Maths Semester 1 2012 JF

# CHAPTER 0

# INTRODUCTION

## (Before Calculus)

Maths

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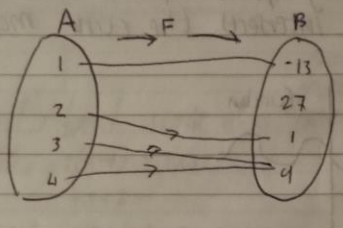
# 1. FUNCTIONS

## 1.1 What is a function?

If a variable $y$ depends on some other variable $x$, such that each $x$ determines exactly one $y$ value, then we say $y$ is a function of $x$

EXAMPLE:



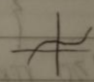

$f$ is a function

Various ways of representing functions

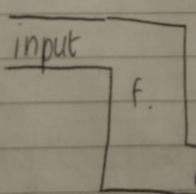
→ Numerically by some data set eg time on treadmill

| | |
|---|---|
| time on treadmill | heartrate |
| 1 min | 60 |
| 2 min | 61 |
| 3 min | 63 |

→ Verbally - "The hypotheneuse is the square root of the sum of the two squares of the two other sides"

→ Geometrically by some function 

→ Algebrically by some $f(x) = x^2 - 1$




output $f(x)$

A function $f$ is a rule that associates exactly one output with each input. For each input $x$, we denote the

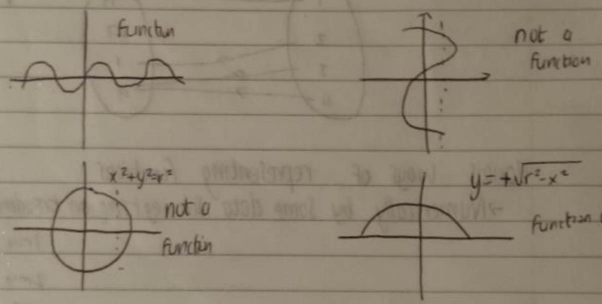
Often we denote the output by y so that $y = f(x)$

y - dependant variable
x - independant variable (argument)

## THE VERTICAL LINE TEST

- Given the graph of some curve in its xy plane, how do we know that the curve represent a function?
- A curve in the xy plane represents a function if and only if no vertical line intersects the curve more than once

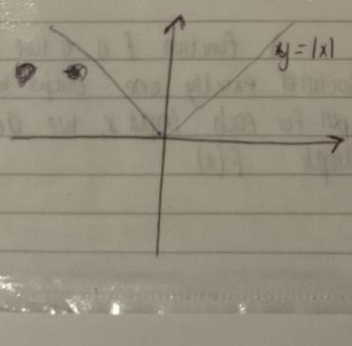
EXAMPLE




## ABSOLUTE VALUE FUNCTION:

Defined as $|x| \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$ example of a piecewise functio

So that $|x| \geq 0$ represents the unit distance of x from the origin



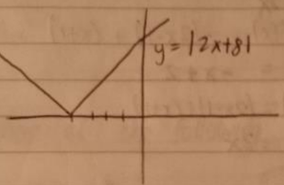
3.

Example: $y = |2x+8|$ Define without using absolute values

when $2x+8 \geq 0$ equ to $x \geq -4$ $y = 2x+8$

if $2x+8 < 0$ equv to $x < -4$ $y = -2x-8$

$$2x+8 = \begin{cases} 2x+8 & x \geq -4 \\ -2x-8 & x < -4 \end{cases}$$




Example: $F(x) = 3|x-1| + |x+1|$ write as a ~~piece~~ piecewise function

$x < -1$ $\quad f(x) = -3(x-1) - (x+1)$

$= -4x+2$

$-1 \leq x < 1$ $\quad F(x) = -3(x-1) + (x+1)$

$= -2x+4$

$x \geq 1$ $\quad f(x) = 3(x-1) + (x+1)$

$= 4x-2$

$$f(x) = \begin{cases} -4x+2 & x < -1 \\ -2x+4 & -1 \leq x < 1 \\ 4x-2 & x \geq 1 \end{cases}$$

PROPERTIES OF ABSOLUTE VALUES:

1. $|-a| = |a|$
2. $|ab| = |a|.|b|$
3. $|a/b| = |a|/|b|$
4. $|a+b| \leq |a| + |b|$ (triangle inequality)

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Example: $f(x) = |2x-1| + |x+1|$ $\quad |x| \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}$

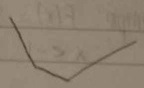
$f(x)$ has turning points at
$x = \frac{1}{2}$ and $x = -1$
Since what's inside the absolute values is zero at these points

$x < -1$ $\quad f(x) \quad -(2x-1)-(x+1)$
$\quad = -3x$

$-1 \leq x < \frac{1}{2}$ $\quad f(x) = -(2x-1) + (x+1)$
$\quad = -x+2$

$x \geqslant \frac{1}{2}$ $\quad = f(x) = (2x-1)+(x+1)$
$\quad = 3x$

$$f(x) = \begin{cases} -3x & x < -1 \\ -x+2 & -1 \leq x < \frac{1}{2} \\ 3x & x \geqslant \frac{1}{2} \end{cases}$$



## Domain and Range of a function

Def: The domain of a function $f$ is the set of all allowable inputs (all possible $x$ values). Denote by $D(f)$

Def: The range of $f$ is the set of all possible outputs as $x$ varies over the domain. Denote range by $R(f)$

The natural domain is the set of real numbers for which the function is a well defined real number.

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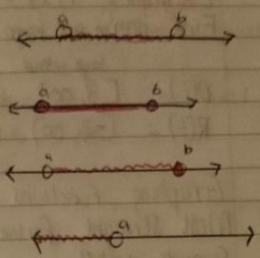
Note on representing intervals

$x \in (a,b) \Leftrightarrow a<x<b$

$x \in [a,b] \Leftrightarrow a \le x < b$

$x \in (a,b] \Leftrightarrow a < x \le b$

$x \in (-\infty, a) \Leftrightarrow$



Example:

Find the domain of the following functions:

1. $F(x) = x^5$

$D(f) = (-\infty, \infty) = \{x \in R\}$  $\boxed{\{x \in R : \ \}}$ such that

2. $F(x) = \frac{1}{(x-2)(x-7)}$

$D(f) = (-\infty, 2) \cup (2,7) \cup (7, \infty)$
$= \{x \in R : x \neq 2, x \neq 7\}$

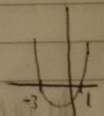
3. $F(x) = \tan x = \frac{\sin x}{\cos x}$  ill defined when $\cos x = 0$

$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$

$D(f) = \{x \in R : x \neq \pm(2n+1)\frac{\pi}{2}\}$

4. $F(x) = \sqrt{x^2+2x-3}$  (such that)

$D(f)$ is all value s.t. $x^2+2x-3 \geq 0$
$(x+1)(x+3) = 0$
root at $x=1$ and $x$-3



$D(f) = (-\infty, -3] \cup [1, \infty)$

Example: $f(x) = -2+\sqrt{x-3}$
Find domain and range

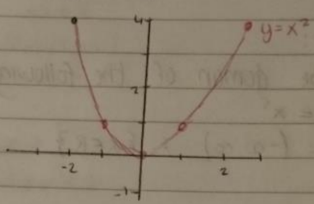
Single interval / Isolated points

$D(f) = [3, \infty] = \{x \in \mathbb{R} : x \geqslant 3\}$
$R(f) = [-2, \infty) \quad \{y \in \mathbb{R} : y \geqslant -2\}$

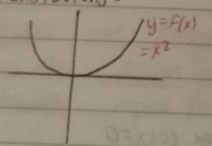
## Graphing Functions

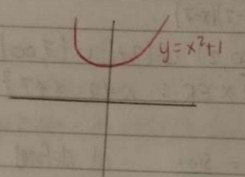
Most straight forward way to plot functions is by taking sample values

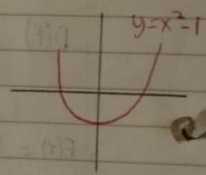
$y = x^2$
$x = -2 \quad y = 4$
$x = -1 \quad y = 1$
$x = 0 \quad y = 0$
$x = 1 \quad y = 1$

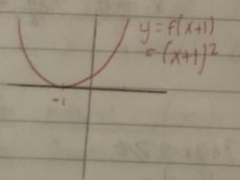


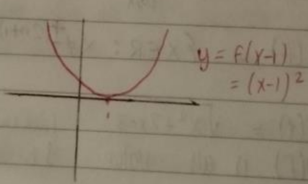

## Translations:




In general vertical shifts - $y = f(x)+k$, shifts graph of f up 'k' units when $k$ is positive and down $|k|$ units if $k<0$

Horizontal Shift - $y = f(x+h)$ shifts graph of $f(x)$ left by h units when h is positive and right by $|h|$ units when h is $h<0$

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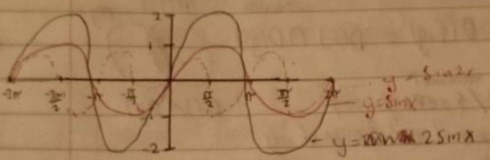
## Scaling graphs:

$y = cf(x)$ stretches the graph vertically if $c>1$ and compresses the graph vertically when $0<c<1$

$y = f(cx)$ stretches the graph of $y=f(x)$ horizontally if $0<c<1$ and compresses the graph horizontally if $c>1$

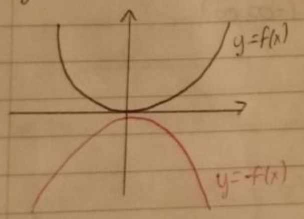
Example - $y = \sin x$ $y = 2\sin x$ $y = \sin 2x$

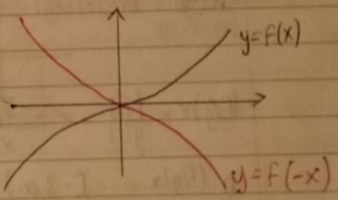



## REFLECTIONS

$y = -f(x)$ reflection of the graph of $f$ across the $x$-axis

$y = f(-x)$ is a reflection of the graph of $f$ across the $y$-axis




If $f(-x) = f(x)$ we say $f$ is an even function eg $f(x) = x^2$

If $f(-x) = -f(x)$ we say that $f$ is odd eg. $f(x) = x^3$

## COMBINING FUNCTIONS:

**Arithmetic Operations:**

Given two functions f, g we define

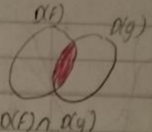
i $(f+g)(x) = f(x)+g(x)$

ii. $(f-g)(x) = f(x)-g(x)$

iii. $(f.g)(x) = f(x).g(x)$

iv $(f/g)(x) = f(x)/g(x)$

$$D(f+g) = D(f-g) = D(f.g) = \overset{\text{intersection}}{D(f) \cap D(g)}$$




$$D(f/g) = D(f) \cap D(g) \overset{\text{(exclude)}}{/} \{x \in R : g(x) = 0\}$$

x has to be in the domain of f and domain of g

but we must exclude all values of x for which $g(x) = 0$, since division by zero isn't well-defined

example: $f(x) = 2+\sqrt{x+3}$ $g(x) = x+2$

$(f+g)x = f(x)+g(x) = x+\sqrt{x+3}$

$D(f+g) = D(f) \cap D(g)$ $[-3, \infty) \cap (-\infty, \infty)$

$= [-3, \infty)$

$(f/g)x = \frac{f(x)}{g(x)} = \frac{-2+\sqrt{x+3}}{x+2}$

$D(f/g)x = [-3, \infty) / \{x = -2)$

$[-3,-2) \cup (-2, \infty)$ or $\{x \in R : x \geq -3 \; x \neq -2\}$

## COMPOSITION OF FUNCTIONS

Def: Given functions f, g the composition of f with g denoted by $f \circ g$ (f after g) (f composed g) is the function $(f \circ g)(x) = f(g(x))$

The domain of $f \circ g$ is all $x \in D(g)$ such that $g(x) \in D(f)$

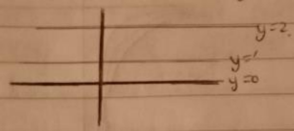
NOTE: $f \circ g \neq g \circ f$

example: $f(x) = x^2+1$ and $g(x) = \sqrt{x}$
Find $(f \circ g)x$ and $D(f \circ g)$
$(f \circ g) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2+1 = x+1$

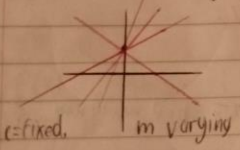
$D(g) = [0,\infty)$  $D(f) = (-\infty,\infty)$
As x varies over $D(g) = [0,\infty)$ then $g(x)$ varies over $[0,\infty) \Rightarrow D(f \circ g) = [0,\infty)$

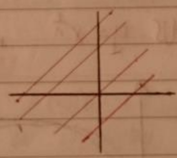
FAMILIES OF CURVES:

1. Constant functions $y=c$



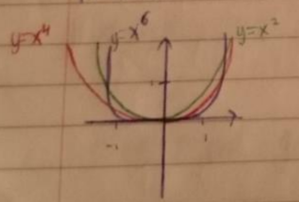

2. Linear functions $y=mx+c$



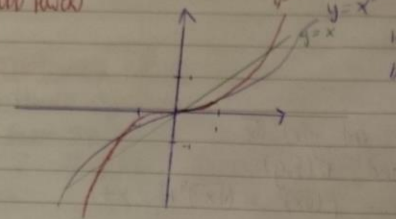




1/10/12 3. POWER FUNCTIONS: $y=x^n$ $n > 0 \in N$

n - even



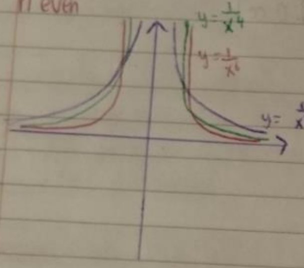
10 ODD POWER



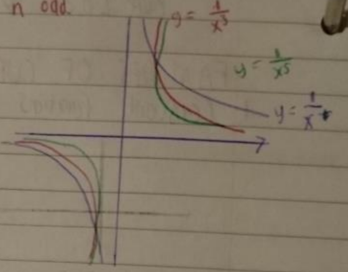

higher power flatter to 1
but then increase greatly

4. POWER FUNCTIONS $y = x^{-n} = \frac{1}{x^n}$ $n > 0$ $\in N$

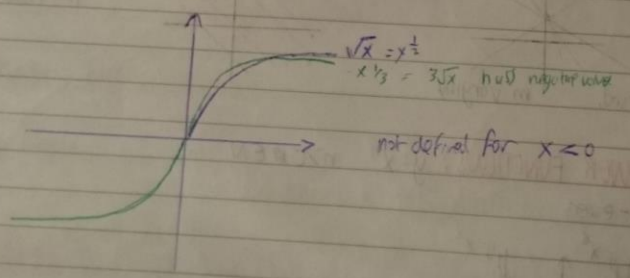
n even




n odd




5. POWER FUNCTIONS $y = x^{\frac{1}{n}}$ $n > 0$ $n \in N$




not defined for $x < 0$

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## 6. POLYNOMIALS

Functions of the following form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots a_1 x + a_0$

for $n \geq 0 \in \mathbb{N}$

n here is called the degree of the polynomial.

domain of polynomial is the entire real line

## 7. RATIONAL FUNCTIONS

let $p(x), q(x)$ be polynomials, then $f(x) = \frac{p(x)}{q(x)}$ is a rational

function domain is entire real line except where $q(x) = 0$

Domain of $f(x)$ is all $x$ except those values for which $q(x) = 0$

## 8. Algebraic Functions

- Functions constructed from polynomials by applying finitely many algebraic operations (addition, subtraction, multiplication, root extraction etc)

e.g. $f(x) = \sqrt{x^2+9}$ $f(x) = (5\sqrt{x})/(2-x)$ $f(x) = x^{3/2}(x+2)^2$ etc.

## 9. TRIGONOMETRIC FUNCTIONS

$f(x) = A\sin(Bx-C)$ $g(x) = A\cos(Bx-C)$

May be graphed by scaling and translating the graph of $\sin x$, $\cos x$

$A$ - amplitude $\frac{2\pi}{B}$ = frequency

## INVERSE FUNCTIONS:

Def$^n$: If two functions $f$ and $g$ satisfy $g(f(x)) = x$ for all $x \in D(f)$

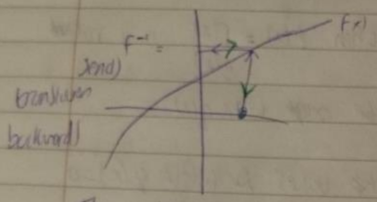
and if $f(g(x)) = x$ for all $x \in D(g)$

then $f$ and $g$ are inverse functions

If f does have an inverse, then the inverse is unique, and we denote it by $f^{-1}$.

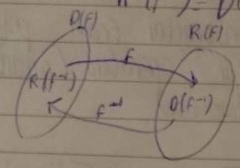
$\therefore \quad f^{-1}(f(x)) = x \qquad$ for all $x \in D(f)$ (cancel each other out)

$\qquad f(f^{-1}(x)) = x \qquad$ for all $x \in D(f^{-1})$




The domain and range of $f^{-1}$ are $D(f^{-1}) = R(f)$

$R(f^{-1}) = D(f)$




example: Show the inverse of $f(x)\ 2x^3-1$ is $f^{-1}(x)\ \sqrt[3]{\tfrac{1}{2}x+1}$

$f^{-1}(f(x)) = f^{-1}(2x^3-1) = \sqrt[3]{\tfrac{1}{2}(2x^3-1+1)} = \sqrt[3]{x^3} = x$

$f(f^{-1}(x)) = f\left(\sqrt[3]{\tfrac{1}{2}(x+1)}\right) = 2\left(\sqrt[3]{\tfrac{1}{2}(x+1)}\right)^3 - 1 = x \quad \blacksquare$

## FINDING THE INVERSE FUNCTION:

theorem: If an eq$^n$ $y=f(x)$ can be solved for $x$ as a function of $y$, $x = g(y)$ say, then f has an inverse and that inverse is $g(y) = f^{-1}(y)$

1. write down the eq$^n$ $y=f(x)$
2. Solve for $x$ as a function of $y$ (if possible)
3. We have $x = f^{-1}(y)$. Interchange $x$ and $y$ to get $y = f^{-1}(x)$

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Example:

$f(x) = \frac{x+1}{x-1}$ Find $f^{-1}(x)$

1. $y = \frac{x+1}{x-1}$

2. $y(x-1) = x+1$

$x(y-1) = y+1$

$x = \frac{y+1}{x-1} = f^{-1}(y)$

3. $f^{-1}(x) = \frac{x+1}{x-1}$

2/10/12 Example

$f(x) = 3 + \sqrt{2x-x}$ find $f^{-1}(x)$, if it exist

$y = 3 + \sqrt{2x-x}$

$(y-3)^2 = 2x-7$

$2x = (y-3)^2 + 7$

$x = \frac{1}{2}((y-3)^2 + 7)$

$y = \frac{1}{2}((x-3)^2 + 7) = f^{-1}(x)$

$\sqrt{x^2} = |x|$

$D(f^{-1}) = R(f) = [3, \infty)$

$R(f^{-1}) = D(f) = [7/2, \infty)$

Existence of Inverse functions:

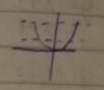
Def$^n$: A function $f$ is injective (one to one)(1-1), if and only if $f(x) \neq f(x_2)$ whenever $x_1 \neq x_2$ (no 2 x values can give you two y values)

Theorm: A function has an inverse if and only if it is 1-1 (injective)

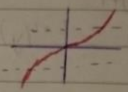
## HORIZONTAL LINE TEST

A function is 1-1 (and therefore has an inverse) if and only if its graph is cut at most once by any horizontal line

eg $x^2$ is not 1-1 and does not have an inverse.

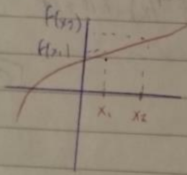


$x^3$ is 1-1 and does have an inverse.



## INCREASING AND DECREASING FUNCTIONS

A function is an increasing function of $x$ if its graph is always rising from left to right, ie for $x_1, x_2$ D(f) if $f(x_1) \leq f(x_2)$ where $x_1 < x_2$ then $f$ is an increasing function ~~$x_1 < x_2$~~




$f(x_1) < f(x_2)$

Similarly for decreasing functions. For $x_1, x_2 \in D(f)$ if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, then $f$ is a decreasing function

Clearly increasing/decreasing functions are 1-1 and hence possess inverses.

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GRAPHING INVERSE FUNCTIONS:

How are the graphs of $f(x)$ and $f^{-1}(x)$ related?

let $(a,b)$ be a point on the graph of $f(x)$

$\Rightarrow b = f(a)$

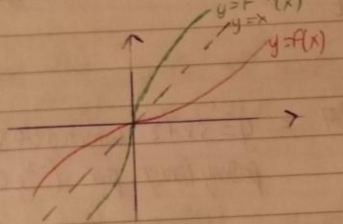
$f^{-1}(b) = a$

$\therefore (b,a)$ is a point on the graph of $f^{-1}(x)$

Reversing coordinates of points on the graph of f we get the coordinates of $f^{-1}(x)$.

Reversing coordinates $\Leftrightarrow$ reflection through line $y=x$

Theorem: If f has an inverse, then the graphs of $y=f(x)$ and $y=f^{-1}(x)$ are reflections of one another through the line $y=x$




Parametric Equations:

A curve C in the x-y plane may be described parametrically by giving the coordinates in terms of some parameter, t say, $x = f(t)$ $y = g(t)$

These are the parametric equations for the curve C

## SKETCHING PARAMETRIC CURVES

There are two ways to sketch these curves

i. Choose various values for the parameter and calculate $(x, y)$ coordinate pairs for each parameter value

ii. Eliminate the parameter $t$ and plot the graph in the usual way (if possible)

**Example:**

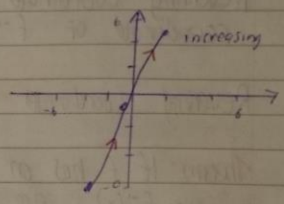
Graph the parametric curve:

$x = 2t - 3 \qquad y = 6t - 7$

$t = 0 \qquad x = -3 \qquad y = -7$

$t = 1 \qquad x = -1 \qquad y = -1$

$t = 2 \qquad x = 1 \qquad y = 5$ etc




Could eliminate $t$

$t = \frac{1}{2}(x+3)$

$\downarrow$

$y = 6\left(\frac{1}{2}(x+3)\right) - 7 = y = 3x + 2$ which is the straight line with slope 3 passing through pts $\left(-\frac{2}{3}, 0\right), (0, 2)$

**Another familiar example:** the unit circle

$x = \cos\theta \qquad y = \sin\theta \qquad 0 \leq \theta \leq 2\pi$

To eliminate $\theta$ we take $x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$

Which is the eq$^n$ of a circle centered $(0,0)$ with radius of 1

The direction in which the graph is traced as the parameter increases is called the orientation

eg the circle is traced out counter-clockwise

4/11/12 Maths

Assignment 1: Friday 5pm
return answers in same form given

Parametric functions: functions and their inverses in parametric form

Given some equation $y=f(x)$, we can always write this parametrically by simply taking $x=t$, $y=f(t)$

example: $y=x^2-2x+3$
$x=t$ $y=t^2-2t+3$

Recall that the inverse of a function is obtained by a reflection in the line $y=x$. ~~Parametrically~~ For a parametric function this compounds to $x=f(t)$ $y=t$

example $f(x)=x^5+x+1$ then $y=f(x) \Leftrightarrow x=t$ $y=t^5+t+1$
$y\ f^{-1}(x)$ $x=5^5+t+1$ $y=t$

Translations:

Given a parametric curve $x=f(t)$ $y=g(t)$ then adding a constant $x_0$ to $f(t)$ translates the curve C $x_0$ units in the x-direction, and adding a constant $y_0$ to $g(t)$ translates C $y_0$ units in the y direction

example circle with centre $(0,0)$ is given parametrically
$x=r\cos\theta$ $y=r\sin\theta$
then add $x_0$ $x=x+r\cos\theta$ $y=y_0+r\sin\theta$ represents the circle of radius r with centre $(x_0,y_0)$

Verify $(x-x_0)+(y-y_0)^2=r^2(\cos^2\theta+\sin^2\theta)=1^2$

SCALING

Given a parametric curve $x=f(t)$ $y=y(t)$
then multiplying $f(t)$ by a constant stretches or compress (in the $x$ direction while multiplying $g(t)$ by a constant stretches or compress in the $y$-direction

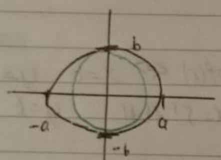
example unit circle centre $(0,0)$

$x = \cos\theta$ $y = \sin\theta$

Multiply $a\cos\theta$ by, $\sin\theta\, b$

$x = a\cos\theta$ $y = b\sin$

This corresponds to an ellipse with centre $(0,0)$ extending from $(-a,a)$ in the $x$ direction and from $(b,b)$ in the $y$ direction




$a$ = semi major axis
$b$ = semi minor axis

Eliminate $\theta$ by taking $(\frac{x}{a})^2+(\frac{y}{b})^2 = \cos^2\theta + \sin^2\theta = 1$