15/16 ALSM 2: EXAM NOTES: BINOMIAL DISTRIBUTION

- n repeated trials
- Each trial has two outcomes: yes/no 1/0
- Trials are independent

$P(y|\theta) = \binom{n}{y}(\theta)^y(1-\theta)^{n-y}$ $\quad y \in (0,1,\dots,N) \quad \theta \in [0,1]$

1. Show it is a distribution

Positive function: $(\pm)(+)^+(+)^+ \rightarrow$ Always positive

$$\Sigma_{y=0}^{N} \binom{n}{y} p^y (1-p)^{n-y} = [p + [1-p]]^n$$
$$= 1^n = 1 \quad \Rightarrow \text{is a distribution}$$

Where we have used the formula for binomial expansions:

$$(a+b)^n = \Sigma_{y=0}^{N} \binom{n}{y} a^y b^{n-y}$$

2. Member of exponential family

Form of: $\exp[a(y)\, b(\lambda) + c(\theta) + d(y)]$

$$= \exp\left[\log\left(\binom{n}{y}(\theta)^y(1-\theta)^{n-y}\right)\right]$$
$$= \exp\left[\log\binom{n}{y} + \log(\theta)^y + \log(1-\theta)^{n-y}\right]$$
$$= \exp\left[\log\binom{n}{y} + y\log(\theta) + (n-y)\log(1-\theta)\right]$$
$$= \exp\left[y(\log\theta - y\log(1-\theta)) + \log\binom{n}{y} + n\log(1-\theta)\right]$$
$$= \exp\left[\underbrace{y}_{a(y)}\ \underbrace{\log\left(\tfrac{\theta}{1-\theta}\right)}_{b(\theta)} + \underbrace{\log\binom{n}{y}}_{d(y)} + \underbrace{n\log(1-\theta)}_{c(\theta)}\right]$$

3. Expectation

$$E[y] = \int_0^\infty y\, P(y|\theta)\, dy$$
$$= \Sigma_{y=0}^{\infty} y \binom{n}{y}\theta^y(1-\theta)^{n-y} \quad \text{Start at } y=1\text{, at } y=0 \text{ expression} = 0$$

$$\text{expand } y\binom{n}{y} = \frac{y\,n!}{(n-y)!\,y!} = \frac{y\,n(n-1)!}{(n-1-(y-1))!\,y.(y-1)!} = n\binom{n-1}{y-1}$$

$$= n\theta\, \underbrace{\Sigma_{y=1}^{N} \binom{n-1}{y-1}\theta^{y-1}(1-\theta)^{(n-1-(y-1))}}_{Bin(\theta, n-1)} = np$$

$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)}$

$= np \sum_{l=0}^{n} \binom{n-1}{l} p^{l} (1-p)^{n-1-l}$ $\quad l = k-1$

$= np \sum_{l=0}^{m} \binom{m}{l} p^{l} (1-p)^{m-l}$ $\quad m = n-1$

$= np(p+(1-p))^{m} = np1^{m} = np$

4 Value of θ that maximises $P(y|\theta)$?

$\frac{d\, P(y|\theta)}{d(\theta)}$ $\quad \binom{n}{y} \overset{u}{\theta^{y}} \overset{v}{(1-\theta)^{n-y}}$

$= \binom{n}{y} \left[\theta^{y}(n-y)(1-\theta)^{n-y-1} + (1-\theta)^{n-y} y \theta^{y-1} \right] = 0$

$\binom{n}{y} \left(\theta^{y-1} (1-\theta)^{n-y-1} \right) \left[y(1-\theta) - \theta(n-y) \right] = 0$

$y(1-\theta) - \theta(n-y) = 0$

$y - y\theta - \theta n + \theta y = 0$

$\theta(y - y - n) = -y$

$\theta = y/n$ $\quad$ or $\quad y = n\theta$

$\prod_{i=1}^{N} \binom{n_i}{y} \left[g(\beta^T x_i) \right]^{y_i} \left[1 - y(\beta^T x_i) \right]^{n_i - y_i}$

↳ logit or probit

$L(\beta) = \prod_{i=1}^{N} \binom{n_i}{y_i} (g^{-1}(\beta_0 + \beta_1 x_i))^{y_i} (1 - g^{-1}(\beta_0 + \beta_1 x_i))^{n_i - y_i}$

$\text{loglik}(\theta_1, \dots, \theta_N) = \sum_{i=1}^{N} \left[\log \binom{n_i}{y_i} + y_i \log\left(\frac{\theta_i}{1-\theta_i}\right) + n_i \log(1-\theta_i) \right]$

$\text{loglik}(\beta_1, \beta_2) = \sum_{i=1}^{N} \left[\log \binom{n_i}{y_i} + y_i (\beta_1 + \beta_2 x_i) + n_i \log(1 + \exp(\beta_1 + \beta_2 x_i)) \right]$

Newton-Raphson Method

$$\begin{pmatrix} \beta_1^{m+1} \\ \beta_2^{m+1} \end{pmatrix} = \begin{pmatrix} \beta_1^{m} \\ \beta_2^{m} \end{pmatrix} - H^{-1}_{\beta_1^m \beta_2^m} \nabla \text{loglik}_{\beta_1^m \beta_2^m}$$

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$$\nabla \text{loglik} \left| \begin{array}{l} d\,\text{loglik}/d\beta_1 = \sum_{i=1}^{N} \left[k_i - n_i \dfrac{\exp(\beta_1+\beta_2 x_i)}{1+\exp(\beta_1+\beta_2 x_i)} \right] \\ d\,\text{loglik}/d\beta_2 = \sum_{i=1}^{N} \left[k_i x_i - n_i x_i \dfrac{\exp(\beta_1+\beta_2 x_i)}{1+\exp(\beta_1+\beta_2 x_i)} \right] \end{array} \right.$$

Second Derivative:

$$\frac{d^2}{\beta_1^2} = -\sum_{i=1}^{N} n_i \theta_i (1-\theta_i)$$

$$\frac{d^2}{\beta_2^2} = -\sum_{i=1}^{N} n_i x_i^2 \theta_i(1-\theta_i)$$

$$\frac{d^2}{\beta_1,\beta_2} = -\sum_{i=1}^{N} n_i x_i \theta_i(1-\theta_i)$$

$$\text{Posterior}(\theta) = \begin{cases} \dfrac{\text{lik}(\theta)}{\int_0^1 \text{lik}(\theta)} & \text{for } \theta \in [0,1] \\ 0 & \text{Otherwise} \end{cases}$$ normalising the likelihood

$$\text{Post}(\theta) = \text{Beta}(n-k+1, k+1)$$

$$S.E = \sqrt{\frac{p(1-p)}{n}}$$