07/12/15 DECISIONS BRETT 1

Game Theory - Making a decision based on decision of others
Proofs which are derived will not be offsetted
Application of it will be (c)ful

BARGAINING: COOPERATIVE GAMES

- Ann and Bob, two players : Each has two cards, (1R, 1B) and each chooses a card
- If cards different colour: no prize
- If both choose red R ⇒ A gets 2 Diamonds, B gets 1 D
- If both choose B ⇒ A gets 1 diamond, B gets 2D.

Both players want to match, but prefer different matches

Suppose: i) If can't reach a bargain, then A plays R, and B plays a B and so get nothing

ii) No side deals - (no bribes)

iii) Can make binding deals

Example of binding deal: both agree to flip a coin. If H both play R, if tails both play B. Seems quite fair.

However, A and B may have different utilities

Eg. $U_A(0D) = 0 = U_B(0D) = 0$

$U_A(1D) = 1 = U_B(1D) = 1$

$U_A(2D) = 2$ $\quad U_B(2D) = 4$ Bob gets more richer in going from 1 to 2 D

This means A is risk neutral and B is risk prone to diamonds, not that B likes diamonds more

Remember $\begin{cases} R \Rightarrow A \text{ gets } 2 \quad B \text{ gets } 1 \\ B \Rightarrow A \text{ gets } 1 \quad B \text{ gets } 2 \end{cases}$

0 otherwise

Bargain is a gamble

Any bargain is a gamble (G) $G = (P_{RR}, P_{BB}, P_{RB}, P_{BR})$

Prob both Red

Prob $A=R, B=B$

P $A=B, B=R$

Ann

Utility $U_A(G) = P_{RR}\, U_A(2) + P_{BB}\, U_A(1) + (P_{BR} + P_{BR})\, U_A(0)$ $= \cancel{2+1+0} = 3$

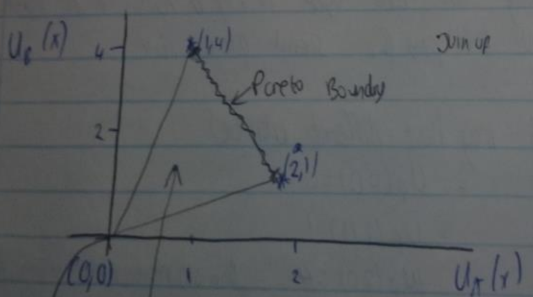
$= 2P_{RR} + 1P_{BB}$

Bob: $U_B(G) = P_{RR} + 4P_{BB}$

$$\begin{pmatrix} U_A(G) \\ U_B(G) \end{pmatrix} = P_{RR}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + P_{BB}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + (P_{BR} + P_{RB})\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any choice for P_{RR} etc provided all are non negative or

$$P_{RR} + P_{BB} + P_{RB} + P_{BR} = 1$$



Join up

Possible joint utilities are 'closed convex hull' of possible outcome pairs (1,4), (2,1), (0,0)

Feasible region for boundary

Status quo point – when there is no agreement

Feasible Region: All possible choices $(U_A(G), U_B(G))$ over all gambles

Pareto Boundary: All points (u,v) in feasible region for which there is no point (μ', v') in feasible region with either $\mu' > \mu$ and $v' \geq v$ or $\mu' \geq \mu$ and $v' > v$ (can't make it better for both players simultaneously).

General Problem: Set of outcomes obtained by 2 people for outcome x, two rewards $r(x)$, $\omega(x)$, with utilities $\mu(x)$ and $v(x)$ for the 2 people. Players may choose a gamble between the outcomes. Possible risk pairs is feasible region S. If no agreement made the status quo decision x^* will be taken with values μ_s^*, v_s^* (status quo point).

Want to find a rule which picks out a fair or 'rational' bargaining point

function $f(\underbrace{S, \mu_s^*, v_s^*}_{\text{status quo}}) \underset{\text{'rational' fair}}{=} \overbrace{\underbrace{(\bar{\mu}_s, \bar{v}_s)}_{\text{rational bargains}}}^{\text{the bargaining point}}$

Nash Bargaining Axioms:

N1: Individual Rationality $(\bar{\mu}_s, \bar{v}_s) \geq (\mu_s^*, v_s^*)$

[N.B $(a,b) \geq (c,d) \Rightarrow a \geq c$ and $b \geq d$]

N2: Feasibility $(\bar{u}_s, \bar{v}_s) \in S$

N3: $(\bar{u}_s, \bar{v}_s)$ is on Pareto Boundary. i.e if $(u,v) \in S$ and $(u,v) \geq (\bar{u}_s, \bar{v}_s)$ then $(u,v) = (\bar{u}_s, \bar{v}_s)$: known as 'Pareto optimality' why choose a point if there is a better point which benefits both parties

4\.

N4: <u>Invariance of Equivalent Representations</u>: If we replace (u,v) by some linear transformation $(\alpha\mu+\beta, \gamma v+\delta)$, i.e. choosing $(\bar{u},\bar{v})$ for old 'S' we should choose $(\alpha\bar{\mu}+\beta, \gamma\bar{v}+\delta)$ for new 'S'. e.g. if $f(S, u_S^*, v_S^*) = (\bar{u}_S, \bar{v}_S)$ then in new representation with feasible region T and status quo u_T^*, v_T^* constructed by reducing each (μ,v) by $(\alpha\mu+\beta, \gamma v+\delta)$ then our requirement is $f(T, \alpha\mu_S^*+\beta, \gamma v_S^*+\delta) = (\alpha\bar{\mu}_S+\beta, \gamma\bar{v}_S+\delta)$ <u>LINEAR ONLY</u>

I.e. Monetary Unit shouldn't matter.

N5: Symetry: Suppose S is ~~Symmetric~~ Symmetric $(\mu,v)\in S$ $(v,\mu)\in S$ and $u_S^*=v_S^*$, then $\bar{u}_S = v_S$ e.g. only the shape S and the values u^*, v^* affect choice, not external circumstances i.e. not status of bargains (shouldn't take into account how much money I already have).

N6: Independence of Irrelevant Alternatives: Suppose $f(T, \mu^*, v^*) = (\bar{\mu}_T, \bar{v}_T)$ (e.g. we jointly choose meat over fish) Suppose $T \overset{\text{subset}}{\subset} S$ (new vegetarian option) Suppose $f(S, \mu^*, v^*) = (\bar{\mu}_S, \bar{v}_S)^{\text{sol. (new)}}$ $\underset{\text{element of}}{\in} T$ then $(\bar{u}_S, \bar{v}_S) = (\bar{v}_T, \bar{v}_T)$ (e.g. either stick with meat or new bargain has some chance of vegetarian (not fish though))

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Nash Bargaining Theorm

There is a unique function defined on all bargaining problems satisfying (N1 - N6). This function gives us $(\bar{u}_S, \bar{v}_S)$ to be the point $(\bar{\mu}_n, \bar{v}_n)$ which uniquely maximises the function $g(\mu,v) = (\mu - \mu_S^*)(v - v_S^*)$ over all $(u,v)\in S$, $(u,v) > (u_S^*, v_S^*)$ provided there are some $(u,v)\in S$ with $u > u_S^*$ and $v > v_S^*$.

$(\bar{u}_n, \bar{v}_n)$ is nash bargaining point. So if both players agree Nash axioms 'fair', then they should agree on nash point (Nash Bargaining)

Alternatively, if can't reach agreement and arbitrator called in who agrees with Nash axioms: this is Nash arbitration

Back to example: Want to max $g(\mu_A, \mu_B) = (u_A - u_A^*)(\mu_B - \mu_B^*)$
$= \mu_A \mu_B$ on S

Equivalently max $\mu_A \mu_B$ on Pareto boundary

Equation of pareto boundary is $\mu_B = a + b\mu_A$

$4 = a + b(1)$ → solve
$1 = a + b(2)$ $\quad b = -3 \quad a = 7$

$\mu_B = 7 - 3\mu_A \quad (1 \le \mu_A \le 2)$

$g(u_A, u_B) = \mu_A \mu_B = \mu_A(7 - 3u_A) = 7\mu_A - 3\mu_A^2$

$\frac{dg}{d\mu_A} = 7 - 6u_A = 0$

$\mu_A = 7/6$ could be a max OR min ⇒ not min, clearly see on graph. MAX

and $1 < \frac{7}{6} < 2$ e.g. is a max on pareto boundary (o/w) max would be at one of end points

Nash Point $\bar{u}_{AN} = 7/6 \quad \bar{u}_{BN} = 7/2$

Nash point is gamble between both R (probability p) and B (probability 1-p)

$\mu_A \quad 2p + (1-p)1 = p+1$

2 → utility to A of R and R

1 → Utility to A for B and B

$\Rightarrow p+1 = 7/6$

$\Rightarrow p = 1/6$ play red 1/6 (no) 5/6 both play black.

So Nash Bargain: Both red (probability 1/6) could use fair die to choose
Both black (probability 5/6)

Comment: Bob seems to benefit more than Anne from Nash Bargain
Nash axioms appear to favour the risk prone

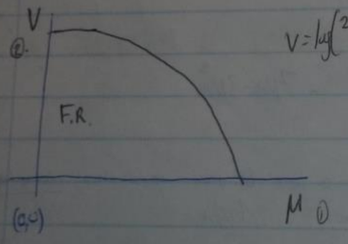
EXAMPLE

2 players offered £100 if they can decide how to divide the money (o/w get 0)

Player 1: risk neutral $\Rightarrow u = (£x) = x$ actuarial fair price

Player 2: risk averse $\Rightarrow v = (£x) = \log\left(\frac{200+x}{100}\right)$ like people who pay insurance

Status quo: (0,0) want nash bargain



$v = \log\left(2 + \frac{\mu}{100}\right)$ equation of the equation

where each point is the division of £μ to £100 − £μ

No probabilities involved here

i.e. no gamble between rewards on Pareto Boundary.

Nash point: max $g(u,v) = u\log\left(\frac{200-\mu}{100}\right)$ $0 < \mu < 100$

$dg/d\mu = 0$ $\frac{u}{200-\mu} = \log\left(\frac{200-\mu}{100}\right)$ $\Rightarrow \mu = 54.4$ $v = 45.6$ Nash point.

Note, risk neutral person gets more than risk averse. i.e. Nash bargaining tends to penalise the risk averse.

Could be typo in +/− of the log function

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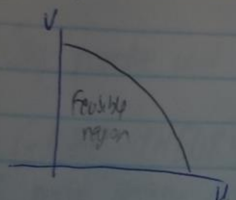
Q From Yesterday

Q2 Players offered £100 if they can decide how to divide the money (o/w nothing)

P1 → risk neutral $u(£x) = x$

P2 → risk averse $u(£x) = \log\left(\frac{100+x}{100}\right)$ ← correction from yesterday

Status quo (0,0) Find nash bargain.



line = $v = \log\left(2 - \frac{u}{100}\right)$ because if P1 take £u has u utility leaving P2 with £100 − u and $\log\left(\frac{100+100-u}{100}\right)$

$= \log\left(2 - \frac{u}{100}\right)$ utility

(plugging x into log function)

Nash pt: $g(u,v) = (u - u^*)(v - v^*)$

$= (u - 0)\left[\log(2 - u/100) - 0\right]$

$= u \log\left[2 - u/100\right]$

Use chain rule: $dy/du = \log\left[200 - u/100\right] \frac{-u}{200-u}$

Set to $0 \Rightarrow u/200-u = \log\left(\frac{200-u}{100}\right)$

$u = £54.4$

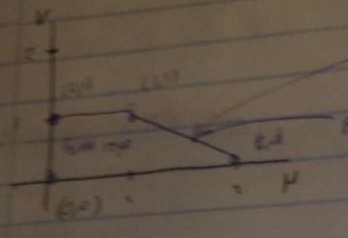
$v = £45.6$

Ann $[0 \to 1 \to 2]$ Scale $0 \to 1$ $[0 \;\; 0.5 \;\; 1]$

Bob $[0 \to 1 \to 4]$ $[0 \;\; 0.25 \;\; 1]$

EXAMPLE

Suppose pure payoffs are (0,0) (0,1) (1,1) (2,0), with (0,0) the status quo



any move on this line will reduce utility for someone

pareto boundary

Line

$v = a\mu + b \qquad 1 = a+b$ and $0 = 2a+b$

$\Rightarrow a = -1 \qquad b = 2$

$v = -1\mu + 2$ (eqn on pareto boundary line)

over $[1 \le \mu \le 2]$

Nash point $g(\mu, v) = (u - \mu^*)(v - v^*)$ μ^* and v^* status quo $(0,0)$

$= \mu v$

$= \mu(2-\mu) = 2\mu - \mu^2$

$dg/d\mu = 2 - 2\mu = 0 \quad \mu = 1$

when $\mu = 1$ $v = 1$ Nash point $(1,1)$ (don't need a mixing strategy)

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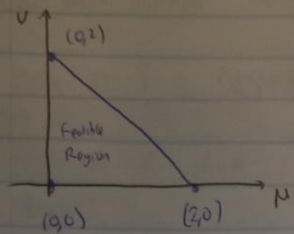
EXAMPLE

Outcomes (0,0) (2,0) (0,2)



Nash point is (1,1) because of symmetry and same status quo for both players
Chose location where each person gets the same. Axiom 5

Only procedure which satisfies the ~~ex~~ all 6 axioms

GAME THEORY

2 person strictly competitive game of the following form:

You make a decision, opponent makes a decision (no co-operation)

Zero-sum games: Your winnings = Opponent Loss

Your aim is to maximise your gain

Example 1

	C1	C2	C3	C4
R1	0	1	7	7
R2	4	1	2	10
R3	3	1	0	25
R4	0	0	7	10

R picks 1-4 and C picks 1-4, R's score as given, C's score as ~~[illegible]~~ minus Given

eg both pick 2, R gets 1, C gets -1 (ie zero sum-game)

- Note a choice A dominates a choice B if all outcomes for A at least as good as for B and some are better (if identical, doesn't count as dominated)

- Don't play dominated choice

a C_4 dominated by C_2 for example : delete C_4

b R_3 dominated by R_2 : delete R_3

c R_4 dominated by R_1 : delete R_4

d C_3 dominated by C_2 : delete C_3

E. R_1 dominated by R_2 : delete R_1

F Finally C_1 dominates C_2 delete C_1

Ordering doesn't matter. if done d before c could do it one step faster.
Useful to look vice-versa, play from each player's point of view.

∴ R and C both play 2. R gets 1 and C gets -1 & this is the value of the game

Note: Neither R nor C can raise the value by playing if opponent plays this choice

EXAMPLE

Ray and Dottie choose campsite. R wants camp high, D wants camp level.
R chooses N.S., D choose E.V. Camp in each section

D → / R ↓	1	2	3	4
1	7	2	5	1
2	2	2	3	4
3	5	3	4	4
4	3	2	1	6

A. R3 dominates R2, delete R2
B. D2 dominates D1, delete D1
No more dominated strategies

Value is 3 maximum minim R can achieve

Note: If R chooses 1 or 4, D might outguess him and chose 3, giving height 2.
If picks 3, guaranteed atleast 3.
D might see this other way around, pick 2 guaranteeing at most 3

Pair (R3, D3) are in equilibrium: If either chooses from the pair the other can do no better than also choosing from the pair.

Game has value 3 (which either party can force)
A point which is the minimum of a row and col is called a saddle point or equilibrium point

So, once dominated choices removed, check for saddle point
If there is such a point, then that is the value of a game

Note: Game may have several saddle points but they all have same value

Example 3

		Clever Student Even	Odd
You	Even	-1	+1
	Odd	+1	-1

Rolling a dice

Protect against being out guessed each time by tossing coin (even if H, odd if T)
(win ½ time - expected value of the game is zero (and you can reveal strategy to your opponent)

Can't force higher expected value as opponent can do the same
e.g. $(\frac{1}{2}, \frac{1}{2})$ is equilibrium strategy and value of game is 0.
odd even

EXAMPLE 4.

Monarchy chilling hans

		Monarchy Contraband	Dover	
Hand	Cont	0	50	← not caught
	Dover	100 escape	0	

no dominating strategy
no saddle point.

Suppose H chose Dover with probability P_H, M chose Dover with probability P_m

Expected payoff to H is $0 \cdot P(H_C \cap M_C) + 0 \cdot P(H_D \cap M_D)$
$+ 100 P(H_D M_C) + 50 P(H_C \cap M_D)$
$= 50(1-P_H)P_m + 100(P_H)(1-P_m)$
$= 50(P_m + 2P_H - 3P_m P_H) = E$ Holmes' wants to move as large as possible

H chose P_H to maximise this, M chose P_m to minimise this

Note: If $P_H = \frac{1}{3}$, $E = 50(P_m + \frac{2}{3} - P_m) = \frac{100}{3}$ whatever P_m is
$\therefore$ H can force a value $\frac{100}{3}$

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Similarly if $p_m = 2/3$ $E = 50(2/3 + 2p_H - 2p_m) = 100/3$ whatever p_H is

So Either Player can force an outcome of 100/3

Say $(1/3, 2/3) = (p_H, p_m)$ are equilibrium strategy. If either deviated, then other can achieve a better outcome than 100/3

IN GENERAL

player chon i and j

Matrix $A = (a_{ij})$ $i = 1, \dots, m$ $j = 1, \dots m$ is payoff matrix for game.
R picks one of rows $1, \dots, m$ and C picks one of columns $1, \dots, n$.

If R picks i and C picks j, payoff a_{ij} to R and $-a_{ij}$ to C, i.e zero sum

Mixed strategy for R is choice $p = (p_1, \dots, p_m)$ (p_i is probability R chooses i)

(Each $p_i \geq 0$, $\sum_{i=1}^{m} p_i = 1$) $q = (q_1, \dots, q_m)$ each $q_i \geq 0$ $\sum_{i=1} q_i = 1$

Payoff a_{ij} chosen with probability $p_i q_j$

Expected payoff to R is $\sum_{i,j} p_i q_j a_{ij}$ (C gets minus this)
$= p^T A q$ R chooses p to make big, C chooses q to make small

(minimax, R needs to maximise this num

Best bet R can guarantee if C can 'outguess' R is $V_R = \max_p(\min_q(p^T A q))$

Best that C can force if R outguesses C is minimax $V_C = (\min_q(\max_p p^T A q))$

THEORM: MINIMAX

For 2 person - zero - sum - game

best R can get $V_R = V_C = V$ (Value of the game) $= p^{*T} A q^*$ for some choice

p^*, q^*, the equilibrium strategy.

Where $p^{*T} A q^*$ $\max_p p^T A q^* = \min_q p^T A q$

Either player can force a result by playing your equilibrium strategy.

Either player can force value V and if one player plays their equilibrium strategy, other player can do no better than playing their equilibrium strategy (but could do worse)

"limits damage by choosing equilibrium" gets rid of opponent probabilities

Solving Game

Eg Find value and minimax strategies of zero-sum games.

1. Delete Dominated strategies
2. Check for saddle points - these are solutions if they exist.
3. In general use simplex algorithm - not covered

Example Special 2x2 Case
No saddle point.

		C 1	2	
R	1	a_{11}	a_{12}	p_1
	2	a_{21}	a_{22}	p_2
		q_1	q_2	

Value of game if p, q are minimax

$V = a_{11}p_1q_1 + a_{12}p_1q_2 + a_{21}p_2q_1 + a_{22}p_2q_2$

$= p_1(a_{11}q_1 + a_{12}q_2) + p_2(a_{21}q_1 + a_{22}q_2)$ NB $[p_2 = 1 - p_1]$

$\therefore$ For minimax, must have $a_{11}q_1 + a_{12}q_2 = a_{21}q_1 + a_{22}q_2 = V$

as $q_1 + q_2 = 1$, this gives values q_1, q_2, V and solve for p_1 p_2 similarly

for finding the $1/3$ and $2/3$ in Sherlock Holmes problem

14/12/15 DECISIONS : GRAPHICAL SOLUTIONS

GRAPHICAL SOLUTIONS FOR 2 x N GAMES

Method 1:

$A = \begin{pmatrix} a_1, & a_2 & \dots & a_n \\ b_1, & b_2 & \dots & b_n \end{pmatrix}$ R wants to maximise f $\min_q$ $p^T A q$

Suppose R chooses $\begin{cases} R_1, & 1-p \\ R_2, & p \end{cases}$

R wants to max $f(p) = \overset{\text{min over } q}{\min_q} p^T A q = \min_q (1-p, p) \begin{pmatrix} a_1 \dots a_n \\ b_1 \dots a_n \end{pmatrix} \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$

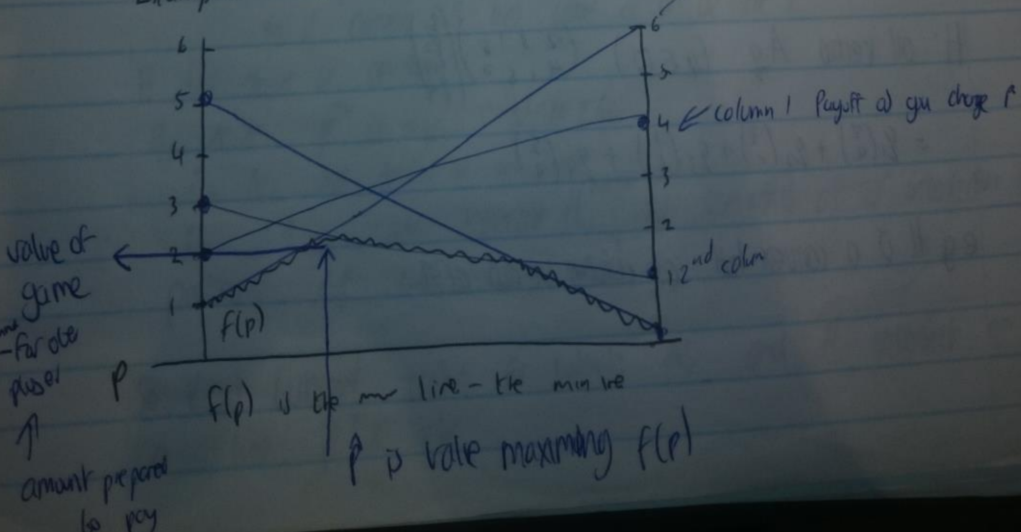
$q_i \geq 0 \quad \sum_{i=1}^{n} q_i = 1$

$= \overset{\text{min over } q}{\min_q} \left[a_1(1-p) + pb_1, \dots\dots, a_n(1-p) + pb_n \right] \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$

$= \min_q \left[q_1(a_1(1-p) + pb_1) + \dots + q_n(a_n + p(b_n - a_n)) \right]$ - one of these will be smallest
- want max probability on this value

$= \min \{ a_1 + p(b_1 - a_1), a_2 + p(b_2 - a_2), \dots, a_n + p(b_n - a_n) \}$ ⇐ linear expressions

Example: $A = \begin{pmatrix} 2 & 3 & 1 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix}$



Col 1: $2+(4-2)p$
Col 2: $3-2p$
Col 3: $1+5p$
Col 4: $5-5p$

Solve the intersection of Col 2 and Col 3 e.g. when $3-2p = 1+5p$ $\hat{p} = 2/7$
$\therefore V = 3-2\hat{p} = 1+5\hat{p} = 17/7$ = value of game
i.e. Probability R plays 2 is 2/7, C plays col 2 and 3, find by solving the 2x2 subgame

$$\begin{array}{c|cc} & C_2 & C_3 \\ \hline R_1 & 3 & 1 \\ R_2 & 1 & 6 \end{array}$$

Knowing $V = 17/7$, C plays $(0, 5/7, 2/7, 0)$

N.B: in a 2xn game always be 2 columns that C can pick between

Method 2:

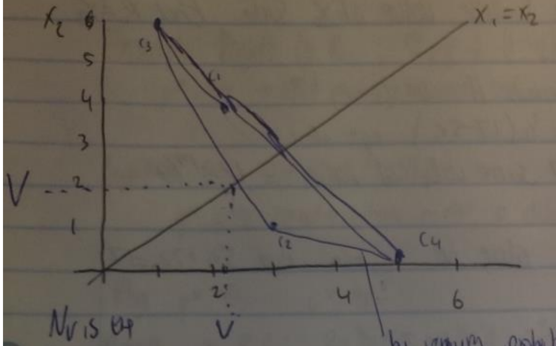
$\hat{p}, \hat{q}$ are minimax R, C : value of game V.
$\hat{p}^T A q \geq V \geq p^T A \hat{q}$ (*) p always does worse if doesn't play $\hat{p}$

p, q are probability vectors, e.g. $q \in Q = (q_1, \dots, q_n)$, $q_i \geq 0$ $\sum_{i=1}^{n} q_i = 1$
$p \in P = (p_1, p_2)$ (2xn games) $\therefore p_1 + p_2 = 1$ $p_i \geq 0$

H: all vectors Aq $(q \in Q)$ $\begin{pmatrix} 2 & 3 & 1 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$

$= q_1\begin{pmatrix} 2 \\ 4 \end{pmatrix} + q_2\begin{pmatrix} 3 \\ 1 \end{pmatrix} + q_3\begin{pmatrix} 1 \\ 6 \end{pmatrix} + q_4\begin{pmatrix} 5 \\ 0 \end{pmatrix}$

e.g H is a convex hull of column vectors of A.



corresponding to q having probability of $q=1$ ⇒ extreme value

by varying probability value can get anything along the line

N_V is the region

C_1 is not on the line ⇒ it is inside the convex hull

$N_V = \{(k_1, k_2) : k_1 \leq V, k_2 \leq V\}$ (k_1 k_2 less than value of game)

$\underline{k} \in N_V \iff \underline{p}^T \underline{k} \leq V$ for all $\underline{p} \in \underline{P}$ (**)

vector k in the region iif

By (*) $\underline{p}^T A \hat{\underline{q}} \leq V$ for all $\underline{p} \in \underline{P}$

By (**) $A\hat{\underline{q}} \in N_V$ ∴ $A\hat{\underline{q}} \in N_V$ $n+1$

all Aq have to be in N_V zone for all Aq element of N_V $n+1$

Consider line $\hat{\underline{p}}^T \underline{x} = V$. By * $\hat{\underline{p}}^T \underline{h} \geq V$ $\forall h \in H$

eg (convex hull lies above or on the line)

H lies above or on line

By ** $\hat{\underline{p}}^T \underline{k} \leq V$ $\forall k \in N_V$

∴ N_V lies below or on line

∴ $\hat{\underline{p}}^T \underline{x} = V$ seperates H (convex hull) and N_V and all of intersection (in particular $A\hat{q}$ lies on the line)

So V is (unique) value for which N_V and H intersect on a line $\hat{\underline{p}}^T \underline{x} = V$

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So we solve by finding smallest value of $\underline{x}$ such that $x_1 = x_2$ intersects H.

e.g. We see $x_1 = x_2$ intersects H on $C_2 - C_3$ line

$\therefore$ $C_2 - C_3$ $=$ $x_2 = \frac{1}{2}(17 - 5x_1)$ eq^n of line

Solve with $x_1 = x_2$ $= 17/7$ same value as before! $= V = 17/7$

Line $\underline{p}^T\underline{x} = V$ with H is above line $C_2 > C_3$ e.g. $5x_1 + 2x_2 = 17$

e.g. $5/7 x_1 + 2/7 x_2 = 17/7$

probability p choose 1 — prob p choose 2

C play C_2 and C_3 and solve: $q\binom{3}{1} + (1-q)\binom{1}{6} = \binom{17/7}{17/7} \Rightarrow q = 5/7$

Non-Zero Sum Games (2 players)

Suppose each player chooses as before, but pay off to R is not necessarily minus the payoff to C.

Example:

		C: 1	2	
R	1	(2,1)	(0,0)	(p)
	2	(0,0)	(1,2)	(1-p)
		(q)	(1-q)	

(R, C)
$(\cdot, \cdot)$ Payoff in utility to R and C respectively

- mixed strategies allowed : R plays 1 with probability p and 2 with prob (1-p).
- c plays 1 with prob q and 2 with prob (1-q)

They determine p and q independently

- Equilibrium: Given one player plays his or her equilibrium strategy, the other can't do better than playing his also! Nash - Such equilibrium strategies exist (uniquely).

Example: Payoff to R : $2pq + (1-q)(1-p)$

$p^* = 2/3$ ⇒ value makes q cancel out

Payoff to C: $1(p)(q) + 2(1-p)(1-q)$

$q^* = 1/3$ ⇒ value which makes p disappear

Play $p^* = 2/3$ $q^* = 1/3$

Payoff to R is: $2 \times 2/3 \times 1/3 + 1/3 \times 2/3 = 2/3$ equilibrium strategy

Payoff to C = $2/3$

cannot be affected by what opponent does but mightn't be the best

But $p = q = 1/2$ is better for both!

Example

		Hugo dont confess	confess
	dont confess	(1,1)	(4,0)
Victor	confess	(0,4)	(3,3)

V and H have committed a crime together. They should confess or not (interviewed independently). The entries in table are years in prison for V, H respectively loss = −utility

V: confess? H: confess?

if H confesses → confess (3 yr) better than not (4 yr)

If H doesn't confess → confess (0 yr) better than not (1 yr)

∴ confessing is always better hence a stationary dominating strategy.

Here it is better for both to play worse individual strategy (both don't confess)

15/12/15. DECISIONS : GROUP DECISION MAKING

Example:
Population want to choose party. Each person gives preference
e.g. SF > FF > FG
How do we combine all votes to get group preference?

Possible preference: Majority rule (e.g. group prefers SF to FF if more than 50% vote as so prefer).

Example:
Votes 1 → SF > FF > FG   > means prefers, not greater than
2 → FF > FG > SF
3 → FG > SF > FF

Majority rule: ∃ For group SF > FF (2:1), FF > FG (2:1)
and FG > SF (2:1)

So Group is a money pump (i.e. willing to pay for a different choice ⇒ pay 3 times and back at start!)

Basic problem of 'social choice' theory Each individual in group has personal preference ranking over a collection of alternatives

How can we combine these into a group preference ranking 'fairly'?

E.g. Collection of rewards: $r_1, \dots, r_n$ and group has m members and person i has preference ranking $\succsim_i$ — prefer or indifferent to.
collection $(\succsim_1, \dots, \succsim_m)$ is preference profile of group

Each persons preference ranking satisfies:

D1 For each pair (r_i, r_j) for each profile $\succsim_k$ we must have one and only one of the following: $r_i \succ_k r_j$, $r_j \succ_k r_i$, $r_i \sim_k r_j$

D2 $r_i \succsim_k r_j$ & $r_j \succsim_k r_s \Rightarrow r_i \succsim_k r_s$ etc (transitivity)
$r_i \sim_k r_j$ and $r_j \sim_k r_s \Rightarrow r_i \sim_k r_s$ etc (must be just as good as choice)

Def^n: A Social Welfare function (SWF) is a function which operates on a collection of individual preference profiles each of which obey (D1) and (D2) and yields a group or social ordering $\succsim_g$ which also obeys (D1) and (D2)

Note: For small number of individuals this is a group decision problem, for many individuals, it is a social choice problem

Arrow Suggests 4 'reasonable' conditions on SWF.

1. Unrestricted domain (U): $\succsim_g$ is defined and obeys D1 and D2 for any collection $(\succsim_1, \dots, \succsim_m)$ which all obey D1 and D2

2. No dictatorship (D): No individual i such that $\succsim_i$ automatically becomes $\succsim_g$

3. Pareto Principle (P): $r_i \succsim_k r_j$ for all k, $r_i \succsim_g r_j$ group decision - if everyone likes cheese, group like likes cheese

4. Independence of Irrelevant Alternatives (I). Suppose some rewards are deleted from the reward set. Then if no individual changes their preference between rewards that remain, the group preferences between remaining alternatives doesn't change.

15/12/15 DECISIONS : GROUP DECISION MAKING

Comment on I: $\frac{1}{2}$ people have FG > FF > SF, $\frac{1}{2}$ have FF > SF > FG
SF drops out now have $\frac{1}{2}$ FG > FF and $\frac{1}{2}$ FF > FG

this ignores the fact that FF in the top 2 preferences always

Compare 2nd constituency: $\frac{1}{2}$ have FG > new FG > FF $\frac{1}{2}$ have FF > FG > new FG
thus: opposite conclusion. manipulated to get FG in top 2 always "consistently vote"
But, possible preference between FG and FF are the same in each constituency. e.g. Arrow argued it is wrong to view FF preference to FG as stronger than FG preference to FF

Can be manipulated by voting strategies - government.

Arrows Impossibility Th^m
Provided there are at least 3 possible rewards, and at least 2 individuals, then there is no SWF which meets all of conditions U, D, P, I

Note: means that for any social welfare function there are sets of preferences which break at least one condition

However, if only 2 rewards, the majority rule does satisfy U, D, P, I

UTILITARIANISM:
Social choices should attempt to maximise well-being, happiness or pleasurable feelings of citizenry

Given a choice between social options, place at top option which produces most pleasure overall and bottom, least pleasure overall

15/12/15

How measure for an individual?
How do you combine individual pleasure? eg Sum or product?

A possible way of measuring individual pleasure is by utility. How can we combine group utilities?

Suppose m citizens r social choices $(x_1, \dots, x_r) = \underline{X}$. Each citizen i individually rational and so has a utility on $\underline{X}$ (μ_i)
Scale each utility so as to lie in $[0,1]$ (0: utility of worst outcome, 1: utility of best) had to standardise not necessarily $[0,1]$

Example:
Council have enough money for exactly 1 of: Swimming Pool (S), library (L), car park (C), museum (M) or nothing (N)
Town has 2 citizens

	S	L	C	M	N
U_1	1	0.5	0	0.5	0
U_2	0.1	0	1	0	0

Introduce Planner: knows citizen utilities and must plan on their behalf.
Planner (P) individually rational, so has utility W on $\underline{X}$
P wants to make a fair choice for community

Suppose P obeys following conditions for 'Social rationality'
A. Anonymity: P doesn't know which citizen made which vote. P would create the same W utility for any permutation of vote $u_1, u_2, \dots$

15/12/15 DECISIONS: UTILITIARNISM

(SP) Strong Pareto: If each citizen indifferent between 2 rewards then so is planner. If nobody prefers a to b, some people prefer b to a then P prefers b to a.

In example, P gets 2 votes (S, L, C, M, N): $(1, 1/2, 0, 1/2, 0)$ $(0.1, 0, 1, 0, 0)$ P doesn't know who made which vote

Note: $u_1(L) = u_1(M) = L \sim M$ $\Rightarrow L \underset{P}{\sim} N$ by (SP)
$u_2(L) = u_2(M) = L \sim M$ planner also indifferent.

Now construct for each option, a vector of utilities $\mu(x_i) = (\mu_1 x_i, u_2 x_i, \dots)$
e.g. $\underline{U}_S = \binom{1}{0.1}$ $U_L = \binom{1/2}{0}$ $U_C = \binom{0}{1}$ $U_M \binom{1/2}{0}$ $U_N \binom{0}{0}$

Note that if $\underline{u}(x_J) = \underline{u}(x_K)$ (same utilities) then $u_i(x_J) = \mu_i(x_K)$ for all i.
So $x_j \underset{i}{\sim} x_K$ for all i, so $x_J \underset{P}{\sim} x_K$ (SP)
So $w(x_J) \sim w(x_K)$ must give same utility
so $w(x_j)$ must be a function of $\underline{\mu}(x_J)$
e.g. $w(x_J) = f(\underline{u}(x_J))$ for some function f.

Th^m Harsanyi
Group of individuals, collection of rewards. Individual i specifies utilities $u_i(x)$ for each x (scaled to $[0,1]$).
Planner wants to construct utility function w, obeying (A) and (SP)
then the only choice ~~...~~ for w is
$W(x) = \mu_1(x) + \dots \mu_n(x)$ for all x

	S	L	C	M	N
e.g. u_1	1	0.5	0	0.3	0
u_2	0.1	0	1	0	0
$W=\mu_1+\mu_2$	1.1	0.5	1	0.5	0

First S then C then L or M