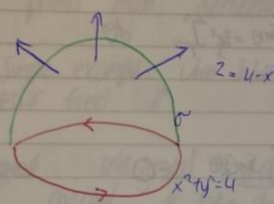
Example: Prove that $\vec{F}(x,y,z) = 2x\vec{i} + 3y\vec{j} + 5y\vec{k}$ defined on the surface which is the portion of the paraboloid $z = 4 - x^2 - y^2$ for $z \geq 0$ with upward orientation satisfies Stoke's theorem if σ is bounded by the positively oriented circle $x^2+y^2=4$ in the xy plane



Solution

Last time: $\oint \vec{F}\cdot d\vec{r} = 12\pi$

Recall Stokes theorem: $\oint_C \vec{F}\cdot d\vec{r} = \iint_\sigma \text{curl}(\vec{F})\cdot\vec{n}\, dS$

$$\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} \quad \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 2z & 3x & 15y \end{vmatrix}$$

$$= 5\vec{i} + 2\vec{j} + 3\vec{k}$$

Since σ is represented by $z = f(x,y)$ and oriented upwards, we use $\vec{n} = -\frac{dz}{dx}\vec{i} - \frac{dz}{dy}\vec{j} + \vec{k}$ (z-component along positive z-axis)

$$\Rightarrow \iint_\sigma (\text{curl}\,\vec{F})\cdot\vec{n}\, dS = \iint_\sigma (5\vec{i}+2\vec{j}+3\vec{k})\cdot\left(-\frac{dz}{dx}\vec{i} - \frac{dz}{dy}\vec{j} + \vec{k}\right)$$

$$= \iint_R (5\vec{i}+2\vec{j}+3\vec{k})\cdot(2x\vec{i}+2y\vec{j}+\vec{k})$$

$$= \iint_R (10x + 4y + 3)\, dA$$

R is a disk of radius 2 so $0 \le \rho \le 2$ and $0 \le \theta \le 2\pi$

$$\Rightarrow \int_0^{2\pi}\int_0^2 (10\underbrace{\rho\cos\theta}_{x} - 4\underbrace{\rho\sin\theta}_{y} + \underbrace{3}_{z})\,\rho\,d\rho\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 (10\rho^2\cos\theta - 4\rho^2\sin\theta + 3\rho)\,d\rho\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{10\rho^3}{3}\cos\theta + \frac{4\rho^3}{3}\sin\theta + \frac{3\rho^2}{3}\right]_{\rho=0}^{2} d\theta$$

$$= \int_0^{2\pi} \frac{80}{3}\cos\theta + \frac{32}{2}\sin\theta + 6$$

$$= \int_0^{2\pi}\cos\theta\,d\theta = \int_0^{2\pi}\sin\theta\,d\theta = 0$$

$$= \left[\frac{80}{3}\sin\theta - \frac{32}{2}\cos\theta + 6\theta\right]_{\theta=0}^{2\pi} = \left[\frac{80}{3}(0) - \frac{32}{3}(1) + 6(2\pi)\right] - \left[\frac{80}{3}(0) - \frac{32}{3}(1) + 6(0)\right]$$

$= 6(2\pi) = 12\pi \Rightarrow$ Same as before $\Rightarrow$ Stoke's Theorem Satisfied

Work can be calculated using Stoke's Theorem

$$W = \oint \vec{F}\cdot d\vec{r} = \iint_S (\mathrm{curl}\,\vec{F})\cdot\vec{n}\,dS$$

LAPL

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LAPLACE TRANSFORMS

Used to solve linear ordinary differential equations (ODE)

They look like:

$$y' + 2y = 0 \quad \text{or} \quad y'' - 3y' + 4y = 0$$

Where the primes mean derivatives

We met a simple example when solving for the potential function ϕ of a vector field $\vec{F}$.

We ~~interrogated~~ integrated $\frac{d(c(y))}{dy} = 2y$ to find $c(y)$ this is a simple method of solving an ODE.

Most ODE's are harder. We introduce Laplace transform to help us

Define Laplace Transform

Laplace Transform is a type of integral transform where we integrate the original function $f(t)$ with ~~the~~ some kernel $k(s,t)$ to produce a new function we will call $F(s)$

$F(s) = \int_0^\infty k(s,t) f(t)\,dt$ is an integral transform

If $f(t)$ is defined for all $t \geq 0$, then the Laplace Transform of f is $F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$.

The kernel is e^{-st}. We also have the inverse transform $f(t) = \mathcal{L}^{-1}(F)$

We will always use f and t for the original function and F and s for Laplace transform

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Function	Transform
$f(t)$	$F(s)$
1	$1/s$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{at} f(t)$	$F(s-a)$
$\delta(t-a)$	e^{-as}
$u(t-a)$	e^{-as}/s
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
$t\, f(t)$	$-\frac{d}{ds} F(s)$
$f(at)$	$\frac{1}{a} F(s/a)$
$u(t-a) f(t-a)$	$e^{-as} F(s)$