MA1E01: Chapter 4 Summary

The Derivative in Graphing and Applications

Definitions

- **Increasing/Decreasing**: For f a continuous function on $[a,b]$ and differentiable on (a,b),

  (a) $f'(x) > 0 \quad \forall\, x \in (a,b) \quad \Longrightarrow \quad f$ is increasing on $[a,b]$,

  (b) $f'(x) < 0 \quad \forall\, x \in (a,b) \quad \Longrightarrow \quad f$ is decreasing on $[a,b]$,

  (c) $f'(x) = 0 \quad \forall\, x \in (a,b) \quad \Longrightarrow \quad f$ is constant on $[a,b]$.

  This definition naturally extends to infinite and semi-infinite intervals.

- **Concavity**: Let f be twice differentiable on an open interval I, then

  (a) $f''(x) > 0 \quad \forall\, x \in I \quad \Longrightarrow \quad f$ is concave up on I,

  (b) $f''(x) < 0 \quad \forall\, x \in I \quad \Longrightarrow \quad f$ is concave down on I.

- **Inflection Points**: The points where f changes direction of concavity are inflection points.

- **Relative Extrema**: If there exists an open interval containing x_0 on which $f(x_0)$ is the largest value, then x_0 is a relative max. of f. Similarly, if there exists an open interval containing x_0 on which $f(x_0)$ is the smallest value, then x_0 is a relative min. of f.

- **Critical/Stationary Points**: A stationary point of f is a point x_0 where the derivative vanishes, $f'(x_0) = 0$. A critical point is either a stationary point or a point of non-differentiability.

- **First Derivative Test**: The following test determines whether or not a critical point x_0 is a relative extremum:

  1. If $f'(x) > 0$ to the left of $x = x_0$ and $f'(x) < 0$ to the right of $x = x_0$, then x_0 is a relative max.
  2. If $f'(x) < 0$ to the left of $x = x_0$ and $f'(x) > 0$ to the right of $x = x_0$, then x_0 is a relative min.
  3. If $f'(x)$ has the same sign on either side of x_0, then x_0 is not a relative extremum.

- **Second Derivative Test**: The following test is used to classify a stationary point x_0 as a relative max/min. Supposing f is twice differentiable at x_0, then

  (a) $f'(x_0) = 0$ and $f''(x_0) > 0 \quad \Longrightarrow \quad$ relative min at x_0,

(b) $f'(x_0) = 0$ and $f''(x_0) < 0 \implies$ relative max at x_0,

(c) $f'(x_0) = 0$ and $f''(x_0) = 0 \implies$ inconclusive.

- **Multiplicity**: A root $x = r$ of a polynomial $p(x)$ has multiplicity m if $(x-r)^m$ divides $p(x)$ but $(x-r)^{m+1}$ does not, i.e., $(x-r)^m$ is the highest power of $(x-r)$ that divides $p(x)$.
- **Absolute Extrema**: Let I be an arbitrary interval in the domain of f, then
  - an absolute maximum at x_0 implies $f(x_0) \geq f(x)$ for all $x \in I$,
  - an absolute minimum at x_0 implies $f(x_0) \leq f(x)$ for all $x \in I$.
- **Linear Motion**: Is $s(t)$ is a position function representing the position of a particle along a coordinate axis at time t, then we define the following:
  - Displacement: The displacement over the interval $[t_1, t_2]$ is $s(t_2) - s(t_1)$.
  - Velocity: The velocity is defined as the rate of change of displacement with respect to time,
  $$v(t) = s'(t) = \frac{ds}{dt}.$$
  - Speed: The speed is the magnitude of the velocity,
  $$\text{speed} = |v(t)| = |s'(t)| = \left|\frac{ds}{dt}\right|.$$
  - Acceleration: The acceleration is the rate of change of the velocity with respect to time,
  $$a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2s}{dt^2}.$$
  - Speeding Up/ Slowing Down: If $a(t)$ and $v(t)$ have the same sign, the particle is speeding up. If $a(t)$ and $v(t)$ have opposite sign, then the particle is slowing down.

Theorems

- **Multiplicity of Roots**: Suppose $p(x)$ is a polynomial with a root of multiplicity m at $x = r$, then

  (a) If m is even, then the graph of $y = p(x)$
    - is tangent to the x-axis at $x = r$,
    - does not cross the x-axis at $x = r$ and
    - does not have an inflection point at $x = r$.

  (b) If m is odd and greater than 1, then the graph of $y = p(x)$
    - is tangent to the x-axis at $x = r$,

    - crosses the x-axis at $x = r$,
    - has an inflection point at $x = r$.
  (c) If $m = 1$ (simple root), then the graph $y = p(x)$
    - is not tangent to the x-axis at $x = r$,
    - crosses the x-axis at $x = r$,
    - may or may not have an inflection point at $x = r$.

- **Extreme Value Theorem**: If a function f is continuous on a finite closed interval $[a, b]$, then f has both an absolute max and an absolute min on $[a, b]$.

- **Absolute Extrema on Open Intervals**: If f has an absolute extremum on an open interval (a, b), then it must occur at a critical point of f. (This result combined with the Extreme Value Theorem tells us that the absolute extrema of f on $[a, b]$ are either located at the critical points of the interior (a, b) or the end points $x = a$ or $x = b$.)

- **Function with One Turning Point**: If f is a continuous function and has exactly one relative extremum (turning point) on an interval I at x_0, then

  (a) If f has a relative min at x_0, then $f(x_0)$ is the absolute min on I.
  (b) If f has a relative max at x_0, then $f(x_0)$ is the absolute max on I.

- **Rolle's Theorem**: Let f be continuous on $[a, b]$ and differentiable on (a, b). If

$$f(a) = 0 \quad \text{and} \quad f(b) = 0,$$

  then there is at least one point $c \in (a, b)$ such that

$$f'(c) = 0.$$

- **Mean Value Theorem**: Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

  i.e., there is at least one point whose derivative at that point is equal to the slope of the line segment from the point on the graph $(a, f(a))$ to the point $(b, f(b))$.

- **Constant Difference Theorem**: If f and g are differentiable on an interval I where

$$f'(x) = g'(x) \quad \forall\, x \in I,$$

  then $f - g$ is a constant on I, i.e.,

$$f(x) = g(x) + k \quad \forall\, x \in I.$$

Miscellaneous Results

- **Applied Max/Min Problems**: The following algorithm can be used to solve applied max/min problems:

  1. If applicable, draw and label all quantities.
  2. Find a formula for the quantity to be extremized.
  3. If the quantity to be extremized in Step 2 depends on two variables, there must be an associated constraint equation which we can use to eliminate one of these variables.
  4. Identify the interval of possible values for the independent variable, based on any physical restrictions.
  5. If the interval is open, then the absolute extrema (if it exists) will be located at a critical point. If the interval is closed, we must also check the value of the function at the endpoints against the value of the function at the critical points to identify the absolute extrema.

- **Newton/Raphson Method**: The Newton/Raphson Method is an iterative method for obtaining an approximation to the root of a function by taking successive x-intercepts of the tangent line at the previous value. The n^{th} approximation to the root is

  $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

  In order for the method to be effective, we require a reasonably good initial guess, which can be obtained to the nearest integer by the Intermediate Value Theorem.