FORECASTING

EXAM PAPER 2012

A i RMSE - Root mean square error: $\sqrt{\frac{1}{n}\Sigma(y_t - f_t)^2} = \sqrt{\frac{SSE}{n}}$

MAPE - Mean Percentage error: $100\,\frac{1}{n}\Sigma_{t=1}^{n}\left|\frac{y_t - \hat{f}_t}{y_t}\right|$
Measure of fit of model. Select lowest one

ii. ACF Autocorrelation Function. Measures correlation between y_t and y_{t-k}

$$r_k = \frac{\Sigma_{t=k+1}^{n}(y_t - \bar{y})(y_{t-k} - \bar{y})}{\Sigma_{t=1}^{n}(y_t - \bar{y})^2}$$

Used to choose the AR or MA of a model

PACF - Partial autocorrelation function
- Extension of auto correlation where the dependence on the intermediate elements is removed
- PACF at lag k is the auto correlation between x_t and x_{t-k} that is not accounted for by lags 1 through to k-1
- Found by fitting a linear regression model of y_t for from k lags
- Then $\alpha_k = \hat{\phi}_{kk}$ the fitted weight from the regression model

iii. ARIMA - Auto regressive integrated moving Average
Trends in TS removed by differencing the TS.
Autoregressive order p, differencing order d and moving average of order q

$$(1-\phi_1 B - \phi_2 B^2 \cdots \phi_p B^p)(1-B)^d y_t = c + (1 - \psi_1 B - \psi_2 B^2 - \cdots - \psi_q B^q)\varepsilon_t$$

B i. $(1-B^2)y_t = \varepsilon_t$

$y_t - y_t B^2 = \varepsilon_t$

$y_t - y_{t-2} = \varepsilon_t$

ii $(1 - \phi_1 B - \phi_2 B^2)(1-B)y_t = (1 - \theta_1 B)(1 - B^{12})\varepsilon_t$

$(1 - B - \phi_1 B + \phi_1 B^2 - \phi_2 B^2 + \phi^2 B^3)y_t = (1 - B^{12} - \theta_1 B + \theta_1 B^{13})\varepsilon_t$

$y_t - y_{t-1} - \phi_1 y_{t-1} + \phi_1 y_{t-2} - \phi_2 y_{t-2} + \phi_2 y_{t-3} = 1 - \varepsilon_{t-12} - \theta_1 \varepsilon_{t-1} + \theta_1 \varepsilon_{t-13}$

c i $y_t - y_{t-12} = \varepsilon_t$

$y_t - B^{12} y_t = \varepsilon_t$

ii $y_t - \phi_1 y_{t-1} - \phi_2 y_{t-12} = \theta_1 \varepsilon_{t-1} + \varepsilon_t$

$y_t - \phi_1 y B \quad - \phi_2 B^{12} y_t = \theta_1 B \varepsilon_t + \varepsilon_t$

D i ARIMA()

ii HoltWinters()

iii forecast() predict()

iv. tsdisplay()

Q 5A Assume $\varepsilon_i \sim N(0, \sigma^2)$ ε_i and ε_j are indep if $i \neq j$.

$y_t = c + \phi_1 \varepsilon_{t-12} + \varepsilon_t$

b $\mathbb{E}[y_t] = \mathbb{E}[c + \phi_1 \varepsilon_{t-12} + \varepsilon_t]$

$= \mathbb{E}[c] + \mathbb{E}[\phi_1 \varepsilon_{t-12}] + \mathbb{E}[\varepsilon_t]$

$= c$

c Assume $\mathbb{E}[y_t] = c = 0$

Model $= y_t = \phi_1 \varepsilon_{t-12} + \varepsilon_t$.

$Var[y_t] = \mathbb{E}[y_t - \mathbb{E}[y_t]]^2$

$= \mathbb{E}[\phi_1 \varepsilon_{t-12} + \varepsilon_t]^2$

$= \mathbb{E}[\phi_1^2 \varepsilon_{t-12}^2 + 2\phi_1 \varepsilon_t \varepsilon_{t-12} + \varepsilon_t^2]$

$= \phi_1^2 \mathbb{E}[\varepsilon_{t-12}^2] + 2\phi_1 \mathbb{E}[\varepsilon_t \varepsilon_{t-12}] + \mathbb{E}[\varepsilon_t^2]$

$= \phi_1^2 s^2 \quad + s^2$

$= (1+\phi_1^2) s^2$

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5 D $Cov[y_t, y_{t-k}] = E[(y_t - E[y_t])(y_{t-k} - E[y_{t-k}])]$

$= E[y_t \cdot y_{t-k}] =$

$= E[(\phi_1 \varepsilon_{t-12} + \varepsilon_t)(\phi_1 \varepsilon_{t-k-12} + \varepsilon_{t-k})]$

$= E[\phi_1^2 \varepsilon_{t-12}\varepsilon_{t-k-12} + \phi_1 \varepsilon_{t-12}\varepsilon_{t-k} + \phi_1 \varepsilon_t \varepsilon_{t-k-12} + \varepsilon_t \varepsilon_{t-k}]$

$= \phi_1^2 E[\varepsilon_{t-12}\varepsilon_{t-k-12}] + \phi_1 E[\varepsilon_{t-12}\varepsilon_{t-k}] + \phi_1 E[\varepsilon_t \varepsilon_{t-k-12}] + E[\varepsilon_t \varepsilon_{t-k}]$

if $k=12 \Rightarrow \phi_1 \sigma^2$

$k=0 \quad \phi_1^2 \sigma^2 + \sigma^2$

if $k \neq 0$ and $k \neq 12 \Rightarrow 0$

E. $Corr[y_t, y_{t-k}] = \dfrac{Cov[y_t, y_{t-k}]}{\sqrt{Var[y_t]\,Var[y_{t-k}]}} = \dfrac{Cov[y_t, y_{t-k}]}{\sqrt{\sigma^2(1+\phi_1^2)^2}}$

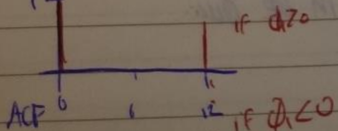
$= \dfrac{Cov[y_t, y_{t-k}]}{\sigma^2(1+\phi_1^2)}$

if $k=0$ $\dfrac{\sigma^2\phi_1^2 + \sigma^2}{\sigma^2(1+\phi_1^2)} = 1$ correlation at lag $0 = 1$

if $k=12$ $\dfrac{\phi_1 \sigma^2}{\sigma^2(1+\phi_1^2)} = \dfrac{\phi_1}{\phi_1^2+1}$

if $k \neq 0$ and $k \neq 12$ then $\dfrac{0}{\sigma^2(1+\phi_1^2)} = 0$

F.



G PACF – no AR component here and exponential or damped sine wave for

H 95% CI for $k = 1 \ldots 12$

Model $\cdot\ y_{n+k} = c + \alpha\varepsilon_{n+k-12} + \varepsilon_{n+k}$

The term ε_{n+k-4} can be expressed as a weighted sum of the past observations y_{n+k-12}, $y_{n+k-24} \ldots$ as shown earlier.
All observations are available to compute for $k \in (1,2,3,4,5,6,7,8,9 \ldots 12)$

Forecast is then $\hat{y}_{n+k} = c + \alpha\varepsilon_{n+k-12}$
95% CI = forecast $\pm 2\sigma$
$= \hat{y}_{n+k} \pm 2\sigma = 95\%$ CI.

I When $12 < k \leq 24$ the expectation is utilised as the forecast $\hat{y}_{n+k}$.

$\mathbb{E}[y_t] = c = 0$ along with error term

Error term will be $\alpha\varepsilon_{n+k-12} + \varepsilon_{n+k}$ which has a variance associated with it of $(1+\alpha^2)\sigma^2$

So 95% CI will be $c \pm 2\sqrt{(1+\alpha^2)\sigma^2}$

Q6 A. SES: $F_1 = y_1$, $0 < \alpha < 1$
$F_{t+1} = (1-\alpha)F_t + \alpha y_t$
$F_{n+k} = F_{n+1}$ for all $k \geq 1$

AR(1) $y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t$
$\varepsilon_t \sim N(0, \sigma^2)$
ε_i indep ε_j if $i \neq j$.

B Suitable when there is no trend or seasonality in the data

C Look at criteria
pick lower RMSE value
AR has lower RMSE value of 501
SES has RMSE of 556, Choose SES

04/15.

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Q 6d S^2 = error of prediction $= \frac{1}{n}\Sigma(y_t - F_t)^2$

E SES: $F_{1981} = (1-\alpha)F_{1980} + \alpha y_{1980}$

$= 0.22(4766) + 0.68(3888) = 4167$

$F_{1982} = F_{n+1} = 4167$

AR(1) $y_{1981} = a(y_{1980}) + b$

$= 0.75(3885) + 1115 = 4028.51$

$y_{1982} = 0.75(y_{1981}) + 115$

$= 0.75(4028.51) + 1115 = 4136$

F 1 step: $\pm 2S$ $\pm(581.6)$ $= 1163.2$

2 step) $\pm 2S\sqrt{1+a^2}$ $\pm 2(581.6)(\sqrt{1+0.75^2})$ $= 1454$

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4 (i Write with backshift oper

$y_t - y_{t-12} = \varepsilon_t$

$y_t - B^{12} y_t = \varepsilon_t \Rightarrow (1-B^{12})y_t = \varepsilon_t$

ii. $y_t - \phi_1 y_{t-1} - \phi_2 y_{t-12} = \theta_1 \varepsilon_{t-1} + \varepsilon_t$

$y_t - \phi_1 B y^t - \phi_2 B^{12} y_t = \theta_1 B \varepsilon_t + \varepsilon_t$

$(1 - \phi_1 B - \phi_2 B^{12}) y_t = (\theta B + 1)\varepsilon_t$

d. ARIMA()
HoltWinters()
forecast()
tsdisplay()

5 A Assumption $\varepsilon_t \sim N(0, \sigma^2)$ ε_i ~~$\neq \varepsilon_j$ if~~ ε_j independ if $i \neq j$.

$y_t = c + \phi_1 \varepsilon_{t-12} + \varepsilon_t$

B. $E[y_t] = E[c + \phi_1 \varepsilon_{t-12} + \varepsilon_t]$

$= c + \phi_1 E[\varepsilon_{t-12}] + E[\varepsilon_t]$

$= c$

C $E[y_t] = 0 \Rightarrow c = 0$

model $y_t = \phi_1 \varepsilon_{t-12} + \varepsilon_t$

Variance $= E[y_t - E[y_t]]^2 = E[y_t^2]$

$= E[\phi_1^2 \varepsilon_{t-12}^2 + 2\phi_1 \varepsilon_{t-12}\varepsilon_t + \varepsilon_t^2]$

$= \phi_1^2 E[\varepsilon_{t-12}^2] + 2\phi_1 E[\varepsilon_{t-13}\varepsilon_t] + E[\varepsilon_t^2]$

$= \phi_1 \sigma^2 + 0 + \sigma^2 = \sigma^2(1+\phi_1^2)$

d Covariance $[y_t, y_{t-k}] = E[(y_t - E[y_t])(y_{t-k} - E[y_{t-k}])]$

$= E[y_t \cdot y_{t-k}] = E[(\phi_1 \varepsilon_{t-12} + \varepsilon_t)(\phi_1 \varepsilon_{t-k-12} + \varepsilon_{t-k})]$

$= E[\phi_1^2 \varepsilon_{t-12}\varepsilon_{t-k-12} + \phi_1 \varepsilon_{t-12}\varepsilon_{t-k} + \phi_1 \varepsilon_{t-k-12}\varepsilon_t + \phi \varepsilon_t \varepsilon_{t-k}]$

$= \phi_1^2 E[\varepsilon_{t-12}\varepsilon_{t-k-12}] + \phi_1 E[\varepsilon_{t-12}\varepsilon_{t-k}] + \phi_1 E[\varepsilon_{t-k-12}\varepsilon_t] + E[\varepsilon_t \varepsilon_{t-12}]$

$E[\varepsilon_i \varepsilon_j] = 0$ if $i \neq j$

$\therefore$

if $k = 12$ σ^2

if $k \neq 12$ 0

if $k=0 \rightarrow \sigma^2$ for first term   Second tr if $k=12$ σ^2 $k \neq 12 \rightarrow 0$ third part $=0$ fourth if $k=0 \Rightarrow \sigma^2$ $k \neq 0 \Rightarrow 0$
if $k \neq 0 \Rightarrow 0$

$Cov[y_t, y_{t-k}]$ if $\begin{cases} k=0 & \phi_1^2\sigma^2+\sigma^2 \\ k=12 & \phi_1\sigma^2 \\ \text{other} & \text{if } k\neq 0 \text{ and } k \neq 12 \Rightarrow 0 \end{cases}$

E. Correlation $[y_t, y_{t-k}] = \dfrac{Cov[y_t, y_{t-k}]}{\sqrt{Var[y_t]\ Var[y_{t-k}]}}$

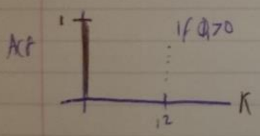
$= \dfrac{Cov[y_t, y_{t-k}]}{\sqrt{(\sigma^2(1+\theta_1^2))^2}} = \dfrac{Cov[y_t, y_{t-k}]}{\sigma^2(1+\theta_1^2)}$

if $k=0$ $\dfrac{\theta_1^2\sigma^2+\sigma^2}{\sigma^2(1+\theta_1^2)} = 1$ correlation at lag $0=1$

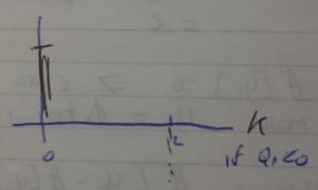
if $k=12$ $\dfrac{\phi_1\sigma^2}{\sigma^2(1+\theta_1^2)} = \dfrac{\theta_1}{1+\theta_1^2}$

If $k\neq 0$ and $k \neq 12$ then $\dfrac{0}{\sigma^2(1+\theta_1^2)} = 0$

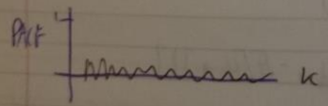
F. Comment on ACF.



Spike at 1. or



G. PACF.
No AR components ⇒ exponential sine wave



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Q6 SES, $F_1 = y_1$

$F_{t+1} = (1-\alpha) F_t + \alpha y_t \quad 0 \le \alpha \le 1$

$F_{n+k} = F_{n+1}$ for all $k \ge 1$

AR(1) $y_t = \alpha_0 + \alpha_1 y_{t-1} + \varepsilon_t$

$\varepsilon_i \sim N(0, \sigma^2)$

$\varepsilon_i \ne \varepsilon_j \quad i = j$

b suitable if no trend or seasonality

c look at criteria
pick lowest RMSE
AR has RMSE of 561
SES has RMSE of 556 Choose SES.

d S^2 error of predicted fitted and actual

E Compute prediction interval for next 2 years based on AR(1) model

SES: $F_{1981} = (1-\alpha) F_{1980} + \alpha y_{1980}$

$= 0.32 (4766) + 0.68 (3885) = 4167$

$F_{1982} = F_{n+1} = 4167$

AR(1) $y_{1981} = a (y_{1980}) + b$

$0.75 (3885) + 1115 = 4028.51$

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F. 1 step ahead $\pm 2S$ $\pm 2(581.6)$ $= 1163.2$

2 steps ahead $\pm 2S\sqrt{1+a^2}$ $\pm 2(581.6)\sqrt{1+0.75^2}$ $= 1454$