2012 MANG SCI PAPER 1 Q6 DAVID WEMBRECHT

6 a Show that the integral below is independant of an integration path:

$$\int_{(-\frac{\pi}{2},\pi)}^{(\pi,\frac{\pi}{2})} (-2x^2+4xy+2\pi^3\cos(x)+3y^2)dx - (-2x^2-6xy+3\pi^3\sin(y))dy$$

If vector field is conservative, it will be path independant.

F conservative $\frac{df}{dy} = \frac{dg}{dx}$

$\frac{df}{dy} = 4x+6y$ $\quad \frac{dg}{dx} = -(-4x-6y) = 4x+6y$

$\frac{df}{dy} = \frac{dg}{dx} \Rightarrow$ conservative $\Rightarrow$ path independant 2

b Find Potential function $\phi(x,y)$

Has potential as it is conservative

$\frac{d\phi}{dx} = (-2x^2+4xy+2\pi^3\cos(x)+3y^2 \qquad \frac{d\phi}{dy} = -(-2x^2-6xy+3\pi^3\sin(y))$

integrate first with respect to x

$\int \frac{d\phi}{dx} = -\frac{2x^3}{3} + 2x^2y + 2\pi^3\sin(x) + 3y^2x + C_1(y) = \phi$

- Where $C_1(y)$ depends on y only
- Now differentiate expression for ϕ with respect to y and equate to 2nd eqn

$\frac{d\phi}{dy} = 2x^2 + 6yx + \frac{dC_1}{dy} = 2x^2+6xy - 3\pi^3\sin(y)$

$\frac{dC_1}{dy} = -3\pi^3\sin(y)$

Integrate both sides $C_1 = 3\pi^3\cos y$

2 $\phi = \frac{-2x^3}{3} + 2x^2y + 2\pi^3 \sin(x) + 3y^2x + 3\pi^2\cos(y) + C$ ✓

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where C is any constant.

C. Use the fundamental Theorm of line Integrals to find the value of the integral.

When $F(x,y) = \nabla\phi(x,y)$.

other way round.

$$\int_C F(x,y)\cdot dr = \int_C \nabla\phi \cdot dr = \phi(x_0,y_0) - \phi(x_1,y_1)$$

$(x_0,y_0) = (-\pi/2, \pi)$ $\quad (x_1,y_1) = (\pi, \frac{\pi}{2})$.

$$\left[-2\frac{(-\frac{\pi}{2})^3}{3} + 2(\tfrac{-\pi}{2})^2(\pi) + 2\pi^3\sin(-\tfrac{\pi}{2}) + 3(\pi)^2(\tfrac{\pi}{2}) + 0\right] - \left[-\frac{2(\pi^3)}{3} + 2(\pi^2)(\tfrac{\pi}{2}) + 0 + 3(\tfrac{\pi}{2})^2(\pi) + 0\right]$$

$\cos\pi = -1$

$$\left[\frac{2\frac{\pi^3}{8}}{3} + \frac{\pi^3}{2} - 2\pi^3 \; \overline{+} \; \frac{3\pi^3}{2} + 0\right] - \left[\frac{-2\pi^3}{3} + \pi^3 + \frac{3\pi^3}{\cancel{2}\,4}\right]$$

$\frac{2(\frac{\pi^3}{8})}{3} = \frac{\pi^3}{(4)(3)} = \frac{\pi^3}{12}$.

$$\left[\frac{3\pi}{4} + \frac{\pi^3}{2} - 2\pi^3 + \frac{3}{2}\pi^3 + 0\right] - \left[\frac{-4\pi^3 + 6\pi^3 + 6\pi^3}{6}\right]$$

$$\left[\frac{3\pi + 2\pi^3 - 4\pi^3 + 6\pi^3}{4}\right] - \left[\frac{8\pi^3}{6}\right]$$

$$\left[\frac{4\pi^3 + 3\pi}{4}\right] - \left[\frac{8\pi^3}{6}\right]$$

$$\frac{12\pi^3 + 9\pi - 16\pi^3}{12}$$

$$\frac{-4\pi^3 + 9\pi}{12}$$

$7\pi^3$.

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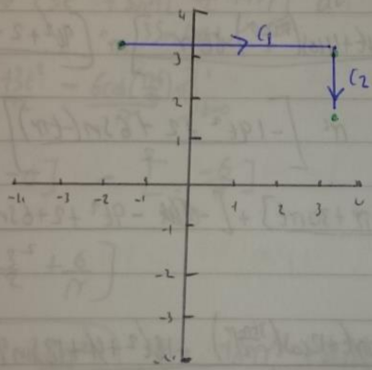
$\frac{1}{2}$

D. Choose the integration path C between the points $(-\frac{\pi}{2}, \pi)$ and $(\pi, \pi/2)$ to be a curve formed from ~~the~~ two line segments C_1 and C_2, where C_1 is joining $(-\frac{\pi}{2}, \pi)$ and (π, π) and C_2 is joining (π, π) and $(\pi, \pi/2)$.

i. Plot the integration path C, and show its orientation on the plot.

$$(-\tfrac{\pi}{2}, \pi) = (-1.57\ \ 3.14)$$
$$(\pi, \pi) = (3.14,\ 3.14)$$
$$(\pi, \pi/2) = (3.14,\ 1.57)$$



✓ 1

$-\frac{\pi}{2} + t\frac{\pi}{2} + t\pi$

ii. Parameterise C_1 and C_2 and evaluate
$$\int (-2x^2 + 4xy + 2\pi^3\cos(x) + 3y^2)\,dx \quad -(-2x^2 - 6xy + 3\pi^3\sin(y))\,dy$$

r_0 to r_1 $= (1-t)r_0 + t(r_1)$ for $0 \le t \le 1$

C_1 $(-\frac{\pi}{2}, \pi)$ to (π, π) $= (1-t)(-\frac{\pi}{2}, \pi) + t(\pi, \pi) = (\frac{3t\pi}{2} - \frac{\pi}{2},\ -t\pi)$ ✓

C_2 (π, π) to $(\pi, \pi/2)$ $= (1-t)(\pi, \pi) + t(\pi, \pi/2) = (\pi,\ -\frac{\pi}{2}t + \pi)$

Integrate in form $\int_C [-2(x(t))^2) + 4x(t)y(t) + 2\pi\cos(x(t)) + 3y(t)^2]\frac{dx}{dt}$
$$+ [-(-2x(t)^2 - 6(x(t)y(t) + 3\pi^3\sin(y(t))\tfrac{dy}{dt}]\ dt.$$

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$C_1: \left(\overset{x}{\frac{3t\pi}{2} - \pi}, \overset{y}{-t\pi}\right)$ π

dy = 0 so this term doesn't contribute

$$\left[-2\left(\frac{3t\pi-\pi}{2}\right)^2 + 4\left(\frac{3t\pi}{2} - \frac{\pi}{2}\right)(-t\pi) + 2\pi^3\cos\left(\frac{3t\pi}{2} - \pi\right) + 3(-t\pi)^3\right]\frac{3\pi}{2} + \left[+2\left(\frac{3t\pi-\pi}{2}\right)^2 + 6\left(\frac{3t\pi}{2} - \frac{\pi}{2}\right)(-t\pi) + 3\pi^3\sin(-t\pi)\right]$$

$$\left[-2\left(\frac{9t^2\pi^2}{4} + \frac{\pi^2}{4} - \frac{3t\pi^2}{2}\right) - 4t\pi\left(\frac{3t\pi - \pi}{2}\right) + 2\pi^3\cos\left(\frac{3t\pi}{2} - \frac{\pi}{2}\right) + 3(-t^3\pi^3)\right]\frac{3\pi}{2} + \left[2\ \frac{9t^2\pi^2}{4} + \frac{\pi}{2} - \frac{3t\pi^2}{2} - 6t\pi\left(\frac{3t\pi - \pi}{2}\right)\right.$$

$$\frac{3\pi}{2}\left[-\frac{9t^2\pi^2}{2} - \frac{\pi^2}{2} + 3t\pi^2 - 6t^2\pi^2 + 2t\pi^2 + 2\pi^3\cos\left(\frac{3t\pi - \pi}{2}\right) - 3t^3\pi^3\right] + \left[\frac{9t^2\pi^2}{2} + \pi^2 - 3t\pi^2 - 9t^2\pi^2 + 3\pi^2 t + 3\pi^3\sin(-t\pi)\right]$$

$$\frac{3\pi^3}{2}\left[-\frac{9t^2}{2} - \frac{1}{2} + 3t - 6t^2 + 2t + 2\cos\left(\frac{3t\pi + \pi}{2}\right) - 3t^3\pi\right] + \pi^2\left[\frac{9t^2}{2} + 1 - 3t - 9t^2 + 3t + 3\sin(-t\pi)\right]$$

$$\frac{3\pi^3}{2}\left[\frac{-9t^2 - 1 + 6t - 12t^2 + 4t + 4\cos\left(\frac{3t\pi - \pi}{2}\right) - 6t^3\pi}{2}\right] + \pi^2\left(\frac{9t^2 + 2 - 18t^2 + 6\sin(-t\pi)}{2}\right)$$

$$3\pi^3\left[\frac{-6t^3\pi - 21t^2 - 1 + 10t}{4}\right] + \pi^2\left[\frac{-19t^2 + 2 + 6\sin(-t\pi)}{2}\right]$$

$$\pi^2\left[\left[\frac{-18t^3\pi^2 - 6\pi t^2 - 3\pi + 30\pi t}{4}\right] + \left[\frac{-18t - 9t^2 + 2 + 6\sin(-t\pi)}{2}\right]\right]$$

$$\pi^2\left[\frac{-18t^3\pi^2 - 6\pi t^2 - 3\pi + 30\pi t + 12\cos\left(\frac{3t\pi - \pi}{2}\right) - 18t^2 + 4 + 12\sin(-t\pi)}{4}\right]$$

$$\pi\int_0^1\left[-18t^3\pi^2 - 6\pi t^2 - 18t^2 + 30\pi t + 4 - 3\pi + 12\cos\left(\frac{3t\pi - \pi}{2}\right) + 12\sin(-t\pi)\right]dt.$$

$$\pi^2\left[-18\pi^2 - 6\pi - 18 + 30\pi + 4 - 3\pi - 12\right] - \left[\frac{8}{\pi} + 0\right]$$

$$\pi^2\left[-18\pi^2 + 21\pi - 38\right]$$

$$\boxed{-18\pi^4 + 21\pi^3 - 30\pi^2}$$

$$-18\pi^4 + 21\pi^3 - 18\pi^2 - \frac{8}{\pi}$$

wrong parameterisation

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$C2: \left(\pi, \frac{-t\pi}{2} + \pi\right)$ ✓

$$\left[-2\pi^2 + 4(\pi)\left(\frac{-t\pi}{2}+\pi\right) + 2\pi^3\cos(\pi) + 3\left(\frac{-t\pi}{2}+\pi\right)^2\right] \, 1 \left[-2\pi^2 - 6(\pi)\left(\frac{-t\pi}{2}+\pi\right) + 3\pi^{3}\sin\left(\frac{-t\pi}{2}+\pi\right)\right]$$

$$\left[-2\pi^2 - 2t\pi^2 + 4\pi^2 - 2\pi^3 + 3\left(\frac{t^2\pi^2}{4} + \pi^2 - t\pi^2\right)\right][0]$$

$$0$$

$$\left[-2\pi^2 + 3t\pi^2 - 6\pi^2 + 3\pi^{3}\sin\left(\frac{-t\pi}{2}+\pi\right)\right] -\frac{\pi}{2}$$

$$-\frac{\pi}{2}\left[-8\pi^2 + 3t\pi^2 + 3\pi^2\sin\left(-\frac{t\pi}{2}+\pi\right)\right]$$

$$\int_0^1 \frac{-\pi^3}{2}\left[-8 + 3t + 3\pi\sin\left(-\frac{t\pi}{2}+\pi\right)\right] dt.$$

$3\sin\left(\frac{\pi t}{2}\right)$ ✓

$$\frac{-\pi^3}{2}\left[-8t + \frac{3t^2}{2} - \frac{6\cos\left(\frac{\pi t}{2}\right)\pi}{\pi}\right]_{t=0}^{1}$$

$$\frac{-\pi^3}{2}\left[-8 + \frac{3}{2} - 0\right] - \left[\frac{-6}{\pi}\right]$$

$$\frac{-\pi^3}{2}\left[-8 + \frac{3}{2} + \frac{6}{\pi}\right]$$

$$\boxed{4\pi^3 - \frac{3\pi^3}{4} + \frac{6\pi^2}{2}} + -18\pi^4 + 21\pi^3 - 18\pi^2 - \frac{8}{\pi}$$

$= 1152.05.$

$\frac{16}{20}$

2.

1 2012 CORRECTION Q6

6 independent and conservative if $\frac{df}{dy} = \frac{dg}{dx}$

$\int \sin = -\cos$
$\int \cos = \sin$
$d\sin = \cos$
$d\cos = -\sin$

$\frac{df}{dy} = 4x + 6y$ $\quad$ $\frac{dg}{dx} = 4x + 6y$

$= \Rightarrow$ conservative

B $\frac{d\phi}{dx} = -2x^2 + 4xy + 2\pi^3\cos(x) + 3y^2$ $\quad$ $\frac{d\phi}{dy} = 2x^2 + 6xy - 3\pi^3\sin(y)$

$\phi = \frac{-2x^3}{3} + 2x^2y + 2\pi^3\sin(x) + 3xy^2 + d(c(y))$

$\frac{d\phi}{dy} = 2x^2 + 6xy + \frac{d(c(y))}{dy} = 2x^2 + 6xy - 3\pi^3\sin(y)$

$\frac{d\,c(y)}{dy} = -3\pi^3\sin(y)$

$c(y) = 3\pi^3\cos(y)$

$\phi = \frac{-2x^3}{3} + 2x^2y + 2\pi^3\sin(x) + 3xy^2 + 3\pi^3\cos(y)$

C $(\pi\ \pi/2) \to (\frac{-\pi}{2}, \pi)$

$-\frac{2\pi^3}{3} + 2(\pi^2(\frac{\pi}{2}) + 2\pi^3\sin(\pi) + 3(\pi)(\frac{\pi}{2})^2 + 3\pi^3\cos(\pi)$

$-\frac{2\pi^3}{3} + \pi^3 + \frac{3\pi^3}{4} - 3\pi^3$

$\left[\frac{-23}{12}\pi^3\right] - \left[\frac{-2\left[\frac{-\pi}{2}\right]^3}{3} + 2\left[-\frac{\pi}{2}\right]^2(\pi) + 2\pi^3\sin\left(-\frac{\pi}{2}\right) + 3\left(\frac{-\pi}{2}\right)(\pi^2) + 3\pi^3\cos(\pi)\right.$

$\frac{\pi^3}{4}\left[\frac{\pi^3}{12} + \frac{\pi^3}{2} - 2\pi^3 - \frac{3\pi^3}{2} - 3\pi^3\right]$

$\frac{-23}{12}\pi^3 - \left[-\frac{71}{12}\right]$

$= 4\pi^3$

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$x \quad y$

$\pi \quad \frac{\pi}{2}$

$$-\frac{2x^3}{3} + 2x^2y + 2\pi^3\sin(x) + 3y^2x + 3\pi^2\cos(y)$$

$$\frac{-2(\pi)^3}{3} + 2(\pi^2)\left(\frac{\pi}{2}\right) + 2\pi^3\sin(\pi) + 3\left(\frac{\pi}{2}\right)^2(\pi) + 3\pi^2\cos\left(\frac{\pi}{2}\right)$$

0

$$\frac{-2\pi^3}{3} + \pi^3 + \frac{3}{4}\pi^3 = \boxed{\frac{13}{12}\pi^3}$$

$x \quad y$

$-\frac{\pi}{2} \quad \pi$

$$\frac{-2\left[\frac{-\pi}{2}\right]^3}{3} + 2\left[\frac{-\pi}{2}\right]^2[\pi] + 2\pi^3\sin\left[\frac{-\pi}{2}\right] + 3[\pi]^2\left[\frac{-\pi}{2}\right] + 3\pi^2\cos(\pi)$$

$$\frac{\pi^3}{12} + \frac{\pi^3}{2} - 2\pi^3 - \frac{3\pi^3}{2} - 3\pi^3$$

$-\frac{29}{12} \quad -3\pi^2.$ $\quad -\frac{65}{12}\pi^3 \quad -\frac{71}{12}$

$$\frac{13}{12} + \frac{71}{12} = \frac{7}{}$$

$G \left(\pi , \frac{2\pi - t\pi}{2}\right)$

$+2x^2 + 6xy + 3\pi^3 \sin(y) \quad dy$

$\left[2(\pi)^2 + 6(\pi)\left(\frac{2\pi - t\pi}{2}\right) + 3\pi^3 \sin\left(\frac{2\pi - t\pi}{2}\right)\right] -\frac{\pi}{2}$

$-\frac{\pi}{2}\left[2\pi^2 + 6\pi^2 - 3t\pi^2 + 3\pi^3 \sin\left[\frac{2\pi - t\pi}{2}\right]\right]$

$\int_0^1 -\pi^3 - 3\pi^3 + \frac{3t\pi^3}{2} - \frac{3\pi^3}{2} \sin\left(\frac{2\pi - t\pi}{2}\right) \, dt$

$-\pi^3 t \quad -3\pi^3 t + \frac{3\pi^3}{2}\frac{t^2}{2} \quad -\frac{3\pi^3}{2}\left(\frac{-2\cos\left(\frac{\pi t}{2}\right)}{\pi}\right] \Big|_0^1$

$\left[-\pi^3 - 3\pi^3 + \frac{3\pi^3}{4}\right] - \left[3\pi^3\right]$

$-\frac{1}{4}\pi^3$

$\frac{67}{16} - \frac{1}{4}\pi^3 = \frac{63}{16}\pi^3?$

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3 2012 CORRECTION

1) C_1 $(-\frac{\pi}{2}, \pi)$ and (π, π)
$(1-t)r_0 + tr_1$

$(1-t)(-\frac{\pi}{2}, \pi) + t(\pi, \pi)$

$(-\frac{\pi}{2} + \frac{t\pi}{2} + t\pi, \ \pi - t\pi + t\pi)$
$(\frac{3t\pi}{2} - \frac{\pi}{2}, \ \pi)$

C_2: (π, π) and $(\pi, \pi/2)$
$(1-t)(\pi, \pi) + t(\pi, \pi/2)$

$(\pi - t\pi + t\pi, \ \pi - t\pi + t\frac{\pi}{2})$
$(\pi, \ \pi - \frac{t\pi}{2})$

C_1: $(\frac{3t\pi}{2} - \frac{\pi}{2}, \pi)$ $\quad$ $dy = 0$, no second term

$-2\left[\frac{3t\pi}{2} - \frac{\pi}{2}\right]^2 + 4\left[\frac{3t\pi}{2} - \frac{\pi}{2}\right][\pi] + 2\pi^3\cos\left[\frac{3t\pi}{2} - \frac{\pi}{2}\right] + 3[\pi]^2$

$-2\left[\frac{9t^2\pi^2}{4} - \frac{6t\pi^2}{4} + \frac{\pi^2}{4}\right] + 4\left[\frac{3t\pi^2}{2} - \frac{\pi^2}{2}\right] + 2\pi^3\cos\left[\frac{3t\pi}{2} - \frac{\pi}{2}\right] + 3\pi^2$

$-\frac{9t^2\pi^2}{2} + 3t\pi^2 - \frac{\pi^2}{4} + 6t\pi^2 - 2\pi^2 + 3\pi^2 \quad + 2\pi^3\cos\left[\frac{3t\pi}{2} - \frac{\pi}{2}\right]$

$\left[-\frac{9t^2\pi^2}{2} + 9t\pi^2 + \frac{3}{4}\pi^2 + 2\pi^3\cos\left[\frac{3t\pi}{2} - \frac{\pi}{2}\right]\right]\frac{3\pi}{2}$

$\int_0^1 -\frac{27t^2\pi^3}{4} + \frac{27t\pi^3}{2} + \frac{9\pi^3}{8} + 3\pi^4\cos\left[\frac{3t\pi}{2} - \frac{\pi}{2}\right]$ 6.

$-\frac{27}{4}\pi^3\frac{t^4}{4} + \frac{27\pi^3}{2}\frac{t^2}{2} + \frac{9\pi^3 t}{8} + 3\pi^4\left(-\frac{2\cos(\frac{3\pi t}{2})}{3\pi}\right)\Big|_{t=0}^{1}$

$-\frac{27\pi^3}{16} + \frac{27\pi^3}{4} + \frac{9\pi^3}{8} \quad - \cancel{2}\pi^3 \quad 0.$

$= \frac{99}{16}\pi^3 \quad - \left[-\frac{\cancel{3}\pi^4(2)(1)}{\cancel{3}\pi}\right.$

$\frac{67}{16}$ $\qquad$ $-2\pi^3$