24/04/16 DECISIONS - BRETT

Game theory: Making a decision based on decision of others
Profits which are demand will not be equal

BARGAINING: COOPERATIVE GAMES

- A and B two players. Each has 2 cards (1R, 1B) and each chooses a card
- If cards different colour: no prize
- If both choose red $R \Rightarrow$ A gets 2D, B gets 1D
- If both choose black $B \Rightarrow$ A gets 1D, B gets 2D
- Both players want to match, but prefer different matches

Suppose:

i. If can't reach a bargain, then A plays R and B plays B and so get nothing
ii. No side deals (bribes)
iii. Can make binding deals.

Example of a binding deal: both agree to flip a coin. If H both play R, if T both play B, same for...

However, A and B may have different utilities

E.g. $U_A(0D) = 0 \quad = U(0D) = 0$

$U_A(1D) = 1 \quad U_B(1D) = 1$ B adds more value in going from 1D to 2D

$U_A(2D) = 2 \quad U_B(2D) = 4$

This means A is risk neutral and B is risk prone to diamonds, not that B likes diamonds more

Remember $\begin{cases} R \Rightarrow A \text{ gets } 2,\ B \text{ gets } 1 \\ B \Rightarrow A \text{ gets } 1,\ B \text{ gets } 2 \\ 0 \text{ otherwise,} \quad \text{Bargain is a gamble} \end{cases}$

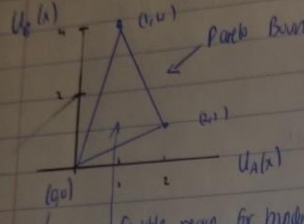
Any bargain is a gamble (G) $\quad G = (\overset{\text{red, red}}{P_{RR}}, \overset{\text{red, black}}{P_{RB}}, P_{BR} P_{BB})$

Utility $\overset{\text{Amt}}{U_A} = P_{rr}U_A(2) + P_{BB}U_A(1) + (P_{BR} + P_{RB})(U_A(0)) = 2P_{rr} + 1P_{BB}$

$U_B = P_{rr} + 4P_{BB}$

$$\begin{pmatrix} U_A(G) \\ U_B(G) \end{pmatrix} = P_{rr}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + P_{BB}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + (P_{BR} + P_{RB})\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any choices for P_{rr} etc are non-negative: $P_{RR} + P_{BB} + P_{RB} + P_{BR} = 1$



Possible joint utilities are 'closed convex hull' of possible outcome pairs (1,4) (2,1) (0,0)

Feasible region: All possible choices $(U_A(G), U_B(G))$ over all gambles.

Pareto Boundary: All points (μ, v) in feasible region for which there is no point (μ', v') in feasible region with either $\mu' \geq \mu$ and $v' \geq v$, $v' \geq v$ or $\mu' \geq \mu$ - can't make it better for both players simultaneously

General Problem: Set of outcomes obtained by 2 people for outcome x, two rewards $r(x)$, $w(x)$ with utilities $\mu(x)$ and $v(x)$ for the 2 people. Players may choose a gamble between the outcomes. Possible risk pairs in the feasible region S. If no agreement is made the status quo decision x^* will be taken with values μ_S^*, v_S^* (status quo point)

Want to find a rule which picks out a fair or 'rational' bargaining point.

Function $F(S, \underbrace{\mu_S^*, v_S^*}_{\text{status quo, 'rational' fair}}) = (\underbrace{\bar{\mu}_S, \bar{v}_S}_{\text{rational bargain}})$

4/04/16 DECISIONS - BRETT

NASH BARGAINING AXIOMS

N1: Individual Rationality $(\bar{\mu}_s, \bar{v}_s) \geq (\mu_s^*, v_s^*)$ [i.e. $(a,b) \geq (c,d) \Rightarrow a \geq c$ and $b \geq d$]

N2: Feasibility $(\bar{\mu}_s, \bar{v}_s) \in S$

N3: $(\bar{\mu}_s, \bar{v}_s)$ is on Pareto boundary. i.e. if $(\mu, v) \in S$ and $(\mu, v) \geq (\bar{\mu}_s, \bar{v}_s)$ then $(\mu, v) = (\bar{\mu}_s, \bar{v}_s)$. Known as 'Pareto Optimality'. Why choose a point if there is a better point which benefits both parties

N4: Invariance of Equivalent Representations

If we ~~could~~ replace (μ, v) by some linear transformation $(\alpha\bar{\mu} + \beta, \gamma\bar{v} + \delta)$ i.e. choosing $(\bar{\mu}, \bar{v})$ for old S, we should choose $(\alpha\bar{\mu} + \beta, \gamma\bar{v} + \delta)$ for new S
e.g. if $f(S, \mu_s^*, v_s^*) = (\bar{\mu}_s, \bar{v}_s)$ then in new representation with feasible region T and status quo μ_T^*, v_T^* constructed by reducing each (μ, v) by $(\alpha\mu + \beta, \gamma v + \delta)$ then our requirement is $f(T, \alpha\mu_s^* + \beta, \gamma v_s^* + \delta) = (\alpha\bar{\mu}_s + \beta, \gamma\bar{v}_s + \delta)$
LINEAR ONLY i.e. monetary unit should not matter.

N5: Symmetry

Suppose S is symmetric $(\mu, v) \in S$ $(v, \mu) \in S$ and $\mu_s^* = v_s^*$ then $\bar{\mu}_s = \bar{v}_s$.
e.g. only the shape S and the values μ^*, v^* affect choice, not external circumstances, i.e. not status of bargains. ⇒ Shouldn't take into account how much money I already have

N6: Independence of Irrelevant Alternatives

Suppose $f(T, \mu^*, v^*) = (\bar{\mu}_T, \bar{v}_T)$ (e.g. we jointly choose meat over fish) Suppose $T \overset{\text{smaller}}{\subset} S$ (now vegetarian option). Suppose $f(S, \mu^*, v^*) = (\bar{\mu}_s, \bar{v}_s)^{\text{(blown)}} \in T$ then $(\bar{\mu}_s, \bar{v}_s) = (\bar{\mu}_T, \bar{v}_T)$ (e.g. either stick with meat or new bargain had some chance of vegetarian not fish though!

NASH BARGAINING THEORY

There is a unique function defined on all bargaining problems satisfying (N1:N6). This function gives us $(\bar{\mu}_s, \bar{v}_s)$ to be the point $(\bar{\mu}_n, \bar{v}_n)$ which uniquely maximises the function $g(\mu, v) = (\mu - \mu_s^*)(v - v_s^*)$ over all $(\mu, v) \in S$, $(\mu, v) \geq (\mu_s^*, v_s^*)$ provided there are some $(\mu, v) \in S$ with $\mu > \mu_s^*$ and $v > v_s^*$

$(\bar{u}_n, \bar{v}_n)$ is nash bargaining point. So if both players agree Nash axioms are 'fair', then they should agree on nash point (Nash Bargaining)

Alternatively, if can't reach agreement and arbitrator called in who agreed with Nash axioms : this is Nash Arbitration

Back to example:

Want to max $g(\mu_A, \mu_B) = (U_A - U_A^*)(\mu_B - \mu_B^*)$

$= \mu_A \mu_B$ on S

Equivalently max $\mu_A \mu_B$ on Pareto Boundary

Equation of Pareto boundary is $\mu_B = a + b\mu_A$

$4 = a + b(1)$

$1 = a + b(2)$ $\quad b = -3,\ a = 7$

$\mu_B = 7 - 3\mu_A \qquad (1 \le \mu_A \le 2)$

$g(\mu_A, \mu_B) = \mu_A \mu_B = \mu_A(7 - 3\mu_A) = 7\mu_A - 3\mu_A^2$

$\frac{dg}{d\mu_A} = 7 - 6\mu_A = 0 \quad \mu_A = 7/6$

- Could be max or min $\rightarrow$ not min, clearly see on graph
- $1 \le 7/6 \le 2$ e.g. is a max on pareto boundary otherwise, max would be at one of the end points
- Plug number back in $\Rightarrow \mu_B = 7/2$

Nash Point $\bar{U}_{AN} = 7/6 \quad \bar{U}_{BN} = 7/2$

Nash point is gamble between both R (probability p) and B (probability 1-p)

$\mu_A = 2p + (1-p)1 = p + 1$

Utility to A for B and B

Utility to A for R and R

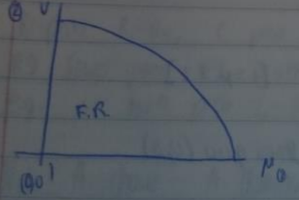
$p + 1 = 7/6 \Rightarrow p = \frac{1}{6}$ play red 1/6 times and 5/6 times play blue.

4/04/16 DECISIONS - BRETT

So Nash Bargain: Both red (probability $\frac{1}{6}$)
Both black (probability $\frac{5}{6}$)
Could use die to choose

Comment: Bob seems to benefit more than Anne from Nash Bargain. Nash axioms appear to favour the risk prone.

Example
2 players offered £100 if they can decide how to divide the money, otherwise get 0
Player 1: risk neutral $\Rightarrow u = (£x) = x$ actual fair price
Player 2: risk averse $\Rightarrow v = (£x) = \log\left(\frac{100+x}{100}\right)$ like people who pay insurance
Status quo: $(0,0)$ want nash bargain



$V = \log\left(1 + \frac{\mu}{100}\right)$

Where each point is the division £μ to £$100-\mu$
No probabilities involved here

i.e. no gambles between rewards on Pareto Boundary
Nash Point: max $G(u,v) = u\log\left(\frac{200-\mu}{100}\right) \quad 0 < \mu < 100$

line $v = \log\left(2 - \frac{\mu}{100}\right)$ because if P_1 told £μ, leaves P_2 with £$100-\mu$ and $\log\left(\frac{100+100-\mu}{100}\right)$
$= \log(2 - \mu/100)$ plugging x into log function

$\log\left(\frac{100+x}{100}\right) \quad x = 100-\mu \quad \log\left(\frac{100+100-\mu}{100}\right) = \log(2 - \mu/100)$

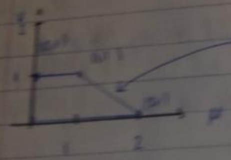
Nash point: $g(\mu,v) = (\mu-\mu^*)(v-v^*)$
$= (\mu - 0)(\log(2-\mu/100))$
$= u\log(2 - \mu/100)$
Differentiate by chain rule: $dy/du = \log(2-\mu/100) - \frac{\mu}{200-\mu}$
Set to 0: $\frac{u}{200-\mu} = \log(2 + \mu/100)$

$p = [illegible]$
$r = [illegible]$

Ann $[0 \to 1 \to 2]$ [illegible] $[0 \; 0.5 \; 1]$
Bob $[0 \to 1 \to 2]$ $[0 \; 0.25 \; 1]$

Example

Suppose [illegible] payoffs are $(0,1)$ $(1,1)$ $(1,1)$ $(2,0)$



any move on this line will reduce utility for [illegible]

Pareto boundary

line $y = mx + c$ $1 = 0 + c$ and $0 = 2m + c$

$r = -p + 2$ (eq[illegible] of pareto bounding line) over $[1 \le p \le 2]$

Nash [illegible] $g(p,r) = (p-p^*)(r-r^*)$ p^* and r^* [illegible] (0,0)

$= pr$

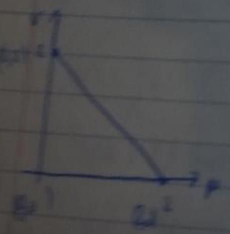
$= p(2-p) = 2p - p^2$

$\frac{dg}{dp} = 2 - 2p$ $p = 1$

When $p = 1$, $r = 1$ [illegible] point (1,1) [illegible] a [illegible] strategy

Example

Outcomes (0,0) (1,0) (0,2)



Nash point is (1,1) because of symmetry and same [illegible] for both players

Check [illegible] where each [illegible] get [illegible] Axiom 5

Only [illegible] (0,0) [illegible]

-4/04/16 DECISIONS - BRETT

GAME THEORY

- 2 Person strictly competitive game of the following form:
  - You make a decision, opponent makes a decision (no co-operation)
  - Zero sum game: Your winnings = Opponent's loss
  - Your aim is to maximise your gain

Example 1

	C_1	C_2	C_3	C_4
R_1	0	1	7	7
R_2	4	1	2	10
R_3	3	1	0	2.5
R_4	0	0	7	10

R picks R_1-R_4, C picks C_1-C_4.
R's score given, C's score given as minus R score
eg. if both pick 2, R gets 1, C gets -1 (ie. zero sum game).

Note: A choice A dominates a choice B if all outcomes for A are atleast as good as for B and some are better. (If identical, doesn't count as dominated.)

- Don't play dominated choices

A. C_4 dominated by C_3 delete C_4
B. R_3 dominated by R_2 delete R_3
C. R_4 dominated by R_1 delete R_4
D. C_3 dominated by C_2 delete C_3
E. R_1 dominated by R_2 delete R_1
F. C_1 dominated by C_2 delete C_1

Ordering doesn't matter. If C_1 here C could do it one step further
Useful to look vice versa, play from each player's point of view.

$\therefore$ R and C both play 2. R gets 1 and C gets -1 and this is the value of the game

Note: Neither R nor C can raise the value by playing if opponent plays this choice

Example
R and D choose campsite. R wants campsite high, D wants Camp low!
R chooses N.S., D chooses E.W. Camp in each section

D	1	2	3	4
R 1	7	2	5	1
2	2	2	3	4
3	5	3	4	4
4	3	2	1	6

- R3 dominates R2 - delete R2
- D2 dominates D1 - delete D1
- No more dominated strategies

Value is 3 - Maximum minimum value R can achieve
Note: If R chooses 1 or 4, D might give him choice 2 giving height 2
If picks option 3, can guarantee at least 3
D might see this the other way around, picking option 2 guaranteeing at most 3

Pair (R3, D3) are in equilibrium: If either deviates from the pair, the other can do no better than also choosing from the pair.
- Game has value 3 (which either party can force).
- A point which is the min of a row and max of a column is called a saddle/equilibrium point.
- Once dominated choices removed, check for saddle points
- If there is no such point, then that is the value of the game

NOTE: Game may have several saddle points but they all have the same value

26/04/16 DECISIONS - BRETT

Example 3

		clever sister	
		even	odd
you	even	-1	+1
	odd	+1	-1

rolling a die

Protect against being outguessed each time by coin (even if H, odd if T)
~~[illegible]~~ Win 50% of time - expected value of game is zero and you can reveal strategy to your opponent

Can't force higher expected value as opponent can do the same
e.g. $\left(\overset{\text{odd}}{\tfrac{1}{2}}, \overset{\text{even}}{\tfrac{1}{2}}\right)$ is equilibrium strategy and value of game is zero

Example 4

		Moriarty	
		Canterbury	Dover
Holmes	Cant	0	50
	Dover	100	0

Moriarty chasing holmes
No dominating strategies
No saddle point.

Suppose H chooses dover with probability p_H, M chooses Dover with prob p_m

Expected payoff to H is $0 \cdot P(H_C \cap M_C) + 0 \cdot P(H_D \cap M_D) + 100\,(H_D M_C) + 50\,P(H_C \cap M_D)$
$= 50\,(1-p_H)p_m + 100\,(p_H)(1-p_m)$
$= 50\,(p_m + 2p_H - 3p_m p_H) = E$ holmes wants to make as large as possible

H chooses p_H to maximise this, M chooses p_m to minimise this

Note: If $p_H = 1/3$ $E = 50(p_m + 2/3 - p_m) = 100/3$ whatever p_m is
$\therefore$ H can force a value of $100/3$

Similarly if $p_m = 2/3$ $E = 50(2/3 - 2p_H - 2p_H) = 100/3$ whatever p_H is.
So either player can force an outcome of $100/3$

Say $(1/3, 2/3)$ (p_H, p_m) are equilibrium strategy. If either deviate, can [illegible] can achieve a better outcome than $10/3$

IN GENERAL:

Matrix $A = (a_{ij})$ $i = 1, \dots, m$ $j = 1, \dots, n$ (player choice i and j) is payoff matrix for game

R picks one of rows $1, \dots, m$ and C picks one of columns $1, \dots, n$.

If R picks i and C picks j, payoff a_{ij} to R and $-a_{ij}$ to C (i.e. zero sum game)

Mixed strategy for R is choice $p = (p_1, \dots, p_m)$ (p_i is probability R chooses i)

Each $p_i \geq 0$ $\sum_{i=1}^{m} p_i = 1$ $\quad q = (q_1, \dots, q_n)$ each $q_i \geq 0$ $\sum_{i=1}^{n} q_i = 1$

Payoff a_{ij} chosen with probability $p_i q_j$

Expected payoff to R is $\sum_{i,j} p_i q_j a_{ij}$ (C gets minus this)

$= p^T A q$ R chooses p to make big, C chooses q to make small

Best bet R can guarantee if C can 'outguess' R is $V_R = \max_p \left(\min_q (p A q) \right)$

C minimises, R needs to maximise this number.

Best that C can force if R outguesses C is minimum of max $V_C = \min_q \left(\max_p (p A q) \right)$

THEORM: MINIMAX

For 2 person-zero sum games

$V_R = V_C = V$ (Value of the game) $= p^{*T} A q^*$ for some choices p^* q^* the equilibrium strategy

Where $p^{*T} A q^*$ $\max_p p^T A q^* = \min_q p^{*T} A q$

- Each player can force a result by playing your equilibrium strategy.

- Each player can force value V and if one player plays their equilibrium strategy, other player can do no better than playing their equilibrium strategy (hard to work)

*limits damage by choosing equilibrium gets rid of opposed probabilities.

26/04/15 DECISION - BRETT

Solving Game:

E.g. Find value and minimax strategies of zero sum game.

1. Delete Dominated Strategies
2. Check for saddle points - these are solutions if they exist
3. In general use simplex algorithm - not covered

Example: Special 2x2 case

No saddle points

C	1	2	
R 1	a_{11}	a_{12}	p_1
2	a_{21}	a_{22}	p_2

Value of game if p q are minimax.

$$V = a_{11}p_1q_1 + a_{12}p_1q_2 + a_{21}p_2q_1 + a_{22}p_2q_2$$
$$= p_1(a_{11}q_1 + a_{12}q_2) + p_2(a_{21}q_1 + a_{22}q_2) \qquad \text{NB } p_2 = 1 - p_1$$

$\therefore$ For minimax, must have $a_{11}q_1 + a_{12}q_2 = a_{21}q_1 + a_{22}q_2 = V$ and $q_1 + q_2 = 1$

this gives values q_1, q_2, V and solve for p_1, p_2 similarly.

Used for finding the 1/3 and 2/3 in Skittles problem

GRAPHICAL SOLUTIONS FOR 2XN GAMES

Method 1:

$$A = \begin{pmatrix} a_1, a_2, \dots & a_n \\ b_1, b_2, \dots & b_n \end{pmatrix}$$ R wants to maximise p $\min_q p^TAq$

Suppose R chooses $\begin{cases} R_1, & 1-p \\ R_2, & p \end{cases}$

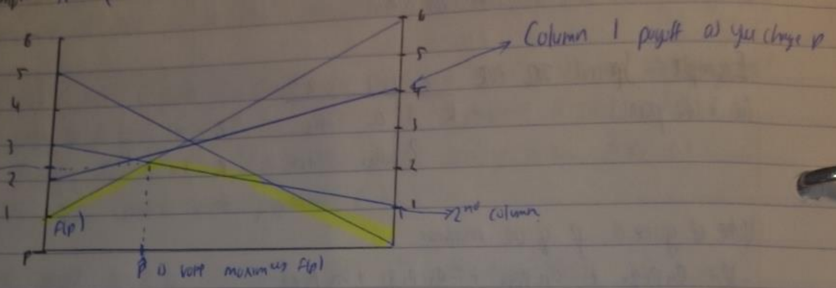
R wants to max $f(p) = \min_q p^TAq = \min_q (1-p, p)\begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix}\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$ $q_i \geq 0$ $\sum_{i=1}^{n} q_i = 1$

$$= \min_q \left[a_1(1-p) + pb_1 + \dots + a_n(1-p) + pb_n\right]\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$= \min_{q} [q_1(a_1(1-p)+b_1 p) + \dots + q_n(a_n + p(b_n - a_n))]$ - one of these will be smallest - want max probability on that value

$= \min \{ a_1 + p(b_1 - a_1), a_2 + p(b_2 - a_2), \dots, a_n + p(b_n - a_n) \}$ ← linear expression

Example $A = \begin{pmatrix} 2 & 3 & 1 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix}$



F(p) is the highlighted line - the minimum line

L Value of game - max for other player ⇔ amount prepared to pay

Col 1 : $2 + (4-2)p$
Col 2 : $3 - 2p$
Col 3 : $1 + 5p$
Col 4 : $5 - 5p$

Solve the intersection of column 2 and 3 e.g. when $3-2p = 1+5p$ $\hat{p} = 2/7$
$\therefore V = 3 - 2\hat{p} = 1 + 5\hat{p} = 17/7 =$ value of the game
ie probability R plays 2 is 2/7, C plays col 2 and col 3, find by solving the 2 × 2 subgame

		C: 2	3
R	1	3	1
	2	1	6

Knowing $V = 17/7$, C plays $(0, 5/7, 2/7, 0)$

NB in a 2xN game, always be 2 columns that C can pick between

28/04/16 DECISIONS : BRETT

METHOD 2

$\hat{p}$ $\hat{q}$ are minimax x, c : value of game V

$\hat{p} A \hat{q} \geq V \geq \hat{p}^T A \hat{q}$ (*) p (always does worse if doesn't play $\hat{p}$)

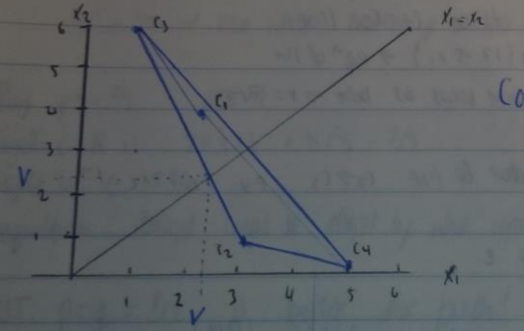
p, q are probability vectors, e.g. $q \in Q = (q_1, \ldots, q_n)$ $q_i \geq 0$ $\sum_{i=1}^{n} q_i = 1$

$p \in P$ (p_1, p_2) (2xN games) $\therefore$ $p_1 + p_2 = 1$ $p_i \geq 0$

H: All vectors Aq $(q \in Q)$ $\begin{pmatrix} 2 & 3 & 1 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$

$= q_1 \binom{2}{4} + q_2 \binom{3}{1} + q_3 \binom{1}{6} + q_4 \binom{5}{0}$

eg. H is a convex hull of column vectors of A.



Corresponds to having probability of $q = 1$

$\Rightarrow$ extreme values

By varying probabilities, can get anything 'along' the line.

C_1 is not on the line $\Rightarrow$ It is inside the convex hull

N_V is the region $N_V = \{(k_1, k_2) : k_1 \leq V, k_2 \leq V\}$ (k_1, k_2 less than value of game)

$\underline{k} \in N_V \iff p^T \underline{k} \leq V$ for all $p \in P$ (**)

Vector k in the region iff ↑

By * $p^T A \hat{q} \leq V$ for all $p \in P$

By ** $A\hat{q} \in N_V$ $\therefore$ $A\hat{q} \in N_V$ n+1

All Aq have to be in N_V zone for all Aq element of N_V n+1

Consider line $\hat{\underline{p}}^T x = v$. By (†) $\hat{\underline{p}}^T b \geq v \;\; \forall\, b \in H$

e.g. Convex hull lies above or on the line

H lies above v or line

By ** $\hat{\underline{p}}^T \underline{k} \leq v \;\; \forall\, k \in N_v$

$\therefore$ N_v lies below or on the line

$\therefore$ $\hat{\underline{p}}^T \underline{x} = v$ separates $H^{\text{convex hull}}$ and N_v and all of intersection - in particular

$A\hat{\underline{q}}$ lies on the line

So v is (unique) value for which N_v and H intersect on a line $\hat{\underline{p}}^T \underline{x} = v$

So we solve by finding smallest value of x such that $x_1 = x_2$ intersects H

e.g. We see $x_1 = x_2$ intersects H on $C_2 - C_3$ line

$\therefore$ $C_2 - C_3$ $= x_2 = \frac{1}{2}(17 - 5x_1) \Rightarrow$ eq^n of line

Solve with $x_1 = x_2$ $= 17/7$ same values as before $\Rightarrow v = 17/7$

Line $\hat{\underline{p}}^T \underline{x} = V$ with H is also the line $C_2 \Rightarrow C_3$ e.g. $5x_1 + 2x_2 = 17$

e.g. $5/7 x_1 + 2/7 x_2 = 17/7$

probability of choosing 1   probability of choosing 2

C plays C_2 and C_3 and solves $\hat{q}\binom{?}{?} + (1-\hat{q})\binom{1}{6} = \binom{17/7}{17/7} \Rightarrow \hat{q} = 5/7$

NON ZERO SUM GAMES

Suppose each player chooses as before, but payoff to R is not necessarily minus the payoff to C.

Example:

		C: 1	C: 2	
R	1	(2,1)	(0,0)	(p)
	2	(4,0)	(1,2)	(1-p)
		(q)	(1-q)	

$(\cdot,\cdot)$ Payoff in utilities to R and C respectively. (R, C)

28/01/16 DECISIONS BRETT

- Mixed strategies allowed: R plays 1 with probability p and 2 with probability $(1-p)$
- C plays 1 with prob q and 2 with prob $(1-q)$
- They determine p and q independently

Equilibrium: Given one player plays his equilibrium strategy, the other one can't do better than playing his also: Nash - Such equilibrium strategies exist (uniquely)

Example (contd): Payoff to R: $2pq + (1-q)(1-p)$
$p^* = 2/3 \Rightarrow$ value which makes q cancel out.

Payoff to C: $1(p)(q) + 2(1-p)(1-q)$
$q^* = 1/3 \rightarrow$ value which makes p disappear

Play $p^* = 2/3$ $q^* = 1/3$
Payoff to R is $2 \times \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} = 2/3$
Payoff to C $: 2/3$
equilibrium strategy - cannot be affected by what opponent does but mightn't be the best

BUT $p = q = 1/2$ is better for both!

Example:

		Hugo dont confess	Hugo confess
Victor	dont confess	$(1,1)$	$(4,0)$
	confess	$(0,4)$	$(3,3)$

V and H have committed a crime together. They should confess or not confess (interval independently). The entries in table are years in prison for V, H respectively, loss = -utility
V: confess? H: confess?

If H confesses $\Rightarrow$ confess (3yr) better than 4yr.
If H doesn't confess $\Rightarrow$ confess (0yr) better than 1yr.

∴ confessing is always the better, hence a strictly dominating strategy
Here it is better for both to play best individual strategy (both choose confess)

28/04/16 DECISIONS : BRETT

GROUP DECISION MAKING

Example:
Population want to choose a party. Each person gives preference
e.g. SF $\succ$ FF $\succ$ FG
How do we combine all votes to get group preference?

Possible preference: Majority rule - e.g. group prefers SF to FF if more than 50% vote as so prefer.

Example:
Votes: 1 : SF $\succ$ FF $\succ$ FG
2 : FF $\succ$ FG $\succ$ SF
3 : FG $\succ$ SF $\succ$ FF

Majority rule $\Rightarrow$ SF $\succ$ FF (2:1) FF $\succ$ FG (2:1) and FG $\succ$ SF (2:1)

So group is a money pump - i.e. willing to pay for a different choice $\Rightarrow$ pay 3 times - end back at the start

Basic problem of 'social choice' theory. Each individual in group has personal preference ranking over a collection of alternatives
How can we combine these into a group preference ranking 'fairly'?

Eg Collection of rewards $r_1, \dots, r_n$ and group has m members and person i has preference ranking $\succsim_i$ (prefer or indifferent to)
Collection $(\succsim_1, \dots, \succsim_m)$ is preference profile of group

Each personal preference ranking satisfies:
O_1 For each pair (r_i, r_k) for each profile $\succsim_k$ we must have one and only one of the following: $r_i \succ_k r_j$, $r_j \succ_k r_i$ or $r_i \sim_k r_j$

D2: $r_i \succsim_k r_j$ and $r_j \succsim_k r_s$ $\Rightarrow r_i \succsim_k r_s$ etc (transitivity)
$r_i \succ r_j$ and $r_j \succsim r_s$ $\Rightarrow r_i \succ_k r_s$ (must be sure is good a choice)

Defⁿ: A social welfare function (SWF) is a function which operates on a collection of individual preference profiles, each of which obeys (D1) and (D2) and yields a group or social ordering $\succsim_g$ which also obeys (D1) and (D2)

Note: For small number of individuals, this is a group decision problem.
For many individuals, it is a social choice problem.

ARROW
Arrow suggests 4 'reasonable' conditions on SWF

1. Unrestricted Domain (U): $\succsim_g$ is defined and obeys D1 and D2 for any collection $(\succsim_1, \dots, \succsim_m)$ which all obey D1 and D2

2. No Dictatorship (D): No individual i such that $\succsim_i$ automatically becomes $\succsim_g$

3. Pareto Principle (P): $r_i \succ_k r_j$ for all R $r_i \succ_g r_j$ Group decision - if everyone likes Check, the group likes Check.

4. Independence of Irrelevant Alternatives (I): Suppose some rewards are deleted from the reward set. Then if no individual changes their preferences between rewards that remain, the group preference between remaining alternatives doesn't change

Comment on I: ½ people have FG > FF > SF ½ have FF > SF > FG
SF drops out, now have ½ FG > FF and ½ FF > FG

This ignores the fact that FF are in the top 2 preferences always

28/04/16

DECISIONS: BRETT

Compare 2nd Constituency: ½ have FG > new FG > FF half have FF ~~FG~~ > new FG thus opposite conclusion, manipulates to get FG in top 2 always "transferrable vote". But possible preference between FG and FF are the same in each constituency. e.g. Arrow argued it is wrong to view FF preference to FG as stronger than FG preferent to FF.

Can be manipulated by voting strategies - government.

Arrow's Impossibility Thm

Provided there are at least 3 possible rewards and at least 2 individuals, then there is no SWF which meets all of conditions U, P, I, I

Note: Means that for any social welfare function there are sets of preferences which break at least one condition

However, if only 2 rewards to majority rule does satisfy U, d, P, I.

Utilitarianism:

Social choices should attempt to maximise 'well-being', happiness or pleasurable feelings of citizenry

Given a choice between social options, place at top option which produces most pleasure overall and bottom least pleasure overall

How to measure for an individual?

How do you combine individual pleasure? Sum or product?

A possible way of measuring individual pleasure is by utility. How can we combine group utilities?

Suppose m citizens, r social choices $(x_1, \ldots, x_r) = X$. Each citizen i individually rational and so, has a utility on X (μ_i)

Scale each utility, so as to lie in $[0,1]$

0: utility of worst outcome, 1 utility of best.

Example:

Council have money for exactly 1 of: Swimming pool (S), library (L), car park (C), museum (M) or nothing (N)

Town has 2 citizens

	S	L	C	M	N
U_1	1	0.5	0	0.5	0
U_2	0.1	0	1	0	0

Introduce Planner: knows citizens utility and makes plan on their behalf.

Planner (P) individually rational, so has utility W on X

P wants to make a fair choice for community

Suppose P obeys following conditions for 'social rationality':

A Anonymity: P doesn't know which citizen made which vote. P would create the same W utility for any permutation of votes $\mu_1, \mu_2, \ldots$

SP Strong Pareto: If each citizen indifferent between 2 rewards, then so is planner. Then so if planner. If nobody prefers a to b, some people prefer b to a then P prefers b to a.

Now: $U_1(L) = \mu_1(M)$ $L \sim M$

$U_2(L) = \mu_2(M)$ $L \sim M$

$\Rightarrow L \sim N$ by (SP)

planner indifferent to the choice

Now construct for each option a vector of utilities $\mu(x_i) = (\mu_1 x_i, \mu_2 x_i, \ldots)$

e.g. ~~$U_s = U_{..}$~~

$u_s = \binom{1}{0.1}$ $u_L = \binom{0.5}{0}$ $u_c = \binom{0}{1}$ $u_m = \binom{0.5}{0}$ $u_n = \binom{0}{0}$

DECISIONS : BRETT

Note that if $u(x_j) = u(x_k)$ ^same weight^ then $u_i(x_j) = u_i(x_k)$ for all i

So $x_j \sim x_k$ for all i, so $x_j \sim x_k$ (SP)

So $w(x_j) \sim w(x_k)$ must give same utility

so $w(x_j)$ must be a function of $\mu(x_j)$

e.g. $w(x_j) = f(u(x_j))$ for some function f.

Th^m^ Harsanyi

Group of individuals, collection of rewards, individual i's perfect utility $\mu_i(x)$ for each x → scaled $[0,1]$

Planner wants to construct utility W, obeying (A) and (SP) then the only choice for W is

$W(x) = \mu_1(x) + \dots \mu_n(x)$ for all x

		S	L	C	M	N
eg	U_1	1	0.5	0	0.5	0
	U_2	0.1	0	1	0	0
		1.1	0.5	1	0.5	0

first S then C then L or M

$S \Rightarrow C \Rightarrow L/M \Rightarrow N$

	C_2	C_3	
R_1	-3	1	1/3
R_2	4	-4	2/3
	q	$(1-q)$	

$$-3(1/3)q + 1/3(1-q) + 4(q)(2/3) - 4(2/3)(1-q) = -2/3$$
$$-q + 1/3 - 1/3q + 8q/3 - 8/3 + 8q/3 = -2/3$$
$$12q/3 - 7/3 = -2/3$$
$$12q/3 = 5/3$$
$$12q = 5$$
$$q = 5/12 \qquad 1-q = 7/12$$

C plays C_2 with probability 5/12 and C_3 with probability 7/12
R plays R_1 with prob 1/3, R_2 with probability 2/3

Q2 Burglar Example
Suppose Betty plays the following strategy: She will never confess to a crime until she knows that Bill has confessed to a crime. As soon as she knows Bill has confessed, she confesses to all crimes from that point onward. This strategy makes it less advantageous for Bill to confess, as he is punished most severely if he does.

~~Supposing Betty plays that strategy, she will never confess to a crime until she knows that Bill has confessed a crime~~

Supposing Betty plays that strategy, should Bill ever confess? If he confesses then on that play he gets r years (as Betty does not confess) and on every further play he gets x years (As Betty confesses, the best Bill can do is confess) so because the probability of playing game k+1 is p^k, Bill's expected payoff from the moment of confessing is

$$G = r + px + p^2x + p^3x + \ldots$$

If Bill does not confess, nor does Antony, then payoffs are μ years and Bill's expected payoff is $C_2 = \mu + \nu p + \nu p^2 + \dots$

It is optimal for Bill not to confess (and hence never confess) if $C_2 < C_1$, so if $p > \frac{\mu - \nu}{x - \nu}$

Therefore we can always find values $p \in [0,1]$ for which not to confess is optimal. It is an equilibrium strategy as no player can improve things for him/her by moving away from this strategy for such values of p.
For other values of p it is harder to analyse 'covert play'

1 Nash Bargaining Axioms

N1 Individual Rationality: $(\bar{\mu}_S, \bar{\nu}_S) \geq (\mu_S^*, \nu_S^*)$ [NB $(a,b) \geq (c,d) \Rightarrow a \geq c,\ b \geq d$]

N2 Feasibility $(\bar{u}_S, \bar{v}_S) \in S$

N3 Pareto Optimality if $u > \bar{u}_S$ or $v > \bar{v}_S$ then $g(\mu, v) > g(\bar{\mu}_S, \bar{v}_S)$. Since the Nash point $(\bar{u}_S, \bar{v}_S)$ maximises $g(\mu, v)$ on S it must be on the Pareto Boundary

N4 Invariance of Equivalent Representations: If we replace (u,v) by some linear transformation $(\alpha\mu + \beta, \gamma v + \delta)$, i.e. choosing $(\bar{u}, \bar{v})$ for old S, we should choose $(\alpha\bar{\mu} + \beta, \gamma\bar{v} + \delta)$ for new S.
e.g. if $F(S, u_S^*, v_S^*) = (\bar{u}_S, \bar{v}_S)$ then in new representation with feasible region T and status quo u_T^*, v_T^* constructed by replacing each (u,v) by $(\alpha\mu + \beta, \gamma v + \delta)$ then our requirement is
$F(T, \alpha\mu_S^* + \beta, \gamma v_S^* + \delta) = (\alpha\bar{\mu}_S + \beta, \gamma\bar{v}_S + \delta)$ LINEAR ONLY i.e. Monetary value

N5 Symmetry suppose S is symmetric $(\mu, v) \in S$ $(v, \mu) \in S$ and $u_S^* = v_S^*$ then $\bar{u}_S = \bar{v}_S$ e.g. only shape S and values μ^*, v^* affect choice, not external circumstances i.e. not skill at bargaining or how much money I already have

29/04/16

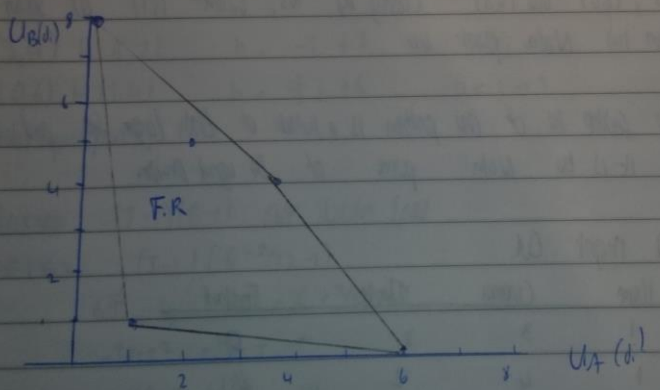
DECISIONS: BRETT EXERCISES

NB: Independance of Irrelevant alternatives: Suppose $f(T, u^*, v^*) = (\bar{u}_T, \bar{v}_T)$ e.g. we jointly chose meat over fish. Suppose $T \subset S$ (new vegetarian option). Suppose $f(S, u^*, v^*) = (\bar{u}_S, \bar{v}_S)^{\text{soln}} \in T$ then $(\bar{u}_S, \bar{v}_S) = (\bar{u}_T, \bar{v}_T)$

e.g. either stick with meat or now bargain for some choice of vegetarian but NOT fish

2. 4 possible joint decisions with status quo d_0

	d_0	d_1	d_2	d_3	d_4
$U_A(d_i)$	1	6	4	0	2
$U_B(d_i)$	1	0	4	8	5



Pareto boundary consists of line segment from (0,8) to (4,4) and (4,4) to (6,0)

$$\max g(u_A, u_B) = (u_A - u_A^*)(u_B - u_B^*)$$
$$= (u_A - 1)(u_B - 1)$$
$$= u_A u_B - u_A - u_B + 1$$

0,8   6,0

$\frac{-8}{6}$ $-4/3$

$u_A = -4/3 u_B$

$6 = -4/3 u_B + 6$

$u_A = -4/3 u_B + 6$

$\overset{A\,B}{(0,8)}$ to $\overset{A\,B}{(4,4)}$ $8 = m(0) + c$ $c = 8$

$4 = m4 + 8$ $u_A = -u_B + 8$

$m = -1$

(4,4) to (6,0) $u_A = -2u_B + 12$

29/04/16

$$g(u_A, u_B) = u_A u_B - u_A - u_B + 1 \qquad u_A = -u_B + 8$$
$$(u_B + 8)(u_B) - (-u_B + 8) - u_B + 1 \qquad u_A = -2u_B + 12$$
$$-u_B^2 + 8u_B + u_B + 8 - u_B + 1 \qquad B + 8 = -2B + 12$$
$$-u_B^2 + 8u_B + 9 \qquad 3B = 4$$
$$\beta = 4/3$$
$$\frac{du}{du_B} = -2u_B + 8 = 0$$
$$u_B = 4$$

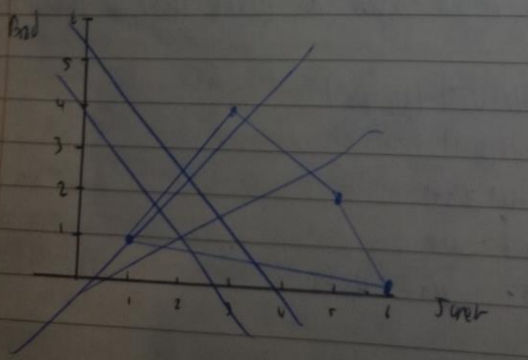
Nash point is d_2 with $(4,4)$
The equilibrium distribution point is 3.5, 4.5 and corresponds to playing d_2 with probability 7/8 and d_3 with probability 1/8

Geometric argument: take a new problem with as feasible set the convex hull of (0,0), (8,0) and (4,8). Clearly by N5, with (1,1) as status quo point this problem has Nash point 4,4

However the feasible set of this problem is a subset of this larger set, and 4,4 is in it, hence by N6 it is the Nash point of the original problem

3 Planning a night out

	Home	Cinema	Theatre	Football
Janet	1	3	5	0
Brad	1	4	2	6



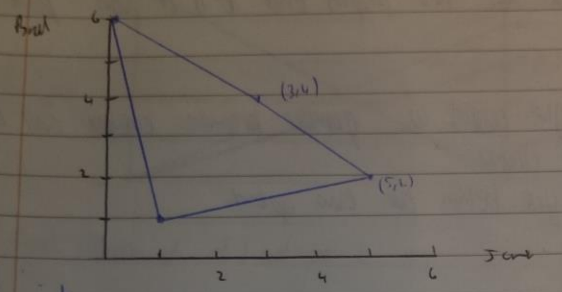
DECISIONS - BRETT

Pareto border (3,4) to (5,2) and (5,2) to (0,6)

(3,4) to (5,2) = $b = 7 - j$ for $3 \le j \le 5$

~~(5,2) to (0,6)~~

(0,0) to (3,4)



j b

(3,4) to (5,2) $b = -j + 7$ $3 \le j \le 5$

(0,6) to (3,4) $b = \frac{-2}{3}j + 6$ $0 \le j \le 3$

Status quo point = (1,1)

Maximise $(J-1)(B-1)$ over solution space

$0 \le j \le 3$ $(J-1)(6 - \frac{2}{3}J - 1)$

$6J - \frac{2}{3}J^2 - J - 6 + \frac{2}{3}J + 1$

$-\frac{2}{3}J^2 + \frac{17}{3}J - 5$

$-\frac{4}{3}J + \frac{17}{3} = 0$

$4J = 17$ $J = 17/4 = 4.25$ for $0 \le j \le 3$.

$(3-1)(5-1) = 6$

$3 \le j \le 5$ $(J-1)(7-J-1)$

$7J - J^2 - \cancel{J} - 7 + \cancel{J} + 1$

$7J - J^2 - 6$

$7 - 2J = 0$

$J = 7/2 = 3.5$ for $3 \le j \le 5$

Max is achieved when closest to 3.5 in both cases

~~Nash point is (3,4)~~ $(3.5-1)(3.5-1) = 6.25$ = nash point

(3.5, 3.5) maximum value of the game

29/04/16

- Geometric solution given by extend feasible region set by taking the line segment $b = 7 - a$ for $0 \le a \le 7$
- Clearly symmetry axiom N5 implies that $7/2, 7/2$ is the Nash point corresponding to this extended feasible set
- But, as this point was also in the original feasible set, which is a subset of this larger set, N6 implies that it is the Nash point of the original problem

- Both (B) and (C) give bargains as gambles between cinema and dance, heavily weighted towards cinema
- So they should play a game between the two options.

B	3	5	p
B	5	2	$1-p$
	q	$1-q$	

$U_B = \cancel{3p + 5p} = 3q + 5 - 5q = 5 - 2q = 6.25$

$U_A =$

$-2q = -1.25$

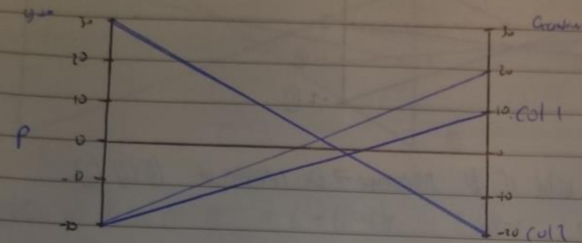
$q = 0.725 \quad 1-q = 0.295.$

29/04/16

DECISIONS EXERCISE

1.

		H	T
You	H	-20	30
	T	10	-20



Find intersection of col 1 and col 2

Col 1: $(-20, 0)$ $(10, 1)$ $y = m(0) + (-20)$ $= y = -20$

$\frac{y_2 - y_1}{x_2 - x_1}$ $\frac{30}{1} = 3$ $y = m(1) + 10$ $-20 = m + 10$ $m = -30$

Col 1 result $= -20 + (+10 - -20)p$

$= -20 + 30p$

Col 2 $r = 30 + (-20 - 30)p$

$= 30 - 50p$

Solve $-20 + 30p = 30 - 50p$ $= \checkmark$

$80p = 50$

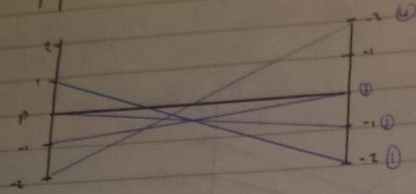
$p = 5/8$

$\rightarrow$ $30 - 50p$ $30 - 50(5/8)$ $= -1.25$ = value of game

- If you are forced to play, play your minimax strategy

1 B

		Gambler HH	HT	TH	TT
you	H	1	0	−1	−2
	T	−2	−1	0	2



Chose the maximum value of the minimum ⇒ the intersection of 1 2 3

Col 1: $1 + (-2-1)p \quad 1-3p$

Col 2: $0 + (-1)p = -p$

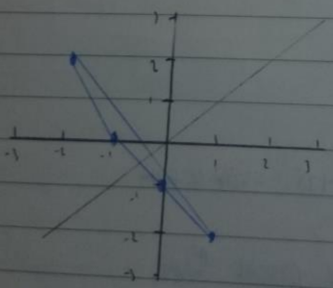
Col 3: $-1 + 0(+1)p = -1+p$

$$-p = -1+p$$
$$-2p = -1 \quad p = 1/2$$

⇒ value $= -p = -1/2$

- Should not play game.
- If you are, pick H with probability 1/2 and T with probability 1/2
- Chose between option 1, 2, 3 not 4

- Gambler can chose any mixture of the 3 choices which includes one line segment which increases as a function of p and one which decreases as a function of p



$$\begin{pmatrix}1\\-2\end{pmatrix}q_1 + \begin{pmatrix}0\\-1\end{pmatrix}q_2 + \begin{pmatrix}-1\\0\end{pmatrix}q_3 + \begin{pmatrix}-2\\2\end{pmatrix}q_4$$

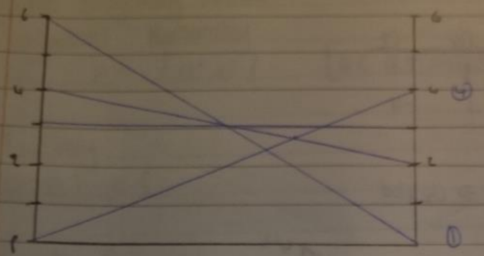
convex hull of vectors at $q = 1$ ⇒ [illegible]

Find value where $x_1 = x_2$ intersection of 2 3

$-p = 1+p \quad p = 1/2 \qquad V = -1.25$

DECISIONS - BRETT EXERCISE

2 Minimax Other Graphical Question



Intersection of ① and ④ ① $= 6 + (0-6)p = 6-6p$

④ $= 0 + 4p = 4p$

$6-6p = 4p$

$10p = 6 \quad p = 6/10$

①	④	
6	0	p
0	4	$(1-p)$

Value $= 6-6\left(\frac{6}{10}\right) = 2.4$

$6p + 4 - 4p = 2.4$

$2p = 2.4 - 4$

	①	④	
①	6	0	p
②	0	4	$1-p$
	q	$1-q$	

$6pq + 4(1-p)(1-q) = 2.4 \quad p = 0.6$

$6(0.6)(q) + 4(0.4)(1-q) = 2.4$

$3.6q + 1.6 - 1.6q = 2.4$

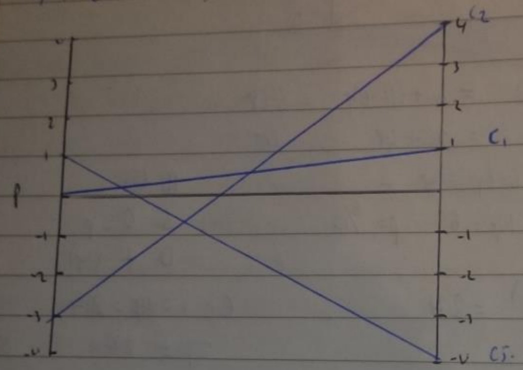
$2q = .8$

$q = 0.4 \quad 1-q = 0.6$

Q 3 Zero sum, payoff to R, negative to C

		C			
	C1	C2	C3	C4	C5
R1	0	−3	6	3	1
R2	1	4	−2	−1	−4

C3, C4 dominated by C5 ⇒ deleted



Intersection of C_2 C_5

C_2 : $-3+7p$

C_5 : $1-5p$

$v = -3+7p = 1-5p$

$12p = 4$

$p = 1/3$

$v = -3+7p = -2/3$ Shouldn't play

	C1	C5	
R_1	0	1	p
R_2	1	4	$1-p$
	q	$1-q$	

$p(1-q) + q(1-p) + 4(1-q)(1-p) = -2/3$

$\frac{1}{3} - \frac{1}{3}q + q - \frac{1}{3}q + (4-4q)(2/3) = -2/3$

$1/3 + 1/3q + 8/3 - 8/3q = -2/3$

$-7/3q = -11/3$

$-7q = 11$

$q = 11/7$

$1/3(1-q) + q(2/3) + 4(2/3)(1-q) = -2/3$

$1/3 - 1/3q + 2/3q + 8/3 - 8/3q = -2/3$

$-9/3q = -11/3$

03/05/16

DECISIONS: BRETT

Nash Bargaining Axioms:

1. Individual Rationality

$(\bar{\mu}_S, \bar{v}_S) \geq [\mu_S^*, v_S^*]$ $\quad$ [NB $(A,B) \geq (C,D) \Rightarrow A \geq C,\ B \geq D$]

2. Feasibility

$(\bar{u}_S, \bar{v}_S) \in S$

3. Pareto Optimality

If $u > \bar{u}_S$ and $v > \bar{v}_S$ then $g(\mu,v) > g(\bar{\mu}_S, \bar{v}_S)$. Since the nash point $(\bar{u}_S, \bar{v}_S)$ maximises $g(\mu,v)$ on S it must be on the pareto boundary

4. Invariance of Equivalent Representations

If we replace (u,v) by some linear transformation $(\alpha\mu+\beta, \gamma v+\delta)$, the choosing $(\bar{u},\bar{v})$ for old S, we should choose $(\alpha\bar{\mu}+\beta, \gamma\bar{v}+\delta)$ for new S

Eg. if $f(S, u_S^*, v_S^*) = (\bar{u},\bar{v})$ then in new representation with feasible region T and status quo u_T^*, v_T^* constructed by reducing each (μ,v) by $(\alpha\mu+\beta, \gamma v+\delta)$ then our requirement is:

$f(T, \alpha\mu_S^* + \beta, \gamma v_S^* + \delta) = (\alpha\bar{\mu}_S+\beta, \gamma\bar{v}_S+\delta)$

Linear only $\Rightarrow$ i.e. monetary value irrelevant.

5. Symmetry

Suppose S is symmetric $(\mu,v) \in S$ $(v,\mu) \in S$ and $U_S^* = V_S^*$ then $\bar{u}_S = \bar{v}_S$

E.g. only shape S and values μ^*, v^* affect choice, not external circumstances. i.e. Doesn't matter the status of bargain or how much money I already have

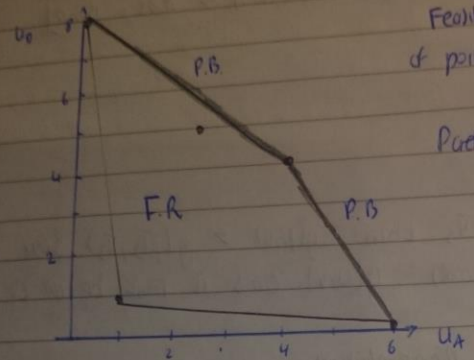
6. Independence of Irrelevant Alternatives

Suppose $f(T, u^*, v^*) = (\bar{u}_T, \bar{v}_T)$ (eg jointly choose meat or fish). Suppose $T \subset S$ (now have new vegetarian option) Suppose $f(S, \mu^*, v^*) = (\bar{\mu}_S, \bar{v}_S) \in T$ then $(\bar{u}_S, \bar{v}_S) = (\bar{u}_T, \bar{v}_T)$

e.g either stick with meat or new bargain has some change but NOT vegetarian

Q2.

	d_0	d_1	d_2	d_3	d_4
$U_A(d_i)$	1	6	4	0	2
$U_B(d_i)$	1	0	4	8	5



Feasible region is convex hull of points $(1,1)$, $(6,0)$, $(4,4)$, $(4,8)$, $(2,5)$

Pareto boundary: line segments: $[(6,0),(4,4)]$
$[(4,4),(0,8)]$

Nash Point: Max $(U_A-1)(U_B-1)$

line $(0,8)$ to $(4,4)$ Slope $= \frac{4-8}{4-0} = -1$ $B = -1A + C$ $\underset{(0,8)}{A\ B} \Rightarrow 8 = C$

line: $B = -A + 8$

line $(4,4)$ to $(6,0)$ Slope $= \frac{0-4}{6-4} = -2$ $B = -2A + C$ $(6,0)$ $0 = -12 + C$ $C = 12$

line: $B = -2A + 12$

For B $4 \le B \le 8$ Max $(U_B - 1)(8 - B - 1)$ $(U_B - 1)(U_A - 1)$

$(U_B - 1)(7 - U_B)$ $AB - B - A + 1$ $B = -A + 8$

$7B - B^2 - 1 + B$ $A(8-A) + (-A+8) - A + 1$

$\frac{dB}{dy} =$ $7 - 2B = 0$ $8A - A^2 - A + 8 - A + 1$

$B = +2/7$ $A =$ $6A - A^2 + 9$

$6 - 2A = 0$

$AB - B - A + 1$ $A = 8 - B$	$AB - B - A + 1$ $A = 12 - \frac{1}{2}B$ $A = 1/3$ $B = 7^{2/3}$
$(8-B)B - B - (8-B) + 1$	$(12 - \frac{1}{2}B)(B) - B - (12 - \frac{1}{2}B) + 1$
$8B - B^2 - \cancel{B} - 8 + \cancel{B} + 1$	$12B - \frac{1}{2}B^2 - B - 12 + \frac{1}{2}B +$
$8 - 2B = 0$	$11.5 - B = 0$
$B = 4$	$B = 11.5$

$B = -A + 8 \quad 4 \leq B \leq 8$

Max $(A-1)(B-1)$

$(A-1)(-A+8-1)$

$-A^2 + 7A + A - 8$

$-2A + 8 = 0$

$A = \cancel{4} \quad B = \cancel{7} \; 4$ vole $= 9$

& $(2/7 - 1)(7\,5/7 - 1)$

~~$(-5/7)(6\,5/7) = -5$ vole~~

$B = -2A + 12$

Max $(A-1)(B-1)$

$(A-1)(-2A+12-1)$

$(A-1)(-2A+11)$

$-2A^2 + 11A + 2A - 11$

$-4A + 13 = 0$

$A = \cancel{4\,1/3} \cancel{\Rightarrow B =} \; 11\,5/13$

$\cancel{A = 6.5 \quad B =}$ $A = 3.25 \quad B = 5.5$

vole $= 2.25 * 4.5 = 10.125$

$0 \leq B \leq 4$ offers higher vote at gove $\Rightarrow (4, 4)$

	d_2	d_3	
	4	0	p 0.5
	4	8	$1-p$ 0.5
	q	$1-q$	

$4pq + 4q(1-p) + 8(1-q)(1-p) = 0$

intersection $-2A + 12 = B \quad -A + 8$

$12 - 8 = A \quad A = 4$

$-2A + 12 = -A + 8$

$-A = -4$

$4p + 4(1-p) + 8(1-p) = 10.125$

$\cancel{4p} + 4 - \cancel{4p} + 8 - 8p = 10.125$

$-8p \quad 1.875$

$p =$

line $0, 8 \quad y = 0 + (8-0)p \quad y = 8p$

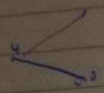
line $(4, 4) \quad y = 4 \quad \boxed{8p = 4 \quad p = 1/2}$

$2q + 2q + \cancel{4-4q}\; ^{4-4q} = 10.25$

d_2	d_3		
~~4~~ 4	0	p	$1/2$
4	8	$1-p$	$1/2$
q	$1-p$		

$4(1/2)q \quad + 0 \quad + 4(1/2)q \quad + 8(1-p)\, 1/2 = 10.125$

$2q \quad + 2q \quad -4 - 4q =$

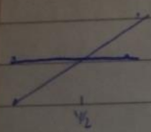


$y = 4 + (0-4)p \qquad y = 4 - 4p$

$y = 4 + (8-4)p \qquad y = 4 + 4p$

$4 - 4p = 4 + 4p$

$4p = p0$



$y = 4$

$y = 0 + 8p \qquad p = 1/2$

$4pq \quad 10p(1-q) \quad + 4(q)(p) \quad + 8(p)(1-q)$

$2q \quad + 2q \quad + 4 - 4q = 0.$

$4p(1/2) + 4p(1/2) + 8(1/2)(1-p)$

$(7 \quad (9,0)p \quad (4,4)(1-p) = 10.125.$

$9p - 9 = 10.125$

$p = 0.125.$

Solution $\quad d2\ \text{vote}\ (p) + d3\ \text{vote}\ (1-p) = 10.125$

$0p \quad + 9(1-p) = 10.125$

$9 - 9p = 10.125$

$p = 0.125$

$1-p = 0.875 = (1/8, 7/8)$

Brad and Jan

Pooler boundary $(3,4)$ to $(5,2)$ $b = -j + 7$ $3 \le j \le 5$

$(0,6)$ to $(3,4)$ $b = -\frac{2}{3}j + 6$ $0 \le j \le 3$

Maximise $(J-1)(\beta-1)$ $3 \le j \le 5$ $(J-1)(7-J-1)$

$(J-1)(7-j-1)$ $7J - J^2 - \cancel{J} - 7 + \cancel{J} + 1$

$(J-1)(6-j)$ $7J - J^2 - 6$

$6J - J^2 - 6 + 6j$ $7 - 2J = 0$ $J = 2/7$ $\beta = 6^{5}/7$

$6 - 2J = 0$ $12j - 2j = 0$ $j = 6$ $\beta = 1$ value $= 0$ value $=$ max

$J = 3$ $\rightarrow \beta = 4$ value $= 2 \times 3 = 6$

Maxy $(J-1)(\beta-1)$

$(J-1)(-\frac{2}{3}j + 6 - 1)$

$\cancel{-\frac{2}{3}J^2 + 5J}$

$-\frac{2}{3}J^2 + 6J - J + \frac{2}{3}J + 1$

$-\frac{4}{3}J + 6 - 1 + \frac{2}{3} = 0$

$-\frac{4}{3}J + \frac{17}{3}$

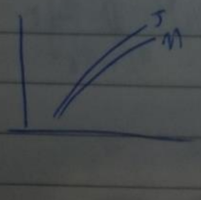
$-4J + 17 = 0$

$J = 17/4 = 4.25$ $\beta =$ value $= 6$.

Dividing a pound

$U_J = x^a$ $U_M = x^\beta$ $0 < a < \beta < 1$

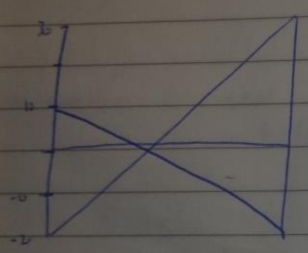
- Both John and Mary are risk averse (they avoid risk) but John is more risk averse than Mary because of the lower β value.

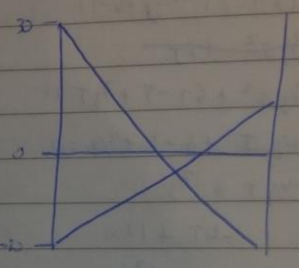


B The given curve is concave
- Clear that gambles between different points on the curve have lower expected utility for both of them and at least one other point on curve - hence such a gamble is no pareto optimal
- Now get 1 - whatever ...
= $(x^a, (1-x)^b)$

A. Gramblin

		Gramblin	
		H	T
You	H	-20	30
	T	10	-20





$(-20, 0)$ to $(1, 10)$ = $30p - 20$
$(0, 30)$ to $(1, -20)$ = $30 - 50p$
Solve $30p - 20 = 30 - 50p$
$-50 = -80p$ $\quad p = 5/8$ value $= -1.25$

$-20q(5/8) + 30(5/8)(1-q) + 10q(3/8) - 20q(3/8) = -1.25$

30/04/16

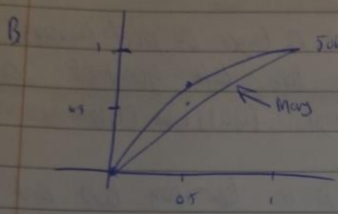
DECISIONS : BRETT

Q3 Dividing the purse

$U_J = (£x) = x^{\alpha}$ $\quad U_M = (£x) = x^{\beta}$

$0 < \alpha < \beta < 1$

Both John and May are risk averse (avoid risk) but John more so than May with a small coefficient - risk not as steep.



$\alpha = 0.5$

$\beta = 0.7$

- The given curve is concave (using example values $\alpha = 0.5$, $\beta = 0.7$)
- Clear that ~~any~~ gambles between two different points on the curve have lower expected utility for both than at least one other point on the curve hence such a gamble will not be pareto optimal

May gets 1 - what John gets

So x and $1-x$ to payoff

For $J \rightarrow x^{\alpha}$ and for M $(1-x)^{\beta}$ and to form, this is the pareto boundary

$g(u) = \cancel{(x^{\beta})(1-x)^{\beta}}$

$\cancel{x^{\beta} - x^{\beta}} =$

$$g(u, v) = x^{\alpha}(1-x)^{\beta}$$

$$\frac{dg}{dx} = (1-x)^{\beta}\alpha(x^{\alpha-1}) - x^{\alpha}\beta(1-x)^{\beta-1}$$

$$= \frac{\alpha(1-x)^{\beta}}{x^{1-\alpha}} - \frac{\beta x^{\alpha}}{(1-x)^{1-\beta}} = 0$$

$$\frac{\alpha (1-x)^{\beta}(1-x)^{1-\beta} - x^{(1-\alpha)}\beta x^{\alpha}}{x^{1-\alpha}(1-x)^{1-\beta}} = 0$$

$$\alpha(1-x) - \beta x = 0$$

$$\alpha = \frac{a}{a+b}$$ also for Mary will be $\frac{b}{a+b}$

D Now the curve is convex, hence it is no longer the pareto boundary. With both J and M risk averse, they both take gambles and the feasible region is now the triangle $(0,0),(1,0),(1,1)$.

By symmetry the Nash point is $(1/2, 1/2)$ so a fair coin toss and winner takes all (whatever values for $\alpha, \beta > 1$ are chosen).