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Maths Semester 2 JF 2012/2013

CHAPTER 7

PRINCIPLES OF INTEGRAL EVALUATION

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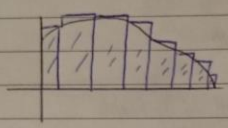
Evaluation of an integral:

$\int_a^b f(x)dx = \lim_{\Delta x_n \to 0} \sum_{i=1} f(x_n)\Delta x_n \qquad (\Delta x_n = x_n - x_{n-1})$



The fundamental theorem of calculus allows integrals to be evaluated easily

Only issue it can be difficult to figure out what $F(x)$ is eg $\cos(x^2)$

For this reason we use three other methods of integration

Three methods:

i. Substitution: Integration is simplified by using a new variable u instead of x. New variable is a function of x itself. $u = g(x)$

ii. Integration by parts: Uses the reverse (integral version) of the product rule to simplify integral

iii. Partial Fractions: Simplifies integral by splitting a single complex fraction into several simpler ones

Examples:

i. $\int x \sin(x^2)\, dx$

ii. $\int x e^x dx$

iii. $\int \frac{x^2+9x+21}{x^3+2x+4}$

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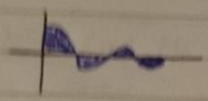
Techniques of integration (question 2) (Three parts)

The integral $\int_a^b f(x)dx$

Meaning: (i.) signed area under graph



$A - B = \int_a^b f(x)dx$

(ii) An average, Average value of $f(x)$ between a and b

$\frac{1}{b-a}\int_a^b f(x)dx$

(iii) Continuous version of a sum

$$\sum_{n=1}^{8} n^2 = \int_1^8 x^2 dx$$

If $f(x)$ was the amount of stress per meter felt at the point x, the $\int_a^b f(x)dx$ is the total amount of stress felt by the interval $[a,b]$

Fundamental Theory of Calculus:

If $f(x) = \frac{dF}{dx}$ then $\int_a^b f(x)dx = F(b) - F(a)$

Integration is the reverse of differentiation

Integration: how to sum up continuous quantities

Differentiation: How to find the slopes of tangent lines

$\Rightarrow \int e^u du$

$\Rightarrow e^{\sin(x)} + c$

iii. $u = e^{\sin x}$, $du/dx = \cos(x) e^{\sin x}$
$du = \cos(x)\, e^{\sin x}\, dx$

$\cos(x) e^{\sin x}$ appear

$\Rightarrow \int du = u + c$
$e^{\sin x} + c$

iii. $\int x^2 \cos(x^3)\, dx$

i. $u = x^2$, $du/dx = 2x$ ✗
ii. $u = x\cos(x^3)$ $du/dx = 3x^2 \sin(x^3)$ ✗
iii. $u = x^2(\cos(x^3))$ $\frac{du}{dx} = 2x\cos(x^3) - 3x^4 \sin(x^3)$ ✗
iv. $u = x^3$ $\frac{du}{dx} = 3x^2$ ✓

(coefficient) /Numbers in front don't matter, just the basic type of function

$\int x^2 \cos(x^3)\, dx$ $u = x^3$ $\frac{du}{dx} = 3x^2$ $x^2 dx = \frac{du}{3}$

$\Rightarrow \frac{1}{3} \int \cos(u)\, du$

$\Rightarrow \frac{1}{3} \sin(u)\ \ + c$

$\frac{\sin(x^3)}{3} + c$

Integration limits and Substitution:

$\int_2^4 x^2 \cos(x^3)\, dx$

Limit change with subsitution

CHANGE: If $u = g(x)$ $a \to g(a)$
$b \to g(b)$ $\int_a^b f(x)\,dx$

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$u = x^3$

$2 \rightarrow (2^3) = 8$

$4 \rightarrow (4)^3 = 64$

$\Rightarrow \displaystyle\int_8^{64} \frac{\cos(u)}{3}\, du$

Example: $\displaystyle\int_1^{e^4} \frac{\ln(x)}{x}\, dx$ $\qquad u = \ln(x) \quad \frac{du}{dx} = \frac{1}{x}$

$1 \rightarrow \ln(1) = 0$

$e^4 \rightarrow \ln(e^4) = 4$

$\Rightarrow \displaystyle\int_0^4 u\,du$

General order for attempting Substitution:

1. Logs — $\ln(x)$
2. Algebraic function — x^3, $\sqrt{x}$
3. Trignometric function — $\sin(x)$
4. Exponetial function — e^4

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Technique 1: Substitution

$\int x\sin(x^2)dx$ ← measure   i. $u = x^2$ the new variable
ii. change the measure

- To do this find the derivative of u
- Derivative $\frac{du}{dx}$ ← u measure, ← x measure

$\frac{du}{dx} = 2x$

$\Rightarrow du = 2x\,dx$

$\Rightarrow \frac{du}{2} = x\,dx$

i. $\int x\sin(x^2)dx$

ii. $\int x\sin(u)dx = \int \sin(u)\,\underline{x\,dx}$ $x\,dx = \frac{du}{2}$

iii. $\Rightarrow \int \frac{\sin(u)}{2}du$

Picking $u = x^2$ simplified $\sin(x^2)$, but still leaves a factor of x. However because $\frac{du}{dx} = 2x$ this is also simplified (it's absorbed into the measure)

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Integration by Substitution:

Examples:

1. $\int \frac{\ln x}{x}$

Substitution: i. $u = \ln(x)$ $\frac{du}{dx} = \frac{1}{x}$

or ii. $u = 1/x$ $\frac{du}{dx} = \frac{-1}{x^2}$

does not appear anywhere in Q

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{dx}{x}$

$\Rightarrow \int u \, du = \frac{u^2}{2} + c$

$\Rightarrow \frac{(\ln(x))^2}{2} + c$

You must chose a substitution $u = g(x)$, such that the derivative also appears in the integral

$\int f(x) \, dx$

integrand measure

$\frac{du}{dx}$ must appear in the integral

2. $\int \cos(x) e^{\sin(x)} dx$

i. $u = \cos(x)$ $\frac{du}{dx} = -\sin x$

ii. $u = \sin(x)$ $\frac{du}{dx} = \cos(x)$

iii. $u = e^{\sin x}$ $du/dx = \cos(x) e^{\sin(x)}$

i. $u = \cos(x)$, $du/dx = -\sin(x)$

Although $\sin(x)$ appears in $e^{\sin x}$, this is a different function, so $\frac{du}{dx}$ does not appear in the question NOT USEFULL

$u = \sin(x)$ $\frac{du}{dx} = \cos x$ $du = \cos x \, dx$

$\Rightarrow \int e^u du$

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Order of Substitution:

$$\int \frac{\ln(\cos(x))}{\cos(x)} \sin(x)\, dx$$

1. Logs
2. Algebraic
3. Trigonometric
4. Exponential

Functions in integrand

a. $\cos(x)$ Trig

b. $\sin(x)$ Trig

c. $\ln(\cos(x))$ log

Substitution:

$u = \ln(\cos(x))$

$\frac{du}{dx} = \frac{1}{\cos(x)} \cdot -\sin x \qquad du = \frac{-\sin(x)\,dx}{\cos x}$

$\Rightarrow \int \frac{\ln(\cos(x))}{\cos(x)} \sin(x)\,dx \Rightarrow \int u \frac{\sin(x)}{\cos}$

$\Rightarrow -\int u\,du$

$\Rightarrow -\int u\,du = \frac{-u^2}{2} + c$

$\frac{(\ln\cos(x))^2}{2} + c$

$\int \frac{\sin(x^2)}{x^3+1}\, dx$

a. $\sin(x^2) = \frac{du}{dx} \quad 2x\cos(x^2)$ X

b. $x^2 = 2x$ X

c. $x^3+1 = 3x^2$ X

Substitution won't work.
Need another method.

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$\int x \sin(x)dx$

Substitution: 1. $x \quad \frac{du}{dx} = 1$ × useless

2 $\sin(x) \quad \frac{du}{dx} = \cos x$ ×

- Substitution will not work
- Need integration by parts

$\Rightarrow \int u dv = u.v - \int v du$

- $u = x$
- $dv = \sin(x)dx$
- Four quantities needed

$u = x$

$\frac{du}{dx} = 1$

$du = dx$

$dv = \sin(x)dx$

$\int dv = \int \sin(x) dx$

$v = -\cos(x)$

Formulas: $\int x\sin(x)dx = -x\cos(x) - \int(-\cos(x))dx$

$\underbrace{\int x \sin(x)dx}_{\text{difficult}} = \underbrace{-x\cos(x)}_{\text{function}} + \underbrace{\int \cos(x)dx}_{\text{easy integral}}$

$\int x \sin(x) dx = -x\cos(x) + \sin x$

Order of Substitution:

1. Log
2. (Inverse functions) $\sin^{-1}(x)$, $\cos^{-1}(x)$
3. Algebraic
4. Trigonometric
5. Exponential LIATE

Second Method: Integration by Parts:

Just the product rule in reverse

Derivation: Two functions, $u(x)$, $v(x)$

$$\frac{d}{dx}u(x)v(x) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Integrate both sides:

$$\int\left(\frac{d}{dx}u.v\right)dx = \int\left(v\frac{du}{dx} + u\frac{dv}{dx}\right)dx$$

$$\int\frac{df}{dx}dx = f + c \quad \text{integration reverses differentiation}$$

$$\Rightarrow u.v + c = \int\left(v\frac{du}{dx} + u\frac{dv}{dx}\right)dx \leftarrow \text{Seperate the integral}$$

$$\Rightarrow u.v + c = \int\left(v\frac{du}{dx}\right)dx + \int\left(u\frac{dv}{dx}\right)dx$$

$$u.v + c = \int\left(v\frac{du}{dx}\right)dx + \int\left(u\frac{dv}{dx}\right)dx$$

$$\frac{du}{dx}dx = du \qquad \frac{dv}{dx}dx = dv$$

$$u.v + c = \int u\,dv + \int v\,du$$

$$\Rightarrow \int u\,dv = u.v - \int v\,du + C$$

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INTEGRATION BY PARTS

Examples:

i. $\int \ln(x) dx$

Cant do substitution:

$u = \ln(x)$ $\frac{du}{dx} = \frac{1}{x}$ derivative doesnt appr.

$\int \ln(x) dx$

$u = \ln(x)$ $dv = dx$

Formula: $\int u dv = uv - \int v du$

du: $\frac{dy}{dx} = \frac{1}{x}$, $dy = \frac{dx}{x}$

v: $dv = dx$ $\Rightarrow v = \int dv = \int dx = x = v$

$\int u dv = uv - \int v du$

$\Rightarrow \int \ln(x) dx = \ln(x) x - \int x \frac{dx}{x}$

$\Rightarrow \int \ln(x) dx = \ln(x) x - \int dx$

$\Rightarrow \int \ln(x) dx = \ln(x) x - x + C$

ii $\int x^2 e^x dx$ Choose algebraic function x^2 first.

$u = x^2$ $dv = e^x dx$

du : $\frac{du}{dx} = 2x$ $du = 2x dx$

v: $v = \int dv = \int e^x dx = e^x = v$

$\int u\,dv = uv - \int v\,du$

$\Rightarrow \int x^2 e^x dx = x^2 e^x - \underbrace{\int 2x e^x dx}$

Needs integration by parts. No substitution works.

New integral: $\int 2x e^x dx$

$u = 2x \qquad dv = e^x dx$

du: $\frac{du}{dx} = 2 \qquad du = 2dx$

v: $v = \int dv = \int e^x dx = e^x = v$

Formula: $\int u\,dv = uv - \int v\,du$

$\Rightarrow \int 2x e^x dx = 2x e^x - \int 2e^x dx$

$\int 2x e^x dx = 2x e^x - 2e^x$

Original answer: $\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx$

worked out: $\int 2x e^x dx = 2x e^x - 2e^x$

Answer: $\int x^2 e^x dx = e^2 e^x - 2x e^x + 2e^x + \underline{\underline{C}}$

METHOD:

i. $\int f(x)\,dx$, can't be performed by substitution
ii. use integration by parts, $\int u\,dv = uv - \int v\,du$
iii. Choose u via the same order used in substitution
iv. Rest of integral is dv.
v. Find du, using $\frac{du}{dx}$
vi. Find v by integrating dv
vii. Now have the four pieces of info: u, v, du, dv
use formula $\int u\,dv = uv - \int v\,du$

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3.

viii. If $\int v\,du$ is a difficult integral, use integration by parts again.

ITERATIONS:

$\int x^3 \sin(x)\,dx$

$\Downarrow$

$\int 3x^2 \cos(x)\,dx$

$\Downarrow$

$\int 6x \sin(x)\,dx$

$\Downarrow$

$\int 6\cos(x)\,dx$

Special case

$\int e^x \sin(x)\,dx$

$u = \sin(x)$ $dv = e^x\,dx$

$du: \frac{du}{dx} = \cos(x)$ $du = \cos(x)\,dx$

$dv: \; v = \int dv = \int e^x\,dx = e^x = v$

$\int u\,dv = uv - \int v\,du$

$\Rightarrow \int \sin(x) e^x\,dx = \sin(x)e^x - \int \cos(x) e^x\,dx$

$\int \cos(x) e^x\,dx$, $u = \cos(x)$, $dv = e^x dx$, $du = -\sin(x)\,dx$, $v = e^x$

Formula: $\int u\,dv = uv - \int v\,du$

$\Rightarrow \int \cos(x) e^x dx = \cos(x)e^x + \int \sin(x) e^x dx$

Original answer: $\int e^x \sin(x)\,dx = \sin(x)e^x - \int \cos(x) e^x dx$

Worked out: $\int \cos(x) e^x dx = \cos(x)e^x + \int e^x \sin(x)\,dx$

Answer: $\int e^x \sin(x)\,dx = \sin(x)e^x - \cos(x)e^x - \int e^x \sin(x)\,dx$

$\Rightarrow 2\int e^x \sin(x)\,dx = \sin(x)e^x - \cos(x)e^x$

$\Rightarrow \int e^x \sin(x)\,dx = \frac{\sin(x)e^x - \cos(x)e^x}{2}$

2.

In integration by parts the limits are the same as those of the original integral

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

Terminology: $[uv]_a^b$ is often known as the boundary term

More difficult examples:

i $\int \cos(\sqrt{x})\,dx$ Substitution will not work. $u = \sqrt{x}$ $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$u = \cos(\sqrt{x})$ $\frac{du}{dx} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$

Integration by parts: $u = \cos(\sqrt{x})$ not allowed choose function ~~not~~ inside other

$\int u\,dv = uv - \int v\,du$

NOT $\int f(u)\,dv$

This will not work, Second integral is just as difficult.

Idea: use substitution to turn the integral into an integration by parts

$\int \cos(\sqrt{x})\,dx$ (use t for substitution variable)

$t = \sqrt{x}$, $\frac{dt}{dx} = \frac{1}{2\sqrt{x}}$ $dt = 2\sqrt{x}\,dt$ $dx = 2t\,dt$

$\Rightarrow \int 2t\cos(t)\,dt$

$\Rightarrow$ Integration by parts: L A T E

i. $u = 2t$ $\frac{du}{dt} = 2$ $\frac{du}{2} = dt$

ii. v: $v = \int dv = \int \cos(t) = +\sin(t) = v$

$\int u\,dv = uv - \int v\,du$

$\Rightarrow \int 2t\cos(t)\,dt = 2t\sin(t) - \int 2\sin(t)\,dt$

$\Rightarrow \int 2t\cos(t)\,dt = 2t\sin(t) + 2\cos(t)$

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More examples

$\int_1^3 r^3 \ln(r)\,dr$ (integral for magnetic field produced from a wire)

Substitution $u = r^3$ $\frac{du}{dr} = 3r^2$ X

$u = \ln(r)$ $\frac{du}{dr} = \frac{1}{r}$

Integration by parts: L A TE

$u = \ln(r)$ $dv = r^3 dr$

Formula: $\int u\,dv = uv - \int v\,du$

$u = \ln(r)$ $v:$ $dv = r^3 dr$

$\frac{du}{dr} = \frac{1}{r}$ $v = \int dv = \int r^3 dr = \frac{r^4}{4}$

$du = \frac{dr}{r}$

$$\int u\,dv = uv - \int v\,du$$

$$\int \ln(r) r^3 dr = \frac{\ln(r) r^4}{4} - \int \frac{r^4}{4}\frac{dr}{r}$$

$$\int \ln(r) r^3 dr = \frac{\ln(r) r^4}{4} - \int \frac{r^3}{4} dr$$

$$\int \ln(r) r^3 dr = \frac{\ln(r) r^4}{4} - \frac{r^4}{16}$$

Limits: $\int_1^3 \ln(r) r^3 dr = \left[\frac{\ln(r) r^4}{4} - \frac{r^4}{16}\right]_1^3$

$$\left(\frac{\ln(3)3^4}{4} - \frac{3^4}{16}\right) - \left(\frac{\ln(1)1^4}{4} - \frac{1^4}{16}\right)$$

$$= \frac{\ln(3)81}{3} \frac{81}{16} - \left(\cancel{\ln(1)1^4} - \frac{1}{16}\right)$$

$$= \frac{\ln(3)81}{4} - \frac{80}{16}$$

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Hence $t = \sqrt{x}$

$2(\sqrt{x})\sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C$

Math Week 5 ①

PARTIAL FRACTIONS

$$\int \frac{x+5}{x^3+x-2}\,dx$$

The integral can't be evaluated using substitution or integration by parts

Substitution: $u = x^2+x-2 \quad \frac{du}{dx} = 2x+1 \quad du = (2x+1)dx$

Integration by parts: $u = \frac{1}{x^2+x-2} \quad dv = (x+5)dx$

$$\frac{du}{dx} = \frac{1-2x}{(x^2+x-2)^2} \qquad v = \frac{x^2}{2}+5x$$

$$du = \frac{(1-2x)\,dx}{(x^2+x-2)^2}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int v\,du = \int \left(\frac{x^2}{2}+5x\right) \cdot \frac{(1-2x)}{(x^2+x-2)^2}\,dx$$

Method of Partial fractions:

Integrand is a rational function

$$f(x) = \frac{P(x)}{Q(x)} \leftarrow \text{Polynomial}$$

You split $f(x)$ into a sum of simpler rational function which are easy to integrate

$$\frac{x+5}{y^2+x-2} = \frac{2}{x-1} \quad \frac{-1}{x+2}$$

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2.

$$\int \frac{x+5}{x^2+x-2} = \int \frac{2}{x-1}\,dx - \int \frac{1}{x+2}\,dx$$

$$= 2\ln(x-1) - \ln(x+2) + c$$

In general: $\int \frac{1}{x \pm a} = \ln(x \pm a)$

Splitting up a rational function:

i. A linear factor appears in the sum with a constant in the numerator.

$$\frac{(\ldots)}{(x+5)(\ldots)} = \frac{A}{(x+5)} + \ldots$$

eg. $\frac{4x+3}{(x+4)(x-2)} = \frac{A}{(x+4)} + \frac{B}{(x-2)}$

example: $\frac{5x^2+2x+3}{(x+4)(x-3)(x+2)} = \frac{A}{(x+4)} + \frac{B}{(x-3)} + \frac{C}{(x+2)}$

ii. If the linear factor is raised to a power then all powers are placed on the right hand side

$$\frac{(\ldots)}{(x+5)^3} = \frac{A}{(x+5)} + \frac{B}{(x+5)^2} + \frac{C}{(x+5)^3}$$

$$\frac{(x+2)}{(x+1)^2(x+3)^3} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+3)} + \frac{D}{(x+3)^2} + \frac{E}{(x+3)^3}$$

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iii If the factor is quadratic then a linear term with two constants is put in the numerator.

$$\frac{(\ldots)}{(x^2+2x+5)(\ldots)} = \frac{Ax+B}{(x^2+2x+5)} + \ldots$$

iv Repeat every power of a quadratic factor.

$$\frac{(\ldots)}{(x^2+2x+5)^2(\ldots)} = \frac{Ax+B}{(x^2+2x+5)} + \frac{(Cx+D)}{(x^2+2x+5)^2} + \ldots$$

Comes up in Exam

Examples: $\dfrac{x^2+2x+5}{(x^2+3x+2)(x+1)^2}$

$$= \frac{Ax+B}{(x^2+3x+2)} + \frac{C}{(x+1)} + \frac{D}{(x+1)^2}$$

Example: $\dfrac{x^3+4x+3}{(x^2-4x+5)^2(x-2)^2}$

$$= \frac{Ax+B}{(x^2-4x+5)} + \frac{Cx+D}{(x^2-4x+5)^2} + \frac{E}{(x-2)} + \frac{F}{(x-2)^2}$$

Note: Sometimes quadratic factors can be factorised into linear factors

$$\frac{(x^2+2x+1)}{(x-1)(x^2-4x+4)} = \frac{x^2+2x+1}{(x-1)(x-2)^2}$$

Makes the question shorter, but not necessary

To check: ax^2+bx+c If $b^2-4ac \geq 0$ it can be factorised

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Maths Week 6 Wednesday

$$\int \frac{2x^2+3x+3}{(x^2+x+2)(x+1)} \, dx$$

can be factorised? $b^2 - 4ac \geq 0$?
$= -7$ cant be factorised

$$\frac{2x^2+3x+3}{(x^2+x+2)(x+1)} = \frac{Ax+B}{(x^2+x+2)} + \frac{C}{(x+1)}$$

Recombine fractions:

$$\frac{Ax+B}{(x^2+x+2)} + \frac{C}{(x-1)} \Rightarrow \frac{Ax+B(x+1)+C(x^2+x+2)}{(x^2+x+2)(x+1)}$$

$$\Rightarrow \frac{(A+C)x^2 + (B+C-A)x + (2C-B)}{(x^2+x+2)(x+1)}$$

Compare numerators:

$$2x^2+3x+3 = (A+C)x^2 + (B+C-A)x + (2C-B)$$

Match coefficients:

i. $2 = A+C$
ii. $3 = B+C-A$
iii. $3 = 2C-B$

Solve: eliminate A
$A = 2-C$ (i)

Use ii and iii
$3 = B+C-A$
$3 = 2C-B$

2

Replace A.

$3 = B + C - (2 - c)$

$3 = 2c - B$

$\Rightarrow 5 = B + 2c$

$3 = B + 2c$

$8 = 4c$

$\Rightarrow \underline{c = 2}$

Find B

$3 = 2c - B$

$3 = 2(2) - B$

$\Rightarrow \underline{B = -1}$

Find A

$2 = A + C$

$2 = A + (2)$

$\Rightarrow \underline{A = 0}$

Original fraction:

$$\frac{2x^2+3x+3}{(x^2+x+2)(x-1)} = \frac{Ax+B}{(x^2+x+2)} + \frac{C}{(x-1)}$$

$$= \frac{1}{(x^2+x+2)} + \frac{2}{(x-1)}$$

Integration: $\int \frac{2x^2+3x+3}{(x^2+x+2)(x-1)} = \int \frac{1}{x^2+x+2}\,dx + \int \frac{2}{x-1}\,dx$

Integrals:

i $\int \frac{2}{x+1}\,dx = 2\ln|x-1|$ modulus to ensure positive

3 maths workd week 6

1. $\int \frac{1}{(x^2+x+2)}\,dx = (\ldots)\tan^{-1}(\ldots)$

Standard Integral

$\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x)$

relate to each other through substitution $u = \sqrt{a}\,x$

Modification:

$\int \frac{1}{ax^2+1}\,dx = \frac{\tan^{-1}(\sqrt{a}\,x)}{\sqrt{a}}$

$\int \frac{1}{x^2+x+2}\,dx$

(x^2+x-2), need to remove linear term

Completing the fraction

$(x^2+x+2) = (x+\frac{1}{2})^2 + \frac{7}{4}$

$\Rightarrow (x+\frac{1}{2})(x+\frac{1}{2}) + \frac{7}{4}$

$\Rightarrow x^2 + \frac{1}{2}x + \frac{1}{2}x + \frac{1}{4} + \frac{7}{4}$

$\Rightarrow x^2+x+2$

$\Rightarrow \int \frac{1}{(x+\frac{1}{2})^2+\frac{7}{4}}$

let $u = x+\frac{1}{2}$

$\Rightarrow \int \frac{1}{u^2+\frac{7}{4}}\,dx$

need to turn $\frac{7}{4}$ into 1:

$\frac{1}{7/4}\int \frac{1}{\frac{4}{7}u^2+1}\,du$

$= \frac{1}{7/4}\left(\frac{\tan^{-1}(\sqrt{4/7}\,u)}{\sqrt{4/7}}\right)$

$= \frac{4}{7}\left(\frac{\tan^{-1}(\sqrt{4/7}\,u)}{\sqrt{4/7}}\right)$

$= \frac{4}{7}\left(\frac{\tan^{-1}(\sqrt{4/7}\,(x+\frac{1}{2}))}{\sqrt{4/7}}\right)$

Answer: $\int \frac{2x^2+3x+3}{(x^2+x+2)(x-1)}\,dx = 2\ln|x-1| + \frac{4}{7}\left(\frac{\tan^{-1}(\sqrt{4/7}(x+\frac{1}{2}))}{\sqrt{4/7}}\right)$

Partial fractions:

1 If fraction has quadratic fraction see if they can be factorise
$ax^2+bx+c, \quad b^2-4ac<0$

2 Split the integrand into a sum of simpler fractions using rules (i) + (iv)

3 Recombine these fractions
Match the coefficients in the numerator with those in the original numerator.
$2x^3 + \ldots = (A+B)x^2 + \ldots$

4. This gives a set of equations
$2 = A+B$
Solve these to find A, B, C

5. Now integrand is sum of easier integrals
$\int \frac{2x^3+3x+3}{(x^2+x+2)(x-1)}\,dx = \int \frac{1}{(x^2+x+2)}\,dx + \int \frac{2}{(x-1)}\,dx$

6 $\int \frac{dx}{x+a} = \ln|x+a| \qquad \int \frac{dx}{ax^2+bx+c} \Rightarrow (\ldots)\tan^{-1}(\ldots)$

7. Algorithm for $\tan^{-1}(\ldots)$ integral

1

Maths Week 6 ~~wednesday~~ friday

7. Integrals of the form: $\int \frac{B}{ax^2+bx+c}\,dx$

i. complete the square.

$$ax^2+bx+c = (x+p)^2+q \quad \text{(letters just numbers)}$$

$p = b/2$

Eg $\quad x^2+4x+6 = (x+2)^2+q$

$\quad x^2-3x+5 = (x-3/2)^2+q$

Choosing q:

Multiply out square term

$x^2+4x+6 = (x+2)^2+q$

$(x+2)^2 = x^2+4x+4 \quad (q=2)$

$x^2-3x+5 = (x-3/2)^2+q$

$(x-3/2)^2 = x^2-3x+9/4$

$q = 11/4$

p: $b/2$ half of term in front of x.

q: error in constant term

$$\int \frac{dx}{(x+p)^2+q}$$

ii. Substitution, $u = x+p \qquad du = dx$

$$\Rightarrow \int \frac{du}{u^2+q}$$

Friday

Maths Week 6 wednesday

7. Integrals of the form. $\int \frac{\beta}{ax^2+bx+c}\,dx$

i. complete the Square.

$ax^2+bx+c = (x+p)^2+q$ (letters just numbers)

$p = b/2$

Eg $x^2+4x+6 = (x+2)^2+q$

$x^2-3x+5 = (x-3/2)^2+q$

Choosing q:

Multiply out Square term

$x^2+4x+6 = (x+2)^2+q$

$(x+2)^2 = x^2+4x+4$ $(q=2)$

$x^2-3x+5 = (x-3/2)^2+q$

$(x-3/2)^2 = x^2-3x+9/4$

$q = 11/4$

p: $b/2$ half of term in front of x.

q: error in constant term

$\int \frac{dx}{(x+p)^2+q}$

[illegible]itution $u = x+p$ $du = dx$

$\Rightarrow \int \frac{du}{u^2+q}$

JF 2012-2013

8/2/13 Maths Week 6

$$\frac{x+3}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$$

To find the constants: Recombine the fractions

$$\frac{A}{(x+1)} + \frac{B}{(x-2} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)} \quad \frac{(A+B)x + (B-2A)}{(x+1)(x-2)}$$

Compare this with the original form

$$\frac{x+3}{(x+1)(x-2)} = \frac{(A+B)x + (B-2A)}{(x+1)(x-2)}$$

$1 = A+B$

$3 = B-2A$

$\Rightarrow A = \cancel{B-1}\ 1-B \quad \Rightarrow 3 = B-2(1-B)$

$3 = B+2B-2$

$3B = 5$

$B = 5/3 \Rightarrow \cancel{A=2}\ A = -2/3$

$$\Rightarrow \frac{x+3}{(x+1)(x-2)} = \frac{-2/3}{(x+1)} + \frac{5/3}{(x-2)}$$

Integral: $\int \frac{x+3}{(x+1)(x-2)} = \int \frac{-2/3\,dx}{(x+1)} + \int \frac{5/2}{x+2}\,dx$

$= -2/3 \ln(x+1) + 5/3 \ln(x+2) + C$

3

iii Standard integral $\int \frac{du}{u^2+q} = \frac{\tan^{-1}\left(\frac{u}{\sqrt{q}}\right)}{\sqrt{q}}$

iv Replace $u = x+p$

$$\frac{\tan^{-1}\left(\frac{x+p}{\sqrt{q}}\right)}{\sqrt{q}}$$

8 Other integrals

a $\int \frac{Ax+B}{x^2+q}\,dx$

$$= \int \frac{Ax}{x^2+q}\,dx + \int \frac{B}{x^2+q}\,dx$$

i $\int \frac{B}{x^2+q}\,dx = \frac{B\tan^{-1}\left(\frac{x}{\sqrt{q}}\right)}{\sqrt{q}}$

ii $\int \frac{Ax}{x^2+q}\,dx$ Substitution: $u = x^2+q$

$$\Rightarrow \frac{A}{2}\int \frac{du}{u} = \frac{A}{2}\ln|u| = \frac{A}{2}\ln|x^2+q|$$

b $\int \frac{Ax}{(x^2+q)^2}\,dx$ Substitution: $u = x^2+q$

$$\Rightarrow \frac{A}{2}\int \frac{du}{u^2} = \frac{A}{2} - \frac{1}{u} = \frac{-A}{2(x^2+q)}$$

Exam: Q1 exponential functions, logs, limits

Q2 Three integrals, substitution, by parts, partial fractions, parts fraction

Q3 Sequences and series

Q4 linear algebra

JF 2012-2013

18/2/13 Maths

Example: $\int \frac{x^2+2x+5}{(x^2+2x+6)(x-2)}\,dx$

Check if quadratic can be factorised:

$a=1 \quad b=2 \quad c=6 \qquad b^2-4ac = -20 < 0$ can't be factorised

Split fraction:

$\frac{Ax+B}{(x^2+2x+6)} + \frac{C}{(x-2)}$

Recombine to find constant:

$\frac{(Ax+B)(x-2) + C(x^2+2x+6)}{(x^2+2x+6)(x-2)}$

$\Rightarrow \frac{(A+C)x^2 + (B-2A+2C)x + (6C-2B)}{(x^2+2x+6)(x-2)}$

Compare to original form:

$A + C = 1$

$B - 2A + 2C = 2$

$6C - 2B = 5$

Solve: $A = 1-C \Rightarrow \cancel{B - 2 + 2C + 2C = 2}$

$\cancel{B - 2 = 2}$

$\cancel{B = 4} \Rightarrow \cancel{6C - 2(4) = 5}$

$\cancel{6C = 13}$

$\cancel{C = 13/6}$

$\cancel{\Rightarrow 4 - 2A + 2\tfrac{13}{6} = 2}$

$\cancel{2A = 4 + 26/6 - 2}$

$\cancel{2A = 42/6}$

$\cancel{A = 21/6}$

$B = 2/7$

$C = \frac{31}{42}$

$A = \frac{1}{6} - \frac{2}{21}$

$\downarrow$

$\int \frac{(\frac{1}{6} - \frac{2}{21})x + 2/7}{(x^2+2x+6)}\,dx + \int \frac{31/42}{(x-2)}\,dx$

3

4

$\Rightarrow$ Integration by Substitution

$= \tan^{-1}(\quad) + \frac{3a}{4+2} \ln(x-2) + C$

3

2

Example: $\int \frac{2x+5}{x^2+3x+2}\, dx$

Check to see if it can be factorised:

$x^2+3x+2 \qquad a=1, \quad b=3, \quad 2=c$

Check: $b^2-4ac = 9-8 = 1 > 0$ ✓ can be factorised

$x^2+3x+2 = (x+2)(x+1)$

$\Rightarrow \int \frac{2x+5}{(x+2)(x+1)}\, dx$

Split up fraction:

$\frac{2x+5}{(x+2)(x+1)} = \frac{A}{(x+2)} + \frac{B}{(x+1)}$

Combine: $\frac{A}{(x+2)} + \frac{B}{(x+1)} = \frac{A(x+1)+B(x+2)}{(x+2)(x+1)} = \frac{(A+B)x + A+2B}{(x+2)(x+1)}$

Match both numbers.

$A+B=2 \qquad A=2-B \Rightarrow 2-B+2B=5$

$A+2B=5 \qquad B=3 \Rightarrow A=-1$

$= \frac{-1}{(x+2)} + \frac{3}{(x+1)}$

Solve integral:

$\int \frac{2x+5}{(x+2)(x+1)} = \int \frac{-1}{x+2}\, dx + \int \frac{3}{x+1}\, dx = -\ln(x+2) + 3\ln(x+1) + C$

EXAM $\int \frac{4x+5}{(x-1)(x+3)}\, dx = \int \frac{A}{(x-1)}\, dx + \int \frac{B}{(x+3)}\, dx$

$= A\ln(x-1) + B\ln(x+3) + C$