25/16 ALSM 2 EXAM NOTES: POISSON DISTRIBUTION

- Model outcome that represents number of events/occurances
- When data is in form of $\{(y_i; x_i; n_i)\}$ $i = 1, \ldots, N$ where the outcome y_i is the number of successes amongst n_i trials, the binomial distribution more suitable
- Probability of a given number of events occuring in a fixed interval of time and or space, if these events occur with a known average rate and independently of the time since the last event
- Rate cannot be higher in some intervals and lower in others
- The probability of an event in a interval is proportional to the length of the interval

$$P(y|\lambda) = \frac{\lambda^y}{y!}\exp(-\lambda) \qquad y \in \mathbb{N} \qquad \lambda > 0$$

1. Show it is a distribution

Show it is always positive $\frac{(+)^{+}(+)}{(+)!}$ $\Rightarrow$ always positive

Show it sums/integrates to 1

Show $\sum_{y=0}^{\infty} P(y|\lambda) = 1$

$$= \sum_{y=0}^{\infty} \lambda^y / y! \exp(-\lambda) \qquad = \exp(-\lambda) \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}$$

Aside: Taylor expansion of $\exp(\lambda) = \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}$

$$= \exp(-\lambda)\exp(\lambda)$$

$$= 1$$

2. Show It Is A Member Of Exponential Family of Distributions

Form of $\exp[a(y)b(\lambda) + c(\lambda) + d(y)]$

$$P(y|\lambda) = \exp[\log(\frac{\lambda^y}{y!}\exp(-\lambda))]$$

$$= \exp[\log(\frac{\lambda^y}{y!}) + -\lambda]$$

$$= \exp[\log(\lambda^y) - \log(y!) - \lambda]$$

$$= \exp[y\log(\lambda) - \log(y!) - \lambda]$$

$a(y)$ $b(\lambda)$ $d(y)$ $c(\lambda)$

3 Expectation

$\mathbb{E}[y] = \int_y p(y|\lambda)dy$ → change to sum as y is discrete R.V.

$= \sum_0^\infty y\,p(y|\lambda)$

$= \sum_{y=0}^\infty y/y!\ \lambda^y \exp(-\lambda)$ take out $\exp(-\lambda)$ as doesn't depend on y

$= \exp(-\lambda) \sum_{y=0}^\infty y\lambda^y/y!$

$= \lambda \exp(-\lambda) \sum_{y\geq 1}^\infty \frac{1}{(y-1)!}\lambda^{y-1}$

$= \lambda \exp(-\lambda) \sum_{j=0}^\infty \frac{\lambda^j}{j!}$ $j = k-1$

$= \lambda \exp^{-\lambda} e^\lambda$ ← Taylor series expansion for exponential function

$= \lambda$

4 Compute the maximum W.R.T. λ of this Distribution

Find λ such that $dp/d\lambda = 0$

$dp/d\lambda = \frac{1}{y!}\left[y\lambda^{y-1}\exp(-\lambda) - \exp(-\lambda)\lambda^y\right]$

$= \frac{1}{y!}\exp(-\lambda)\lambda^{y-1}[y-\lambda] = 0$ 0 when $y = \lambda$

Saturated solution of maximum likelihood is $\lambda = y$

- y is an integer ($\mathbb{N}$), λ is in $\mathbb{R}^+$
- Use log to map $\mathbb{R}$ onto $\mathbb{R}^+$, use as link between our function and expectation

$\mathbb{E}[y] = \lambda \quad \in \mathbb{R} \quad \underset{\lambda = \exp(x^T\beta)}{\overset{\log \lambda = x^T\beta}{\rightleftarrows}} \quad x^T\beta$

Instead in binomial we use $\log\left[\mathbb{E}[y]/n\right] = \log\left[\frac{\lambda}{n}\right] = x^T\beta$

$= \log(\lambda) = \underbrace{\log(n)}_{\text{offset}} + x^T\beta$ → like normalising for size of group

n is called exposure, $\log(n)$ is the offset → only affects β_0 (intercept) in model

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05/16 ALSM 2 EXAM NOTES: POISSON DISTRIBUTION

Relationship between Poisson and Binomial when $n \to +\infty$

Show that $\lim_{n\to\infty} \frac{n!}{(n-y)!\,y!}\theta^y(1-\theta)^{n-y} = \frac{\lambda^y \exp(-\lambda)}{y!}$ with $\lambda = n\theta$

Binomial(n,θ) Poisson(λ)

Change $\theta = \frac{\lambda}{n}$ into binomial:

$$\frac{n!}{(n-y)!\,y!}\left(\frac{\lambda}{n}\right)^y\left(1-\frac{\lambda}{n}\right)^{n-y} = \frac{n!}{(n-y)!\,y!}\left(\frac{\lambda}{n}\right)^y\underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{A_n}\underbrace{\left(1-\frac{\lambda}{n}\right)^{-y}}_{B_n}$$

$$= \frac{\lambda^y}{y!}\frac{n!}{n^y(n-y)!}\left(\frac{\lambda}{n}\right)^y\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-y}$$

$\frac{\lambda^y}{y!} \to$ stays the same, not dependent on n

$$\lim_{n\to\infty}\frac{n!}{(n-y)!\,n^y} = \frac{n(n-1)(n-2)\cdots(n-y+1)}{n\cdot n\cdot n} = 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots = 1$$

$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n$ we know $(1+s)^n = \sum_{k=0}^{\infty}\binom{n}{k}s^k \Rightarrow$ Taylor expansion of $\exp(-\lambda) = \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}$

$= \exp(-\lambda)$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-y} = 1$$

$$\lim_{n\to\infty}\frac{n!}{(n-y)!\,y!}\theta^y(1-\theta)^{n-y} = \frac{\lambda^y\exp(-\lambda)}{y!} \quad \text{(POISSON)}$$