Maths paper 1 2011

Does not determine a single [illegible] for each day

1 a Not a function - for every x value there is more than one y-value ✓

b If $f(f^{-1}) = x$ for every x in domain of f
and $f^{-1}(f) = x$ for every x in domain of f^{-1}

It is simply interchanging the x and y values of the function and then re-writing it for y ✓

As $x \to 0$ $\frac{1}{x} \to \infty$

c $f(x) = \sin\frac{1}{x}$ $\lim_{x \to 0} \sin\frac{1}{x}$

$-\frac{1}{x} < \sin\frac{1}{x} < \frac{1}{x}$ ✓

Limit fails to exist, oscillates between -1 and 1.

d The function $f'(x)$ defined by $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ is called the derivative of f in respect to x

The domain of $f'(x)$ consists of all $x \in D(f)$ for which the limit above exists ✓

e $f(x) = |x|$ $f'(x) = \lim_{h \to 0} \frac{(f(x+h) - f(x))}{h} \rightarrow \lim_{h \to 0} \frac{|x+h|-|x|}{h}$

$x=0 \rightarrow f(0) : \frac{|h|}{h}$ $\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$

$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} = \frac{-h}{h} = -1$

Two sided limit does not exist. $f(x)$ is non differentiable at $x=0$ ✓

2 a For $f(x)$ to be continuous at $x=c$, the following three conditions must satisfy

- The limit of $f(x)$ as x approach c exists
- The value of the function and the value of the limit at c are the same
- The function $f(x)$ is defined at c

— $f(c)$ is defined

— $\lim_{x \to c} f(x)$ exists

— $\lim_{x \to c} f(x) = f(c)$ ✓

2 b i $y = \frac{d(\sin((x^2+1)^4))^{1/2}}{d\sin((x^2+1)^4)} \cdot \frac{d\sin((x^2+1)^4)}{d((x^2+1)^4)} \cdot \frac{d(x^2+1)^4}{d(x^2+1)} \cdot \frac{d(x^2+1)}{dx}$

$\frac{1}{2}(\sin((x^2+1)^4))^{-1/2} \cos((x^2+1)^4)(4)(x^2+1)^3(2x)$ ✓

ii $\underset{u}{x^2\sin x} \quad \underset{v}{\cos x} \qquad v\frac{du}{dx} + u\frac{dv}{dx}$

$x^2\cos x + 2x\sin x \qquad \cos x(x^2\cos x + 2x\sin x) + (x^2\sin x)(-\sin x)$

$x^2\cos^2 x + 2x\sin x\cos x + x^2\sin^2 x$ ✓

iii $x^3y + xy^3 = \frac{1}{x}$

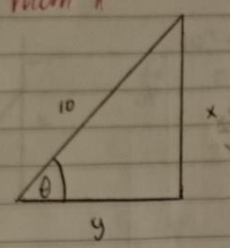
$x^3\frac{dy}{dx} + 3x^2y + 3y^2x\frac{dy}{dx} + y^3 = -1\left(\frac{1}{x^2}\right) - x^{-2}$

$\frac{dy}{dx}(x^3+3y^2x) = -x^{-2} - 3x^2y - y^3$

$\frac{dy}{dx} = \frac{-x^{-2} - 3x^2y - y^3}{x^3+3y^2x}$ ✓

2011 Math 1

Q 2c



$\frac{dy}{dt} = \frac{-1}{2}$ ft/sec when $y = 2$

$x = \sqrt{96}$

$x^2 + y^2 = 10$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \qquad \frac{dx}{dt} = \frac{-y}{x}\frac{dy}{dt}$

$\tan\theta = \frac{x}{y} \qquad \theta = \tan^{-1}\left(\frac{x}{y}\right)$

$$\frac{d\theta}{dt} = \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{d\left(\frac{x}{y}\right)}{dt}$$

$$= \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{dx}{dt}\cdot\frac{1}{y} - \frac{x}{y^2}\frac{dy}{dt}\right)$$

$\rightarrow \frac{dx}{dt} = \frac{-y}{x}\frac{dy}{dt}$ $$= \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\left(\frac{-y}{x}\frac{dy}{dt}\right) - \frac{x}{y^2}\frac{dy}{dt}\right)$$

$y=2 \rightarrow$ $$= \frac{1}{1+\frac{96}{4}}\left(\frac{1}{\sqrt{24}+\sqrt{24}}\right)$$

$$= \frac{1}{\sqrt{24}} \text{ radian / sec.}$$

length =

3b

$2\pi r + 3x = L \qquad r = \frac{L-3x}{2\pi}$

$\text{Area} = \overset{\text{circle}}{\pi r^2} + \overset{\text{Grey}}{\frac{x^2}{2}}$

$\text{Area} = \frac{(L-3x)^2}{4\pi} + \frac{x^2}{2} \qquad 0 \le x \le \frac{L}{3}$

$\frac{dA}{dx} = \frac{-3}{2\pi}(L-3x) + x = 0$

$\left(\frac{9}{2\pi}+1\right)x = \frac{3L}{2\pi}$

$x = \frac{3L}{2\pi} \cdot \frac{2\pi}{9+2\pi}$

$x = \frac{3L}{9+2\pi}$

$0 \le \frac{3L}{9+2\pi} \le \frac{L}{3}$ three points to test:

$x=0 \rightarrow A = \frac{L^2}{4\pi}$

$x = \frac{3L}{9+2\pi} \quad A(x) = \frac{\left(L-\frac{9L}{9+2\pi}\right)^2}{4\pi} + \frac{\left(\frac{3L}{9+2\pi}\right)^2}{2}$

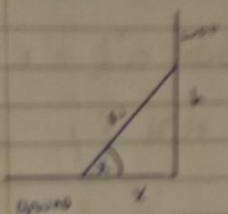
At $x = \frac{L}{3}$ $\quad A = \frac{L^2}{18} \qquad \frac{d^2A}{dx^2} = \frac{9}{2\pi} + 1 > 0$ local min

at min $x = \frac{3L}{9+2\pi}$ and $\frac{L^2}{4\pi} > \frac{L^2}{18}$

max at $x=0$

2011 Maths 1

2c



$10^2 = h^2 + x^2$

$\frac{d}{dt} \quad 0 = 2h\frac{dh}{dt} + 2x\frac{dx}{dt}$

$2x\frac{dx}{dt} = -2h\frac{dh}{dt}$

$\frac{dx}{dt} = -\frac{h}{x}\frac{dh}{dt}$

When $x = 2$
$h = \sqrt{96}$

$\frac{-\sqrt{96}}{2}\left(\frac{1}{2}\right) = 2.54$
2.55

-3

2d $f(x) = \sqrt[5]{x}$ $\quad x = 31$

$f'(x) = \frac{1}{5(x)^{4/5}}$ $\quad x_0 = 32$

$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$

$\sqrt[5]{32} + \frac{1}{5(2)^{4/5}}(31-32)$

$2 + (-0.0125) = 1.9875$ for x closer to x_0 it should be better

We are approximating the curve $y = f(x)$ by the tangent line $x = x_0$

- If a function is differentiable at x_0 then a sufficiently magnified portion of the graph of f centered at point $P(x_0, f(x_0))$ takes on appearance of a straight line
- For this reason, if a function is differentiable at x_0 it is sometimes said to be locally linear at x_0

3a $f(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2$

$f'(x) = 3x^2 - 3$

Critical points $= f'(x) = 0$ $\quad x^3 - 3x = 0$ $\quad x = 0 \quad - \quad$ max

$x(x^2 - 3) = 0$ $\quad x = \sqrt{3} \quad + \quad$ min

$x = 0 \quad x = \pm\sqrt{3} \quad x = -\sqrt{3} \quad + \quad$ min

Inflection $\frac{d^2y}{dx^2} = 0$ $\quad 3x^2 - 3 = 0$

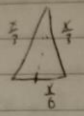
$3x^2 = 3$

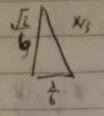
$x^2 = 1$

$x = \pm 1$

increasing/decreasing	concavity
$-\sqrt{3}$ ... 0 ... $\sqrt{3}$	-1 ... 1
inc $(-\sqrt{3}, 0]$ and $[\sqrt{3}, \infty)$	up $(-\infty, -1)$ $(1, \infty)$
dec $(-\infty, -\sqrt{3}]$, $[0, \sqrt{3}]$	down $(-1, 1)$

3b L into two parts x and y





$$y = 3.14r$$

$$r = \frac{y}{3.14}$$

$$\text{area} = \pi r^2$$

$$= \pi \frac{y^2}{9.8596}$$

$$\text{area} = \frac{1}{2}\left(\frac{\sqrt{2}}{6}\right)\left(\frac{x}{3}\right)$$

$$\frac{x^2}{9} = \frac{x^2}{6} + y^2$$

$$\frac{\sqrt{2}x}{36} = \text{area}$$

$$y^2 = \frac{x^2}{9} - \frac{x^2}{6}$$

$$\frac{\sqrt{2}x}{36} + \frac{\pi y^2}{9.8596}$$

$$\frac{\sqrt{2}}{36} + \frac{\pi}{9.8596} 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\sqrt{2}}{36} \cdot \frac{9.8596}{\pi\, 2y}$$

3c A function f is said to integrable on a finite closed interval $[a, b]$ if the limit:

$$\lim_{\max \Delta x_k \to \infty} \sum_{k=1}^{n} f(x_k^*)(\Delta x_k)$$

exists and does not depend on the choice of partition (Δx_k) or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol which is called the definite integral of f from a to b

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)(\Delta x_k)$$

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(a) 3d i $\int_0^1 x^2(x^3+1)^{10}dx$ $\quad u = x^3+1$
$du = 3x^2 dx$

$\frac{1}{3}\int_1^2 u^{10}\,du$

$\frac{1}{3}\frac{u^{11}}{11} = \left[\frac{1}{3}\frac{(x^3+1)^{11}}{11}\right]_0^1$

$\left[\frac{1}{3}\frac{(2)^{11}}{11}\right] - \left[\frac{1}{3}\frac{(0+1)^{11}}{11}\right]$

$\frac{2048}{33} - \frac{1}{3} = \frac{674}{11}$ square units

ii $\int x\cos x^2\,dx$ $\quad u = x^2 \quad du = 2x\,dx$

$\frac{1}{2}\int \cos u\,du$

$\frac{1}{2}\sin u = \frac{\sin(x^2)}{2} + c$

3e Part one of fundamental theorem of calculus

If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Part two

If f is continuous on an interval, then f has an antiderivative on that interval. In particular if a is any point in the interval, then the function F is defined by:

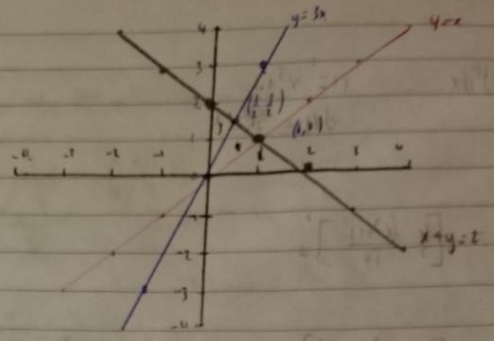
$F(x) = \int_a^x f(t)\,dt$

is an antiderivative of f, that is $F'(x) = f(x)$

for each x in the interval, or in an alternate notation

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

4a



Area betw $y=3x$ and $x+y=2$ $-$ area $x+y=2$ $y=x$

Total area large triang $= \frac{1}{2} b \times h = \frac{1}{2} 2 \times 2 = \frac{1}{2} 4 = 2$

area $2 = 1$

area $3 = \frac{1}{2}(2)(\frac{1}{2})$

$\frac{1}{2} y = \frac{3}{2}$

$y = 3x \quad 3x - y = 0$

$x + y = 2 \quad x + y = 2$

$3x = 0 \quad 2x = 2 \quad 4x = 2$

$-x = -2 \quad x = 2 \quad x = 2$

area $= \frac{1}{2}$

$y = 3x \quad y - 3x = 0$

$x + y = 2 \quad y + x = 2$

$-4x = -2$

4b

$\frac{1}{b-} \quad \frac{1}{4-1} \int_1^4 x^2 + 2x - 4$

$\frac{1}{3} \left[\frac{x^3}{3} + x^2 - 4x \right]_1^4$

$\frac{1}{3} \quad \left[\frac{64}{3} + 16 - 16\right] - \left[\frac{1}{3} + 1 - 4\right]$

$= 8$

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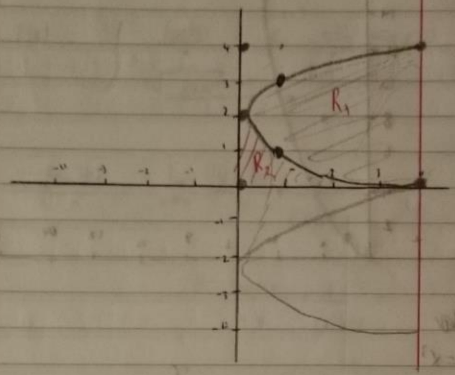
Q 4c. $x = (y-2)^2$ $\quad x = y^2 - 4y + 4$ $\quad \sqrt{x} = y - 2$ $\quad y = \sqrt{x} + 2$

$x = 4$ $\quad y^2 - 4y = x - 4$

$$\boxed{\begin{array}{l} y = -\sqrt{x} + 2 \\ y = \sqrt{x} + 2 \end{array}}$$

$x = 1$ $\quad y = \begin{array}{c}1\\3\end{array}$



Area $= \int_0^4 \pi (\sqrt{x}+2)^2$

$\pi \int_0^4 x + 4\sqrt{x} + 4$

$\pi \int_0^4 \frac{x^2}{2} \frac{(2)4x^{3/2}}{3} + 4x$

$\left[\frac{x^2}{2} + \frac{8x^{3/2}}{3} + 4x\right]_0^4$

$R_2 - R_1$

$= \int_0^4 \pi \left((\sqrt{x}+2)^2 - (-\sqrt{x}+2)^2\right)$

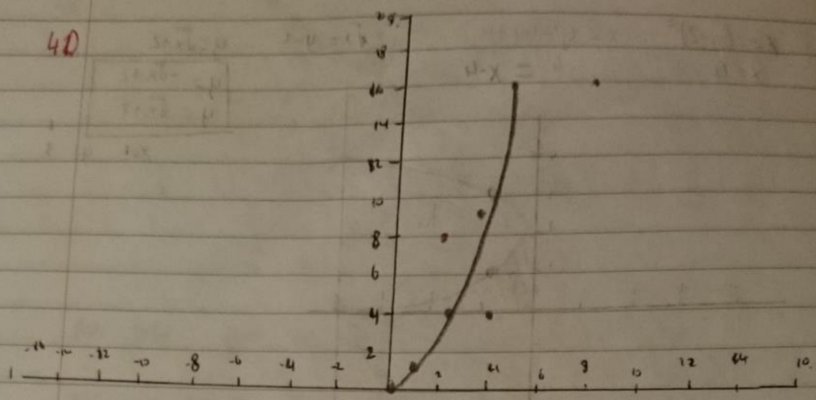
$\int_0^4 \pi \left(\cancel{(2\sqrt{x}+4)}\right)(x + 4\sqrt{x} + 4) - (x - 4\sqrt{x} + 4)$

$\pi \int_0^4 8\sqrt{x}$

$8\pi \left[\frac{2x^{3/2}}{3}\right]_0^4$

$8\pi \left(\frac{16}{3}\right) = \frac{128}{3}\pi$

4D



Point of intersection

$x^2 = x^3$

$x = 0,\ x = 1$ limits are $\int_0^1$

$R_1 - r_2$ $\quad R_1 = x^3 \quad y = x^3 \qquad R_2 = y = x^2$

$x = y^{1/2}$ $\qquad x = y^{1/2}$

$y^{2/3}$

$\pi \int_0^1 (y^{1/3})^2 - (y^{1/2})^2 \, dy$

$\pi \int_0^1 \cancel{y^{1/3} - y^{1/2}}$

$\pi \int y^{2/3} - y \, dy$

$\pi \left[\frac{3y^{5/3}}{5} - \frac{y^2}{2} \right]_0^1$

$\frac{3}{5} - \frac{1}{2} = \frac{1}{10}\pi$ ✓