19/2/14 MATHS 1.

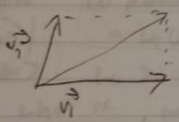
Used Vector Operation
1. Addition
2 Multiplication

Now (3) dot product:

$\vec{u}\cdot\vec{v} = u_1v_1 + \dots + u_nv_n$

$(1, 0, -5, 3)\cdot(0, -1, 3, 0) = -15.$

\# $\vec{v} = k_1\vec{v_1} + k_2\vec{v_2}$



(k_1, k_2) non-standard coordinates

LINEAR PRODUCT Ch. 6.1

$\vec{u} = (u_1, u_2)$ $\vec{v} = (v_1, v_2)$ $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2$

$\langle \vec{u}.\vec{v} \rangle$ $u_1v_1 + 2u_2v_2$ – linear product

$(\vec{u},\vec{v}) = w_1u_1v_1 + w_2u_2v_2$ 2-vector

$(\vec{u},\vec{v}) = w_1u_1v_1 + \dots + w_nu_nv_n$ n-vector

$w_1 \dots w_n$ weight

$(\bar{u},\bar{v}) = u_1v_1 + \dots + (u_1-u_2)(v_1-v_2)$ $(*)$

$\bar{u}\cdot\bar{v} = u_1v_1 + u_2v_2.$

$u_1 \Rightarrow u_1$

$u_2 \Rightarrow u_1 - u_2$

RULE:

$(\bar{u},\bar{v})$ is inner product if:

1. $(\bar{u}, \bar{v}) = \langle v, u \rangle$ Symmetry

2

2. $(\bar{u}+\bar{v}, \bar{w}) = \langle \bar{w}, \bar{u} \rangle + \langle \bar{w}, \bar{v} \rangle$

think: $(a+b)c = ab + bc$

Additivity

3. $(k\vec{u}, \vec{v}) = k(\bar{u}, \bar{v})$ (homogeneity)

Think: $(ka)b = k(ab)$

4. Positivity $\langle \bar{v}\bar{v} \rangle \geq 0$. $\bar{v} \neq 0$ $\langle \bar{u}, \bar{v} \rangle > 0$

$a \cdot a = a^2 \geq 0$

weighted dot product:

$\langle \bar{u}, \bar{v} \rangle = 2u_1v_1 + u_2v_2$

1. ✓

2. $(\bar{u}+\bar{v}, \bar{w}) = 2(u_1+v_1)w_1 + (u_2+v_2) \cdot w_2 =$
$2u_1w_1 + u_2w_2 + 2v_1w_1 + v_2w_2 = \langle \bar{u}, w \rangle$

3. $(k\bar{u}, \bar{v}) = 2ku_1v_1 + ku_2v_2 =$
$k(2u_1v_1 + u_2v_2) = k(\bar{u}, \bar{v})$

4. $(\bar{v}, \bar{v}) = 2v_1^2 + v_2 > 0$
for $\bar{v} = (v_1, v_2) \neq \vec{0}$

Not linear product:

$\langle \bar{u}, v \rangle = u_1v_1 - u_2v_2$

1 2 3 hold

Not 4 $(\bar{v}, \bar{v}) = \bar{v}_1^2 - \bar{v}_2$ not always positive

$\bar{v}_1^2 - \bar{v}^2 > 0$

$v_n^2 > \bar{v}^2 \Leftrightarrow |v_1| > |v_2|$

for (*) $\langle \bar{v}\ \bar{v} \rangle = v_1^2 - (v_1 - v_2)^2 > 0$

$\Leftrightarrow v_2 \neq 0$ or $v_1 - v_2 = 0$

19/2/14 3 Norms

If not $\Rightarrow v_1 = 0$ & $v_1 - v_2 = 0 \Rightarrow v_2 = 0$ $v = (v_1, v_2) = \vec{0}$

Symmetry $(\bar{u}, \bar{v}) = (\bar{v}, \bar{u})$

$(\bar{u}, \bar{v}) = u_1 v_1 + (u_1 - u_2)(v_1 - v_2)$

$(\bar{v}, \bar{u}) = v_1 u_1 + (v_1 - v_2)(u_1 - u_2)$

Not inner product:

$(\bar{u}, \bar{v}) = u_1 v_1 + (u_1 - v_1) u_2$

$(\bar{v}, \bar{u}) = v_1 u_1 + (v_1 - u_1) v_2$

Not Symmetric

Weighted dot product.

$(\bar{u}, \bar{v}) = w_1 u_2 v_1 + \ldots + w_n u_n v_n$ & inner product

$\Leftrightarrow w_1 > 0 \ldots w_n > 0$

what if $w_1 > 0$; $w_2 = ?$

$(\bar{u}, v) = u_1 v_1 + 0 u_2 v_2$

Positivity $(\bar{v}, \bar{v}) > 0 \Leftrightarrow v_1^2 > 0 \Leftrightarrow v_1 \neq 0$

Does not hold for $\bar{v} = (0, 1)$

$u_1 v_1 + u_2 v_2 = \bar{u} \cdot \bar{v}$

$u_1 v_1 + (u_n - u_1)(v_1 - u_1) = (\bar{u}, \bar{v})$

$(\bar{u}, \bar{v}) = (A\bar{u}) \cdot (A\bar{v})$ A invertible or non-singular $\Leftrightarrow \det \neq 0$

$\Rightarrow$ Inner product

21/2/14 Maths 1

Inner Product $(\bar{u}, \bar{v})$

Dot product: $\bar{u}\cdot\bar{v} = u_1v_1 + \dots + u_nv_n$

Inner Product: $\langle \bar{u}, \bar{v} \rangle = w_1u_1v_1 + \dots + w_nu_nv_n$ w-weight

$\langle \bar{u}, \bar{v} \rangle = u_1v_1 + 2u_2v_2$

General $(\bar{u}, \bar{v}) = (A\bar{u})\cdot(A\bar{v})$

Length and Distance with respect to Inner Product.

length or norm w.r.t. $(\bar{u}, \bar{v})$.

$\|\bar{u}\| = \sqrt{(\bar{u}, \bar{u})}$

$\langle \bar{u}, \bar{v}) = 3u_1v_1 + 2u_2v_2$

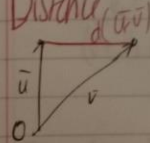
$\|\bar{u}\| = \sqrt{(\bar{u}, \bar{v})} = \sqrt{3u_1^2 + 2u_2^2}$

$\bar{u} = (1,2)$ norm $\|\bar{u}\| = \sqrt{3\cdot1^2 + 2\cdot2^2}$

length with respect to $(\bar{u}, \bar{v}) = 3u_1v_1 + 2u_2v_2$

// bridge across a river

Distance



points distance between u and V

$$d(\bar{u}, \bar{v}) = \|\bar{v} - \bar{u}\|$$

$$= \sqrt{(\bar{v} - \bar{u}, \bar{v} - \bar{u})}$$

$(\bar{u}, \bar{v}) = 3u_1v_1 + 2u_2v_2$

$d(\bar{u}, \bar{v}) = \sqrt{(\bar{u} - \bar{v}, \bar{u} - \bar{v})}$

2

$\neq \quad \bar{u}=(1,2) \quad \bar{v}=(2,-3)$

$d(\bar{u},\bar{v}) = \sqrt{\langle(1-2),(2+3),(1-2,2+3)\rangle}$

$= \sqrt{3(1-2)^2+2(2+3)^2}$

Angle between $\bar{u},\bar{v}$, w.r.t inner produl

$\measuredangle(\bar{u},\bar{v}) = \cos^{-1} \dfrac{\langle\bar{u},\bar{v}\rangle}{\|\bar{u}\|\ \|\bar{v}\|}$

$\bar{u}=(1,2) \quad \bar{v}=(2,-3)$

$\theta = \cos^{-1} \dfrac{(\bar{u},\bar{v})}{\|\bar{u}\|\|\bar{v}\|}$

$\dfrac{3u_1v_1+2u_2v_2}{\sqrt{3u_1^2+2u_2^2}\ \sqrt{3v_1^2+2v_2^2}} = \dfrac{3(1)(2)+2(2)(-3)}{\sqrt{3(1)^2+2(2)^2}\ \sqrt{3(2)^2+2(-3)^2}}$

Example $(\bar{u},\bar{v}) = u_1v_1+(u_1-v_2)(u_2-v_1)$

$= A\bar{u} \cdot A\bar{v}$

$A\bar{u}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}v_1\\v_1-v_2\end{pmatrix} \quad = \begin{pmatrix}1&0\\-1&1\end{pmatrix}$

$A\bar{u}\cdot A\bar{v} = (u_1, u_1-u_2)\cdot(v_1, v_1-v_2)$

$\|\bar{u}\| = \sqrt{\langle\bar{u},\bar{u}\rangle} = \sqrt{u_1^2+(u_1-u_2)^2}$

$\bar{u},\bar{v}$ orthogonal w.r.t $(\bar{u},\bar{v})$ if $(\bar{u},\bar{v})=0$

$\bar{u} = (1,0,-1,2)$

$\bar{v} = (-1,3,0,1)$

21/2/14 3 Maths 3.

$\langle \bar{u}, \bar{v} \rangle = u_1 v_1 + 2u_2 v_2 + 3u_3 v_3 + 4u_4 v_4$

$\langle \bar{u}, \bar{v} \rangle = 1(-1) + 2(0)(3) + 3(-1)(0) + 4(2)(1)$

$-1 + 8 \neq 0$

not orthogonal $\neq 0$

Basis is set of vectors $\{\bar{v}_1, \ldots, \bar{v}_n\}$

Orthogonal Basis w.r.t inner product $(\bar{u}, \bar{v})$

if any two elements are orthogonal if $(\bar{v}_i, \bar{v}_j) = 0$ if $i \neq j$.

$\{\bar{v}_1, \bar{v}_2\}$ orthogonal if and only if $\langle \bar{v}_1, \bar{v}_2 \rangle = 0$

$\{\bar{v}_1, \bar{v}_2, \bar{v}_3\} \Rightarrow \langle v_1, v_3 \rangle = 0$

$\langle \bar{v}_2, \bar{v}_3 \rangle = 0$ all have to $= 0$

$\langle \bar{v}_1, \bar{v}_2 \rangle = 0$

Question is a basis orthogonal w.r.t inner product?

$\bar{v}_1 = (0, 1, 0)$ $\bar{v}_2 = (-4, 0, 3)$ $\bar{v}_3 = (3, 0, 4)$

$\langle \bar{u}, \bar{v} \rangle = \bar{u} \cdot \bar{v}$

Each pair: $(v_1, v_2) = 0+0+0 = 0$ ✓ All zero

$(v_1, v_3) = 0+0+0 = 0$ ✓

$(v_2, v_3) = -12+0+12 = 0$ ✓

$\Rightarrow$ orthogonal

$\bar{w}_3 = (3, 0, 3)$

Is new basis $(\bar{v}_1, \bar{v}_2, \bar{w}_3)$ orthogonal?

$\langle v_1, v_2 \rangle = 0$ ✓

$\langle v_1, \bar{w}_3 \rangle = 0+0+0 = 0$ ✓

$\langle \bar{v}_2, w_3 \rangle = -12+0+9 = -3$ X Not orthogonal.

7/3/14 Maths

Eigenvalues And Eignvectors of Matrices

Matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$ → Transformation

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{1m} \\ a_{n1} & m \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 5 \end{pmatrix} \longleftrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

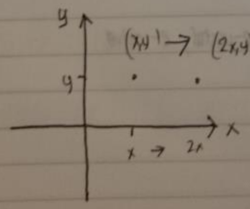
Matrices → Only Square Matrices

$\begin{pmatrix} 1 & 3 \\ 0 & 5 \end{pmatrix}$ square   not square $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \underset{\text{Given}}{\square} \longmapsto \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Square Matrix ⇒ number colums = number rows

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \xrightarrow{\boxed{\text{transformation}}} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$



$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 2x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

or $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2x \\ y \end{pmatrix}$

or $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$(0,y) \to (0,y)$
$(x,0) \to (2x,0) \quad = 2(x,0)$

Vector : $\bar{e}_1 = (1,0) \quad \bar{e}_2 = (0,1)$

by $\overset{A}{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}} \therefore \Rightarrow A\bar{e}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

or $A\bar{e}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Result: $\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \bar{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$A\bar{e}_1 = 2\bar{e}_1$
$A\bar{e}_2 = 1\bar{e}_2$

$\bar{e}_1, \bar{e}_2$ are eiginvectors of matrix A
2, and 1 are eiginvalues

A = Square $n \times n$ matrix
Eiginvector is a vector $\bar{v}$ not equal to zero vector $\bar{v} \neq \bar{0}$ (no direction)
Such that $A\bar{v} = \lambda\bar{v}$ for some real $\lambda \in \mathbb{R}$ i.e. stretch vector.

Eigenvalue $^{\text{of } \bar{v}}$ is λ ~~of~~ $\bar{v}$.

Goal: Given A find all eigen e-vectors, e-value

$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$ Solve $A\bar{v} = \lambda\bar{v}$

7/3/14 Math 3

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix}$$

System not linear

STEP 1:

Eliminate $\bar{v}$ to get λ

Characteristic polynomial

$p(\lambda) = \det(\lambda I - A)$ Sounds Crazy

$$= \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

has to be $A\bar{v}$ = matrix

$$A\bar{v} = (\lambda I)\bar{v}$$

I is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ the indentity matrix

Makes matrix out of scaler λ

Can solve Now:

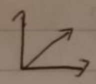
$\Rightarrow (A - \lambda I)\bar{v} = 0.$

has Solution $\bar{v} \neq \bar{0}$

$\Leftrightarrow$ if and only if $\underline{\det(A - \lambda I) = 0}$

3/3/14 Maths. 1. Week 8.

orthogonal
all 90° angle

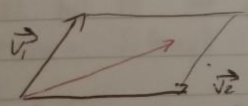


not orthogonal
not all 90° angle

We have Basis $\{v_1 \dots v_n\}$ is orthogonal if
all combinations of dot products $= 0$ $(v_i, v_j) = 0$ $i \neq j$ inner product
inner product

~~is orthogonal~~ is orthonormal if it is
orthogonal and $\|\bar{v}_i\| = 1$ for all i.

Coordinates of vector with respect to orthogonal basis
$\bar{v} = c_1\bar{v}_1 + \dots + c_n\bar{v}_n$
$\{v_1 \dots v_n\}$ Basis



$\bar{v} = c_1\bar{v}_1 + c_2\bar{v}_2$ c are free
can make any vector on plane

Q ~~Find~~ Given $\bar{v}_1$ find its coordinates $(c_1 \dots c_n)$

Assume: basis is orthogonal wrt inner product $(\bar{u}, \bar{v})$

Multiply $(*)$ by $\bar{v}_1$: $\langle \bar{v}, \bar{v}_1 \rangle = c_1 \langle v_1, v_1 \rangle + c_2 \langle v_2, v_1 \rangle$
$\dots + c_n \langle v_n, v_1 \rangle$

because of assumption all $= 0$ except for $\langle v_1, v_1 \rangle$
$= \|v_1^2\|$

$c_1 = \frac{\langle v_1, v_1 \rangle}{\|v_1^2\|}$ inner product
norm

2

$C_2 = \dfrac{\langle V_2, V_2 \rangle}{\|\vec{V_2}\|^2}$ inner / norm²

$\ldots \quad C_n = \dfrac{\langle V_n, V_n \rangle}{\|\vec{V_n}\|^2}$

Example:

Orthogonal basis $\vec{V_1} = (0, 1, 0)$

$\vec{V_2} = (-4, 0, 3) \quad \vec{V_3} = (3, 9, 5)$

Want coordinates of $\vec{v} = (1, 2, 3)$.

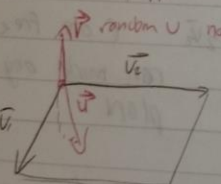
$C_1 = \dfrac{\langle (1,2,3), (0,1,0) \rangle}{\|(0,1,0)\|^2} = \dfrac{1\cdot 0 + 2\cdot 1 + 3\cdot 0}{0^2+1^2+0^2} = 2$

$C_2 = \dfrac{\langle (1,2,3)(-4,0,3) \rangle}{-4^2, 0^2, 3^2} = \dfrac{-1(4) + 2(0) + 3(3)}{4^2+0^2+3^2} = \dfrac{15}{25}$

$C_3 = \dfrac{\langle (1,2,3)(3,0,5) \rangle}{3^2+0^2+5^2} = \dfrac{18}{34} = \dfrac{9}{17} =$

For orthonormal basis $\|\vec{v_i}\| = 1$ all i.

$C_i = \dfrac{\langle V, V_i \rangle}{\|V_i\|^2} = \langle \vec{v}, v_i \rangle$



$C_1 = \dfrac{\langle V, V_1 \rangle}{\|V_1\|^2}$ same formula

$C_2 = \dfrac{\langle V, V_2 \rangle}{\|V_2\|^2}$

$\vec{u}$ is projection of span $\{V_1, V_2\}$ of $\vec{V}$

$= C_1\vec{V_1} + C_2\vec{V_2} = \mu$

scaler / scaler

$= V_1 \dfrac{\langle V, V_1 \rangle}{\|V_1\|^2} + \dfrac{\langle V, V_2 \rangle}{\|V_2\|^2} V_2$

3/3/14 3 Math Week 8 monday

Q Find orthogonal projection of $\bar{v} = (1,2,3)$ onto Span $\{v_1, \bar{v}_2\}$ $\bar{v}_1 = (0,1,0)$ $\bar{v}_2 = (-1,0,2)$

$\{v_1, v_2\}$ is orthogonal $\langle \bar{v}_1, \bar{v}_2 \rangle = 0$ project $w^{\bar{v}}$

$W = \text{Span}\{v_1, v_2\}$

$$\text{projection}_W \bar{v} = \frac{\langle (1,2,3),(0,1,0) \rangle}{\|(0,1,0)\|^2}(0,1,0) + \frac{\langle (1,2,3),(-1,0,2) \rangle}{\|(-1,0,2)\|^2}(-1,0,2)$$

$$= \frac{1(0)+2(1)+3(0)}{0^2+1^2+0^2}(0,1,0) + \frac{1(-1)+2(0)+3(2)}{-1^2+0^2+2^2}(-1,0,2)$$

$= 2(0,1,0) + +1(-1,0,2)$

$= (0,2,0) + (1,0,-2)$

$= \cancel{(1,2,-2)}$

$= (-1,2,2)$

Now.

Basis has to be orthogonal

Gram - Schmidt Process chapter 6.3

Given $\{\bar{u}_1 \dots \bar{u}_n\}$ any basis

Want orthogonal basis $\{v_1, \dots, v_n\}$

step 1: $\bar{v}_1 = u_1$

step 2: $\bar{v}_2 = u_2 - \text{projection}_{W_1} \bar{u}_2$

$W_1 = $ Span of u_1 only

$$\bar{v}_2 = \bar{u}_2 - \frac{\langle u_2, v_1 \rangle}{\|\bar{v}_1\|^2}\bar{v}_1$$

u

$$V_3 = \bar{U_3} - \frac{\langle \bar{U_3}, \bar{V} \rangle}{\|V_1\|^2} V_1 - \frac{\langle \bar{U_3}, \bar{V_2} \rangle}{\|V_2\|^2} \bar{V_2}$$

$\Rightarrow \{ \bar{V_1} \ldots \bar{V_n} \}$ orthogonal basis

Example $\bar{U_1} = (1,1,1)$ $\bar{U_2} = (0,1,1)$ $\bar{U_3} = (0,0,1)$

Not orthogonal

Step 1: $\bar{V_1} = U_1 = (1,1,1)$

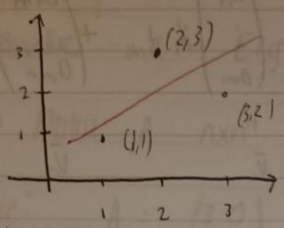
Step 2: $\bar{V_2} = U_2 -$

$(0,1,1) - \frac{\langle (0,1,1),(1,1,1) \rangle}{\|(1^2,1^2,1)\|^2} (1,1,1)$

Step 3 $\bar{V_3} = \bar{U_3} = \bar{U_3} (0,0,1) - \frac{\langle (0,0,1)(1,1,1) \rangle}{1^2+1^2+1^2} (1,1,1)$ - that.

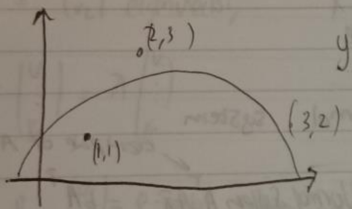
$$\frac{\langle (0,0,1)(\bar{V_2}) \rangle}{\|\bar{V_2}\|^2} \bar{V_2}$$

5/3/14 Maths 1



$y = ax + b$

(x,y)		
$(1,1)$	$\rightarrow$	$1 = a \cdot 1 + b$
$(2,3)$	$\rightarrow$	$3 = a \cdot 2 + b$
$(3,2)$	$\rightarrow$	$2 = a \cdot 3 + b$



$y = ax^2 + bx + c$

$(1,1) \rightarrow 1 = a \cdot 1^2 + b \cdot 1 + c$ still a linear system even though quadratic!

$(2,3) \rightarrow 3 = a \cdot 2^2 + b \cdot 2 + c$

$(3,2) \rightarrow 2 = a \cdot 3^2 + b \cdot 3 + c$

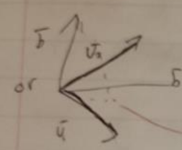
Best Approximation By Method of least Squares

$A\bar{x} = \bar{b}$ System in matrix form

$$\underset{A}{\begin{pmatrix} & & \end{pmatrix}} \underset{\bar{x}}{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \underset{\bar{b}}{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}$$

2

$$A\bar{x} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\vec{\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}}}_{\bar{v}_1} x_1 + \dots + \underbrace{\vec{\begin{pmatrix} a_{1m} \\ \vdots \\ a_{mn} \end{pmatrix}}}_{\bar{v}_n} x_n = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$



$\bar{b} = x_1\bar{v}_1 + x_2\bar{v}_2$

projection onto plane

Goal: Find approximate solution $(x_1, \dots, x_n)$ minimising the error.
$\|A\bar{x} - \bar{b}\|$ as close to zero as possible.

Method: Solve associated normal system

transpose of A.

Original System = $A\bar{x} = \bar{b}$ $\rightsquigarrow$ Normal System $A^T A\bar{x} = \bar{b}A^T$

$$\begin{cases} x - y = 5 \\ x + y = 3 \\ x - 2y = 7 \\ 2x - y = -1 \end{cases} \qquad \overset{A}{\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -2 \\ 2 & -1 \end{pmatrix}} \overset{\bar{x}}{\begin{pmatrix} x \\ y \end{pmatrix}} = \overset{\bar{b}}{\begin{pmatrix} 5 \\ 3 \\ 7 \\ -1 \end{pmatrix}}$$

ex.

$$\overset{A^T}{\begin{pmatrix} 1 & 1 & 1 & 2 \\ -1 & 1 & -2 & -1 \end{pmatrix}} \overset{A}{\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -2 \\ 2 & -1 \end{pmatrix}} \overset{\bar{x}}{\begin{pmatrix} x \\ y \end{pmatrix}} = \overset{A^T}{\begin{pmatrix} 1 & 1 & 1 & 2 \\ -1 & 1 & -2 & -1 \end{pmatrix}} \overset{\bar{b}}{\begin{pmatrix} 5 \\ 3 \\ 7 \\ -1 \end{pmatrix}}$$

$$\begin{pmatrix} 7 & -4 \\ -4 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ -15 \end{pmatrix} \qquad \begin{matrix} 7x - 4y = 13 \\ -4x + 7y = -15 \end{matrix} \qquad x = \frac{13}{7} + \frac{4}{7}y$$

Solve and
= find x and y

10/3/14

Maths 1

Eigenvalues and Eigenvectors

Square Matrix A. $n \times n$

Example: $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

Eigenvectors

$$\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$\bar{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ eigenvector, λ eigenvalue

$$A \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$\bar{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ e-vector eigenvectors

λ e-value

Characteristic polynomial

Same thing

$\det(A - \lambda I) = \det(\lambda I - A)$

$P(\lambda) = \det(A - \lambda I)$

$$= \det\left[\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right]$$

$$= \det \begin{pmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{pmatrix}$$

$$(3-\lambda)(-1-\lambda) - 8 \cdot 0 = (\lambda-3)(\lambda+1)$$

$P(\lambda) = (\lambda-3)(\lambda+1)$ characteristic polynomial of A

$P(\lambda) = |\lambda I - A|$

E-values are roots of characteristic polynomial

10/3/14

Maths 2

$P(\lambda) = (\lambda-3)(\lambda+1) = 0$

$\lambda_1 = 3$ or $\lambda_2 = -1$

→ Eigen values are 3 and -1.

→ Eigenvectors for $\lambda_1 = 3$ and $\lambda_2 = -1$

$A \quad \lambda = 3$

$$\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 3\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow \begin{cases} 3v_1 = 3v_1 \\ 8v_1 - v_2 = 3v_2 \end{cases} \quad \begin{cases} 0 = 0 \\ 8v_1 = 4v_2 \end{cases} \Rightarrow \begin{cases} 0 = 0 \\ 2v_1 = v_2 \end{cases}$$

free depend

Solution for $\lambda_1 = 3$.

v_1 = free $\quad v_1 = 1 \Rightarrow v_2 = 2$

Solution $\quad \bar{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ Solution of system → is eigenvector of $\lambda_1 = 3$.

→ Eigen vector for $\lambda_2 = -1$.

A.

$$\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (-1)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow \begin{cases} 3v_1 = -1v_1 \\ 8v_1 \cancel{-v_2} = \cancel{-v_2} \end{cases} \quad \begin{cases} v_1 = 0 \\ 0 = 0 \end{cases}$$

v_2 is free

$v_1 = 0$

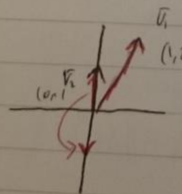
$v_1 = 0 \Rightarrow$ solution of is $\bar{v} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ← eigen vector

Matrix A has 2 e-values $\lambda_1 = 3 \quad \lambda_2 = -1$

2 e-vectors $\bar{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \bar{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$A\bar{v}_1 = \lambda\bar{v}_1$

$A\bar{v}_2 = \lambda\bar{v}_2$



Flip and stretch going on

3 × 3 Matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$$

10/3/14

Maths

Characteristic polynomial

$$P(\lambda) = \det(\lambda I - A)$$

$$= \det\left[\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}\right]$$

$$= \det\left[\begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda-8 \end{pmatrix}\right]$$

$$= \overset{+}{\lambda}\begin{bmatrix} \lambda & -1 \\ 17 & \lambda-8 \end{bmatrix} \overset{-}{+1}\begin{bmatrix} 0 & -1 \\ -4 & \lambda-8 \end{bmatrix} \overset{+}{+0}\begin{bmatrix} 0 & \lambda \\ -4 & 17 \end{bmatrix}$$

$$= \lambda\left(\lambda(\lambda-8) + 1(17)\right) + 1\left[(0(\lambda-8) - -1(-4)\right]$$

$$= \lambda^3 - 8\lambda^2 + 17\lambda - 4 = P(\lambda)$$

Need roots of polynomial

Look for roots among divisors of constant term (here -4)

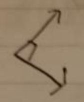
Divisors of -4 ± 1 ± 2 ± 4 Try these possible divisors

$\Rightarrow \lambda = 4$ is a root. is first even value

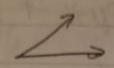
Remains: find other roots $\rightarrow$ other e-values then get e-vectors

4. Maths

Orthogonal

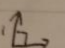


Not orthogonal

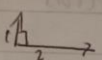


$\measuredangle \neq 90°$

Orthonormal = orthogonal + norms (all) are 1.



orthonormal



not orthonormal

Q. Is $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ orthonormal?

It is orthogonal, Basis.

$||v_1|| = \sqrt{0^2+1^2+0^2} = 1$

$||v_2|| = \sqrt{-4^2+0^2+3^2} = 5 \neq 1$ NOT orthonormal Basis.

Normalisation:

Replace $\bar{v}_1$ by $\frac{\bar{v}_1}{||\bar{v}_1||}$ gets norm of 1.

$\bar{v}_2 = (-4,0,3) \Rightarrow \bar{w}_2 = \frac{\bar{v}_2}{||v_2||} \quad \frac{(-4,0,3)}{\sqrt{-4^2+0^2+3^2}} = \frac{(-4,0,5)}{5}$

$\bar{w}_2 \left(-\frac{4}{5}, 0, \frac{3}{5}\right) \quad ||w_2|| = \sqrt{-\frac{4}{5}^2+0^2+\frac{3}{5}^2} = 1$

$\bar{v}_3 = (3,0,4) \Rightarrow \bar{w}_3 = \frac{\bar{v}_3}{||v_3||} \quad \frac{(3,0,4)}{5}$

$= \left(\frac{3}{5}, 0, \frac{4}{5}\right) \quad ||\bar{w}_3|| = \sqrt{\frac{3}{5}^2+0^2+\frac{4}{5}^2} = 1$

Now an orthonormal basis

$\{\bar{v}_1, \bar{w}_2, \bar{w}_3\}$

12/3/14 Maths 1

Square Matrix (#rows = #columns)

A matrix look for solutions of
$A\bar{v} = \lambda \bar{v}$ $v \neq \bar{0}$

$\bar{v}$ eigenvector λ eigenvalue

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$$

Characteristic polynomial
$P(\lambda) = \det(\lambda I - A) = \lambda^3 - 8\lambda^2 + 17\lambda - 4$

Find roots for divisors of constant term 4.
Divisors are ± 1 ± 2 ± 4

1st root $\lambda = 4$

Now divide $P(\lambda)$ by $(\lambda - 4)$

$$\begin{array}{l} \cancel{\lambda^3} - 8\lambda^2 + 17\lambda - 4 \quad \big| \; \lambda - 4 \ (\text{divisor}) \\ -\cancel{\lambda^3} - 4\lambda^2 \quad\quad\quad\quad \big| \; \lambda^2 - 4\lambda + 1 \\ \hline \quad -4\cancel{\lambda^2} + 17\lambda - 4 \\ (-) \quad -4\cancel{\lambda^2} + 16\lambda \\ \hline \quad\quad \lambda - 4 \\ (-) \quad \lambda - 4 \\ \hline \quad\quad 0 \end{array}$$

$\lambda^2 - 4\lambda + 1$
$P(\lambda) = (\lambda - 4)(\lambda^2 - 4\lambda + 1)$

2.

Look for roots of $\lambda^2 - 4\lambda + 1$

$\Rightarrow \lambda_{2,3} = 2 \pm \sqrt{3}$ $\qquad \lambda_1 = 4$

Eigenvalues $\lambda_1 = 4$ $\quad \lambda_{2,3} = 2 \pm \sqrt{3}$

→ Find eigenvectors for each λ_i one for each

$\lambda_1 = 4$
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 4\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad v_2 =$$

identity vector for 4

$$\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}\right]\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \bar{0}$$

$$\begin{pmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 4 & -17 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 0 & -16 & 4 \end{pmatrix} \times 4 \rightarrow \begin{pmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\begin{aligned} -4v_1 + v_2 &= 0 \\ -4v_2 + v_3 &= 0 \end{aligned} \rightarrow \begin{cases} v_2 = 4v_1 \\ v_3 = 16v_1 \end{cases} \qquad v_1 = 1,\ v_2 = 4\ \ v_3 = 16$$

Non zero solution $\vec{v} = (1, 4, 16)$

$\lambda_2 = 2 + \sqrt{3}$

$A\bar{v} = (2+\sqrt{3})(\bar{v})$

3

12/3/14 Math

$$\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix} - \begin{pmatrix} 2+\sqrt{3} & 0 & 0 \\ 0 & 2+\sqrt{3} & 0 \\ 0 & 0 & 2+\sqrt{3} \end{pmatrix}\right]$$

$$\begin{pmatrix} -2+\sqrt{3} & 1 & 0 \\ 0 & -2-\sqrt{3} & 1 \\ 4 & -17 & 6-\sqrt{3} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution: $\bar{v} = (v_1, v_2, v_3) = (1,\ 2-\sqrt{3},\ 7-4\sqrt{3})$

$\lambda_3 = 2-\sqrt{3} \quad \Rightarrow \bar{v} = (v_1, v_2, v_3) = (1,\ 2+\sqrt{3},\ 7+4\sqrt{3})$

Eigen values	Eigen Vectors
$\lambda_1 = 4$	$w_1 = (1, 4, 16)$
$\lambda_2 = 2+\sqrt{3}$	$w_2 = (1, 2-\sqrt{3}, 7-4\sqrt{3})$
$\lambda_3 = 2-\sqrt{3}$	$w_3 = (1, 2+\sqrt{3}, 7+4\sqrt{3})$

~~Dia~~ Diagonalisation of Matrix

Given matrix A.

Want: Find P invertible matrix (nonsingular)

such that $D = P^{-1}AP$ is diagonal

$$\begin{array}{ccc} x & \xrightarrow{A} & x' \\ P\uparrow & & \downarrow P^{-1} \\ y & \xrightarrow{P^{-1}AP} & y \end{array}$$

Rule: Read P and D from e-vectors and e-values

For P, write all eigen vectors as columns

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 2-\sqrt{3} & 2+\sqrt{3} \\ 16 & 7-4\sqrt{3} & 7+4\sqrt{3} \end{pmatrix}$$

7

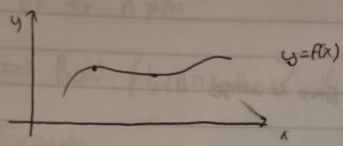
4

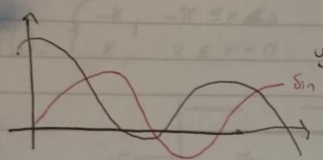
$$D = P^{-1}AP = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2+\sqrt{3} & 0 \\ 0 & 0 & 2-\sqrt{3} \end{pmatrix}$$ Diagonal using e-values

14/3/14. Maths

Fourier series (Kreyszig Ch 10/11)

Vectors $\bar{v} = (v_1, v_2, \ldots, v_n)$





smooth function

$\sin x = \cos(x - \frac{\pi}{2})$

Fourier series

$f(x) = \cancel{a_0 + \sum (a_n \cos(nx))}$

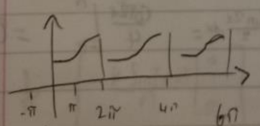
$a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$

Basis $\bar{e}_1, \ldots \bar{e}_n$ $\quad \bar{v} = k_1\bar{e}_1 + \ldots + k_n\bar{e}_n$

$f(x) = a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots + (a_3 \cos 3x + b_3 \sin 3x) \ldots +$

look at f 2π-periodic

$f(x + 2\pi) = f(x)$ repeats itself every 2π



$f(x)$ given for $x \in [-\pi, \pi]$

2.

Problem: Given $f(x)$ want to expand it into furrer seri F.S.

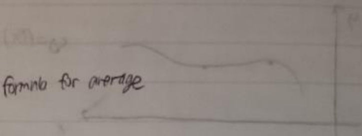
→ find a_n, b_n

Basis: for a_0 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, ... $\cos nx$, $\sin nx$

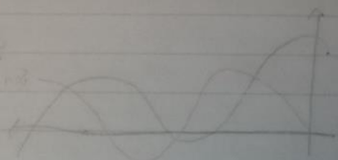
Use Euler Formula:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

formula for average



$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx$$



$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx$$

Vectors:

$\bar{e}_1, \ldots \bar{e}_n$ assume orthogonal basis $\left(\langle \bar{e}_i, \bar{e}_j \rangle = 0 \text{ for } i \neq j\right)$

$\bar{v} = c_1 \bar{e}_1 + \ldots + c_n \bar{e}_n$

$$c_i = \frac{\langle \bar{v}, \bar{e}_i \rangle}{\|\bar{e}_i\|^2}$$

Inner Product for functions: $\langle f, g \rangle$

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\,dx$$

$\langle \bar{v}, \bar{w} \rangle = v_1 w_1 + \ldots + v_n w_n$

$$\langle \cos x, \sin x \rangle = \int_{-\pi}^{\pi} \cos x \sin x\,dx = \int_{-\pi}^{\pi} \frac{\sin 2x}{2}\,dx = \frac{\cos 2x}{4}\Big|_{-\pi}^{\pi} = 0$$

$\langle \cos x, \sin x \rangle = 0$ orthogonal

All inner product of $\cos x$ $\sin x$ will $= 0$.

$$\langle 1, \cos nx \rangle = \int_{-\pi}^{\pi} \cos nx\,dx = 0$$

$\langle 1, \sin(nx) \rangle = 0$

14/3/14 Maths 3

$\langle \cos nx, \sin mx \rangle = 0$

$\langle \cos nx, \cos mx \rangle = 0$

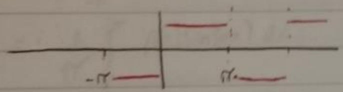
$\langle \sin nx, \sin mx \rangle = 0$

0 for $n \neq m$

$\Rightarrow$ Basis $\{1, \cos nx, \sin nx\}$ is orthogonal.

Example

$$f(x) = \begin{cases} -k, & -\pi \le x < 0 \\ k, & 0 \le x < \pi \end{cases}$$



$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} \int_{0}^{\pi}\right) f(x)dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} -k\,dx + \int_{0}^{\pi} k\,dx\right]$$

$$\frac{1}{2\pi} \quad (-kx)\Big|_{-\pi}^{0} \quad (kx)\Big|_{0}^{\pi}$$

$$= 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\cos(nx)\,dx \quad \int_{0}^{\pi} k\cos(nx)\,dx\right]$$

$$\frac{1}{\pi}\left[-\frac{k\sin(nx)}{n}\Big|_{-\pi}^{0}\right] + \frac{1}{\pi}\left[\frac{k\sin(nx)}{n}\Big|_{0}^{\pi}\right]$$

$$= 0$$

C4 Math

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$

$\frac{1}{\pi} \int_{-\pi}^{0} -k \sin(nx)\, dx + \frac{1}{\pi} \int_{0}^{\pi} k \sin(nx)\, dx$

$= \frac{2k}{n\pi} \left(1 - \cos(n\pi)\right) = b_n$

$\cos(n\pi) = \begin{cases} -1, & n \text{ odd} \\ 1, & n\text{-even} \end{cases}$

$= (-1)^n$

$= \frac{2k}{n\pi} \left(1 - (-1)^n\right)$

FOURIER $\quad f(x) = \sum b_n \sin(nx)$

$= \sum \frac{2k}{n\pi} \left(1 - (-1)^n\right) \sin(nx)$

$= \frac{4k}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \ldots + \right)$

19/3/14 Math 1

Fourier Series

$f(x) = a_0 + \sum \left(a_n \cos(nx) + b_n \sin(nx) \right)$

$-\pi \le x \le \pi$

Problem: Given $f(x)$, find a_n b_n

Euler's formula's: $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$ (average)

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$

LAST TIME LECTURE

$f(x) = \begin{cases} K & x \ge 0 \\ -K & x < 0 \end{cases}$

$\Rightarrow a_n = 0$

F Series $f(x) = \sum b_n \sin(nx)$

EVEN AND ODD FUNCTIONS

$f(x)$ even if $f(-x) = f(x)$

$f(x)$ odd if $f(-x) = -f(x)$

$\cos x$ is even $\cos(-x) = \cos(x)$

$\cos nx$ even $\cos(-nx) = \cos(nx)$

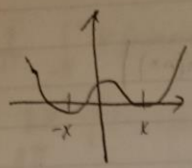
$\sin x$ odd $\sin(-x) = -\sin(x)$

$\sin(nx)$ odd $\sin(nx) = -\sin(nx)$ for $\forall n$

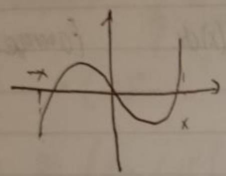
2

In series cos - even   Sin → odd

even



$f(-x) = f(x)$



$f(-x) = -f(x)$

KEY FACTS:

1. f, g both even or both odd $\Rightarrow$ fg even (f multiplied by g.)

2. if f even, g odd, product fg is odd

3. f odd $\Rightarrow$ $\int_{-L}^{L} f(x)\,dx = 0$ Cancels itself out

Main Conclusion:

- f even $\Rightarrow$ Even Fourier Series $f(x) = a_0 + \sum a_n \cos nx$
  cosine F.S
- f odd $\Rightarrow$ Odd F.S $f(x) = \sum b_n \sin(nx)$

f even $\Rightarrow$ $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overset{\text{even}}{f(x)}\, \overset{\text{odd}}{\sin(nx)}\,dx$ Odd $= 0$

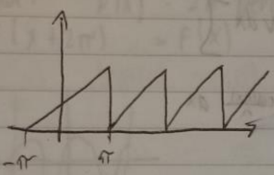
f odd $\Rightarrow$ $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overset{\text{odd}}{f(x)}\, \overset{\text{even}}{\cos(nx)}\,dx$ Odd $= 0$

19/3/14 Maths 3

General $f(x)$ can be written as $f(x) = g(x) + h(x)$
where : g even, h odd.

$g(x) = \frac{f(x) + f(-x)}{2}$ even part

$h(x) = \frac{f(x) - f(-x)}{2}$ odd part



Sawtooth wave
$f(x) = x + \pi \quad -\pi \le x \le \pi$

$f = g + h$ given h odd

$g(x) = \frac{x + \pi + -x + \pi}{2} = \pi$ even part.

$h(x) = \frac{x + \pi - (-x + \pi)}{2} = x$ odd part

For g $\quad g(x) = \underset{F.}{\pi} + \sum \cancel{(0 \cos(nx)}$ 0

for h, $\quad h(x) = \sum b_n \sin(nx)$ odd part

Sin fourier series!

4 Maths

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} h(x) \sin(nx)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, dx \quad \text{integration by parts}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} uv'\, dx \qquad u = x \qquad u' = \sin(nx) \quad v = -\frac{\cos(nx)}{n}$$

$$= \frac{1}{\pi} \left[(uv)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u'v\, dx \right]$$

$$= \frac{1}{\pi} \left[-\frac{x\cos(nx)}{n} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\cos(nx)}{n}\, dx \right]$$

$$= -\frac{2\pi\cos(n\pi)}{n}$$

$$-\frac{2\pi\cos(n\pi)}{n} - \underbrace{\frac{\sin(nx)}{n^2}\Big|_{-\pi}^{\pi}}_{=0}$$

$$= \frac{-2\pi\cos(n\pi)}{n}$$

$$\cos \pi n = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$$

$$b_n = \begin{cases} \frac{2}{n}\pi & n \text{ even} \\ -\frac{2}{n}\pi & n \text{ odd.} \end{cases}$$

$$F.S = f(x) = g(x) + h(x)$$

$$= \pi + \sum b_n \sin(nx)$$

$$= \pi + 2\pi\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots \right)$$

$$\pi + 2\pi \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{n} \sin(nx) \right)$$

21/3/14 MATH 1

No lect next monday and wedn...

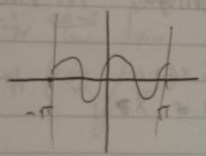
Fourier Integral

Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

Assumption: $f(x)$: 2π periodic

$f(x+2\pi) = f(x)$



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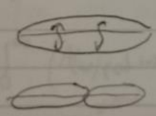
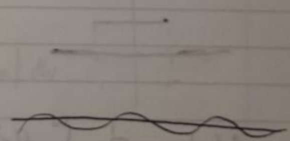


up down up dn



f not periodic $\Rightarrow$ need any $\cos wx$, $\sin wx$ for any $\underline{w}$.

Euler's formulas:

$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$

$a_n \; \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx$

$b_n \; \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx$

2.

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)\cos wx \, dx$$

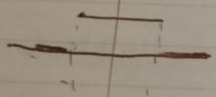
$$B(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)\sin(wx)\, dx$$

$$f(x) = \int_0^{\infty}\left(A(w)\cos(wx) + B(w)\sin(wx)\right) dw$$

Fourier Integral

Problem: Given $f(x)$ Find Fourier Integral, i.e. Find $A(w)$, $B(w)$

$$f(x) = \begin{cases} 1 & |x| \le 1 \iff -1 \le x \le 1 \\ 0 & |x| > 1 \iff x \le -1 \text{ or } x \ge 1 \end{cases}$$



$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(wx)\, dx$$

$$\frac{1}{\pi}\left[\int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{+\infty}\right]$$

$$\frac{1}{\pi}\int_{-\infty}^{-1} \overset{f(x)}{0}\cos(wx)\,dx + \int_{-1}^{1} \overset{\text{function } f(x)}{1}\cos(wx)\,dx + \int_{1}^{+\infty} \overset{f(x)}{0}\cos(wx)\,dx$$

$$0 + \frac{1}{\pi}\int_{-1}^{1}\cos(wx)\,dx + 0$$

$$= \frac{1}{\pi}\left.\frac{\sin(wx)}{w}\right|_{-1}^{1} = \frac{1}{\pi}\left(\frac{\sin(w)}{w} - \frac{\sin(-w)}{w}\right)$$

$$= \frac{2\sin(w)}{\pi w} = A(w)$$

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?

3

MATHS

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(wx)\, dx$$

$$= \frac{1}{\pi} \left(\int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{+\infty} \right)$$

$$= \frac{1}{\pi} \left[\underbrace{\int_{-\infty}^{-1} 0 \sin(wx)\, dx}_{0} + \int_{-1}^{1} 1 \sin(wx)\, dx + \underbrace{\int_{1}^{+\infty} 0 \sin(wx)\, dx}_{0} \right]$$

$$= \frac{1}{\pi} \int_{-1}^{1} \sin(wx)\, dx$$

$$= \frac{-1}{\pi} \frac{\cos wx}{w} \Big|_{x=-1}^{1}$$

$$= \frac{-1}{\pi} \underbrace{\left(\frac{\cos w}{w} - \frac{\cos -w}{w} \right)}_{0}$$

$$= 0 = B(w)$$

2

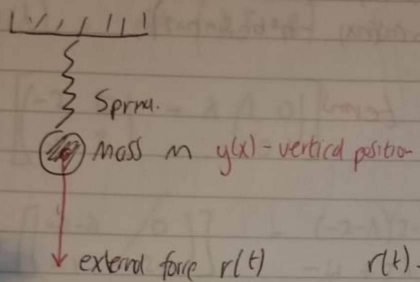
$$f(x) = \int_0^{\infty} \left(A(w) \cos wx + B(w) \sin wx \right) dw$$

$$\Rightarrow f(x) = \int_0^{\infty} \underbrace{\frac{2 \sin w}{\cancel{\pi} w}}_{A(w)} \cos(wx)\, dw$$

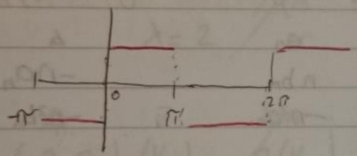
$$f(x) \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

28/3/14 Maths 1

Application of Fourier Series to forced oscillation



$r(t) = \begin{cases} 1 & 0 \le x \le \pi \\ -1 & -\pi \le x \le 0 \end{cases}$



Basic Equation $r(x) = m y''_{(x)} + c y'_{(x)} + k y_{(x)}$ m - mass

c - damping constant

K - Spring modulus

Simplify: $y''(x) + y'(x) + y(x) = r(x)$

Method: Expand y, r in Fourier Series

F.S for $r(x)$

$r(x) = \frac{4}{\pi}\left(\sin + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right)$

$\frac{4}{\pi} \sum_{s=0}^{\infty} \frac{1}{2s+1} \sin(2s+1)x$

General expansion: $y(x) = a_0 + \sum (a_n \cos nx + b_n \sin nx)$

2

Substitute y, r in equation

$$y'(x) = \sum\left(-n a_n \sin nx\right) + \left(n b_n \cos nx\right)$$

$$y''(x) = \sum\left(-n^2 a_n \cos(nx) - n^2 b_n \sin(nx)\right)$$

Substitute & identify terms

constant

cos nx

sin nx

	constant	cos nx	sin nx (n-even)	sin nx (n-odd)
r	0	0	0	$4/\pi n$
y	a_0	a_n		b_n
y'	0	$n b_n$		$-n a_n$
y''	0	$-n^2 a_n$		$-n^2 b_n$

$1/n\,(4/\pi$

Constant terms $a_0 + 0 + 0 = 0 \quad a_0 = 0$

cosine term n) $a_n + n b_n - n^2 a_n = 0$

$$a_n(1-n^2) + n b_n = 0$$

$$b_n = \left(\frac{n^2-1}{n}\right) a_n$$

Sin terms: $b_n - n a_n - n^2 b_n = \begin{cases} 0 & n\text{-even} \\ 4/\pi n & n\text{-odd.} \end{cases}$

$$(1-n^2)\frac{n^2-1}{n} a_n - n a_n = \begin{cases} 0 & \text{even} \\ 4/\pi n & \text{odd} \end{cases}$$

$$\frac{-n^4-n^2+1}{n} a_n = \begin{cases} 0 \\ \frac{4}{\pi/n} \end{cases} \quad \Big| \quad a_n = \begin{cases} 0 & n\text{-even} \\ \therefore -\frac{4}{\pi(-n^4-n^2+1)} \end{cases}$$

$$b_n = \frac{n^2-1}{n} a_n = \begin{cases} 0 & n\text{-even} \\ \frac{-4(n^2-1)}{\pi n(n^4-n^2+1)} & n\text{-odd} \end{cases}$$

$$y(x) = a_0 + \sum(a_n \cos nx + b_n \sin nx) = \frac{4}{\pi}\sum_{n\text{ odd}} \frac{1}{(n^4-n^2+1)}\left(\cos nx + \frac{n^2-1}{n}\sin nx\right)$$

31/3/14 Maths

Fourier series with arbitrary periods

F.S $f(x) = a_0 + \Sigma(a_n \cos nx + b_n \sin nx)$

f 2π - periodic $f(x+2\pi) = f(x)$

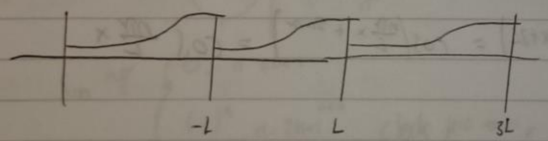
Euler's formulas

$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$

$f(y)$ with period $p = 2L$



$g(y)$ $-L \leq y < L$
$f(x)$ $-\pi < x < \pi$

$x = \frac{\pi}{L} y$
$y = L \Rightarrow x = \pi$
$y = -L \Rightarrow x = -\pi$

$a_0 = \frac{1}{2\pi} \int_{-L}^{L} f(\frac{\pi}{L} y) \, d(\frac{\pi}{L} y)$

$= \frac{1}{2L} \int_{-L}^{L} \underbrace{f(\frac{\pi}{L} y)}_{g(y)} dy$

MATH

2

$$= \frac{1}{2L}\int_{-L}^{L} g(y)\,dy$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} g(y)\,dy$$

$$a_n = \frac{1}{\pi}\int_{-L}^{L} f\left(\frac{\pi}{L}y\right)\cos\left(\frac{n\pi}{L}y\right)\,d\left(\frac{\pi}{L}y\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} g(y)\cos\left(\frac{n\pi}{L}\right)y\,dy$$

$$b_n = \frac{1}{L}\int_{-L}^{L} g(y)\sin\left(\frac{n\pi}{L}\right)y\,dy$$

F.S. $g(y) = a_0 + \Sigma\left(a_n\cos\left(\frac{n\pi}{L}\right)x + b_n\sin\left(\frac{n\pi}{L}\right)x\right)$

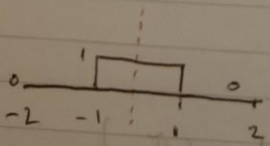
$\cos\left(\frac{n\pi}{L}\right)x$ is $2L$-periodic

$$\cos\frac{n\pi}{L}(x+2L) = \cos\left(\frac{n\pi}{L}x + 2n\pi\right) = \cos\frac{n\pi}{L}x$$

Example

$$g(y) = \begin{cases} 0 & -2<y<-1 \\ 1 & -1<y<1 \\ 0 & 1<y<2 \end{cases}$$

$g(-y) = g(y)$ even



$$a_0 = \frac{1}{L}\int g(y)\,dy = \frac{1}{L}\left(\int_{-2}^{-1}\int_{-1}^{1} + \int_{1}^{2}\right)$$
$\qquad 0 \qquad\qquad 0$

$$\frac{1}{2L}\int^{1} dy$$

31/3/14 Maths 3

$= 1/2 \quad = a_0.$

1 -1

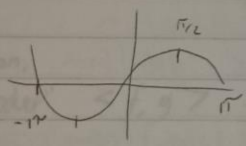
$a_n = \frac{1}{2}\int_{-2}^{L=2} g(y)\cos\frac{n\pi}{2}y\,dy$

$= \frac{1}{2}\left(\underbrace{\int_{-2}^{-1}}_{0}\int_{-1}^{1}\underbrace{\int_{1}^{2}}_{0}\right)$

$= \frac{1}{2}\int_{-1}^{1}\cos\frac{n\pi}{2}y\,dy \quad = \frac{1}{2}\left.\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}}\right|_{-1}^{1}$

$= \frac{1}{2}\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{-n\pi}{2}\right)\right)$

$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$



$\sin\frac{\pi}{2} \quad \sin\pi \quad \sin\frac{3\pi}{2} \quad \sin 2\pi$

$+1 \quad 0 \quad -1 \quad 0$

$\sin\frac{n\pi}{2}\begin{cases}0, & n \text{ even}\\ (-1)^k, & n = 2k+1 \text{ odd}\end{cases}$ check $k=0 \Rightarrow n=1$ ✓ ✓

$g(y) = a_0 + \sum a_n\cos\frac{n\pi}{2}y$ 2L periodic

$= \frac{1}{2} + \frac{2}{\pi}\left(\sum_{n=1}^{\infty}\cos\frac{\pi}{2}y\right.$

$= \frac{1}{2} + \frac{2}{\pi}\left(\cos\frac{\pi}{2}y - \frac{1}{3}\cos\frac{3\pi}{2}y + \frac{1}{5}\cos\frac{5\pi}{2}y \ldots\right)$

$= a_0 + \sum a_n\cos\frac{n\pi}{2}y$

$\frac{1}{2} + \frac{2}{\pi}\sum_{k}\frac{(-1)^k}{2k+1}\cos\frac{(2k+1)\pi}{2}y$

2/4/14 Maths $\boxed{1}$

Vectors

Expand vectors $\bar{v} = k_1\bar{v}_1 + \dots + k_n\bar{v}_n$

Given: $\underbrace{\bar{v}_1 \quad \bar{v}_i \quad \bar{v}_n}_{\text{basis}}$

Find $k_1, \dots k_n$.

Basis of orthogonal wrt $\langle \bar{u}, \bar{v} \rangle$

$\Rightarrow k_i = \dfrac{\langle \bar{v}, \bar{v}_i \rangle}{\|\bar{v}_i\|^2}$

Fourier series

Expand $f(x)$ into: $f(x) = a_0 + \Sigma(a_n\cos nx + b_n\sin nx)$

Given: $f(x)$ "basis" $1, \cos nx, \sin nx$

Find a_n, b_n,

"Inner product" $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$.

Euler's formula for a_n b_n

Fourier Integral

Expand $f(x) = \int_0^\infty (A(w)\cos wx + B(w)\sin wx)\,dw$.

"Basis" $\cos wx$, $\sin wx$ $0 \leq w \leq \infty$

$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos wx\,dx$

$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin wx\,dx$

Fourier Transform

$A(w)$ $B(w)$

Put cos and sin together.

$\hat{f}(w) = \sqrt{\frac{\pi}{2}}\,(\underset{\uparrow\text{ real part}}{A(w)} - i\,\underset{\uparrow\text{ imaginary part}}{(B(w))})$

2.

Fourier transform

$$\hat{f}(w) = \sqrt{\tfrac{\pi}{2}} \quad \tfrac{1}{\pi} \int_{-\infty}^{\infty} f(x)\,(\cos wx - i\sin wx)\,dx$$

Euler's formula

$\cos u + i\sin u = e^{iu}$

$\cos(u+v) + i\sin(u+v) = e^{i(u+v)}$

$(\cos u + i\sin u)(\cos v + i\sin v) = e^{iv}e^{iu}$

$\cos(u+v) = \cos v \cdot \cos u - \sin u \cdot \sin v$

$\cos u - i\sin u = e^{-iu}$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ixw}\, dx$$

Fourier Transform $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw$

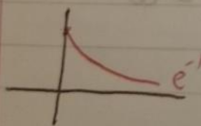
Inverse Fourier Transform

$$F(x) = \int_0^{\infty} \underbrace{A(w)\cos wx}_{\text{even}} + \underbrace{B(w)\sin wx}_{\text{odd}}\, dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \sqrt{\tfrac{\pi}{2}}\, (A(w) - iB(w))(\cos wx + i\sin wx)\, dx$$

extra terms $\int_{-\infty}^{\infty} \underbrace{A(w)\, i\sin wx}_{\text{odd}} - \underbrace{i B(w)\cos wx}_{\text{odd}} = 0$

$$F(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \geq 0 \end{cases}$$



Find F transform $\hat{f}(w)$ $\quad \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ixw}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x(1+iw)}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-x(1+iw)}}{-1+iw} \right]_0^{\infty} = \frac{1}{\sqrt{2\pi}} \frac{1}{1+iw}$$

Inverse F. T.

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{F}(w)\, e^{iwx}\, dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+iw}\, e^{iwx}\, dx$$