02/02/16 ALSM2

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POISSON DISTRIBUTION

- Suitable to model outcomes that represent numbers of events/occurances
- When data is in form of $\{(y_i, x_i, n_i)\}$ $i=1,\dots,n$ where the outcome y_i is the number of successes amongst n_i trials, the binomial distribution more suitable
- The relation between binomial and poisson distribution is when $n \to \infty$

$$P(y|\lambda) = \frac{\lambda^y}{y!} \exp(-\lambda) \qquad y \in N \quad \lambda > 0$$

1. Show it is a distribution

Show it is always positive $P(y|\lambda) > 0 \Rightarrow$ Positive Function

Show it sum/integrates to 1

Show $\sum_{y=0}^{\infty} P(y|\lambda) = 1$

$$= \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} \exp(-\lambda) = \exp(-\lambda) \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}$$

Aside: Taylor expansion $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Taylor $\exp(\lambda) = \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}$

2. Show it is a member of exp family of distributions

Form of: $\exp[a(y)b(\lambda) + c(\lambda) + d(y)]$

$$P(y|\lambda) = \exp[\operatorname{Log}(\lambda^y)] \exp[\operatorname{Log}\tfrac{1}{y!}] \exp(-\lambda)$$

$$= \exp[y \log(\lambda) - \operatorname{Log}(y!) - \lambda]$$

$a(y)\ b(\lambda) \qquad d(y) \qquad c(\lambda)$

3. Compute the expectation of y and verify this is a function of λ

$\mathbb{E}[y] = \int y\, P(y|\lambda)\, dy \rightarrow$ change to sum as y is discrete rv

$$= \sum_{0}^{\infty} y\, P(y|\lambda)$$

$$= \sum_{y=0}^{\infty} \frac{y}{y!} \lambda^y \exp(-\lambda)$$ take out $\exp(-\lambda)$ as doesn't depend on y

$= \exp(-\lambda) \sum_{y=0}^{\infty} \frac{y\lambda^y}{y!}$

$= \exp \sum_{y=0}^{\infty} \frac{y\lambda^y}{y(y-1)!}$ Start at $y=1$ as at $y=0$ sum=0

$= \exp(-\lambda)(\lambda) \sum_{y=1}^{\infty} \frac{y\lambda^{y-1}}{y(y-1)!}$

$\sum_{y=1}^{\infty} \frac{\lambda^{y-1}}{(y-1)!} = \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = \exp(\lambda)$ taylor expansion

$= \exp(-\lambda)(\lambda)\exp(\lambda) = \lambda =$ expectation

4. Compute the maximum wrt λ of this distribution
Find λ such that $dp/d\lambda = 0$

$$\frac{dp}{d\lambda} = \frac{1}{y!}\left[y\lambda^{y-1}\exp(-\lambda) - \exp(-\lambda)\lambda^y\right]$$
$$= \frac{\lambda^{y-1}\exp(-\lambda)}{y!}[y-\lambda] \quad = 0 \text{ when } \lambda = y$$

Saturated solution of maximum likelihood is $\lambda = y$

- y is an integer ($\mathbb{N}$), λ is in $\mathbb{R}^+$
- use log to map $\mathbb{R}$ onto $\mathbb{R}^+$ can use it as a link function between a[…]
function and expectation

$$\mathbb{E}[y] = \lambda \in \mathbb{R} \underset{\lambda = \exp(x^T\beta)}{\overset{\log(\lambda) = x^T\beta}{\rightleftarrows}} x^T\beta$$

Same link function for poisson and binomial

ALSM 2

BINOMIAL DISTRIBUTION

- n repeated trials
- Each trial has 2 outcomes, success/failure yes/no
- Trials independent

$p(y|\theta) = \binom{n}{y} (\theta)^y (1-\theta)^{n-y} \quad y \in (0,1,..,n) \quad \theta \in [0,1]$

1. Show it is a distribution

Positive function $\binom{+}{+} (+)^+ (+)^+ \Rightarrow$ always positive

$\binom{n}{y} = \frac{n!}{(n-y)!\,y!}$

$\sum \binom{n}{y} \theta^y (1-\theta)^{n-y}$
$= (\theta + 1 - \theta)^n \leftarrow$ Pascal triangle
$= 1^n = 1$

2. Member of Exponential family?

Form of: $\exp[a(y) b(\lambda) + c(\theta) + d(y)]$

$= \exp[\log\binom{n}{y} + \log(\theta)^y + (n-y)\log(1-\theta)]$

$= \exp[y[\log(\theta) - \log(1-\theta)] + \log\binom{n}{y} + n\log(1-\theta)]$

$= \exp[\underbrace{y}_{a(y)} \underbrace{\log(\tfrac{\theta}{1-\theta})}_{b(\theta)} + \underbrace{\log\binom{n}{y}}_{c(y)} + \underbrace{n\log(1-\theta)}_{d(\theta)}]$

3. Value of θ that maximises $p(y|\theta)$?

$\frac{d\,p(y|\theta)}{d(\theta)} = \binom{n}{y} [y\theta^{y-1}(1-\theta)^{n-y} - \theta^y (n-y)(1-\theta)^{n-y-1}] = 0$

$\binom{n}{y} \theta^{y-1} (1-\theta)^{n-y-1} [y(1-\theta) - \theta(n-y)] = 0$

$y(1-\theta) - \theta(n-y) = 0$

$y - \theta y - \theta n + \theta y = 0$

$\theta(y - y - n) = -y \quad \theta = y/n \quad$ or $\quad y = n\theta$

4. Expectation

$E[y] = \sum_{y=0}^{n} y \binom{n}{y} \theta^y (1-\theta)^{n-y}$

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$= \sum_{y=0}^{\infty} y \binom{n}{y} \theta^y (1-\theta)^{n-y}$ at $y=0$ expression $=0$

$\hookrightarrow y\binom{n}{y} = \frac{y\,n!}{(n-y)!\,y!} = \frac{y\,n\,(n-1)!}{(n-1-(y-1))!\,y\,(y-1)} = n\binom{n-1}{y-1}$

$= n\theta \underbrace{\sum_{y=0}^{n-1} \binom{n-1}{y-1} \theta^{y-1} (1-\theta)^{(n-1-(y-1))}}_{Bin\ (\theta_{n-1}}$ $= n\theta$

Insted in binomial we use $\log\left[\frac{E[y]}{n}\right] = \log\left[\frac{\lambda}{n}\right] = x^T\beta$

$= \log(\lambda) = \underset{\text{offset}}{\log(n)} + x^T\beta$ $\Rightarrow$ like normalising for size of group

n is called exposure, $\log(n)$ is the offset $\Rightarrow$ only affects β_0 (intercept) in model

Relationship Between Poisson and Binomial When $n \Rightarrow +\infty$

Show that $\lim_{n\to\infty} \frac{n!}{(n-y)!\,y!} \theta^y (1-\theta)^{n-y} = \frac{\lambda^y \exp(-\lambda)}{y!}$ with $\lambda = n\theta$

Binomial$(n\theta)$ Poisson (λ)

Change $\theta = \frac{\lambda}{n}$ into binomial:

$\frac{n!}{(n-y)!\,y!} \left(\frac{\lambda}{n}\right)^y \left(1-\frac{\lambda}{n}\right)^{n-y} = \frac{n!}{(n-y)!\,y!} \underbrace{\left(\frac{\lambda}{n}\right)^y}_{A_n} \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{B_n} \left(1-\frac{\lambda}{n}\right)^{-y}$

$= \frac{\lambda^y}{y!} \frac{n!}{n(n-y)} \left(\frac{\lambda}{n}\right)^y \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-y}$

$\frac{\lambda^y}{y!}$ $\Rightarrow$ stays the same, not dependent on n

$\lim_{n\to\infty} \frac{n!}{(n-y)!\,n^y} = \frac{n(n-1)\cdots(n-y+1)}{n\,n\cdots n} = 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) = 1$

$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n$ we know $(1+s)^n = \sum^{n} \binom{n}{k} s^n$ $\rightarrow$ Taylor expansion of $\exp(-\lambda) = \sum_{k=0}^{n} \left(\frac{-\lambda}{k!}\right)^{k}$

$\lim_{n\to\infty} = \exp(-\lambda)^{\infty}$

$\lim_{n \to \infty} (1 - \frac{\lambda}{n})^y = 1$

$\lim_{n \to \infty} \frac{n!}{(n-y)!\,y!} \theta^y (1-\theta)^{n-y} = \frac{\lambda^y \exp(-\lambda)}{y!}$ (Poisson)

Binomial and Posterior Odds

- Example village, dirty water, how many infected?

$n = 10$ people tested $y = 7$ infected

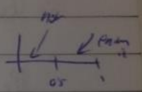
H_0: $\theta \leq 0.5$ v H_1: $\theta > 0.5$

(not endemic) (infection endemic)

Binomial : $\binom{10}{7} \theta^7 (1-\theta)^3$ $\hat{\theta}_b = y/n = 0.7$

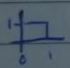
Posterior Distribution of θ given y: $P(\theta|y) = \frac{P(y|\theta).\, p(\theta)}{\int p(y|\theta)\, p(\theta)\, d\theta}$

"normalising constant" $p(y)$ $\Rightarrow$



$\mathbb{P} = \int_{0.5}^{1} p(\theta|y)\, d\theta \geq 0.95$ then H_1 is true

Belief Ⓐ :



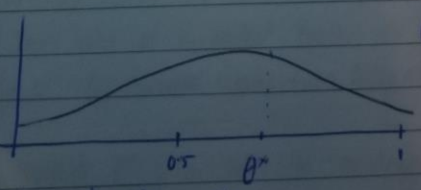
θ can be anywhere between 0 and 1

"Uniform probability for value of θ"

Has to integrate to 1

$\mathbb{P} = \frac{\int_{0.5}^{1} \binom{10}{7} \theta^7 (1-\theta)^3\, d\theta}{\int_0^1 \binom{10}{7} \theta^7 (1-\theta)^3\, d\theta}$ $p(\theta) = 1$

$p(\theta|y)$



Checking that integration for 0.5 to 1 is 95% to prove H_1

$\theta^7 (1-\theta)^3 = \frac{-\theta^{11}}{11} + \frac{3\theta^{10}}{10} - 3\theta^9/9 + \theta^8/8$

$(1-\theta)^3 = 1 - 3\theta + 3\theta^2 - \theta^3$

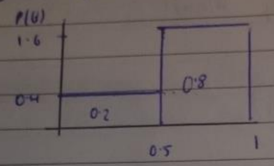
$\theta^7 (1-\theta)^3 = \theta^7 - 3\theta^8 + 3\theta^9 - \theta^{10}$ then integrate

$$\frac{[\ldots]_{0.5}^{1}}{[\ldots]_{0}^{1}} = 0.88 = \theta^{*}$$

Reject H_1, not > 0.95

Belief (B) : expert says 80% sure infection is endemic

Prior



Compute the prior again (integral function)

$$\mathbb{P}_0 = \frac{\int_{0.5}^{1} \binom{10}{7} \theta^7 (1-\theta)^3 (1.6)\, d\theta}{\int_0^{0.5} \binom{10}{7} \theta^7 (1-\theta)^3 (0.4) + \int_{0.5}^{1} \binom{10}{7} \theta^7 (1-\theta)^3 (1.6)}$$ ← Split to larger uniform

$< 50\%$ $\qquad$ > 0.5

Check if ≥ 0.95 to confirm H_1

Result is 0.9691

→ Expert saves money by not needing extra tests or larger n size to gain significant results