MA1E01: Chapter 1 Summary

Functions

Definitions

- **Functions**: If a variable y depends on some other variable x such that each value of x determines *exactly one* value of y, then we say that y is a function of x.
- **Domain**: The domain of a function f is the set of all allowable inputs, we usually denote this set by $\mathcal{D}(f)$.
- **Range**: The range of f is the set of all possible outputs $f(x)$ as x varies over the domain, denoted by $\mathcal{R}(f)$.
- **Even/Odd function**: We say a function $f(x)$ is even if $f(-x) = f(x)$, whereas we say a function is odd if $f(-x) = -f(x)$.
- **Arithmetic Operations**: Given functions f and g, we define the following arithmetic combinations:

$$
\begin{array}{ll}
(i) & (f+g)(x) = f(x) + g(x) \\
(ii) & (f-g)(x) = f(x) - g(x) \\
(iii) & (f \cdot g)(x) = f(x)g(x) \\
(iv) & (f/g)(x) = f(x)/g(x).
\end{array}
$$

  The domain of (i)-(iii) is $\mathcal{D}(f) \cap \mathcal{D}(g)$, i.e., x must be in both the domain of f and the domain of g, whereas the domain of (iv) is $\mathcal{D}(f) \cap D(g)/\{x \in \mathbb{R} : g(x) = 0\}$.
- **Composition**: The composition of two function is defined as

$$(f \circ g)(x) = f(g(x)),$$

  and the domain of $f \circ g$ is all $x \in \mathcal{D}(g)$ such that $g(x) \in \mathcal{D}(f)$.
- **Injectivity**: We say a function is injective or one-to-one (1-1) if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.
- **Inverse functions**: If there exists a function, $g(x)$, such that

$$
\begin{aligned}
f(g(x)) &= x, \qquad \text{for all } x \in \mathcal{D}(g), \\
g(f(x)) &= x, \qquad \text{for all } x \in \mathcal{D}(f),
\end{aligned}
$$

  then we say that $g(x)$ is the inverse of $f(x)$, and we denote it by $g = f^{-1}$.

- **Increasing/Decreasing functions**: We say a function is increasing if

$$f(x_1) < f(x_2) \quad \text{whenever} \quad x_1 < x_2.$$

  Similarly, we say a function is decreasing if

$$f(x_1) > f(x_2) \quad \text{whenever} \quad x_1 < x_2.$$

- **Parametric curves**: A parametric curve in the xy-plane is a curve whose coordinates are given in terms of functions of some parameter,

$$x = f(t) \quad y = g(t).$$

- **Orientation**: The direction in which the graph of a parametric curve is traced as the parameter increases is called the orientation.

Theorems

- **Existence of inverse functions**: A function has an inverse if and only if it is injective (1-1).

- **Finding inverse functions**: If an equation $y = f(x)$ can be solved for x in terms of y, $x = g(y)$ say, then f has an inverse given by $f^{-1}(x) = g(x)$.

- **Domain/Range of inverse functions**: If f has an inverse, then the domain and range are given by

$$\begin{aligned} \mathcal{D}(f^{-1}) &= \mathcal{R}(f) \\ \mathcal{R}(f^{-1}) &= \mathcal{D}(f). \end{aligned}$$

- **Graphs of inverse functions**: If f has an inverse, then the graph of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another through the line $y = x$.

Miscellaneous Results

- **Vertical Line Test**: A curve in the xy-plane represents a function if and only if no vertical line intersects the curve more than once.

- **Translations**: The graph of $y = f(x) + k$ may be obtained by a vertical shift of the graph of $y = f(x)$, up k units if $k > 0$ and down $|k|$ units if $k < 0$. Similarly, the graph of $y = f(x + h)$ may be obtained by a horizontal shift of the graph of $y = f(x)$, to the left h units if $h > 0$ and to the right $|h|$ units if $h < 0$.

- **Scalings**: Assuming $c > 1$, the graph $y = cf(x)$ stretches the graph $y = f(x)$ vertically by a factor of c and the graph $y = \frac{1}{c}f(x)$ compresses the graph $y = f(x)$ by a factor of c. Similarly, the graph $y = f(cx)$ compresses the graph of $y = f(x)$ horizontally by a factor of c and the graph $y = f(\frac{1}{c}x)$ stretches the graph $y = f(x)$ horizontally by a factor of c.

- **Horizontal Line Test**: A function is injective (and therefore invertible) if and only if its graph is cut at most once by any horizontal line.

- **Inverses in parametric form**: The inverse of an invertible function whose parametric form is

  $$x = f(t) \qquad y = g(t)$$

  can be obtained by simply interchanging the x and y, i.e., the parametric form of the inverse is

  $$x = g(t) \qquad y = f(t).$$