Maths Semester 1 JF 2012

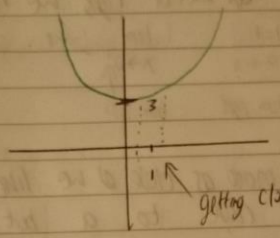
CHAPTER 1

Limits and Continuity

4/10/12 CHAPTER 2 LIMITS AND CONTINUITY

An intuitive approach: Describe how functions behave as the independent variable approaches a given value

example: $F(x) = 2x^2 - 2x + 3$



getting closer to 1 as x gets close to 3

$\lim_{x \to 1} 2x^2 - 2x + 3$

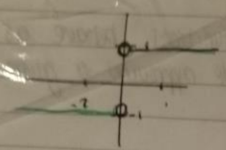
$f(0.9) = 2.82$	$f(1.1) = 3.22$
$f(0.95) = 2.905$	$f(1.05) = 3.105$
$f(0.999) = 2.9982$	$f(1.01)$ 3.002002

Def (INFORMAL):
If the value of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), we write $\lim_{x \to a} F(x) = L$ or $F(x) \to L$ as $x \to a$

Note: $F(x)$ need not be defined at a for the limit to exist there

One-sided Limits:
Some functions exhibit different limiting behaviour on the two sides of an x value

example: $f(x)$ $\frac{|x|}{x} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$ not defined at $x = 0$



As x approaches 0 from the left ~~or then~~ or right we have different limiting behaviour.

$\lim_{x \to 0^+} f(x) = 1$ $\lim_{x \to 0^-} = -1$

approach from right   from left

Def^n (informal)
one sided limit:

If the value of $f(x)$ can be made as close as we like to L by taking x-values sufficiently close to a but greater than (less than) a, we write $\lim_{x \to a^+} f(x) = L$

(less) for approach from left

Theorem: The two sided limit of $f(x)$ exists at $x=a$ if and only if, both one-sided limits exist at a and are equal i.e. $\lim_{x \to a} f(x) = L$ iff (if and only if) $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$
right   left

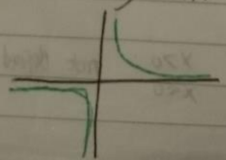
$x \in \mathbb{R} : x \neq a$

Infinite Limits and Vertical Asymptotes:

There are functions that increase or decrease without bound as x approaches some value

eg $f(x) = 1/x$ is unbounded as $x \to 0$

As x approaches 0 from the right $f(x)$ increases without bound, as x approaches ~~from the left~~ 0 from the left $f(x)$ is decreasing without bound



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Limits

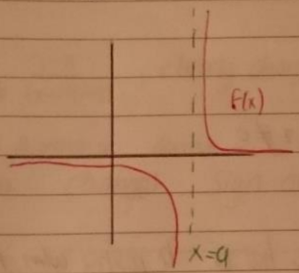
example: Consider the limits of $f(x) = \frac{1}{x-a}$ and $g(x) = \frac{1}{(x-a)^2}$ $x \to a$

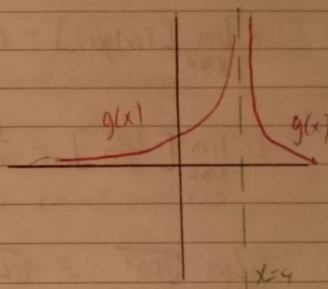
$\lim\limits_{x \to a^+ \text{ (from right)}} f(x) = +\infty$ (function always positive) $\lim\limits_{x \to a^-} f(x) = -\infty$

$\lim\limits_{x \to a} f(x)$ does not exist

$\lim\limits_{x \to a^+} g(x) = +\infty$ (always positive) $\lim\limits_{x \to a^-} g(x) = +\infty$ (always positive) square

$\lim\limits_{x \to a} g(x) = +\infty$





$x = a$ is called the vertical asymptote

Computing: Limits and limit laws:

Basic Limits: $\lim\limits_{x \to a} k = k \quad k \in \mathbb{R}$

$\lim\limits_{x \to a} x = a$

$\lim\limits_{x \to 0^+} \frac{1}{x} = +\infty$

$\lim\limits_{x \to 0^-} \frac{1}{x} = -\infty$

Limit Laws: $\lim_{x \to a} f(x) = L_1$

$\lim_{x \to a} g(x) = L_2$

1. $\lim_{x \to a} [f(x)+g(x)] = L_1 + L_2$

2. $\lim_{x \to a} [f(x) - g(x)] = L_1 - L_2$

3. $\lim_{x \to a} [f(x)g(x)] = L_1 L_2$

4. $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{L_1}{L_2}, \quad L_2 \neq 0$

5. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L_1}$, L_1 has to be positive when n is even

Result (iii) implies $\lim_{x \to a} [\underbrace{f(x).f(x)...f(x)}_{n \text{ product}}] = \left(\lim_{x \to a} f(x)\right)^n$

eg $\lim_{x \to a} x^n = \left(\lim_{x \to a} x\right)^n = a^n$

example Find $\lim_{x \to -1} 2x^3 - 4x + 9 = 2\lim_{x \to -1} x^3 - 4\lim_{x \to -1} x + \lim_{x \to -1} 9$

$2\left(\lim_{x \to -1} x\right)^3 - 4\lim_{x \to -1} x + \lim_{x \to -1} 9$

$2(-1)^3 - 4(-1) + 9$

$= 11$

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8/10/17

Limits of Polynomials and Rational Functions:

$p(x) = c_n x^n + c_{n-1} x^{n-1} + \dots c$

$$\Rightarrow \lim_{x \to a} p(x) = c_n a^n + c_{n-1} a^{n-1} + \dots + c_0 = p(a)$$

Let $p(x)$, $q(x)$ be polynomials

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}, \quad q \neq 0$$

example: $\lim_{x \to 2} \frac{2x^2+7}{x+2} = \frac{2(2)^2+7}{2+2} = \frac{15}{4}$

$\lim_{x \to 4} \frac{2-x}{(x-4)(x+3)}$, clearly diverges (increase/decrease without bound)

Does it increase or decrease without bound as $x \to 4$

The function changes sign at $x=4$, $x=-3$ and $x=2$

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−3 0 2 4

Take $x=0$ $f(0) < 0$

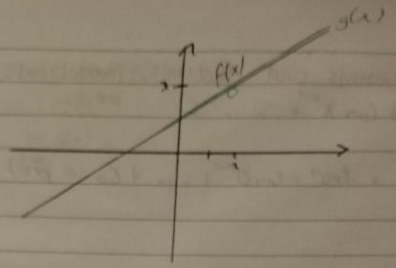
$\lim_{x \to 4^-} = +\infty$ $\lim_{x \to 4^+} = -\infty$ Two sided limit doesn't exist

example: $\lim_{x \to 2} \frac{x^2-x-2}{x-2}$ $\frac{2^2-2-2}{2-2} = \frac{0}{0}$ undefined - has to be a common factor

If $p(a) = 0 = q(a)$ then there is a common factor, and we factorise

$$\lim_{x \to 2} \frac{x^2-x-2}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+1)}{(x-2)} \quad \lim_{x \to 2} x+1 = 3$$

NB $f(x) = \frac{x^2-x-2}{x-2}$ and $g(x) = x+1$ are everywhere equal except at $x=2$



Limits Involving Radicals: $a + b\sqrt{c}$

example: Find $\lim\limits_{x\to 4} \frac{4x - x^2}{2-\sqrt{x}} = \frac{4(4)-4^2}{2-\sqrt{4}} = \frac{0}{0} \Rightarrow$ undefined $\Rightarrow$ common

multiply above and below by radical conjugate

$$\lim_{x\to 4} \frac{(4x-x^2)}{(2-\sqrt{x})} \cdot \frac{2+\sqrt{x}}{2+\sqrt{x}} = \lim_{x\to 4} \frac{(4x-x^2)(2+\sqrt{x})}{(4-x)} = \lim_{x\to 4} \frac{x\cancel{(4-x)}(2+\sqrt{x})}{\cancel{(4-x)}} \quad \lim_{x\to 4} x(2+\sqrt{x}) = 16$$

* $(a+\sqrt{b})(a-\sqrt{b}) = a^2 - b$ (difference of two squares)

Limits of Piece-wise Defined Functions:

example: $f(x) = \begin{cases} \frac{1}{(x+4)} & x < -4 \\ 2x^2+1 & -4 \le x \le 2 \\ \sqrt{20x^2+1} & x > 2 \end{cases}$

a. $\lim\limits_{x\to -4} f(x)$ b. $\lim\limits_{x\to 2} f(x)$

a. We compute the two one-sided limits: $\lim\limits_{x\to -4^-} = \lim\limits_{x\to -4^-} \frac{1}{(x+4)} = -\infty$

$\lim\limits_{x\to -4^+} f(x) = \lim\limits_{x\to -4^+} 2x^2+1 = 2(-4)^2+1 = 33 \rightarrow$ two sided limit does not exist

b. $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} 2x^2+1 = 2(2)^2+1 = 9$

$\lim\limits_{x\to 2^+} = \lim\limits_{x\to 2^+} \sqrt{20x^2+1} = \sqrt{81} = 9$

The two one-sided limit exist and are equal $\therefore$ two sided limit exists

$\lim\limits_{x\to 2} f(x) = 9$

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9/10/12

LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES

Involves the 'end' behavior of a function as $x \to \pm\infty$

eg $\lim_{x \to \pm\infty} \frac{1}{x \neq 0} = 0$

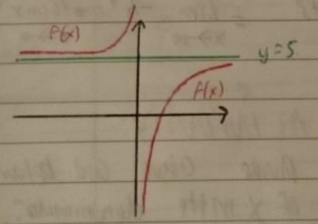
Def^n: Limits at infinity (informal approach)
$f(x)$ get arbitrarily close to L as x increases/decreases without bound, then

$$\lim_{x \to \pm\infty} f(x) = L$$

If either limit exists $y = L$ (horizontal asymptote) is a horizontal asymptote for $f(x)$

Example: Find the horizontal asymptote of $f(x) = 5 - \frac{1}{x}$

$\lim_{x \to \pm\infty} 5 - \frac{1}{x} = 5$ $\quad y = 5$ is the horizontal asymptote



Usual Limit Laws hold for limits at infinity.

- $\lim_{x \to \pm\infty} (f(x))^n = \left(\lim_{x \to \pm\infty} f(x)\right)^n$

- $\lim_{x \to \pm\infty} k f(x) = k \lim_{x \to \pm\infty} f(x)$

Infinite Limits at infinity (as x is increasing/decreasing without bound no horizontal asym.)

$f(x)$ may be unbounded as $x \to \pm\infty$, then we write

$$\lim_{x \to \pm\infty} f(x) = \infty \quad \text{(increases without bound)}$$

or $$\lim_{x \to \pm\infty} f(x) = -\infty \quad \text{(decreases without bound)}$$

e.g. $\lim_{x \to \infty} x^n = \infty$ $\qquad \lim_{x \to -\infty} x^n = \begin{cases} \infty & n \text{ even} \\ -\infty & n \text{ odd} \end{cases}$

End behaviour of polynomial is equivalent to the end behaviour of highest degree term.

eg $$\lim_{x \to \pm\infty} (c_n x^n + c_{n-1} x^{n-1} + \dots) = \lim_{x \to \pm\infty} c_n x^n$$

odd power

eg $$\lim_{x \to -\infty} -7x^7 + 2x^6 + 11x^5 + 3 = \lim_{x \to -\infty} -7x^7 = -7 \lim_{x \to -\infty} x^7 = +\infty$$

RATIONAL FUNCTION AT INFINITY:

eg. $\lim_{x \to \infty} \frac{2x^2 + 7x}{4x^2 - 7}$ $\qquad$ Divide above and below by the highest power of x in the denominator.

$$\lim_{x \to \infty} \frac{2 + \frac{7}{x}}{4 + \frac{7}{x^2}} = \frac{2}{4} = \frac{1}{2}$$

well defined non-0 limit
equal x^n on above + below

LIMITS AT INFINITY INVOLVING RADICALS:

eg Find $\lim_{x \to \pm\infty} \frac{\sqrt{5x^2 - 2}}{x + 3}$ $\qquad$ Divide above and below by $|x|$ and use the fact that $|x| = \sqrt{x^2}$

$$\lim_{x \to \pm\infty} \frac{\sqrt{5x^2 - 2}/|x|}{x/|x| + 3/|x|} = \lim_{x \to \pm\infty} \frac{\sqrt{5 - 2/x^2}}{\frac{x}{|x|} + \frac{3}{|x|}}$$

9/10/12

2

For $\lim_{x \to +\infty} \dfrac{\sqrt{5 - 2/x^2}}{1 + 3/x} = \dfrac{\sqrt{5}}{1} = \sqrt{5}$ (x is positive)

$\lim_{x \to -\infty} \dfrac{\sqrt{5 - 2/x^2}}{-1 - 3/x}$ (x is negative) $\dfrac{\sqrt{5}}{-1} = -\sqrt{5}$

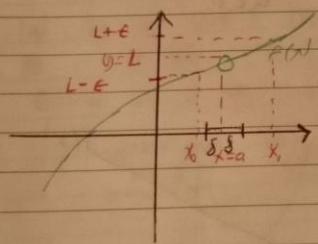
LIMITS AT INFINITY OF TRIGANOMETRIC FUNCTIONS

$\lim_{x \to \infty} \sin(x)$ $\qquad$ $\lim_{x \to \pm\infty} \cos(x)$ etc

do not exist, trig functions oscillate between -1 and 1 indefinitely in either direction

LIMITS (A RIGOROUS APPROACH)

Need to make statement such as "as close as we like" and sufficiently close" more precise.



How close does x have to be to a in order for the values of f(x) be within ϵ units of L:

Consider the following interval $(a - \delta, a + \delta)$ for some $\delta > 0$ which is entirely contained in (x_0, x_1)

- When $a - \delta < x < a + \delta$ then $L - \epsilon < f(x) < L + \epsilon$

or expressed in terms of absolute values, we say that when $0 < |x - a| < \delta$ the $|f(x) - L| < \epsilon$

Defn: Formal Defn of a Limit:

Let f(x) be defined for all x in some open interval containing a, with the exception that f(x) need not be defined at

$x=a$. We write $\lim_{x \to a} f(x) = L$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $|f(x) - L| < \epsilon$

if $0 < |x - a| < \delta$ "epsilon-delta" formalism

Example: Prove that $\lim_{x \to 2} 6x - 7 = 5$

We must show that given any $\epsilon > 0$, there exist a positive δ such that $|6x - 7 - 5| < \epsilon$ if $0 < |x - 2| < \delta$ (A)

$\Leftrightarrow |6x - 12| < \epsilon$ if $0 < |x - 2| < \delta$

$\Leftrightarrow 6|x - 2| < \epsilon$ if $0 < |x - 2| < \delta$

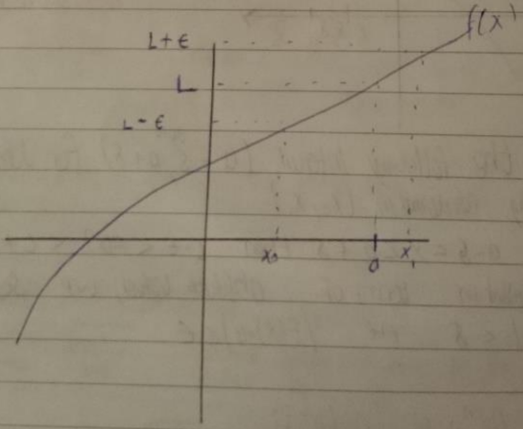
$\Leftrightarrow |x - 6| < \epsilon/6$ if $0 < |x - 2| < \delta$ (B)

(B) is true for δ $\epsilon/6$

(B) $\Leftrightarrow$ A

(A) is true for $\delta = \epsilon/6$ Q.E.D.

10/12 Epsilon Delta formalise



(1)

11/10/12

If $x \in (x_0, x_1)$

then $L - \epsilon < (f(x)) < L + \epsilon$

Take some open interval x $(a-\delta, a+\delta)$ entirely contained in (x_0, x_1) then $L - \epsilon < f(x) < L + \epsilon$

or $|f(x) - L| < \epsilon$ if $|x-a| < \delta$

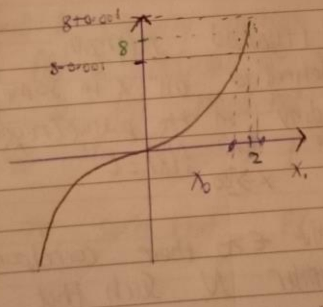
If we can always find a δ for which this statement is true, then we say $\lim_{x \to a} f(x) = L$

Example

$\lim_{x \to 2} x^3 = 8$

To prove this we would have to show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ if $0 < |x-a| < \delta$

Instead, let's say we're given a particular ϵ value, $\epsilon = 0.001$ say we want a δ such that if $0 < |x-2| < \delta$ then $f(x) = x^3$ is within $\epsilon = 0.001$ units of the limit $L = 8$



$x^3 = 8 - 0.001$
$= 7.999$

$x_0 = 1.99991666\overline{3}$

$x_1^3 = 8 + 0.001$
$x^3 = 8.001$
$x = 2.0000083333$

If x (x_0, x_1) then $f(x)$ is always within 0.001 units of $L = 8$

$|x_0 - 2| = 2 - x_0 = 0.00003337$ Take, $\delta = 0.00008$ Then if $|x-2| < 0.00008$ then $|f(x) - 8| < 0.001$

$|x_1 - 2| = x_1 - 2 = 0.000083333$

②

Example: Prove $\lim_{x \to 1^+} \sqrt{x-1} = 0$

Domain of $f(x) = \sqrt{x-1}$ is $D(f) = [1, \infty)$
So only the one sided limit makes sense

We want to prove that for any $\epsilon > 0$, there exists a $\delta > 0$ such that
$|\sqrt{x-1} - \underbrace{0}_{\text{"L"}}| < \epsilon$ ~~whenever~~ if $0 < |x-1| < \delta$ Ⓐ + side

value approach from right drop absolute value

$\Leftrightarrow \sqrt{x-1} < \epsilon$ if $0 < x-1 < \delta$

$\Leftrightarrow x-1 < \epsilon^2$ if $0 < x-1 < \delta$ Ⓑ

Statement Ⓑ is true if $\delta = \epsilon^2$
Statement Ⓑ is equivalent to Statement Ⓐ
Statement Ⓐ is true if $\delta = \epsilon^2$ Q.E.D.

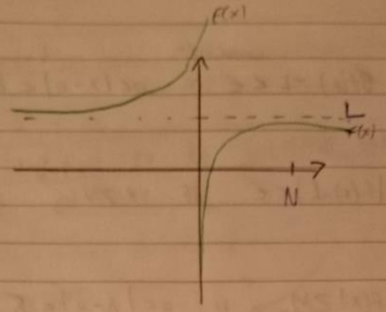
1. Write out what we want to prove and label it Ⓐ.
2. Manipulate Ⓐ so that left column looks like right column
3. Label this statement as Ⓑ
4. Choose δ value usually be a function of ϵ
- Statement Ⓐ is true for this value of δ

Limits at Infinity: A rigorous Approach

Def^n: Let $f(x)$ be defined for all x in some infinite open interval extending in the positive (negative) x direction. This case we write $\lim_{x \to \pm\infty} f(x) = L$

If given any positive $\epsilon > 0$, there corresponds a number positive (negative) number N, such that
$|f(x) - L| < \epsilon$ if $x \gtrless N$

11/10/12 Limits



Example: $\lim_{x \to \infty} \frac{1}{x} = 0$

We must show that for $\epsilon > 0$, there exists some positive N such that

$\left|\frac{1}{x} - 0\right| < \epsilon$ if $x > N$ (A)

$\Leftrightarrow \left|\frac{1}{x}\right| < \epsilon$ if $x > N$

$\Leftrightarrow \frac{1}{x} < \epsilon$ if $x > N$

$\Leftrightarrow x > \frac{1}{\epsilon}$ if $x > N$ (B)

(B) is true for $N = \frac{1}{\epsilon}$

But (B) $\Leftrightarrow$ (A)

(A) is true for $N = \frac{1}{\epsilon}$ Q.E.D.

Infinate Limits: A rigourous approach

Def^n: Let $f(x)$ be defined for x in some open interval containing a, but need not be defined at a, we say:

$\lim_{x \to a} f(x) = \infty$ ($-\infty$) if for any positive (negative) number M, we can

find a $\delta > 0$, such that $f(x) > M$ ($< M$) if $0 < |x - a| < \delta$

15/10/12

Recap

1. $\lim_{x \to a} f(x) = L$ $\quad |f(x) - L| < \epsilon$ if $0 < |x-a| < \delta$

2. $\lim_{x \to \infty} f(x) = L$ $\quad |f(x)-L| < \epsilon$ if $x > N$

3. $\lim_{x \to a} f(x) = \infty$ $\quad f(x) > M$ if $0 < |x-a| < \delta$

Example (3) Prove:

Prove $\lim_{x \to 0} \frac{1}{x^2} = \infty$

We must show that, given some $M > 0$ we can find a δ such that $\frac{1}{x^2} > M$ if $0 < |x-0| < \delta$ (A)

$\Leftrightarrow$ $x^2 < \frac{1}{M}$ if $0 < |x| < \delta$

$\Leftrightarrow$ $|x| < \frac{1}{\sqrt{M}}$ if $0 < |x| < \delta$ (B)

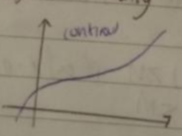
$\Leftrightarrow$ (B) is true for $\delta = \frac{1}{\sqrt{M}}$

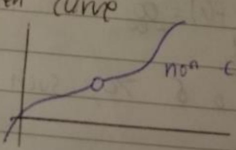
(B) $\Leftrightarrow$ (A)

(A) is true for $\delta = \frac{1}{\sqrt{M}}$ Q.E.D

Continuity:

Intuitively: Continuity $\rightarrow$ an "unbroken" curve





Def^n: A function f is said to be continual at $x=c$ provided

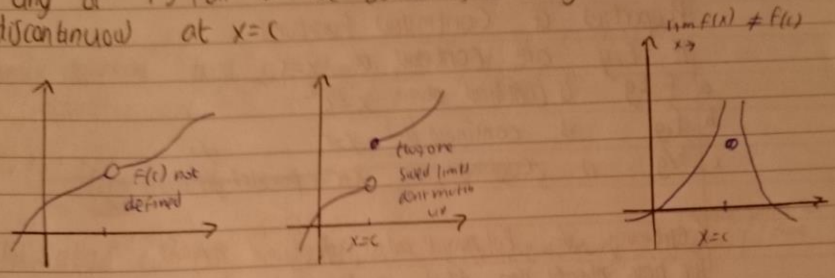
(1) $f(c)$ is defined

(2) $\lim_{x \to c} (f(x))$ exists

15/12/12 Math 2

3. $\lim_{x \to c} f(x) = f(c)$

If any of 1-3 fail to be satisfied, we say $f(x)$ is discontinuous at $x = c$



Example: Let $f(x) = \begin{cases} \frac{x^2-4}{x-2} & x \neq 2 \\ k^2 & x = 2 \quad k \in \mathbb{R} \end{cases}$

What value of k is the function continuous at $x=2$

For $f(x)$ to be continuous at $x=2$, we must have $\lim_{x \to 2} f(x) = f(2)$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{(x-2)} = \lim_{x \to 2} (x+2) = 4$$

and $f(2) = k^2$

for continuity we must have $k^2 = 4 \Rightarrow k = \pm 2$

Continuity in an interval

If f is continuous at each point in an open interval (a,b), we say f is continuous on (a,b)

For closed intervals $[a,b]$, we say that f is continuous at its endpoints if it values there matched the one sided limits

f is CTS on $[a,b]$ if:

1 f is ct on (a,b)
2 f is ct from right at $x=a$ $\lim_{x\to a^+} f(x) = f(a)$
3 f is ct from the left at $x=b$ $\lim_{x\to b^-} f(x) = f(b)$

Properties of Continuous Functions:
If f, g are continuous at $x=c$
a $f \pm g$ is continuous at $x=c$
b $f.g$ is continuous at $x=c$
c f/g is continuous at $x=c$ provided $g(c)\neq 0$

Continuity of Polynomials and Rational functions
We have already see that $\lim_{x\to a} p(x) = p(a)$ $p(x)$ is a polynomial function
Hence, polynomials are continuous everywhere

And, rational functions are continuous everywhere except when the denominator is 0

example $f(x) = \dfrac{x^2-7}{x^2+2x-3}$ is everywhere continuous

except when $x^2+2x-3=0 \Rightarrow (x-1)(x+3)=0$
$\Rightarrow x=1, x=-3$
$f(x)$ is ct everywhere except $x=1, x=-3$

example: Prove that $f(x) = |x|$ is continuous everywhere
$|x| = \begin{cases} x & x \geq 0 \\ -x & < 0 \end{cases}$

Clearly on $(-\infty,0)$ and $(0,+\infty)$
Show that it's continuous at $x=0$
We must show that $\lim_{x\to 0} |x| = f(0) = 0$
$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$ Hence $\lim_{x\to 0} |x| = 0$ Hence $f(x) = |x|$
$\lim_{x\to 0^-} |x| = -\lim_{x\to 0^-} x = 0$ is continuous everywhere

3

15/10/12 Math Continuity

Continuity of Compositions:

Theorm: If $\lim_{x \to c} g(x) = L$ and if F is continuous at L, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L)$$

Similar definitions hold for $\lim_{x \to c^+}$, $\lim_{x \to c^-}$, $\lim_{x \to \infty^{\pm}}$

16/10/12 ~~If $\lim_{x \to c} g(x) = L$ and if F is cts at L~~

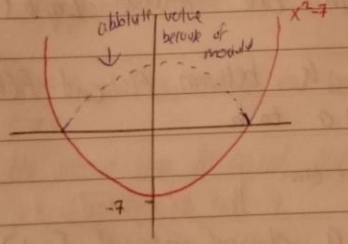
(1) This allows us to consider the continuity of $|g(x)|$ absolute value of func

i.e if $f(x) = |x|$, the $f(gx) = |g(x)|$

But we know that $f(x) = |x|$ is continuous everywhere

$\lim_{x \to c} |g(x)| = \left|\lim_{x \to c} g(x)\right|$ provided the $\lim_{x \to c} g(x)$ exists

example: $\lim_{x \to 2} |x^2 - 7| = \left|\lim_{x \to 2} (x^2 - 7)\right| = |2^2 - 7| = |-3| = 3$



Theorm:

A If the function g is continuous at c and the function f is continuous at ~~[illegible]~~ g(c), then fog is continuous at c

B If the function g is continuous everywhere, and the function f is continuous everywhere, then fog is continuous everyw

16/10/18 ②

Continuity of Inverse functions:

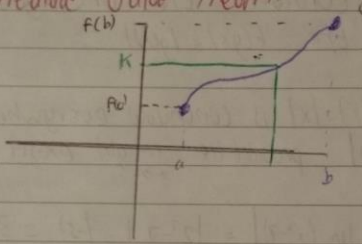
If f is 1-1, it has an inverse, which is a reflection of $y=f(x)$ through the line $y=x$

∴ If the graph of x has no breaks neither does the graph of f^{-1}

Theorm:

If f is a one-to-one function that is continuous at each point of its domain, then f^{-1} is continuous over each point of the domain of f^{-1}, which is the range of f

Intermediate Value Theorm:



f is continuous over $[a, b]$

For $f(a) < k < f(b)$, the line $y=k$ will cross the curve $y=f(x)$ at least once in the interval $[a, b]$

f must take every value k between $f(a)$ and $f(b)$ at least once as x varies from a to b.

Theorm IVT:

intermediate value theo.

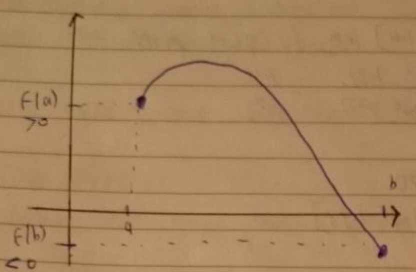
Proof not required

If f is continuous on a closed interval $[a,b]$ and k is any number between $f(a)$ and $f(b)$ inclusive, then there is at least one number x in the interval $[a,b]$ such that $f(x)=k$

③

16/10/12

Approximating Roots using IVT:



If f is cts on $[a,b]$ and $f(a)$, $f(b)$ are non-zero but have different signs, then there is at least one solution to the equation $f(x)=0$ in (a,b)

Errors in Approximations:

If x approximates x_0 then $|x-x_0|$ is the absolute error.

- If $|x-x_0| \leq 0.1$, then x approximates x_0 with an error of not 0.1 etc
  - $|x-x_0| \leq 0.5$, then x approximates to the nearest integer.
  - $|x-x_0| \leq 0.05$ then x approximates x_0 to 1 decimal place of accuracy
  - $|x-x_0| \leq 0.005$ " " 2 decimal " "

Example: Given $f(x) = x^3-x-1$, has a root in the interval $[1,2]$ approximate the root to 2 decimal points of accuracy

$f(1) = -1 < 0$ IVT - Sign change root between both points.
$f(2) \; 5 > 0$

Divide the interval into 10 equal parts

$x = 1, 1.1, 1.2, 1.3, 1.4, \ldots 1.9, 2$

$f(x) = -1, -0.77, -0.47, -0.10, 0.34$ ← change in sign

$f(1.3) < 0$ $\quad f(1.4) > 0$ $\therefore$ there is a root in $[1.3, 1.4]$. The length of this interval is 0.1 so the midpoint of the interval is 1.35

is at most 0.05 away from the actual root (1 d.p.)

Divide the interval [1.3, 1.4] into 10 equal parts

$x =$ 1.3, 1.31, 1.32, 1.33, 1.4

$f(x) =$ −0.110, −0.062, −0.012, 0.023, 0.344

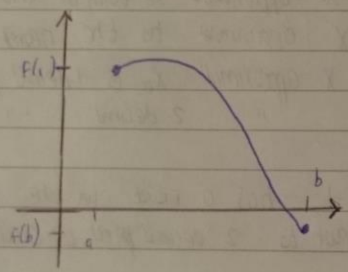
$f(1.32) < 0$, $f(1.33) > 0$

$\therefore$ There is a root in [1.32, 1.33]

The midpoint of this interval $x_0 = 1.325$ approximates the root with an absolute error of $|x - x_0| < 0.005$ (i.e. an approx to 2 d.p.

Actual root: 1.32472

18/10/12



curve has to pass through 0

Example:

Show that $f(x) = x^3 + x^2 - 2x - 1$ has at least one root [−1, 1]

Approximate root to 2 decimal place of accuracy

$f(-1)$ $f(1)$

$= 1 > 0$ $= -1 < 0$ hence there is a root in the interval between (−1, 1)

Divide the interval into 10 equal parts

$x =$ −1 −0.8 −0.6 −0.4 −0.2 ...

$f(x) =$ 1 0.73 0.34 −0.104

18/10/12

So $f(-0.6) > 0$ $f(-0.4) < 0$
Hence, there is a root in $(-0.6, -0.4)$
Divide this into 10 equal parts

$x = -0.6 \quad -0.58 \quad -0.56 \quad -0.54 \ldots -0.44$

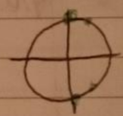
$f(-0.46) > 0$
$f(-0.44) < 0$ hence there is a root in $(-0.46, -0.44)$

We take the midpoint to approximate the root, $x = -0.45$

Continuity of Trignometric functions:
Sinx, cosx are continuous functions

$\Rightarrow \lim_{x \to a} \sin x = \sin a \qquad \lim_{x \to a} \cos x = \cos a$

Other trigonometric functions ~~[illegible]~~ have discontinuous eg $\tan x = \frac{\sin x}{\cos x}$
dis continuous at $\cos x = 0$



$x = \frac{\pi}{2}(2n+1)$
$n = 0, 1, 2, 3$

Similarly for other trig. functions such as $\operatorname{cosec}(x) = \frac{1}{\sin x}$

Example: $\lim_{x \to 1} \cos\left(\frac{x^2-1}{x-1}\right) = \cos\left(\lim_{x \to 1} \frac{x^2-1}{x-1}\right)$ [can only do if continuous] Since $\cos x$ is everywhere continuous

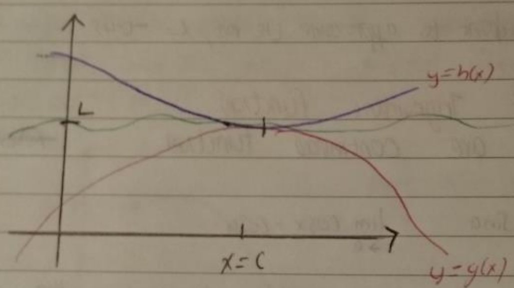
$\cos \lim_{x \to 1} \left(\frac{(x+1)(x-1)}{(x-1)}\right) = \cos 2$

2

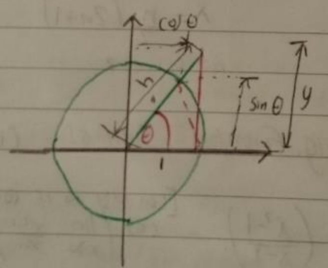
Squeezing Theorem: Let f, g, h be functions, satisfying $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing some point c with the possible exception that the inequality need not hold at c

If g and h have the same limit as $x \to c$, say

$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$ then f has the limit as $x \to c$ i.e. $\lim_{x \to c} f(x) = L$



Theorem $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$



Clearly the area of the sector is between the area of the small triangle and the area of the large triangle

Area of sector $= \frac{1}{2} r^2 \theta$ $A = \frac{1}{2}\theta$ on unit circle

Area of the small triangle $= \frac{1}{2} b \times h$

$= \frac{1}{2} \sin\theta$

Area of large triangle $= \frac{1}{2} b h$

$h^2 = 1^2 + y^2 \Rightarrow (y)^2 = h^2 - 1$

18/10/12 (3)

Sine rule $\frac{\sin\theta}{y} = \frac{1}{h} \Rightarrow h = \frac{y}{\sin\theta}$

$\Rightarrow y^2 = \frac{y^2}{\sin^2\theta} - 1 \Rightarrow y^2 = \frac{\sin^2\theta}{\cos^2\theta} \Rightarrow y = \tan\theta$ $\quad 0 < \theta < \frac{\pi}{2}$

Area of large triangle $= \frac{1}{2}\tan\theta$

$\frac{1}{2}\sin\theta \leq \frac{1}{2}\theta \leq \frac{1}{2}\tan\theta \qquad 0 < \theta < \frac{\pi}{2}$

$(\times \frac{2}{\sin\theta})$ which is positive in the first quadrant

$\Rightarrow 1 \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}$

$1 \geq \frac{\sin\theta}{\theta} \geq \cos\theta$

Since $\lim_{\theta\to 0} (1) = 1$ and $\lim_{\theta\to 0} \cos\theta = 1$ by the squeezing theorem we must have $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ QED.

Corollary: $\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$ $\qquad \lim_{x\to 0} \frac{\sin(h(x))}{g(x)}$ where $h(0) = g(0) = 0$

$\lim_{x\to 0} \frac{\sin h(x)}{h(x)} \cdot \frac{h(x)}{g(x)} \Rightarrow \lim_{x\to 0} \frac{\sin h(x)}{g(x)} \cdot \lim_{x\to 0} \frac{h(x)}{g(x)} \Rightarrow \lim_{x\to 0} \frac{h(x)}{g(x)}$

Example: $\lim_{x\to 0} \frac{1-\cos x}{x} = 0 \qquad = \lim \frac{(1-\cos x)(1+\cos x)}{x(1-\cos x)}$

$= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} \Rightarrow \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} \Rightarrow \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim \frac{\sin x}{1+\cos x}$

$(1)\cdot(0) = 0$

22/10/12

1

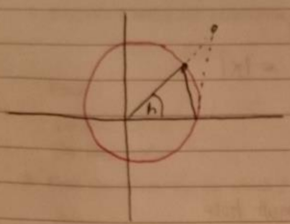
Maths continuity

Recap:

Squeezing theorm

$\Rightarrow \lim_{h \to 0} \frac{\sin h}{h} = 1$ h must be measured in radians

Corollary $\lim_{h \to 0} \frac{\sin f(h)}{f(h)} = 1$ if $f(h) = 0$



area of sector $= \frac{1}{2} r^2 h$

Example: $\lim_{x \to 0} \frac{\sin 5x}{x} \cdot \frac{5}{5} = \lim_{x \to 0} \frac{\sin 5x}{5x} \cdot \lim_{x \to 0} 5$

$= (1) \cdot (5) = 5$

Example: $\lim_{x \to 0} \frac{\sin(x^2 - 2x)}{x} = \lim_{x \to 0} \frac{\sin(x^2-2x)}{x^2-2x} \cdot \frac{x^2-2x}{2} = \lim_{x \to 0} \frac{\sin(x^2-2x)}{x^2-2x} \cdot \lim_{x \to 0} \frac{x^2-2x}{2}$

$= (1) \cdot \lim_{x \to 0} \frac{\cancel{x}(x-2)}{\cancel{x}}$

$(1) \cdot (-2) = -2$

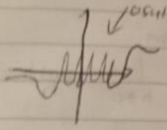
Example: $\lim_{x \to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \frac{3x}{5x} \cdot \frac{5x}{\sin 5x}$

$= \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \lim_{x \to 0} \frac{3\cancel{x}}{5\cancel{x}} \cdot \lim_{x \to 0} \frac{5x}{\sin 5x}$

$(1) \cdot \left(\frac{3}{5}\right) (1) = \frac{3}{5}$

Limits Involving $\sin(1/x)$

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ - does not exist



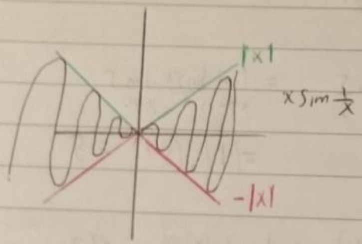
$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$ $\qquad -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

$\Rightarrow \quad -|x| \leq x\sin\left(\frac{1}{x}\right) \leq |x|$

but $\lim_{x \to 0} -|x| = 0$ and $\lim_{x \to 0} |x| = 0$

By the Squeezing theorm, we must have

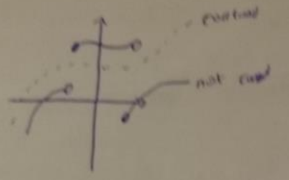
$\lim_{x \to 0} x\sin\left(\frac{1}{x}\right) = 0$

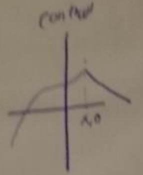


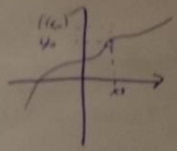
More generally, we have $\lim_{x \to 0} x^n \sin\frac{1}{x} = 0$ for $n \geq 1$

$|f x - L| < \epsilon$ if $0 < |x - a| < \delta$

2







$f(x) = \{ \; x \leq 4 \quad x > 4$

$\lim_{x \to x_0^-} f(x) = y_0$

$\lim_{x \to x_0^+} f(x) = y_0$

$\lim_{x \to x_0} f(x) = y_1$

but $f(x_0) \neq y_0$] not cont

$\lim_{x \to 1^-} f(x) \overset{?}{=} \lim_{x \to 1^+} f(x)$

$14 - 10 \overset{?}{=} k$

$4 = k$

$\lim_{x \to 2} 2^2 + 2 \quad \lim_{x \to 2} m(2+1) + 4$

$6 = 3m + 4$

$4 = 3m \quad m = \frac{4}{3}$

$4x^2 \quad 10x + 4 \quad x \cdot 1$

$2(-1)^3 + (1) + 5 = m(-1+1) + k$

$-2 + 1 + 5 = k$

$2 = k$

$4x^2 \quad x \leq 2$

$10x + 4 \quad x > 2$

$k = 2$

$4k = 4 + 4$

$4k = 4 + 4$

$3k = 4$

$k = \frac{4}{3}$

$4(2^2) \quad 4(4) \quad 16$

$10(2) + 4 \quad 16 + 4 \quad 20$

$(-1 \quad 1)$

$2k = 20 + 4$

$-1 + 4 + 2$

$k = 20$

$\lim_{x \to 2^+}$

$7 = k$

$6 = 3m + 4$

$2 = k$

$-2 - 1 + 5$

$x^2 - 4 = 0$

$(x+2)(x-2)$

$x = -2 \quad x = +2$

$6 = 3m + 2$

$4 = 3m \quad m = \frac{4}{3}$