DECISIONS
08/12/17

DECISION THEORY EXERCISES 6 — W

1. Show that the Nash bargaining point does satisfy the 6 axioms (N1)-(N6) of the Nash bargaining theorem.

2. Brad and Janet are planning a night out. Their choices are to go to the cinema, the theatre or a football match. If they can't decide where to go together then they will both stay at home. Their utilities for the various alternatives are

	Home	Cinema	Theatre	Sports
Brad	1	4	2	6
Janet	1	3	5	0

   (a) Treating this as a bargaining problem, sketch the feasible region. Identify the Pareto boundary and suggest a natural status quo point.

   (b) Find the Nash arbitration point. (i) directly + (ii) by geometric argument.

   (c) Find the equitable distribution point.

   What should Brad and Janet do?

3. An eccentric economist offers John and Mary one pound, provided that they can come to a mutual agreement as to how to divide it (otherwise, they get nothing). Suppose that John and Mary have utility functions for money of the form

$$U_J(£x) = x^\alpha,\ U_M(£x) = x^\beta,$$

   where $0 < \alpha < \beta < 1$.

   (a) Compare and discuss John and Mary's attitudes to risk.

   (b) Show that the Pareto boundary is the curve $(x^\alpha, (1-x)^\beta)$
   [so, what you have to show is that there is no gamble over possible divisions of the pound that has a higher utility for John and Mary than all possible simple divisions of the pound].

   (c) Find the Nash arbitration point. Comment on the effect of attitudes to risk upon the solution.

   (d) Now suppose instead that $1 < \alpha < \beta$. Discuss risk attitudes and find the Nash point. [Warning: if you just use the same formula that you derived in (b) above, you will get the wrong answer (why?).]

08/12/15

DECISION THEORY
Exercises 7

1. (a) You are approached by a gambler who suggests that you play the following game. You each toss a coin. If you get heads and he gets tails, you win £30. If you get tails and he gets heads, you win £10. If the coins match, you lose £20. You find that you have no coins. "Never mind," says the gambler, "we can still play the game. Each time, I'll write down heads or tails and you do the same. We score as before." Should you play? (i.e. find the value of the game.)

   (b) Suppose that you decline to play the game. The gambler offers you the following alternative. "You write down the outcome of one toss, I'll write down the outcome of two tosses. Payoffs will be as follows ":

HIM / YOU	HH	HT	TH	TT
H	1	0	-1	-2
T	-2	-1	0	2

   (so, for example, if you play H and he plays TH then you pay a pound)
   Analyse this game using each of the two graphical methods (i.e. find the value, strategies for both players, whether you should play, by each method).

2. This example explores what happens in the repeated prisoners' dilemma when the number of games is random.

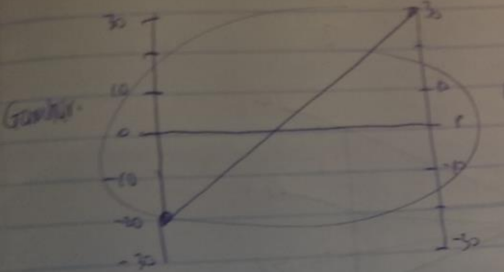
   Burglar Bill and Burglar Betty are arrested. The police have enough evidence to lock them up for a Small Job. They know, but cannot prove, that they also committed a Big Job. Bill and Betty are separated and each is offered the following deal. If neither confesses, then each will serve u years for the small crime. If one confesses, and the other does not, then that burglar will serve v years, while the other will serve w years. If both confess, they will each do x years. The values are such that $v < u < x < w$.

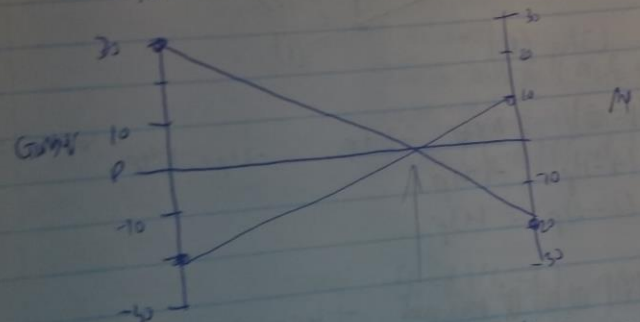
   Suppose further that the police have evidence to convict Bill and Betty of several such jobs. For each job, there is a small version for which they can convict both burglars, and a big version which they can only convict on if one burglar confesses. Each crime is handled in sequence with 'scoring' as above. Bill and Betty are not allowed to communicate, but they are told, after each crime is treated, whether or not their partner confessed.

   Suppose also that Bill and Betty do not know how many crimes they will be accused of. Suppose in particular that after each crime has been settled the police toss a coin which has probability p of landing heads. If the coin lands heads, then Bill and Betty are accused of the next crime, while if the coin falls tails the process stops. [Not particularly realistic - but it gives us a stopping rule which is easy to analyse.]

   Analyse the game. In particular, show that for any $v < u < x < w$, there are certain values of p for which never confessing is an equilibrium strategy for Bill and Betty (i.e. devise a strategy such that if either burglar follows the strategy then both burglars can do no better in expected payoff than never confessing).

		Gambler H	T		
Me	H	−20	+30	p	5/8
	T	10	−20	$(1-p)$	3/8
		q	$(1-q)$		





col 1: $\overset{G,M}{(-20,0)}\ (1,10)$ ~~y=mx~~ $G = \text{slope}\, M + C$ $\quad -20 = c$

$1 = 10M - 20 \quad M = 1.1$

$G = 1.1M - 20. \quad G = 1.1M - 20$

Col 1: $-20 + (10 - -20)p$

$= -20 + 30p$

Col 2: $30 + (-20 - 30)$

$= \cancel{30}\ 30 - 50p$

50p

$-20 + 30p = 30 - 50p \quad 80p = 50$

$p = 5/8$ = ~~value~~ degrees

Probability I play heads

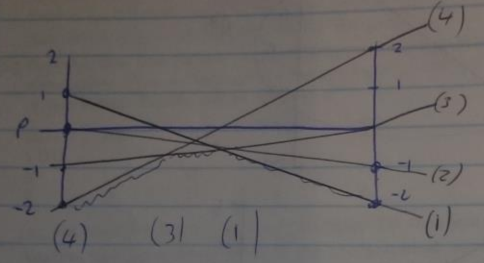
3/8 probability I play tails

Value: $-20 + 30(5/8)$ $-20\frac{150}{8}$ $\frac{75}{4}$ $-20\ \frac{75}{2}$

$= -1.25$ = value of ...

q plays H with probability 0.5
and T with 0.5 probability

	You	HH	HT	TH	TT
B	T	1	0	-1	-2
ME	H	-2	-1	0	2



(4) $= -2 + (1 - -2)p = \cancel{1+3p}$ $-2 + p$ $-2+p = -1-4p$ $5p = 1$ $p = 1/5$

(3) $= -1 + (-1-3)p = -1-4p$

(1) $= 1 + (1 - -2)p = 1+3p$

Use (3) and (1) as has the max point $-1-4p = 1+3p$ $7p = -2$ $p = -2/7$

(3) (1) (2)
(-1,0) (1,-2) (0,-1)

3: $-1 + (-1-0)p$ 3: $= -1-p$ $-1-p = 1+3p$ $p = 1/2$

1: $1 + (1 - -2)p$ $= 1+3p$ $-1-p = -p$

2: $0 + (0-1)p$ $= -p$ $1+3p = -p$ $p = -1/4$

2/15 DECISIONS: NASH BARGAINING POINT AXIOM PROOFS

N4 Invariance Suppose we have transformed space $(S';d')$ and normal space (S,d)

$S' = \{(\alpha_1 \mu_1 + \beta_1, \alpha_2 v_1 + \beta_2) \in \mathbb{R}^2 : \mu_1, v_1 \in S\}$

and $d'_i = \alpha_i \underset{u \text{ or } v}{s_i} + \beta_i \quad i = 1,2$

$s' \in S'$ if and only if there exists $s \in S$ such that
$s_i' = \alpha_i s_i + \beta_i$ for $i = 1, 2$, $\quad s_i = u$ or v

Therefore if $(s_1, s_2) \in S'$ we have:

$$(s_1' - d_1')(s_2' - d_2') = (\alpha_1 s_1 + \beta_1 - \alpha_1 d_1 - \beta_1)(\alpha_2 s_2 + \beta_2 - \alpha_2 d_2 - \beta_2)$$
$$= (\alpha_1 s_1 - \alpha_1 d_1)(\alpha_2 s_2 - \alpha_2 d_2)$$
$$= \alpha_1 \alpha_2 (s_1 - d_1)(s_2 - d_2)$$

For some $(s_1, s_2) \in S$

Now (s_1^*, s_2^*) maximises $(s_1 - d_1)(s_2 - d_2)$ over S, if and only if
$(s_1^* - d_1)(s_2^* - d_2) \geq (s_1 - d_1)(s_2 - d_2) \quad \forall (s_1, s_2) \in S$
if and only if:
$\alpha_1 \alpha_2 (s_1^* - d_1)(s_2^* - d_2) \geq \alpha_1 \alpha_2 (s_1 - d_1)(s_2 - d_2) \quad \forall (s_1, s_2) \in S$
if and only if:
$(s_1'^* - d_1')(s_2'^* - d_2') \geq (s_1' - d_1')(s_2' - d_2') \quad \forall (s_1, s_2) \in S$

where $s_i'^* = \alpha_i s_i^* + \beta_i$

Namely ~~Earth~~ $(\alpha_1 s_1^* + \beta_1, \alpha_2 s_2^* + \beta_2)$ Maximises $(s_1' - d_1')(s_2' - d_2')$ over S'

AXIOM 5: SYMETRY

Let $H(s_1, s_2) = (s_1 - d_1)(s_2 - d_2)$.

Let $(S;d)$ be a symetric bargaining problem.

Assume that $(s_1^*, s_2^*) \in S$ maximises H over S namely:

$(s_1^* - d_1)(s_2^* - d_2) \geq H(s_1, s_2) \quad \forall (s_1, s_2) \in S$ ie. better than (or =) solution

Since $(S;d)$ is symetric, $d_1 = d_2$. Therefore,

$(s_2^* - d_1)(s_1^* - d_2) \geq H(s_1, s_2) \quad \forall (s_1, s_2) \in S \quad (1)$

Since S is symetric $(s_2^*, s_1^*) \in S$. Thus (1) means that (s_2^*, s_1^*) also maximises H over S.

But since the maximiser is unique, it must be that $(s_1^*, s_2^*) = (s_2^*, s_1^*)$ which implies $s_1^* = s_2^*$

AXIOM 6: Independance of Irrelevant Alternatives

Assume $S \subset T$ and that $(s_1^*, s_2^*) \in S$ maximises H over T

Namely: $(s_1^* - d_1)(s_2^* - d_2) \geq (s_1 - d_1)(s_2 - d_2) \quad \forall (s_1, s_2) \in T$

In particular: $(s_1^* - d_1)(s_2^* - d_2) \geq H(s_1, s_2) \quad \forall (s_1, s_2) \in S$

Since $s^* \in S$, the result follow

Axiom 3: Pareto Boundary

Since $H(s_1,s_2)$ is increasing both in s_1 and s_2 - in the sense that if $s_1' > s_1$ and $s_2' > s_2$ then $H(s') > H(s) \Rightarrow (s_1,s_2)$ cannot maximise H if there exists $(t_1,t_2) \in S$ with $t_1 > s_1$ and $t_2 > t_2$.

We show that f^n is the only bargaining solution that satisfies all 6 axioms.

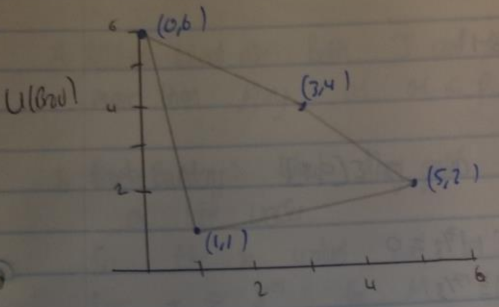
Axiom 1: Individual Rationality

This is common sense. Why would a player choose a solution (s_1,s_2) if there exists a solution (s_1^*,s_2^*) such that $s_1^* \geq s_1$ and $s_2^* \geq s_2$? The player should choose the higher value - after all the goal is to maximise one's utility

Axiom 2: Feasibility

The solution (s_1,s_2) must be an element of the solution space. i.e. it is impossible to win 10 diamonds if there is only for example 2 diamonds at stake. This solution doesn't exist with the solution space and hence, is infeasible

Q2 Brad and Janet.



(1,1) home
(3,4) Cinema
(5,2) Theatre
(0,6) Sports

Pareto boundary is made up of line segment connecting (0,6), (3,4) and (5,2)

$\overset{B\ J}{(0,6)}$ (3,4) $B = mJ + c$ $6 = 0m + c$

$-\ 4 = 3m + c$

$-2 = 3m$ $m = -2/3 \rightarrow c = 6$

(0,6) (3,4) $b = -2/3 J + 6$

(3,4) to (5,2) $b = mJ + c$ $4 = 3m + c$

$-\ 2 = 5m + c$ $\rightarrow 2 = -2m$ $m = -1$

$\hookrightarrow c = 7$

(0,6) to (3,4) $b = -2/3 J + 6$ $0 \le J \le 3$

(3,4) to (5,2) $b = -J + 7$ $3 \le J \le 5$

Status quo point = (1,1)

Maximise $(J-1)(b-1)$ over solution space

$0 \leq J \leq 3 \quad b = -\frac{2}{3}J + 8$

$(J-1)\ (-\frac{3}{2}J + 6 - 1)$

$(J-1)(-\frac{3}{2}J + 5)$

$-\frac{3}{2}J^2 + 5J + \frac{3}{2}J - 5$

$-\frac{2}{3}J^2 + \frac{17}{3}J - 5$ over $J \in [0,3]$

Diff and $= 0$

$\cancel{-3/J + 17/3 = 0}$ $\quad -\frac{4}{3}J + \frac{17}{3} = 0$

$\cancel{3J = 17/3}$ $\quad \frac{4}{3}J = \frac{17}{3}$

$\cancel{J = 17/9}$ $\quad J = \frac{17}{4} \approx 4.25.$

4.25 is outside of $[0,3] \Rightarrow$ max is achieved closest to 4.25 which is 3.

In first segment, solution is $(3,5)$ with value $(3-1)(5-1) = 6$

For $J \in [3,5] \Rightarrow b = -J + 7$

max $(b-1)(J-1)$

$(J-1)\ (-J + 7 - 1)$

$(J-1)(-J + 6)$

$-J^2 + 6J + J - 6$

$-J^2 + 7J - 6$

diff $\frac{d}{dJ} = -2J + 7 = 0$

$J = +\frac{7}{2} = 3.5$

$\hookrightarrow$ 3.5, with $b = 3.5$ $= (3.5-1)(3.5-1) = 6.25$ value $J \in [3,5]$

Nash point is given as $[\frac{7}{2}, \frac{7}{2}]$ it is the max of both line segments

- both B and C give bargains as gambles between cinema and theatre, heavily weighted towards cinema. so they should play a game between these two options.

13/12/15 DECISIONS: Nash BARGAINING POINTS EXERCISES 1

Q3 $U_J(£x) = x^{\alpha}$ $U_m(£x) = x^{\beta}$ $0<\alpha<\beta<1$

A. Attitude towards risk: Both J and M are risk averse, but John more than May as $\alpha < \beta$.

B. Pareto Boundary: What May gets John gets 1 − what may gets or vice versa.
So the x values will be x and 1-x.
For $J \Rightarrow x^{\alpha}$ for M $(1-x)^{\beta}$ and so forth, this is the pareto boundary.

The given curve is concave, then it is clear that gambles between two different points on the curve have lower expected utility for both than at least one other point on the curve, hence such a gamble will not be pareto optimal.

C $U_J = x^{\alpha}$ $U_M = x^{\beta}$ $-(1-x)^{\beta}$ $m = (1-x^{\beta})^{\beta}$

Nash pt = $g(u,v) = (\mu-\mu^*)(v-v^*)$
$= (x^{\beta}-0)((1-x)^{\beta}-0)$
$= \underset{\mu}{x^{\beta}}\ \underset{v}{((1-x)^{\beta})}$

$x^{\alpha}(1-x)^{\beta}$ $x^{\alpha}(1-x)2x$

$\alpha x(1-x\beta):0$

$x^{\beta}(\beta(1-x)(-1)) + (1-x)^{\beta}$

$x^{\alpha}\beta(1-x)^{\beta-1}(-1) + (1-x)^{\beta}\alpha x = 0$

$$\overset{u}{x^{\alpha}}\overset{v}{(1-x)^{\beta}}$$

$(1-x)^{\beta}\alpha\,(x^{\alpha-1}) + x^{\alpha}\beta\,(1-x)^{\beta-1}$ ⊖ From the $(1-x)$

$$\frac{\alpha(1-x)^{\beta}}{x^{(1-\alpha)}} - \frac{\beta x^{\alpha}}{(1-x)^{1-\beta}} = 0$$

$$\frac{\alpha(1-x)^{\beta}(1-x)^{1-\beta} - \beta x^{\alpha}x^{(1-\alpha)}}{x^{(1-\alpha)}(1-x)^{1-\beta}} = 0$$

cancel bottom line

$\alpha(1-x)^{\beta}(1-x)^{1-\beta} - \beta x^{\alpha}x^{(1-\alpha)} = 0$

$(1-x)^{\beta}(1-x)^{1-\beta}$

$(1-x)^{\beta+1-\beta}$

$= (1-x)^{1}$

$\alpha(1-x) - \beta x = 0$

$\alpha - \alpha x - \beta x = 0$

~~$x(\beta - \alpha) = \alpha$~~ $\underline{\alpha}$

$(a+b)x = a$

$x = a/a+b$

$$\frac{d()}{dx}\ \overset{u}{x^{\alpha}}\overset{v}{(1-x)^{\beta}}$$

$= (1-x)^{\beta}\alpha(x^{\alpha-1}) - x^{\alpha}\beta(1-x)^{\beta-1}$

$$\frac{\alpha(1-x)^{\beta}}{x^{(1-\alpha)}} - \frac{\beta x^{\alpha}}{(1-x)^{1-\beta}} = 0$$

$$\frac{\alpha(1-x)^{\beta}(1-x)^{1-\beta} - (x^{(1-\alpha)})\beta x^{\alpha}}{x^{(1-\alpha)}(1-x)^{1-\beta}} = 0$$

$\alpha(1-x) - \beta x = 0$

$$\alpha = \frac{a}{a+b} \rightarrow \text{a/b for M will be } \frac{b}{a+b} \therefore$$

C Nash arbitration point is achieved at a division of $\frac{\alpha}{\alpha+\beta}$ to John and $\frac{\beta}{\alpha+\beta}$ to Mary.

D $x^{\alpha}(\cancel{1}-x)^{\beta}$

$\cancel{\alpha}\ \alpha(1-x)^{\beta}x^{\alpha-1} - \beta(1-x)x^{\alpha}$

$x^{\alpha}(-1)(1-x)^{\beta-1} + (1-x)^{\beta}\alpha x^{\alpha-1} = 0$

$(1-x)^{\beta}\alpha x^{\alpha-1} - (1-x)^{\beta-1}x^{\alpha}$

$(1-x)^{\beta}(\alpha x^{\alpha-1} - (1-x)^{-1}x^{\alpha})$

$(1-x)^{\beta}(\cancel{x}^{\alpha})(\alpha x^{-1} \cancel{+}(1-x)^{-1}) = 0$

$\cancel{\alpha}$

Now the curve is convex, hence it is no longer the Pareto Boundary. With both John and Mary risk prone, they both take gambles, and the feasible region is now the triangle (0,0) (1,0) (1,1).
By Symmetry the Nash point is $(\frac{1}{2},\frac{1}{2})$, so toss a fair coin and the winner takes all (for whatever values for $\alpha, \beta > 1$ are chosen).

Extra for Q3 Decision Exercise

$\alpha < \beta$

$U_J = x^{\alpha}$ $U_m = x^{\beta}$

When $U_J = x$ $U_J = 1-x$

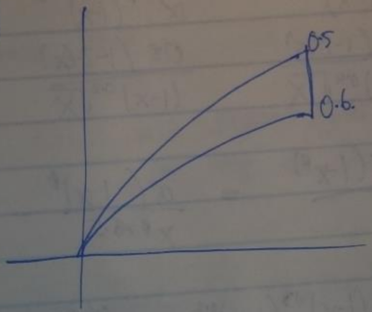
$x^{0.5}$ $x^{0.6}$

$(x)^{\alpha}(1-x)^{\beta}$

$m = x^{0.6}$

$m = (1-x)^{0.6}$ $1-x^{0.6}$

$x^{0.5}\,\cancel{x}(1-x^{0.5})$

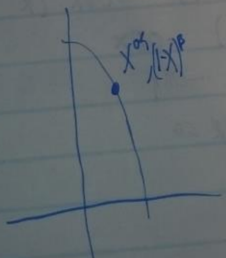


$x^{0.5}(1-x)^{0.5}$

$\frac{x^{0.5}}{\sqrt{x}\sqrt{1-x}}$

$\sqrt{x(1-x)}^{\frac{1}{2}}$

$\frac{1}{2}\frac{1-2x}{\sqrt{x(1-x)}} = 0$



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$1-x^{\alpha}$

x^{α}

$x(1-x)$ $x-x^2$

$1-2x=0$

$x=\frac{1}{2}$

$x^{\alpha}-x^{\alpha+b}$

$\alpha x^{\alpha-1} - (\alpha+b)x^{\alpha+b-1} = 0$

$x^{0.5}-x^{1}$

$0.5x^{-0.5} - 1 = 0$

$0.5x^{-0.5} = 1$

$x^{-0.5} = 2$

$x = 4$

$x^{0.5}(1-x)^{0.6}$

$\sqrt{x}\,(1-x)^{0.6}$

$0.5 - 1.1x$

$= 1.1x = 0.5$

$x = 0.5/1.1$

$x^{0.5}(1-x)^{0.6}$ $\qquad$ $x^{0.5}(1-x)^{0.8}$ $\qquad$ $x^{0.6}(1-x)^{0.8}$

$\dfrac{0.5(1-2.2x)}{(1-x)^{0.4}\sqrt{x}}$ $\qquad$ $\dfrac{0.5(1-2.6x)}{(1-x)^{0.2}\sqrt{x}}$ $\qquad$ $\dfrac{0.6(1-x)^{0.8}}{x^{0.4}} - \dfrac{0.8x^{0.6}}{(1-x)^{0.2}}$

$$\underline{x^{\alpha}(1-x^{\beta})} = \frac{\alpha(1-x)^{\beta}}{x^{\beta-0.5}} - \frac{\beta x^{\beta-\alpha}}{(1-x)^{\beta-\alpha}}$$

$0.6(1-x)^{0.8}(1-x)^{0.2} - 0.8x^{0.6}(x^{0.4})$

$= 0.6(1-x) - 0.8x$

$0.6 - 0.6x - 0.8x$

$0.6 - 1.4x = 0$

$6/1.4$

$x^{\alpha}(1-x^{\beta})$ $\qquad$ $\dfrac{\cancel{x^{\alpha}}(1-x)^{\beta}\alpha}{x^{\alpha-1}}$

$(1-x)^{\beta}$

DECISION : NASH BARGAINING POINTS.

1(a)

		Gambler H	T
Me	H	−20	+30
	T	+10	−20

$U_M = -20_{HH} + 30_{HT} + 10_{TH} - 20_{TT}$
$U_G = 20_{HH} - 30_{HT} - 10_{TH} + 20_{TT}$

Game Theory. ⇒ No dominant Strategy!

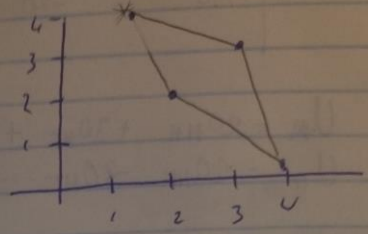
H $-20(0.5) + 30(0.5) = 5$ $= 5 =$ expected payoff to Me
T $+10(0.5) - 20(0.5) = 0$ the value of the game

B $0.5(0.25)(1) + 0.5(0.25)(0) + 0.5(0.25)(-1) + 0.5(0.25)(-2) = -1/2$
$0.5(0.25)(-2) + 0.5(0.25)(-1) + 0.5(0.25)(0) + 0.5(0.25)(2) = -1/8$
$= -5/8$ value of game shouldn't play?

		Betty	
		Confess	Deny
Bill	Confess	(x,x)	(v,w)
	Deny	(w,v)	(μ,μ)

$$\begin{matrix} 1 & 2 & 3 & 4 \\ v < & \mu < & x < & w \end{matrix}$$

		B	
		(N,N)	Y
A		$(3,3)$	$(1,4)$
		$(4,1)$	$(2,2)$



$A = mb + c$

Boundary $(1,4)\ (2,2)$ $\quad 1 = 4m + c \quad 2 = 2m + c \qquad -1 = 2b = m = -\frac{1}{2} \qquad 1 = 4(-\frac{1}{2}) + c \quad c = 3$

$(1,4)\ (2,2)\quad A = -\frac{1}{2}b + 3$

$(2,2)\ (4,1)\quad 2 = 2m + c \quad 4 = m + c \qquad -2 = m \Rightarrow 4 = -2 + c \quad c = 6$

$A = -2b + 6 \qquad A \in [2,4]$

$A = -\frac{1}{2}b + 3 \qquad A \in [1,2]$

max $(A-2)(B-2)\quad (A-2)(1$

- Default strategy is to defect, if the other player has ~~defected~~ confessed, it is best option to defect.
- If they know next game is last game they will confess as it is the last game and trying to get away with as much as they can
- Cooperate on first round then go tit for tat after?
- Players don't know which game will be the last and so, will defect