LP DUALITY (Simplex)

Every LP has an associated LP called the dual problem we refer to original problem as primal problem.

We see how the primal can be converted into its corresponding dual. We solve dual problem.

Fundamental property of the primal dual relationship is that the optimal solution to either the primal or dual provide same answer.

If one is harder to solve than the other, we pick the easier one.

Hyhtech example Max $50x_1 + 40x_2$

s.t: $3x_1 + 5x_2 \leq 150$ assembly time

$1x_2 \leq 20$ portable display

$8x_1 + 5x_2 \leq 300$ warehouse $x_1, x_2 \geq 0$

A maximisation problem will all less than or equal to constraints and nonnegativity is said to be in canonical form.

Dual problem is: Min $150u_1 + 20u_2 + 300u_3$

ST: $3u_1 + 8u_3 \geq 50$

$5u_1 + 1u_2 + 5u_3 \geq 40$ $u_1, u_2, u_3 \geq 0$

The canonical form for a minimization problem is a min problem with all $\geq$ constraints and non neg RHS.

Thus the dual of a max problem in canonical form is a min problem in canonical form. $u_1\ u_2\ u_3$ are refered to as dual variables

With the preceding example in mind we make the following general statements about dual of a max problem in canonical form

1. The dual is a min problem in canonical form.
2. When primal has n decision variables ($n=2$ for hightech), the dual will have n constraints. First constraint of dual is associated with variable x_1 and 2nd with variable x_2 and so on
3. When the primal has m constraints ($m=3$ for hightech), the dual will have m decision variables. Dual variable U_1 associated with the first primal constraint, dual variable U_2 associate with second constraint and so on
4. The right hand sides of the primal constraints become the objective function coefficient in dual
5. The objective function coef of the primal become the rhs of dual constraints
6. The constraint coefs of the ith primal variable become the coefficient in the ith constraint of the dual

We have formulated the high tech dual linear p problem. Solve using simplex. After subtracting surplus variable S_1 and S_2 to obtain standard form, adding artificial variables a_1 and a_2 to obtain the tableau form, and multiplying the o.f by -1 to convert the dual problem to an equivalent max problem, we arrive at initial tableau:

Basis	C_B	u_1	u_2	u_3	S_1	S_2	a_1	a_2	
		-150	-20	-300	0	0	$-M$	$-M$	
a_1	$-M$	3	0	8	-1	0	1	0	50
a_2	$-M$	5	1	5	0	-1	0	1	40
Z_j		$-8M$	$-M$	$-13M$	M	M	$-M$	$-M$	$-90M$
$C_j - Z_j$		$-150+8M$	$-20+M$	$-300+13M$	$-M$	$-M$	0	0	

U_3 into basis, a_1 removed and second iteration U_2 into basis, a_1 out

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Duality

Basis	C_B	U_1	U_2	U_3	S_1	S_2	
		-150	-20	-300	0	0	
U_3	-300	0	-3/25	1	-5/25	3/25	26/5
U_1	-150	1	8/25	0	5/25	-8/25	14/5
Z_J		-150	-12	-300	30	12	-1980
$C_J - Z_J$		0	-8	0	-30	-12	

All entries in net evaluation row are $= 0$ or $(-)$. Optimum solution reached.

$U_1 = 14/5 \quad U_2 = 0, \quad U_3 = 26/5 \quad S_1 = 0 \quad S_2 = 0$

- Objective function $-(-1980) = 1980$

Optimal value is the same as primal problem solution

Property 1: If the dual problem has an optimum solution, the primal problem has an optimal solution and vice versa. Values of the optimal solution are equal.

Economic Interpretation of the dual variables

We said O.F. values for primal and dual must be equal.

$50x_1 + 40x_2 = 1980$ or $150u_1 + 20u_2 + 300u_3 = 1980$

Using first equation. With x_1 = desktop and x_2 = portable.

$\left(\begin{matrix}\text{dollar value}\\\text{per unit desktop}\end{matrix}\right)\left(\begin{matrix}\text{units of}\\\text{desktop}\end{matrix}\right) + \left(\begin{matrix}\text{dollar value}\\\text{p/u portable}\end{matrix}\right)\left(\begin{matrix}\text{units of}\\\text{portable}\end{matrix}\right) = \begin{matrix}\text{total dollar value}\\\text{of production}\end{matrix}$

From 2nd eq^n, can be interpreted as number of units of resource available:

$\left(\begin{matrix}\text{units resource}\\1\end{matrix}\right)u_1 + \left(\begin{matrix}\text{units resource}\\2\end{matrix}\right)u_2 + \left(\begin{matrix}\text{units resource}\\3\end{matrix}\right) = \begin{matrix}\text{total dollar value}\\\text{of production}\end{matrix}$

Thus we see that the dual variables must carry the interpretation of being value per unit of resource (DUAL Price)

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u_1 = \$ value of assembly time = \$ 2.80
U_2 \$ value of portable display = \$ 0.00
U_3 \$ value of warehouse space = \$ 5.20
For min problem dual prices are negative of the dual variables

Primal problem: Given a per unit value of each product, determine how much of each should be produced to max value of total production. Constraints require the amount of each resource used to be ≤ amount of variable

Dual problem: Given the availability of each resource, determine the per unit value such that the total value of the resources is minimized. Constraints require the resource value per unit be ≥ value of each unit of output

Using the dual to identify the primal solution
If we solve only the dual problem can we identify the optimal values for the primal variables?

Property 2: Given the simplex tableau corresponding to the optimal dual solution, the optimal values of the primal decision variables are given by the Z_J entries for the surplus variables; furthermore the optimal values of the primal slack variables are given by the negative of the $C_J - Z_J$ entries for the U_J variables

This property enables us to use simplex from dual problem to determine the optimal primal solution of $X_1 = 30$, $X_2 = 12$ and the values for slack variables are negative of $C_J - Z_J$ row
$S_1 = 0$ $S_2 = 8$ $S_3 = 0$

5. Duality

Finding the dual of any primal problem:

If primal problem is a minimisation problem in canonical form the dual is a maximisation problem in canonical form

$$\begin{array}{ll} \text{min } 6x_1+2x_2 & \\ \text{ST: } 5x_1-1x_2 \geq 13 & \Rightarrow \\ 3x_1+7x_2 \geq 9 & \\ x_1, x_2 \geq 0 & \end{array} \qquad \begin{array}{l} \text{Max } 13u_1+9u_2 \\ 5u_1+3u_2 \leq 6 \\ -1u_1+7u_2 \leq 2 \\ u_1, u_2 \geq 0 \end{array}$$

It is easier to convert any primal problem into an equivalent problem in canonical form.

Example

$$\begin{array}{l} \text{Min } 2x_1-3x_2 \\ \text{ST: } 1x_1+2x_2 \leq 12 \\ 4x_1-2x_2 \geq 3 \\ 6x_1-1x_2 = 10 \quad x_1, x_2 \geq 0 \end{array}$$

For min form we need to convert all constraint to $\geq$ form

Step 1: convert constraint 1 into $\geq$ by multiplying by -1

$$-x_1-2x_2 \geq -12$$

Step 2: Constraint 3 is an equality constraint. For equality constraint we first create two inequalities, one with $\leq$ form, the other with $\geq$ form

$$6x_1-1x_2 \geq 10$$
$$6x_1-1x_2 \leq 10$$

Multiply the $\leq$ by -1 to get the $\geq$ constraint

$$6x_1-1x_2 \geq 10$$
$$-6x_1+1x_2 \geq -10$$

Original problem is now :

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Min $2x_1 - 3x_2$

ST: $-1x_1 - 2x_2 \geq -12$

$4x_1 - 2x_2 \geq 3$

$6x_1 - 1x_2 \geq 10$

$-6x_1 + 1x_2 \geq -10$ $\quad x_1\ x_2 \geq 0$

Change to Max dual problem becomes

Max $-12u_1 + 3u_2 + 10u_3' - 10\hat{u}_3$

ST: $-1u + 4u_2 + 6u_3' - 6\hat{u}_3 \leq 2$

$-2u_1 - 2u_2 - 1u_3' + 1\hat{u}_3 \leq -3$ $\quad u_1, u_2, u_3', u_3^{\wedge} \geq 0$

The equality contrant required two $\geq$ constraints so we denoted the dual variables associated with these constraints as u_3' and $u_3^{\wedge}$

This notation reminds us that u_3' and $u_3^{\wedge}$ both refer to the third constraint in mid primal problem.

Because two dual variables are associated with an equality contrat, the interpretation of the dual variable must be modified slightly

The dual variable for the equality contrant $6x_1 - 1x_2 = 10$ is given by the value $u_3' - u_3^{\wedge}$ in the optimum solution of the dual. Hence the dual variable for an equality contraint can be negative

7. Dually examp

Produr mix, $x_1\ x_2\ x_3\ x_4$ indiwal units of produr 1, 2, 3, 4

Q 17. Max $4x_1 + 6x_2 + 3x_3 + 1x_4$

ST: $1.5x_1 + 2x_2 + 4x_3 + 3x_4 \le 550$ Machen A hour

$4x_1 + 1x_2 + 2x_3 + 1x_4 \le 700$ m B H

$2x_1 + 3x_2 + 1x_3 + 2x_4 \le 200$

Min $550u_1 + 700u_2 + 200u_3$

ST: $1.5u_1 + 4u_2 + 2u_3 \geq 4$

$2u_1 + 1u_2 + 3u_3 \geq 6$.

$4u_1 + 2u_2 + 1u_3 \geq 3$

$3u_1 + 1u_2 + 2u_3 \geq 1$ $\quad u_1\ u_2\ u_3 \geq 1$

Standar Max $-550u_1 - 700u_2 - 200u_3$

ST. $1.5u_2 + 4u_2 + 2u_3 - S_1 + a_1 = 4$

$2u_1 + 1u_2 + 3u_3 - S_2 + a_2 = 6$

$4u_1 + 2u_2 + 1u_3 - S_3 + a_3 = 3$

$3u_1 + 1u_2 + 2u_3 - S_4 + a_4 = 1$

		U_1	U_2	U_3	S_1	S_2	S_3	S_4	
Basis	C_B	+550	+700	+200	0	0	0	0	
U_3	+200	1	0.5	1	0	-0.5	0	0	3
S_3	0	-3	-1.5	0	0	-0.5	1	0	0
S_4	0	-1	0	0	0	-1	0	1	5
S_1	0	0.5	-3	0	1	-1	0	0	2
Z_J		-200	+100	200	0	-100	0	0	+600
$C_J - Z_J$		+350	600	0	0	100	0	0	

$X_1 =$ $\quad X_2 =$ $\quad X_3 =$ $\quad X_4 =$

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Final table is

Basis	C_B	U_1	U_2	U_3	S_1	S_2	S_3	S_4	
		550	700	200	0	0	0	0	
U_3	200	0	0	1	0	-0.4	0.2	0	1.8
S_1	0	0	-3.25	0	1	-0.65	-0.05	0	0.05
S_4	0	0	0.5	0	0	-0.5	-0.5	1	3.5
U_1	550	1	0.5	0	0	0.1	-0.3	0	0.3
Z_j		550	275	200	0	-25	-125	0	525
$C_j - Z_j$		0	425	0	0	25	125	0	

$X_1 = 0$ $X_2 = 25$ $X_3 = 125$ $X_4 = 0$

$S_2 = 425$ $S_3 = 0$

$U_1 = 0.3$ $U_2 = 0$ $U_3 = 1.8$

Machine A and C ($U_i > 0$) are operating at capacity, machine (1) gen[illegible] priority since each hour (1) worth 1.8.

Duality

Q19 Max $3x_1 + 1x_2 + 5x_3 + 3x_4$

ST : $3x_1 + 1x_2 + 12x_3 = 30$

$2x_1 + 1x_2 + 3x_3 + 1x_4 \geq 15$

$2x_2 + 3x_4 \leq 12$ $\qquad x_1\ x_2\ x_3\ x_4 \geq 0$

$3x_1 + 1x_2 + 12x_3 \leq 30$ $\quad$ ① $\quad 3x_1 + 1x_2 + 12x_3 \leq 30$

$3x_1 + 1x_2 + 12x_3 \geq 30$ $\quad -3x_1 - 1x_2 - 12x_3 \leq -30$

② $-2x_1 - 1x_2 - 3x_3 - 1x_4 \leq -15$

Max $3x_1 + 1x_2 + 5x_3 + 3x_4$

ST: $3x_1 + 1x_2 + 12x_3 \leq 30$

$-3x_1 - 1x_2 - 12x_3 \leq -30$

$-2x_1 - 1x_2 - 3x_3 - 1x_4 \leq -15$

$2x_2 + 3x_4 \leq 12$

$3u_1' - 3u_1'' - 3u_2 \geq 3$

$1u_1' - 1u_1'' - 1u_2$

$$\boxed{\begin{array}{l} 3u_1' - 3u_1'' - 2u_2 \geq 3 \\ 1u_1' - 1u_1'' - 1u_2 + 2u_3 \geq 1 \\ 12u_1' - 12u_1'' - 3u_2 \geq 5 \\ -1u_2 + 3u_3 \geq 3 \\ \text{min } 30u_1' + 30u_1'' - 15u_2 + 12u_3 \end{array}}$$