Maths Semester 2 JF 2012-2013

CHAPTER

10

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Math Sem 2 JF 2012-2013

CHAPTER 11

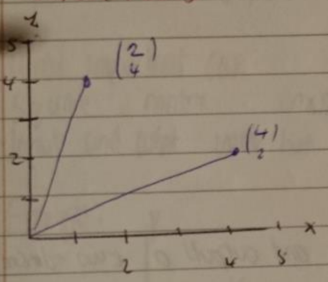
VECTORS

13/3/13 Linear Algebra Week 9

i. Vectors, Matrices
ii. Solving equations using matrices
iii. Fundamental theory of linear algebra

Vectors:
A list of numbers: (1, 2, 4); (2, -3) (1, -2, 4, 1)

Geometric meaning



Arrows pointing from the origin to the point given by the vector components

Adding and subtracting Vectors:

i. $\binom{2}{4} + \binom{4}{2} = \binom{6}{6}$

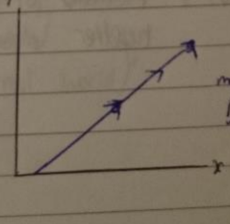
$\binom{3}{1} + \binom{-2}{4} = \binom{1}{5}$

Add components seperately

ii. $\binom{3}{1} - \binom{4}{3} = \binom{-1}{-2}$

Multiplication
$2 \times \binom{1}{3} = \binom{2}{6}$

Multiply components seperately



multiplying by 2, Direction fixed

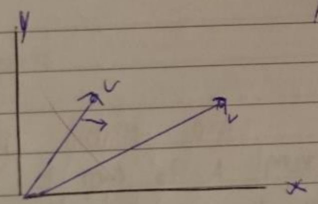
13/3/13. 3 Maths Week 9

Examples: (i)
$$\begin{pmatrix} 4 & 1 \\ 2 & 3 \\ 1 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4(1) + 1(2) \\ 2(1) + 3(2) \\ 1(1) + -1(2) \\ -2(1) + 1(2) \end{pmatrix} \begin{pmatrix} 6 \\ 8 \\ -1 \\ 0 \end{pmatrix}$$

(ii)
$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1(1) + 2(3) + -1(2) \\ 3(1) + 1(3) + 1(2) \\ -2(1) + 1(3) + 3(2) \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}$$

Most important case:
- Square matrix $n \times n$ matrix
- Input and output vector have the same dimension (number of components)

Example:



$A = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$ square

Matrices will stretch and rotate a vector

(only for square matrices)

Determinant:
det (A) = amount A stretches vector
det (A) < 1 = shrinks vector
det (A) > 1 = lengthens vector
det (A) = 1 = only rotates

Matrices with det (A) = 1 are pure rotations
(known as SO(n), n is the dimension)
(SO(3) is the rotation in the real world)

(ii)

Maths Week 9

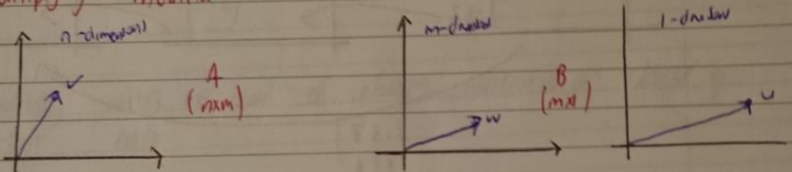
Vectors and Matrices:

$$m\{\overbrace{\begin{pmatrix} & & \end{pmatrix}}^{n} \quad \begin{pmatrix} \\ \end{pmatrix}\}n = \begin{pmatrix} \\ \end{pmatrix}\}m$$

matrix vertical

Changes a n-dimensional vector into a m-dimensional one

Multiplying matrices:



$B \cdot A$ represents

To Multiply Matrices B·A:

Number of rows of A must equal the number of columns of B
(output of A is input of B)

Multiplication:

$$\underset{B}{\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \end{pmatrix}} \quad \underset{A}{\begin{pmatrix} C_1, C_2, C_3 \end{pmatrix}} = \begin{pmatrix} R_1C_1, & R_1C_2 & \dots \\ R_2C_1, & R_2C_2 & \dots \\ \dots & & \end{pmatrix}$$

Example:

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 4 & 1 \\ -1 & 2 \end{pmatrix}$$

$R_1 = (3\ 2,1)$
$R_2 = (1\ 3\ 1)$

$C_1 = (2\ 4\ -1)$
$C_2 = (1\ 1\ 2)$

2.

Matrices: "blachs of numbers"

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}; \begin{pmatrix} 1 & 2 \\ 4 & 1 \\ -1 & 2 \end{pmatrix}; \begin{pmatrix} 2 & 1 & 3 & 4 \\ 5 & 9 & -2 & 8 \end{pmatrix}.$$

Geometrically: n rows
m columns

A $m \times n$ matrix convents a m-dimensional vector into n-dimensland one

Example:

$$\underbrace{\begin{pmatrix} 2 & 1 & 4 \\ -1 & 2 & 3 \end{pmatrix}}_{3 \text{ columns}} \Big\} \text{2 rows}$$

Takes in a three-dimensional veltor and outputs a two-dimbnd one

Example:

$$\begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2.2 + 1.3 + 1.1 \\ -1(2) + 2(3) + 1(1) \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ 5 \end{pmatrix}$$

Multiplying vectors by matrices:

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Inputvetr — output vector

w_1 = elements of R_1 and the vector multiplied in order and added together etc.

5/3/13 3 matrices facts

Uses:

- four dimensional vectors, and 4x4 matrices used in computer graphics

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{position} \\ \} \text{extra information} \end{matrix}$$

- State of a physical object

Six dimensional:

$$\begin{pmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{pmatrix}$$

- essentially vectors are chunks of information, matrices are changes in that info

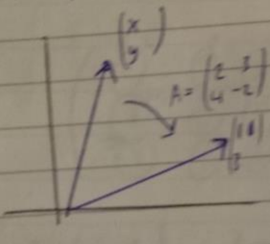
Matrices and equations

$2x + 3y = 11$

$4x - 2y = 3$

Matrix representation:

$$\begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} 11 \\ 3 \end{pmatrix}$$



Solved by find A^{-1}, the matrix which reverses the action of A.

22/3/18 Maths Week 10

(Gaussian elimination)

$4x + 2y + 3z = 2$

$2x - y + 2z = 1$

$x + 3y + z = 3$

$$\left(\begin{array}{ccc|c} 4 & 2 & 3 & 2 \\ 2 & -1 & 2 & 1 \\ 1 & 3 & 1 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1/2 & 3/4 & 1/2 \\ 0 & 1 & 0 & 20/28 \\ 0 & 0 & 1 & 20/7 \end{array}\right)$$

Subtract $R_1 - 3/4 R_3$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & -14/28 \\ 0 & 1 & 0 & 5/7 \\ 0 & 0 & 1 & 20/7 \end{array}\right)$$

Subtract $R_1 - 1/2 R_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -56/28 \\ 0 & 1 & 0 & 5/7 \\ 0 & 0 & 1 & 20/7 \end{array}\right)$$

$x = -56/28 = -2$

$y = 5/7$

$z = 20/7$

Order:

$$\left(\begin{array}{ccc|c} 1 & 9 & 8 & \\ 2 & 4 & 7 & \\ 3 & 5 & 6 & \end{array}\right)$$

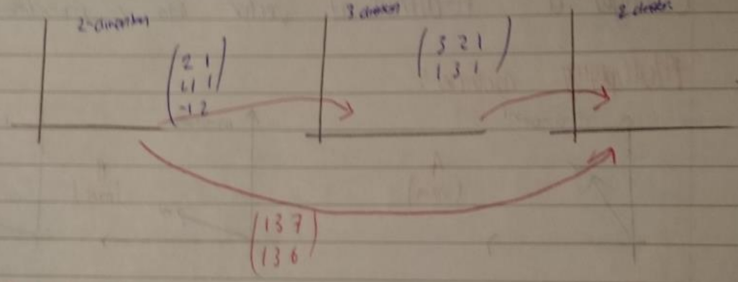
2, 3 : R_1 | 7, 8 : R_3

5 : R_2 | 9 : R_2

2

$R_1 \cdot C_1 = (3 \cdot 2) + (2 \cdot 4) + (1 \cdot -1) = 13$
$R_1 \cdot C_2 = (3 \cdot 1) + (2 \cdot 1) + (1 \cdot 2) = 7$
$R_2 \cdot C_1 = (1 \cdot 2) + (3 \cdot 4) + (1 \cdot 1) = 13$
$R_2 \cdot C_2 = (1 \cdot 1) + (3 \cdot 1) + (1 \cdot 2) = 6$

$$= \begin{array}{cc} \begin{pmatrix} 13 & 7 \\ 13 & 6 \end{pmatrix} & \begin{matrix} R_1 \\ R_2 \end{matrix} \end{array}$$



Non commutative:
For Matrices: $AB \neq BA$ order sometimes matter

Example: $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 1 \\ 4 & 1 \end{pmatrix}$

$A \cdot B = \begin{pmatrix} 10 & 3 \\ 14 & 4 \end{pmatrix}$

$B \cdot A = \begin{pmatrix} 3 & 7 \\ 5 & 11 \end{pmatrix}$

Think of matrices with determinant 1:
- Pure rotations
- A is one rotation (90° about x-axis)
- B is another rotation (90% about y axis)

- Rotations don't commute, Order of two rotations affects the outcome

Two other cases:

1. Infinite solutions

$$4x + 3y - z = 2$$
$$8x + 6y - 2z = 4$$
$$x - y + 3z = 7$$

$$\left(\begin{array}{ccc|c} 4 & 3 & -1 & 2 \\ 8 & 6 & -2 & 4 \\ 1 & -1 & 3 & 7 \end{array}\right)$$

$R_2 - 2R_1$

$$\left(\begin{array}{ccc|c} 4 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 3 & 7 \end{array}\right)$$

Infinite numbers of solutions

Really only two equations
There isn't a unique solution

Eliminate second equation:

i $$\left(\begin{array}{ccc|c} 4 & 3 & -1 & 2 \\ +1 & -1 & 3 & 7 \end{array}\right)$$

Divide by 4 in R_1:

ii $$\left(\begin{array}{ccc|c} 1 & 3/4 & -1/4 & 1/2 \\ 1 & -1 & 3 & 7 \end{array}\right)$$

$R_2 - R_1$

iii $$\left(\begin{array}{ccc|c} 1 & 3/4 & -1/4 & 1/2 \\ 0 & -7/4 & 13/4 & 13/2 \end{array}\right)$$

Multiply $-4/7(R_2)$

$$\left(\begin{array}{ccc|c} 1 & 3/4 & -1/4 & 1/2 \\ 0 & 1 & -13/7 & -26/7 \end{array}\right)$$

Equation: $x + \frac{3}{4}y - \frac{1}{4}z = \frac{1}{2}$
$y - \frac{13}{7}z = -\frac{26}{7}$

No equation fixing the value of z.

20/3/13 Maths Week 10

Solving Simultaneous equations using matrices

example: $2x + 4y = 3$
$x - 2y = 7$

Three methods: Gaussian elimination
Gaussian - Jordan elimination } Similar
Matrix Inverse

Convert to matrix form: $2x + 4y = 3$
$x - 2y = 7$

$$\begin{pmatrix} 2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

Write down as augmented matrix.

$$\left(\begin{array}{cc|c} 2 & 4 & 3 \\ 1 & -2 & 7 \end{array}\right)$$

1. Gaussian elimination $\left(\begin{array}{cc|c} 1 & \cdot & \cdot \\ 0 & 1 & \cdot \end{array}\right)$ $\cdot$ = anything

Subtract $R_1 - 2R_2$

$$\left(\begin{array}{cc|c} 1 & 0 & 17/4 \\ 0 & 1 & -11/ \end{array}\right)$$

To this row multiplication and subtraction

i. E.G. = $\left(\begin{array}{cc|c} 2 & 4 & 3 \\ 1 & -2 & 7 \end{array}\right)$

Divide R_1 by 2

ii. $\left(\begin{array}{cc|c} 1 & 2 & 7/2 \\ 1 & -2 & 7 \end{array}\right)$

iii. Subtract $R_1 - R_2$

$$\left(\begin{array}{cc|c} 1 & 2 & 3/2 \\ 0 & -4 & 11/2 \end{array}\right)$$

Divide R_2 by -4

iv. $\left(\begin{array}{cc|c} 1 & 2 & 3/2 \\ 0 & 1 & -11/8 \end{array}\right)$ $\Rightarrow$ (Row Echelon) form

Matrix can be read as equation

$$x + 2y = 3/2$$
$$y = -11/8$$

2. Gauss-Jordan .. convert matrix into:

$$\left(\begin{array}{cc|c} 1 & 0 & : \\ 0 & 1 & : \end{array}\right)$$

i. $\left(\begin{array}{cc|c} 1 & 2 & 3/2 \\ 0 & 1 & -11/8 \end{array}\right)$

Subtract $R_1 - 2R_2$

ii. $\left(\begin{array}{cc|c} 1 & 0 & 17/4 \\ 0 & 1 & -11/8 \end{array}\right)$ $\quad x = 17/4$, $y = -11/8$

Example

$$4x + 2y + 3z = 2$$
$$2x - y + 2z = 1$$
$$x + 3y + z = 3$$

matrix form: $\left(\begin{array}{ccc|c} 4 & 2 & 3 & 2 \\ 2 & -1 & 2 & 1 \\ 1 & 3 & 1 & 3 \end{array}\right)$

convert into $\left(\begin{array}{ccc|c} 1 & \cdot & \cdot & : \\ 0 & 1 & \cdot & : \\ 0 & 0 & 1 & : \end{array}\right)$

Steps:
1. $R_1 \div 4$
2. $R_2 - 2R_1$
3. $R_3 - R_1$
4. $R_2 \div -2$
5. $R_3 - 5/2 R_2$
6. $R_3 \div 2/10$

$$\left(\begin{array}{ccc|c} 1 & 1/2 & 3/4 & 1/2 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 1 & 20/7 \end{array}\right) \Rightarrow \quad z = 20/7, \; y = 5/7, \; x = -2$$

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22/3/13 Maths Week 10

$y = -^{26}/_{7} + ^{13}/_{7}z$

$x = ^{1}/_{4}z - ^{3}/_{4}y \quad ^{1}/_{2} \qquad = ^{1}/_{4}z - ^{3}/_{4}y\left(^{-26}/_{2} + ^{13}/_{4z}\right) + \frac{1}{2} = x$

2 No Solutions:

$4x + 3y - z = 2$

$8x + 6y - 2z = 3$

$x - y + 3z = 7$

$$\left(\begin{array}{ccc|c} 4 & 3 & -1 & 2 \\ 8 & 6 & 2 & 3 \\ 1 & -1 & 3 & 7 \end{array}\right)$$

$R_2 - 2R_1$

$$\left(\begin{array}{ccc|c} 4 & 3 & -1 & 2 \\ 0 & 0 & 0 & -1 \\ 1 & -1 & 3 & 7 \end{array}\right)$$

Equations: $4x + 3y - z = 2$

$0 = -1 \leftarrow$ Impossible

$x - y + 3z = 7$

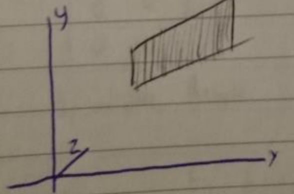
No choice of x, y, z will work

Geometric Picture

$x + 2y + 3z = 11$

$4x + 6y - 2z = 12$

$8x + 5y + 3z = 6$



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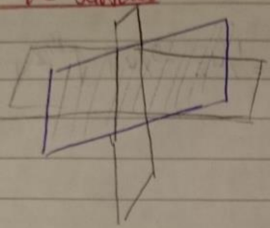
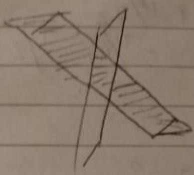
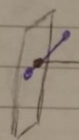
First the Surfaces



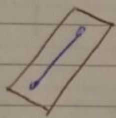


Third Surface

i.  one intersection

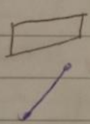
OR:

Third Surface parallel to line of intersection

ii 

Line lies on Surface

Infinite number of Solution

iii OR: 

no intersection

no Solution

5/3/13 Maths Week 11

Solution using inverse matrix

$2x + 3y = 11$
$4x - 2y = 9$

$$\underbrace{\begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 11 \\ 9 \end{pmatrix}}_{B}$$

$Ax = B$

Gauss Jordan:

$$A = \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix} \xRightarrow{\text{Gauss Jordan}} \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Inverse: Of a matrix is another matrix A^{-1} such that,

$A^{-1}A = \mathbb{I}$

Gauss - Jordan:

$$\underbrace{E_n \; E_{n-1} \ldots \; E_2 \; E_1}_{\text{Gauss Jordan steps}} A = \mathbb{I}$$

E_i - Elementary Matrices:

e.g. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ By multiplying R_2 by 4 $\qquad \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $R_1 + 2R_2$

$$\underbrace{E_n, E_{n-1} \ldots E_2, E_1}_{= A^{-1}} A = 1$$

$A^{-1}A = 1$

Inverse Matrix

$E_n, E_{n-1} \dots E_2 E_1 A = E_n, E_{n-1} \dots E_1 I$

$I = A^{-1}$

Example: $\begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 11 \end{pmatrix}$

i. $\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & -2 & 0 & 1 \end{array}\right)$

Divide R_1 by 2

ii. $\left(\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 4 & -2 & 0 & 1 \end{array}\right)$

Subtract $R_2 - 4R_1$

iii. $\left(\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & -8 & -2 & 1 \end{array}\right)$

Divide R_2 by -8

iv. $\left(\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & 1/4 & -1/8 \end{array}\right)$

Subtract $R_1 - 3/2 R_2$

v. $\left(\begin{array}{cc|cc} 1 & 0 & 1/8 & 3/16 \\ 0 & 1 & 1/4 & -1/8 \end{array}\right)$

Inverse $A^{-1} = \begin{pmatrix} 1/8 & 3/16 \\ 1/4 & -1/8 \end{pmatrix}$

ordinary - only one variable

$$\frac{d^2y}{dt^3} + \frac{d^2y}{dt^2} + t^2\frac{dy}{dt} + y^2 = 0$$

$y(t)$

Partial - Several variables

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0$$

$\phi(x, y)$

Example:

Electromagnetism: 4 2^{nd}-order Linear Partial differential equations

Newton's Gravity: 1 2^{nd} order linear partial diff. eqn

Population growth: Several 1^{st} order ordinary diff eqns.

General relativity 10 2^{nd} order non linear partial diff eqn's

Fluid-flow (navier-stokes): 3 2^{nd} order non-linear partial differential equns

This course: First order linear ordinary differential equations

Basic form - $\frac{dy}{dt} + p(t)y = F(t)$

eg $\frac{dy}{dt} + t^2y = \sin(t)$

$\frac{dy}{dt} + t^4y = e^t$

3.

Week 11

7/3/13 Maths Week 11 replace all ([ ], " ")

She'd

Q2 Differential equations. $\frac{d^2y}{dt} + t^2\frac{dy}{dt} + \sin(t)y = \cos(t)$

Solution: A function $y(t)$

Algebraic equation: $x^2 + 2x + 3 = 0$ Solution: x = Some real number.

$x^2 - 2 = 0$ $x = \pm\sqrt{2}$

Example: $\frac{dy}{dt} = y$

Solution: $y = Ae^t$

↑ constant

Existence and uniqueness

Differential equations are most common application of higher mathematics

Definitions:

order - highest derivative

$\frac{d^2y}{dt^2} + t^2y = 0$ 2nd order

$\frac{dy}{dt} + e^ty = 0$ 1st order

Linear - Only y and its derivatives appear, no power of y or its derivative

e.g. $\frac{d^2y}{dt^2} + d^4\frac{dy}{dt} + y = 0$

Non-Linear - Powers of y or the derivative are present

$\left(\frac{d^2y}{dt^2}\right)^2 + \frac{dy}{dt} + y^3 = 0$

25/3/13 Math W11

3.

Original expression:

$$\begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 11 \end{pmatrix}$$

$Ax = b$

$A^{-1}Ax = A^{-1}b$

$\mathbb{1}\, x = A^{-1}b$

$x = A^{-1}b$

$$A^{-1}b = \begin{pmatrix} 1/8 & 3/16 \\ 1/4 & -1/8 \end{pmatrix} \begin{pmatrix} 9 \\ 11 \end{pmatrix} = \begin{pmatrix} 9/8 & 27/16 \\ 9/4 & -9/8 \end{pmatrix}$$

$$= \begin{pmatrix} 18/16 & 33/16 \\ 22/8 & 11/8 \end{pmatrix} = \begin{pmatrix} 52/16 \\ 7/8 \end{pmatrix} = \begin{pmatrix} 26/8 \\ 7/8 \end{pmatrix} = \begin{matrix} x = 26/8 \\ y = 7/8 \end{matrix}$$

Solution using invert matrix

i.

$3x + 2y + 5z = 11$

$2x - y + 3z = 4$

$x + 3y - z = 6$

convert into matrix

ii.

$$\begin{pmatrix} 3 & 2 & 5 \\ 2 & -1 & 3 \\ 1 & 3 & -1 \end{pmatrix}$$

iii.

$$\left(\begin{matrix} 3 & 2 & 5 \\ 2 & -1 & 3 \\ 1 & 3 & -1 \end{matrix} \;\middle|\; \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & & 1 \end{matrix}\right)$$

iv. Use gauss-Jordan to convert left hand side into identity

$$\left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \;\middle|\; \underbrace{\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}}_{A^{-1}}\right)$$

6.

v. Act on $\begin{pmatrix} 11 \\ 4 \\ 6 \end{pmatrix}$ with A^{-1}

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 11 \\ 4 \\ 6 \end{pmatrix} = \left.\begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} \begin{matrix} =x \\ =y \\ =z \end{matrix}\right\} \text{answers}$$

Comes up every year

Multiply vector of right hand of equation with A^{-1}.

EXAM

Fundamental theorm of linear Algebra

- Conditions equivalent to the existence of A^{-1} (making it possible to solve a set of equations)

i. $\det(A) \neq 0$ (determinant)
ii. $A\underline{x} = 0$ only has the solution $x = 0$
iii. $Ax = b$ only has one solution
iv. The rows of A are linerly independent.
v. The columns of A are linerly independent (Rows arent multiples of each other)
vi. A is a product of elementary matrices

Determinant:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 6 & 4 \\ 2 & 1 & 3 \end{pmatrix}$$

Gaussian elimination $\begin{pmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 0 & 0 & \cdot \end{pmatrix}$ without setting diagonal to 1

multiply diagonal element $= \det(A)$

3

27/3/12. Maths Week 11

Method: $\frac{dy}{dt} + 3t^2 y = e^{-t}t$

i. Integrate the function in front of y, p(t)

$p(t) = 3t^2$

$\int p(t) = \int 3t^2 dt = t^3$

ii. Integrating factor $I(t) = t^3$

iii. Multiply across by $e^{I(t)}$

$$\underbrace{e^{t^3}\frac{dy}{dt} + 3t^2 e^{t^3} y}_{\text{This is just a simple derivative.}} = t$$

iv. $\frac{d}{dt}(e^{t^3} y) = t$

always same form (mostly)

v. Integrate both sides

$$\int \frac{d}{dt}(e^{t^3}y)\,dt = \int t\,dt$$

$$e^{t^3}y = \frac{t^2}{2} + C$$

vi. Only y on left hand side.

$$y = \frac{t^2 e^{-t^3}}{2} + Ce^{-t^3}$$

Usually sure exponent to cancel $\underline{e^{It}}$

example $\frac{d^2y}{dt} + 2y = e^{-t^2-t}$

$t^2 \to e^{t^2} \to e^{-t}$

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3/4/13 Week 12

EXAM

1 Limits (L'hopitals rule)

2 Integration
- Substitution
- By parts
- Partial fraction

Differential equation

3 Series
- Geometric
- Integral test
- Ratio test

4 Linear Algebra
- Gaussian, Gauss + Jordan elimination
- Inverse matrix
- Fundamental theorem

$$\begin{matrix} 1 & 2 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{matrix}$$

DPAS LIIY

Q1 LIMITS

$\lim_{x \to a} F(x) = ?$
$x \to 0$
$x \to \infty$

i. Substitute value: $F(a)$
ii. $\frac{0}{0}$, $\frac{\infty}{\infty}$
iii. $0 \cdot \infty$
iv. $\infty - \infty$
v. 0^{∞}, 1^{∞}, ∞^{0}

Technique: ii. L'hopitals rule
iii, iv, v convert to type ii

Q2. INTEGRATION

i Substitution

$\int_a^b F(x)dx$

a Pick $u = g(x)$
b Check if du/dx appear
c Change limits

$$\int \frac{1}{x^2+2}\,dx = \frac{\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)}{\sqrt{2}}$$

$$\int \frac{2x}{x^2+2} \text{ "=" } \ln|x^2+2| \qquad (u = x^2)$$

c. Differential Equations:

$$\frac{dy}{dt} + p(t)y = f(t) \qquad \text{Solving for } y$$

i. $I(t) = \int p(t)\,dt$

ii. Multiply by $e^{I(t)}$ and the equation becomes:

$$\frac{d}{dt}\left(e^{I(t)}y\right) = e^{I(t)} f(t)$$

Q3 Series

a. Geometric Series:

$$\sum_{n=0}^{\infty} ar^n, \qquad \sum_{n=0}^{\infty} \frac{3}{10^n}, \qquad \sum_{n=0}^{\infty} \frac{4^n}{10^n}$$

$r = \frac{1}{10}$, $a = 3$ $\qquad$ $r = \frac{4}{10}$, $a = 1$

converges if $-1 < r < 1$

b. TESTS

i. Ratio Test

$\sum_{n=1}^{\infty} a_n$

a. a_n, a_{n+1}

b. $\frac{a_{n+1}}{a_n}$

c. $\left|\frac{a_{n+1}}{a_n}\right|$

d. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ → < 1 converges; → 1 ?; → > 1 diverges

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ii. By Parts:

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

a Choose u, dv

b work out du and v

$v = \int dv$

Choose u in the order:

Logs	$\ln(x)$ $\ln(x^3)$
Algebraic	x^2, x^3
Trigonometric	$\sin(x)$
Exponential	e^{-x^2}

Note: $\sin(x^2)$

Substitution: x^2, $\sin(x^2)$

By parts: $\sin(x^2)$ can choose function inside function

iii. Partial Fractions

$$\int \frac{x^2+2x+3}{(x^2+4)(x+3)}\,dx$$

a Integrand is split up into a sum of fractions

$$\frac{x^2+2x+3}{(x^2+4)(x+3)} = \frac{Ax+B}{(x^2+4)} + \frac{C}{(x+3)}$$

b Check if quadratic (x^2) factors can be factorised.

Use: b^2-4ac e.g. x^2+4 $a=1$, $b=0$, $c=4$

$b^2-4ac = -16 \leq 0$

cannot be factorised

>0 then be factorised.

c $\int \frac{A}{x+c}\,dx = A\ln(x+c)$

ii. Integral Test.

$$\sum_{n=1}^{\infty} a_n \Rightarrow \int_1^{\infty} f(x)\,dx$$

f(x): Change n to x.

Note: Can not use if $n!$ or $(-1)^n$ appear.

Change limit

a. evaluate integral

not: $-\infty$ or ∞ $\qquad$ ∞ or $-\infty$ = diverge

$\Rightarrow$ series) converges

b. (if asked) Bound

$$\sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x)\,dx$$

Qu. LINEAR ALGEBRA

(a) Gauss-Jordan, Gaussian

$3x + 2y + z = 4$

$2x + 0 + 2z = 8$

$x - 4y + 9z = 2$

$\Downarrow$

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 2 & 0 & 2 & 8 \\ 1 & -4 & 9 & 2 \end{array}\right)$$ Augmented matrix

(b) Inverse Matrix

$$\left(\begin{array}{ccc|c|ccc} 3 & 2 & 1 & 4 & 1 & 0 & 0 \\ 2 & 0 & 2 & 8 & 0 & 1 & 0 \\ 1 & -4 & 9 & 2 & 0 & 0 & 1 \end{array}\right)$$

$\Downarrow$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \end{array}\right)$$

A^{-1}

(c) State theorem

$$A^{-1}\begin{pmatrix} 4 \\ 8 \\ 2 \end{pmatrix} = \begin{pmatrix} \vdots \end{pmatrix}\begin{matrix} 8 \\ 2 \end{matrix}$$