2012 Maths 4 Exam Part

Q1 a Reflect x-axis, ~~x~~ y value change

$$(2,-1)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = (2,1)$$

b onto y axis, x=0 $$(2,-1)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0\\ -1\end{pmatrix}$$

c. Rotation $$(2,-1)\begin{pmatrix}\cos\frac{\pi}{4} & -\sin(\frac{\pi}{4})\\ \sin\frac{\pi}{4} & \cos(\frac{\pi}{4})\end{pmatrix}$$

$$(2,-1)\begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix} \quad \left(\sqrt{2}-\frac{\sqrt{2}}{2},\ -\sqrt{2}-\frac{\sqrt{2}}{2}\right)$$

$$\left(\frac{\sqrt{2}}{2},\ -\frac{3\sqrt{2}}{2}\right)$$

3a $$\begin{pmatrix}0 & 4 & -2\\ 0 & -2 & 1\end{pmatrix} \div 4 \quad \begin{pmatrix}0 & 1 & -\frac{1}{2}\\ 0 & -2 & 1\end{pmatrix} \; R_2+2R_1 \quad \begin{pmatrix}0 & 1 & -\frac{1}{2}\\ 0 & 0 & 0\end{pmatrix}$$

Dim Row(A) = Dim col(A) = 1

Dim null Space = 3-2 = 2

Basis of row Space = Span$\{(0,1,-\frac{1}{2})\}$

Basis of column Space = $\begin{pmatrix}4\\ -2\end{pmatrix}$

Null Space $A\vec{x} = \vec{0}$ $$\begin{pmatrix}0 & 1 & -\frac{1}{2}\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{matrix}0\\0\\0\end{matrix}$$

$y - \frac{1}{2}z = 0$

$y = \frac{1}{2}z$

$z = 2y$ x = free

$x_1 = s_1$ $y = s_2$

$$s_1\begin{pmatrix}1\\0\\0\end{pmatrix} + s_2\begin{pmatrix}0\\1\\2\end{pmatrix}$$

Null (A) = Span$\{(1,0,0),\ (0,1,2)\}$

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3b. $\begin{pmatrix} 2 & 1 \\ 3 & 0 \\ -2 & 1 \end{pmatrix}$ $r_1 + r_3 =$ $\begin{pmatrix} 0 & 2 \\ 3 & 0 \\ -2 & 1 \end{pmatrix}$ $r_2 \div 3$ $r_1 \div 2$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 1 \end{pmatrix}$ $r_3 - r_2 + 2r_1$ $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

Dim row(A) = Dim colum(A) = 2.
Dim null = 2-2 = 0.

Row basis = $\{(1, \tfrac{1}{2}), (0, 1)\}$
colum basis = span $\left\{ \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\begin{pmatrix} 2 & 1 \\ 3 & 0 \\ -2 & 1 \end{pmatrix}$ $R_1 \div 2$ $= \begin{pmatrix} 1 & \tfrac{1}{2} \\ 3 & 0 \\ -2 & 1 \end{pmatrix}$ $R_2 - 3r_1$ $\begin{pmatrix} 1 & \tfrac{1}{2} \\ 0 & -\tfrac{3}{2} \\ -2 & 1 \end{pmatrix}$ $R_2 / -\tfrac{3}{2}$ $\begin{pmatrix} 1 & \tfrac{1}{2} \\ 0 & 1 \\ -2 & 1 \end{pmatrix}$

$R_3 - 2r_1$ $= \begin{pmatrix} 1 & \tfrac{1}{2} \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

Null Spa $\begin{pmatrix} 1 & \tfrac{1}{2} \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$ $x + \tfrac{1}{2}y = 0$ $\quad$ $x = -\tfrac{1}{2}$ $\quad$ $x = 0$
$y = 0$ $\quad$ $y = 1$ $\quad$ $y = 0$

Null Space Span $\{ \cancel{(-\tfrac{1}{2}, 1)} \}$ $0, 0$ $\quad$ $-\tfrac{1}{2}, 1$

Q 2A $\begin{pmatrix} 3 & -6, & 0 \\ -2 & 4, & 0 \end{pmatrix}$ $R_1 \div 3$ $\begin{pmatrix} 1, & -2, & 0 \\ -2 & 4, & 0 \end{pmatrix}$ $R_2 + 2r_1$ $\begin{pmatrix} 1, & -2, & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Rank = 1 $\qquad$ Rank = # leading 1s
Nullity = 3 - 1 = 2 $\qquad$ nullity # colum - rank

b $\begin{pmatrix} 3 & -6 & 0 \\ -2 & 4 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & -1 \\ -2 & 4 & 0 \\ 3 & -6 & 0 \end{pmatrix}$ $r_2 + 2r_1$ $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 4 & -2 \\ 3 & -6 & 0 \end{pmatrix}$ $r_2 \div 4$ $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\tfrac{1}{2} \\ 3 & -6 & 0 \end{pmatrix}$ $r_3 - 3r_1$ $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\tfrac{1}{2} \\ 0 & -6 & 3 \end{pmatrix}$

$r_3 + 6r_1$ $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\tfrac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$ $\quad$ rank = 2
nullity = 3 - 2 = 1.

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Q4 $u_1 = (0,-1,0)$ $u_2 = (0,1,1)$ $u_3 = (1,2,0)$

$v_1 = u_1 = (0,-1,0)$

$v_2 = u_2 - \frac{v_1 \cdot u_2}{v_1 \cdot v_1} v_1$ $(0,1,1) - \frac{(0,-1,0)(0,1,1)}{0+1+0}(0,-1,0)$

$(0,1,1) + (0,-1,0)$

$(0,0,1)$

$v_2 = (0,0,1)$

$v_3 = u_3 - \frac{v_2 \cdot u_3}{v_2 \cdot v_2} v_2 - \frac{v_1 \cdot u_3}{v_1 \cdot v_1} v_1$

$(1,2,0) - \frac{(0,-1,0)\cdot(1,2,0)}{1}(0,-1,0) - \frac{(0,0,1)(1,2,0)}{(0,0,1)}(0,0,1)$

0

$(1,2,0) + 2(0,-1,0)$

$(1,2,0) + (0,-2,0)$

$v_3 = (1,0,0)$

5. $A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix}$ det $\left(\begin{pmatrix} 3 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right) = 0$

det $\begin{bmatrix} 3-\lambda & -1 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & -1 & -2-\lambda \end{bmatrix} = 0$ $(3-\lambda)\left[(1-\lambda)(-2-\lambda) - (2)(-1)\right] - (-1)\left[\cancel{0}\right] + 1\left[\cancel{0}\right]$

$3-\lambda\ [-\cancel{2} - \lambda + 2\lambda + \lambda^2 + \cancel{2}]$

$(3-\lambda)(\lambda^2+\lambda) = 0$

$\lambda = -3$ or $\lambda = 0$ or $\lambda = -1$

Eigen value -3, 0, -1

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$\lambda_1 = 0$ $\begin{pmatrix} 3 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $r_3 + r_2$ $\begin{pmatrix} 3 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $r_1 + r_2$

$\begin{pmatrix} 3 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $3x + 3z = 0$ $\Rightarrow$ $3x - 3y = 0$

$y + z = 0$ $z = -y$ $x = y$

$z = -y$

$x = -z$

$y = -2\cancel{z}$ $z = -1$

$\bar{v}_1 = (1, 2, -1)$

$\lambda = -1$ $\begin{pmatrix} 3 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$\begin{pmatrix} 4 & -1 & 1 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $r_2 + 2r_3$ $\begin{pmatrix} 4 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}$ $r_1 r r_3$ $\begin{pmatrix} 4 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$

$4x - 2y = 0$

$-y - z = 0 \Rightarrow -y = z \Rightarrow 4x = -2z$ $x = \frac{-1}{2}z$

$-y = z$

$\bar{v}_2$ $(1, 2, -2)$

$\lambda = -3$ $\begin{pmatrix} 6 & -1 & 1 \\ 0 & 4 & 2 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $r_1 + r_3$ $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & -1 & 1 \end{pmatrix}$ $6x = 0$ $x = 0$

$4y + 2z = 0$ $x = 0.$

$-y + z = 0$ $z = y$

$\begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{0}$

$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$ $-y + z = 0$ $-6z = 0$ $z = 0$ $-y = 0$

$\bar{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

5.

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Q5h $D = P^{-1}AP$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ -1 & -2 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 \end{pmatrix}$$

The matrix $P^{-1}AP$ will be diagonal and have eigenvalues $\lambda_1 \lambda_2 \lambda_n$ corresponding to the eigenvectors $v_1, v_2 \dots v_n$ as its diagonal entries

6a $f(x) = \begin{cases} 1+x & \text{if } -\pi \leq x < 0 \\ 0 & 0 \leq x \leq \pi \end{cases}$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_0^{\pi} 0\,dx \int_{-\pi}^{0} 1+x\,dx = x + \frac{x^2}{2}\Big|_{-\pi}^{0}$$

$$= \frac{1}{2\pi}\left(\pi + \frac{\pi^2}{2}\right) = \cancel{\frac{1}{2} + \frac{\pi}{2} = \frac{1+\pi}{2}}$$

$$\frac{1}{2} \cdot -\frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} (1+x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx$$

$$\frac{1}{\pi}\left[\frac{\sin(nx)}{n}\Big|_{-\pi}^{0} + \frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[(0 + 0 + \frac{1}{n^2}) - (0 + 0 + \frac{(-1)^n}{n^2}\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right) = \frac{1-(-1)^n}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} (1+x)\sin(nx)dx$$

$$\frac{1}{\pi}\int_{-\pi}^{0} \sin nx\,dx + \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos nx}{n} - \frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_{-\pi}^{0}$$

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$$= \frac{1}{\pi}\left[\left(-\frac{1}{n} - 0 + 0\right) - \left(-\frac{(-1)^n}{n} - \left(-\frac{\cancel{\pi}(-1)^n}{2} + 0\right)\right)\right]$$

$$= \frac{1}{\pi}\left(-\frac{1}{n} + \frac{(-1)^n}{n} - \frac{\pi(-1)^n}{n}\right)$$

$$= \frac{-1 + (-1)^n}{\pi n} - \frac{(-1)^n}{n}$$

B $$f(x) = \frac{1}{2} - \frac{\pi}{4} + \sum\left[\frac{1-(-1)^n}{\pi n^2}\cos(nx) + \left(\frac{-1+(-1)^n}{\pi n} - \frac{(-1)^n}{n}\right)\sin nx\right]$$