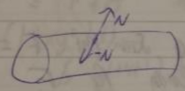
Maths 4 Notes Chapter 6

Recall that for a parametric surface σ traced out by the graph of
$$r(u,v) = x(u,v)\,i + y(u,v)\,j + z(u,v)\,k$$

the principle unit normal vector is:
$$n = \frac{\frac{dr}{du} \times \frac{dr}{dv}}{\left\| \frac{dr}{du} \times \frac{dr}{dv} \right\|}$$

This vector defines the positive orientation of the surface σ with n pointing in the positive direction.
$-n$ defines the negative orientation, with $-n$ pointing in negative direction



Example find orientation of cylinder $r = \cos u\, i + v j - \sin u\, k$

$\frac{dr}{du} = -\sin u\, i - \cos u\, k \qquad \frac{dr}{dv} = j.$

$\frac{dr}{du} \times \frac{dr}{dv} = \cos u\, i - \sin u\, k \qquad \left\| \frac{dr}{du} \times \frac{dr}{dv} \right\| = 1$

$n = \frac{dr}{du} \times \frac{dr}{dv} = \cos u\, i - \sin u\, k$

this points outwards, $-n$ points inwards

Flux

If positive $\Rightarrow$ out of surface   negative $\Rightarrow$ into surface

$$\text{Flux} = \iint_\sigma F \cdot n \, dS = \iint_R F \cdot \left\| \frac{dr}{du} \times \frac{dr}{dv} \right\|$$

$$\Rightarrow \iint_\sigma F \cdot \left(\frac{dr}{du} \times \frac{dr}{dv} \right)$$

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2c

Divergence theorem

$$\text{Div } F = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}$$

Get flux using divergence

$$\text{flux} = \iiint \text{div} F \; dS$$

Acting on a vector field, Gradient, Divergence, and Curl.

Gradient $\nabla\phi = \frac{d\phi}{dx} i + \frac{d\phi}{dy} j + \frac{d\phi}{dz} k$

Gradient field of ϕ, points in direction in which ϕ increases most

Vector field is conservative if there exists a ϕ such that $F = \nabla\phi$ where ϕ is called the potential function for F.

Divergence of a vector field $F(x,y,z) = f \cdots$

is $\text{div } F = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}$ results in a number

$$\text{curl } F = \left(\frac{dh}{dy} - \frac{dg}{dz}\right) i + \left(\frac{df}{dz} - \frac{dh}{dx}\right) j + \left(\frac{dg}{dx} - \frac{df}{dy}\right) k$$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ f & g & h \end{vmatrix}$$

$$\text{Div } F = \nabla \cdot F = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}$$

$$\nabla = \frac{d}{dx} i + \frac{d}{dy} j + \frac{d}{dz} k$$

$$\text{Curl } F = \nabla \times F = \left(\frac{dh}{dy} - \frac{dg}{dz}\right) i + \left(\frac{df}{dz} - \frac{dh}{dx}\right) j + \left(\frac{dg}{dx} - \frac{df}{dy}\right) k$$

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Line Integral

Integrating a function along a curve C, denoted by $\int_C f(x,y,z)\,ds$

If C is smoothly parametrised by $r(t) = xi + yj + zk$ then the line integral of $f(x,y,z)$ along C is given by

$$\int_C f(x,y,z)\,ds = \int_a^b f(x,y,z)\,\|r'(t)\|\,dt$$

If we want to integrate wrt x direction we would simply replace r' with x' and ds with dx.

Example: Evaluate $\int_C (1+xy^2)\,ds$ along curve $r(t) = ti + 2tj$, $0 \le t \le 1$.

Solution: $r' = \overset{x=t}{i} + \overset{y=2t}{2j}$ hence $\|r'\| = \sqrt{5}$. Therefore line integral becomes

$$\int_C (1+xy^2)\,ds = \int_0^1 [1 + t(2t)^2]\sqrt{5}\,dt$$

$$\int_0^1 (1+4t^3)\sqrt{5}\,dt$$

$$= \sqrt{5}\,[t + t^4]_0^1 = 2\sqrt{5}.$$

Can find the line integral for each component of a vector field in same way as before. However if we have a small length in each direction, this can be represented by

$$dr = dx\,i + dy\,j + dz\,k$$

Then the line integral ~~for each component in the sum~~ of F along C:

$$\int_C F\cdot dr = \int_C f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz$$

Which we can calculate by

$$\int_C F\cdot dr = \int_a^b F(r(t))\cdot r'(t)\,dt.$$

Example: Evaluate $F = \cos t\, i + \sin t\, j$ along curve given by $r = t i + t^2 j$ in the interval $-\pi/2 \le t \le \pi$

Solution: We have $r' = i + 2t j$

$$\int_C F \cdot dr = \int_{-\pi/2}^{\pi} (\cos t\, i + \sin t\, j) \cdot (i + 2t j)\, dt$$

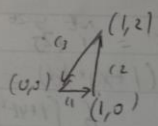
$$\int_{-\pi/2}^{\pi} (\cos t + 2t \sin t)\, dt = 2\pi + 3$$

Line Integral along Smooth piecewise curve

Consider a line integral along a series of smooth curves joined end to end

$$\int_C F \cdot r = \int_{C_1} F \cdot r + \int_{C_2} F r + \dots + \int_{C_n} F r$$

Example: Evaluate $\int_C (x^2 y\, dx + x dy)$ on the



Solution: line segment connecting r_0 to r_1 given by $r = (1-t) r_0 + t r_1$ for $0 \le t \le 1$

Parametrise: $C_1: r(t) = (1-t)(0,0) + t(1,0) = (t, 0)$

$C_2: r(t) = (1-t)(1,0) + t(1,2) = (1, 2t)$

$C_3: r(t) = (1-t)(1,2) + t(0,0) = (1-t, 2-2t)$

Now integrate along each curve from 0 to 1, noting each integral is broken into x and y components

On C_1: $y = 0$, $y'(t) = 0$ so we see:

$$\int_{C_1} (x^2 y dx + x dy) = 0.$$

C_2: $x'(t) = 0$ $y'(t) = 2$, so first term vanishes

$$\int_{C_2} (x^2 y dx + x dy) = \int_{C_2} x dy = \int_0^1 1 \frac{d}{dt}(2t)\, dt = 2$$

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C_3: $x'(t) = -1$ $y'(t) = -2$ So we have

$\int_{C_3} x^2y\,dx + x\,dy$ $\int_0^1 (1-t)^2(2-2t)\frac{d}{dt}(1-t)\,dt + \int_0^1 (1-t)\frac{d}{dt}(2-2t)\,dt$

$= 2\int_0^1 (t-1)^3\,dt + 2\int_0^1 (t-1)\,dt$

$= -3/2$

Therefore $\int (x^2y\,dx + x\,dy) = 0 + 2 - \frac{3}{2} = \frac{1}{2}$

Path Independence

The path of integration, i.e. the curve along which the integral is taken can affect the result of the integral

If vector field is conservative, it will be path independent
Any path chosen will give the same result which is determined by the end points.

Fundamental Theorem of Line Integrals

$\int_C F(x,y,z)\cdot dr = \int_C \nabla\phi\cdot dr = \phi(x_0,y_0,z_0) - \phi(x_1,y_1,z_1)$

Testing for a conservative vector field

If $\frac{df}{dy} = \frac{dg}{dx}$ $\Rightarrow$ conservative

Example

Is $F = 2xy^3 i + (1+3x^2y^2)j$ conservative? Find ϕ.

Solution $\frac{df}{dy} = 6xy^2$ $\frac{dg}{dx} = 6xy^2$

F is conservative $\frac{d\phi}{dx}$ $2xy^3$ $\frac{d\phi}{dy} = 1+3x^2y^2$

$\int \frac{d\phi}{dx}\, dx = \int 2xy^3\, dx \qquad \phi = x^2y^3 + C_1(y)$ 3

where $C_1(y)$ depends on y only and is therefor a constant of integration wrt integration in x.

$\frac{d\phi}{dy} = \frac{d}{dy}(x^2y^3 + C_1(y)) = 3x^2y^2 + \frac{dC_1}{dy}$

$3x^2y^2 + \frac{dC_1}{dy} = 1 + 3x^2y^2$

$\frac{dC_1}{dy} = 1 \qquad C_1(y) = y + C$

where C is any constant which appeared from integrating the equation over y, $\phi = x^2y^3 + y + C$

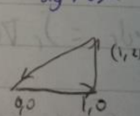
Green's Theorm

Allows us to rewrite an area integral as a line integral.

Theorm: Let R be a simply connected plane region whose boundary is piecewise smooth curve C oriented anticlockwise

$$\int_C f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{dg}{dx} - \frac{df}{dy}\right) dA$$

Example: Evaluate $\int_C x^2y\,dx + x\,dy$



Solution line from $(0,0)$ to $(1,2)$ has eqn $y = 2x$, which will be upper limit for the y integration

$$\int_C x^2y\,dx + x\,dy = \iint_R \left[\frac{d}{dx}(x) - \frac{d}{dy}(x^2y)\right] dA$$

$$= \int_0^1 \int_0^{2x} (1 - x^2)\, dy\, dx$$

$$\int_0^1 [y - x^2y]_{y=0}^{2x}\, dx = \int_0^1 (2x - 2x^3)\, dx = \frac{1}{2}$$

If line integral is taken around a closed curve, indicated by $\oint f(x,y)dx$.

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Chapter 6.

Surface Integrals

Idea is to integrate a function $f(x,y,z)$ over a surface σ.

Surface integral of $f(x,y,z)$ over σ is

$$\iint_{\sigma} f(x,y,z)\,dS = \iint_{R} f(x(u,v), y(u,v), z(u,v)) \left\| \frac{dr}{du} \times \frac{dr}{dv} \right\| dA$$

Note the surface area comes from using $f(x,y,z) = 1$ giving

$$S = \iint_{R} \left\| \frac{dr}{du} \times \frac{dr}{dv} \right\| dA$$

For a surface $z = f(x,y)$ this reduces to simpler form:

$$\iint_{\sigma} f(x,y,z)\,dS = \iint_{R} f(x,y,f(x,y)) \sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2}\, dA$$

Orientation of a smooth parametric surface

principal unit normal vector is $n = \dfrac{\frac{dr}{du} \times \frac{dr}{dv}}{\left\| \frac{dr}{du} \times \frac{dr}{dv} \right\|}$

This vector defines the positive orientation and $-n$ the negative orientation

Stokes Theorem

$$\oint F \cdot dr = \iint_{\sigma} (\operatorname{curl} F) \cdot n\, dS$$

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Chapter 6.

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Surface integral of $f(x,y,z)$ over σ is

$$\iint_{\sigma} f(x,y,z)\,dS = \iint_{R} f(\,x(u,v), y(u,v), z(u,v)) \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dA$$

Note the surface area comes from using $f(x,y,z) = 1$ gives

$$S = \iint_{R} \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dA$$

For a surface $z = f(x,y)$ this reduces to simpler form:

$$\iint_{\sigma} f(x,y,z)\,dS = \iint_{R} f(x,y, f(x,y)) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Orientation of a smooth parametric surface

principle unit normal vector is $n = \dfrac{\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}}{\left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\|}$

This vector defines the positive orientation and $-n$ the negative orientation

Stokes Theorem

$$\oint F \cdot dr = \iint_{\sigma} (\operatorname{curl} F) \cdot n\, dS$$

2013 Q5 A $F(x,y) = (1 + ye^{xy})i + (2y + xe^{xy})j$

$\frac{df}{dy} \stackrel{?}{=} \frac{dg}{dx}$ $\quad yxe^{xy} + e^{xy} \quad = \quad xe^{xy} + e^{xy}$

Is a conservative vector field

B $\frac{d\phi}{dx} = 1 + ye^{xy}$ $\qquad \frac{d\phi}{dy} = 2y + xe^{xy}$

$\phi = \int 1 + ye^{xy}\, dx$

$x + \frac{y}{y} e^{xy} + C_1(y)$

$\frac{d\phi}{dy} = xe^{xy} + \frac{dC_1(y)}{dy} = 2y + xe^{xy}$

constant of integration w.r.t. x

$\frac{dC_1(y)}{dy} = 2y$

$C_1(y) = y^2$

$\phi = x + e^{xy} + y^2 + C$ where C is any con

C $\phi(1,1) - \phi(0,1)$

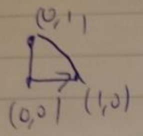
$(1 + e^1 + 1) - (+ e^0 + 1$

$(e^1 + 2) - (1+1)$

$= e^1 = e.$

D
$C_1 : (0,1)$ to $(0,0)$ $\quad r(t) = (1-t)(0,1) + t(0,0) = (0, 1-t)$
$C_2 : (0,0)$ to $(1,0)$ $\quad r(t) = (1-t)(0,0) + t(1,0) = (t, 0)$
$C_3 : (1,0)$ to $(0,1)$ $\quad r(t) = (1-t)(1,0) + t(0,1) \quad (1-t, t)$

Greens Theorem $\iint_R \left(\frac{dg}{dx} - \frac{df}{dy}\right) dA.$



$(1,1,1)$ $0 \le x \le 1$

$x \le y \le 1$

$$\int_0^1 \int_x^1 1 - 3y^2x^2 \; dy\,dx$$

$$y - y^3x^2 \Big|_x^1$$

$$\int_0^1 (1 - x^2) - (x - x^5)$$

$$\int_0^1 1 - x^2 - x + x^5 \; dx$$

$$x - \frac{x^3}{3} - \frac{x^2}{2} + \frac{x^6}{6} \Big|_0^1$$

$$\frac{1}{4} - \frac{1}{2} - \frac{1}{12} + \frac{1}{8} = 0.$$

11 Q3 $\int_{(-1,0)}^{(1/2,1)} (-3x^2+x+2y^2)\,dx - (3y-4xy)\,dy$

$\frac{dg}{dy} \stackrel{?}{=} \frac{df}{dx}$ $\quad 4y \stackrel{?}{=} +4y$

✓ indep of path

B. $\frac{d\phi}{dx} = -3x^2+x+2y^2$

$\phi = \int -3x^2+x+2y^2 \, dx$

$\phi = -x^3 + \frac{x^2}{2} + 2xy^2 + C(y)$ where $C(y)$ is constant of integration

$\frac{d\phi}{dy} = 4xy + \frac{dC(y)}{dy} = +3y+4xy$

$\frac{dC(y)}{dy} = 3y \quad C(y) = \frac{3y^2}{2}$

$\Phi = -x^3 + \frac{x^2}{2} + 2xy^2 + \frac{3y^2}{2} + C$

C $\phi(1/2,1) - \phi(-1,0)$

$\cancel{3/2} \quad - 3/2$

$5/2 - 3/2 = 2/2 = 1 \qquad = -2$

D C_1: $(-1,0)$ to $(1/2,0)$ $\quad (1-t)(-1,0) + t(1/2,1) = (3/2 t - 1, 0)$

C_2: $(1/2,0)$ to $(1/2,1)$ $\quad (1-t)(1/2,0) + t(1/2,1)$ $(1/2, t)$

$t = -1 + 3/2 t$

for C1: $y=0$, $dy=0$ $\quad C_1$: $x=t$, $y=0$ $\quad -1 \le t \le 1/2$

$t - 3/2 t = -1$

$0.5 t$

$\int_0^1 (-3x^2+x+2y^2)\,dx$

~~$\int_0^1 [-3(3/2 t-1)^2 + (3/2 t - 1)]$~~ $[3/2]\,dt$

$\int_{-1}^{1/2} -3t^2 + t \, dt = -t^3 + \frac{t^2}{2} \Big|_{-1}^{1/2}$

$0 - [1 + 1/2] = -3/2$

$C_2 \quad (1/2, t)$

$\int_0^1 -3y + 4xy$

$\int_0^1 \left[-3t + 4(\tfrac{1}{2})(t)\right] 1 \; dt$

$-3t + 2t$

$\int_0^1 -t \; dt$

$\frac{-t^2}{2}\Big|_0^1 = -\frac{1}{2}$

$-\frac{1}{2} - \frac{3}{2} = \frac{-4}{2} = -2$

$(1-t)(-\frac{\pi}{2}, \pi) \quad + t(\pi, \pi$

$-\frac{\pi}{2} + \frac{t\pi}{2} + t\pi \quad , \quad \pi - t\pi + t\pi$

$\pi.$

$\left(-\frac{\pi}{2} + \frac{3}{2}t\pi \quad , \pi\right)$

$t=0 \quad \frac{-\pi}{2} \qquad t=1 = \pi$