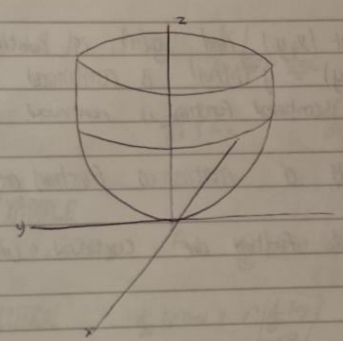
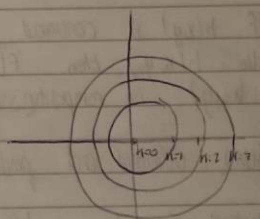
2/10/13 Maths

LEVEL CURVES

For a surface $z = f(x,y)$?

We define a level curve of height k to be the shape the surface makes on the plane $z=k$. If we project a series of such plots onto the x-y plane we get a contour plot





eg. ordnance survey maps

LIMITS AND CONTINITY IN FUNCTIONS OF SEVERAL VARIABLE

We define the limit of function $f(x,y,z)$ along a curve C (as) (x,y,z) approaches $(x_0,y_0,z_0) = (x(t_0), y(t_0), z(t_0))$ to be

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = \lim_{t\to t_0} f(x(t), y(t), z(t))$$

A function $f(x,y,z)$ is continuous at (x_0,y_0,z_0) if $f(x_0,y_0,z_0)$ and

$$\lim_{t\to t_0} f(x,y,z) = f(x_0,y_0,z_0)$$

If $f(x,y)$ is continuous at every point in a region D, it is continuous on D.

Similarly, if it is continuous for all x, y it is continuous everywhere

PROPERTIES

1. If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x,y) = g(x)h(y)$ is continuous at (x_0, y_0)

2. If $h(x,y)$ is continuous at (x_0,y_0), and $g(u)$ is continuous at $u_0 = h(x_0,y_0)$, then $f(x,y) = g(h(x,y))$ is continuous at (x_0,y_0) i.e composition of continuous functions is continuous.

3. Sums, differences and products of continuous functions are continuous

4. Quotients of two differentiable functions are continuous unless the denominator is zero

PARTIAL DERIVATIVES

We have a function of two or more variables, we need to understand how it acts in each variable. To do this we fix all other variables and look at the behaviour of one of them

This is the idea behind partial derivatives

Take $z = f(x,y)$ Imagine we can fix $y = y_0$

The derivative in respect to x of $f(x,y_0)$ is $\frac{d}{dx} f(x, y_0)$

We define partial derivatives as $f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}$

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x,y)}{\Delta y}$$

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Other notation $\frac{df}{dx}$, $\frac{dz}{dx}$

the partial derivative of f with respect to x.

To find a partial derivative at (x_0, y_0) take the derivative and then substitute (x_0, y_0)

This will be denoted $\frac{df}{dy}\Big|_{x_0 = x_0,\ y = y_0}$ or:

$\frac{df}{dx}\Big|_{x_0, y_0}$ or: $\frac{df}{dx}(x_0, y_0)$

EXAMPLE

let $z = x^2 \sin y$ find $\frac{dz}{dx}\Big|_{(\pi,\pi)}$ and $\frac{dz}{dy}\Big|_{(\pi,\pi)}$

SOLUTION $\frac{d}{dx}(x^2)\sin y + x^2 \boxed{\underbrace{\frac{d}{dx}\sin y}_{=0}}$

$\frac{dz}{dx} = 2x \sin y$

$\Rightarrow \frac{dz}{dx}\Big|_{(\pi,\pi)} = 2\pi + \underbrace{\sin \pi}_{=0} = 0$

$\frac{dz}{dy} = x^2 \cos y$ $\quad \frac{dz}{dy}\Big|_{(\pi,\pi)} = \pi^2 \cos(\pi) = -\pi^2$

HIGHER ORDER PARTIALS

We have mixed partials, and use the following notation

$f_{xy}(x,y) = \frac{d^2 f}{dy\,dx} = \frac{d}{dy}\frac{df}{dx}$

We differentiate from left to right

Also $f_{xx}(x,y) = \frac{d^2 f}{dx^2}$.

$f_{yy}(x,y) = \frac{d^2 f}{dy}$ etc

4

EXAMPLE

Find $f_{xy}(x,y)$ for $f(x,y) = x^2(y^2-y)$

SOLUTION

$$f_{xy}(x,y) = \frac{d}{dy}\left(\frac{d}{dx} x^2(y^2-y)\right)$$

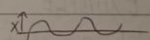
$$= \frac{d}{dy}(2x(y^2-y))$$

$$= 2x(2y-1) = \text{ans.}$$

SLOPE

A function of one variable $f(x)$ has slope $= \frac{df}{dx}$. If a function has 3 variables, $f(x,y,z)$, it has slope $\frac{df}{dx}$ in the x direction, slope $\frac{df}{dy}$ in y-direction and slope $\frac{df}{dz}$ in the z-direction

ONE DIMENSIONAL WAVE EQUATION



A string oscillating in one dimension (up and down), the position at any point on the string depends on coordinates x and t.

displacement from resting — time

and it can be described by a function $u(x,t)$

It can be shown to satisfy the wave equation

$$\frac{d^2u}{dt^2} = c^2 \frac{d^2u}{dx^2}$$

c^2 depends on the properties of the string

It appears in hooke's law, and a general form describing electromagnetic radiation, x-ray etc

LAPLACE'S EQUATION

In three dimensions, Laplace's equation is $\frac{d^2f}{dx^2} + \frac{d^2f}{dy^2} + \frac{d^2f}{dz^2} = 0$

appears in fluid dynamics

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1. Write in form of ellipse

$$\left(\frac{x-x_1}{a}\right)^2 + \left(\frac{y-y_1}{b}\right)^2 = 1$$

$2x^2+4x+y^2+2y$

$2(x+1)^2-2$ and so on solve to this form

2. Keep y constant
product rule between x func

$\frac{dz}{dx}$ slope in x direction

$\frac{dz}{dy}$ slope in y direction

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If a function $f(x,y,z)$ is differentiable at a point (x_0,y_0,z_0) and we consider shifting away to a nearby point

$(x = x_0+\Delta x, y = y_0+\Delta y, z = z_0+\Delta z)$ we can approximate

$f(x,y,z) \approx f(x_0,y_0,z_0)$

difference

$+ f_x(x_0,y_0,z_0)\Delta x$
$+ f_y(x_0,y_0,z_0)\Delta y$
$+ f_z(x_0,y_0,z_0)\Delta z$

and since $\Delta x = x-x_0$
$\Delta y = y-y_0$
$\Delta z = z-z_0$ we define the local linear approximation

$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0)$
$+ f_z(x_0,y_0,z_0)(y-y_0)$

Example: Find the local linear approximation of
$f(x,y) = x^\alpha y^\beta + \frac{y^\alpha}{x^\beta}$ at $(1,1)$.

Solution: We need $L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$f(1,1) =$ sub in! $1^\alpha 1^\beta + \frac{1}{1} = 1+1 = 2$

diff $f_x(x,y) = \alpha x^{\alpha-1} y^\beta - \frac{\beta y^\alpha}{x^{\beta-1}}$

subin $f_x(1,1) = \alpha - \beta$

diff $f_y(x,y) = \beta x^\alpha y^{\beta-1} + \alpha\frac{y^{\alpha-1}}{x^\beta}$

sub in $f_y(1,1) = \alpha + \beta$

$\Rightarrow L(x,y) = 2 + (\alpha-\beta)(x-1) + (\alpha+\beta)(y-1)$

The Chain Rule

In general a function $f(x,y,z)$ depends on a parameter t via $f(x(t), y(t), z(t))$.

Varying t will change x, y and z

Recall, if a function $v(u(t))$ is defined, we have

$$\frac{dv}{dt} = \frac{dv}{du} \cdot \frac{du}{dt}$$

In the more general case we have;

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt} + \frac{df}{dz}\frac{dz}{dt}$$

If $z = f(x,y)$ has variables that depend on two parameters u and v ie $x(u,v)$, $y(u,v)$ we have the chain rule for partial derivatives

$$\frac{df}{du} = \frac{df}{dx}\frac{dx}{du} + \frac{df}{dy}\frac{dy}{du} \qquad \frac{df}{dv} = \frac{df}{dx}\frac{dx}{dv} + \frac{df}{dy}\frac{dy}{dv}$$

Example

Use the chain rule to find $\frac{dz}{du}$ and $\frac{dz}{dv}$ for $\overset{f(x,y)}{z = x^2 y}$ $\quad x = 2u+v$, $y = u - v^2$

$\frac{dz}{dx} = 2xy \qquad \frac{dz}{dy} = x^2$

$\frac{dx}{du} = 2 \qquad \frac{dz}{dv} = 1$

$\frac{dy}{du} = 1 \qquad \frac{du}{dv} = -2v$

3.

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$$\frac{dz}{du} = \frac{dz}{dx}\frac{dx}{du} + \frac{dz}{dy}\frac{dy}{du}$$

$= (2xy)2 + (x^2)(1)$

$= x^2 + 4xy$

$(2u+v)^2 + 4(2u+v)(u-v^2)$

$$\frac{dz}{dv} = \frac{dz}{dx}\frac{dx}{dv} + \frac{dz}{dy}\frac{dy}{dv}$$

$= (2xy)(1) + (x^2)(-2v)$

$= -2v(2u+v)^2 + 2(2u+v)(u-v^2)$

Directional Derivatives

For $f(x,y,z)$, we could move in many different directions from the original point.

Any unit vector $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$ where $a^2+b^2+c^2=1$ define a direction, away from (x_0, y_0, z_0). In terms of the arc length parameter we have the equation for a subsequent point.

$x = x_0 + as \qquad y = y_0 + bs \qquad z = z_0 + cs$

$s \to 0$ gives us the original point.

Differentiation with respect to s gives the slope in the direction of $\vec{u}$ when we set $s=0$

We define the directional derivative of f in the direction of $\vec{u}$ to be

$$D_{\vec{u}} f(x_0, y_0, z_0) = \frac{d}{ds}\left(f(x_0+as), y_0+bs, z_0+zs)\right)$$

$= f_x(x_0, y_0, z_0)a + f_y(x_0, y_0, z_0)b + f_z(x_0, y_0, z_0)c$

$=$

This can be regarded as the slope of the surface $w = f(x, y, z)$ in the direction of $\vec{u}$.

The gradient

The gradient is denoted by ∇, called "nabla" or "del".

$$\nabla f(x, y, z) = f_x(x, y, z)\vec{i} + f_y(x, y, z)\vec{j} + f_z(x, y, z)\vec{k}$$

Any direction derivative is given by

$$D_{\vec{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \vec{u}$$

Properties

1. $z = f(x, y, z)$ has maximum slope in direction of the gradient $\|\vec{\nabla} f(x, y, z)\|$
2. $z = f(x, y, z)$ has minimum slope in direction opposite the gradient $-\|\vec{\nabla} f(x, y, z)\|$
3. If $\vec{\nabla} f = 0$, all directional derivatives are zero at that point
4. Level curves $z = f(x, y)$ have gradient normal to the curve $\Rightarrow \vec{\nabla} \cdot \vec{T} = 0$ on level curve

Example: Find the unit vector in the direction in which $f(x, y) = 10 - 2x^2 - y^2$ increases most quickly at $(1, 1)$ and find the rate of change

Solution: The direction f increases most quickly is $\vec{\nabla} f|_{(1,1)} = \frac{df}{dx}\vec{i} + \frac{df}{dy}\vec{j}|_{(1,1)}$

$= 4x\vec{i} - 2y\vec{j}|_{(1,1)}$

$= 4\vec{i} - 2\vec{j}$

$\Rightarrow \|\vec{\nabla} f\||_{(1,1)} = \sqrt{4^2 + 2^2} = 2\sqrt{5}$

$\Rightarrow \vec{u} = \frac{-2}{\sqrt{5}}\vec{i} - \frac{1}{\sqrt{5}}\vec{j}$ The rate of change is $+\|\vec{\nabla} f\| = +2\sqrt{5}$

7/10/13 Maths

Tangent Plane and Normal Vectors

A tangent plane, is the surface that contains all possible tangent lines of all curves at point P of a surface $F(x(t), y(t), z(t))$

At $P(x_0, y_0, z_0)$, the surface F has value $c = F(x_0, y_0, z_0)$.

We assume that the surface is continuous at P, and its partial derivatives are also continuous.

Then at P we have

$$
\begin{aligned}
0 = \; & F_x(x_0, y_0, z_0)\, x'(t_0) \quad \text{derivative} \\
& + F_y(x_0, y_0, z_0)\, y'(t_0) \\
& + F_z(x_0, y_0, z_0)\, z'(t_0)
\end{aligned}
$$

Consider a curve C parameterised by $\vec{r}t = (x'(t), y'(t), z'(t))$

The tangent line for C is parallel $\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$

Therefore $(*)$ becomes $0 = (F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0)) \cdot \vec{r}\,'(t_0)$

$\Rightarrow 0 = \vec{\nabla} F(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0)$

So $\vec{\nabla} F(x_0, y_0, z_0)$ is normal to the tangent line of C at P.

Since C was arbitrary $\vec{\nabla} F(x_0, y_0, z_0)$ is normal to any tangent line at P, and hence is normal to the tangent plane

We define the tangent plane to be the plane with normal vector given by $\vec{n} := \vec{\nabla} F(x_0, y_0, z_0) = (F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0))$

with the tangent plane given by

2

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

It is a plane that touches the surface $F(x,y,z)$ at (x_0,y_0,z_0) in analogy to the tangent line

The normal line is parallel to the normal vectors with parametric form

$x = x_0 + F_x(x_0,y_0,z_0)t$
$y = y_0 + F_y(x_0,y_0,z_0)t$
$z = z_0 + F_z(x_0,y_0,z_0)t$
or $\vec{r}t = \vec{r_0} + \vec{n}t$

If instead we have $z = f(x,y)$ at the point $(x_0,y_0, f(x_0,y_0))$ we get the tangent plane

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

The normal vector is now

lower case f

$$\vec{n} = (-f_x(x_0,y_0), -f_y(x_0,y_0), 1)$$

If we have $F(x,y,z) = z - f(x,y)$.

The normal line may be written $\vec{r}(t) = \vec{r_0} + t((-f_x(x_0,y_0))\vec{i} - f_y(x_0,y_0)\vec{j} + \vec{k})$

In this form, the equation of the tangent plane is the same as the local linear approximation of $z = f(x,y)$ at P.

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

3

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Example: Find the tangent plane and normal line of $z = f(x,y) = -(x^2+y^2)$ the surface at the point $(1,1,-2)$

Solution 1 First notice $f(x_0,y_0) = f(1,1) = -(1^2+1^2) = -2$
So the point is on the surface.
Next we find the derivatives

$$\frac{df}{dx}\Big|_{(1,1)} = -2x\Big|_{(1,1)} = -2$$

$$\frac{df}{dy}\Big|_{(1,1)} = -2y\Big|_{(1,1)} = -2$$

$\Rightarrow$ tangent plane is
$z = (-2) + (-2)(x-1) + (-2)(y-1)$
$z = 2(1-x-y)$ tangent equation

$\vec{n} = (-(-2), -(-2), 1) = (2,2,1)$

The normal line is $\vec{r} = \vec{r_0} + \vec{n}t = (1,1,-2) + (2,2,1)(t)$
$\Rightarrow (1+2t, 1+2t, t-2)$

Solution 2 Take $F(x,y,z) = z - f(x,y)$
$= z + x^2 + y^2$

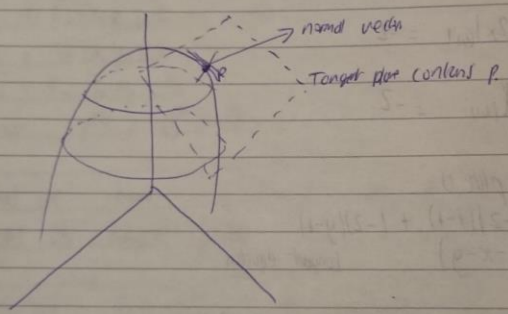
Normal vector: $\vec{\nabla} F(1,1,-2)$
$= F_x(1,1,-2), F_y(1,1,-2), F_z(1,1,-2)$
$(2x, 2y, 1)\big|_{(1,1,-2)} = (2,2,1)$

The normal line is now $x = x_0 + F_x t = 1+2t$
$y = y_0 + F_y t = 1+2t$
$z = z_0 + F_z t = -2+(1)t = t-2$
This is the component form of $\vec{r}$

4

The tangent plane is (from (*))

$$F_x(x-x_0) + F_y(y-y_0) + F_z(z-z_0) = 0$$
$$2(x-1) + 2(y-1) + 1(z-(-2)) = 0$$
$$2x + 2y + z - 2 = 0$$
$$z = 2(1-x-y) \quad \text{(the same as solution 1)}$$



NOTE: Some textbooks might use a parameterisation with opposite sign for t

They might write the following $(1-t,\ 2-t,\ t+1)$
but if take $t \to -t$ we recover
$(1+t,\ 2+t,\ t-1)$

These ~~happen~~ are the same lines with different parameter → change of parameter. Any $t \to at+b$ ok for all a, b const.

11th

7/10/13 Maths

Minima and Maxima of function of two variable

Consider $f(x,y)$. We define the concept of minima and maxima as follows.

f has a relative (or local) $\begin{cases} \text{maximum} \\ \text{minimum} \end{cases}$

at (x_0, y_0) if $\begin{cases} f(x_0, y_0) \geq f(x,y) \\ f(x_0, y_0) \leq f(x,y) \end{cases}$ for all points that lie in some disk centred on (x_0, y_0).

f has an absolute (or global) $\begin{cases} \text{Maximum} \\ \text{minimum} \end{cases}$ at (x_0, y_0)

if $\begin{cases} f(x_0, y_0) \geq f(x,y) \\ f(x_0, y_0) \leq (x,y) \end{cases}$ for all points for which f is defined

Both minima and maxma are types of extrema, i.e. points for which f takes on extreme value

A set is bounded is there is a box that can be drawn around the entire set.

A closed set contains its boundry, an open set does not

$\Rightarrow$ A disk including its boundry is closed and bounded.

An infinite line, however, is open because the end points are at $\pm\infty$ and is unbounded because no box is bigger than infinite size

However the interior of a disk, i.e. without the boundary is bounded but open

2

Extreme-Value Theorm
If $f(x,y)$ is continuous on a closed bounded set, then it has an absolute maximum and absolute minimum on that set.

The position of a stationary point is shown by the fact that the first derivatives vanish. In other words, there is a stationary point at (x_0, y_0) if

$$F_x(x_0, y_0) = F_y(x_0, y_0) = 0$$

A critical point is either a stationary point or where one or more derivatives dont exist.

The Second Partial Derivative Test.
Let $f(x,y)$ be continuous, with continuous second order partial derivatives in disk centred around a critical point (x_0, y_0)

Define $D = F_{xx}(x_0, y_0)F_{yy}(x_0, y_0) - F_{xy}(x_0, y_0)^2$

$D > 0$ and $F_{xx}(x_0, y_0) > 0$ then $F(x,y)$ has a relative minimum at (x_0, y_0)

$D > 0$ and $F_{xx}(x_0, y_0) < 0$ then $f(x,y)$ has a relative maximum at (x_0, y_0)

$D < 0$ then $F(x,y)$ has a saddle point at (x_0, y_0) (stationary point)

$D = 0$ then no conclusion can be drawn

A saddle point is a stationary point that is not a relative or absolute extremum.

An example is $f(x,y) = x^2 - y^2$ at $(0,0)$

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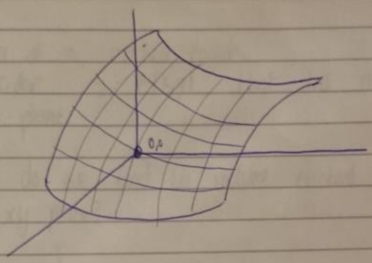

Both direction change their slope

Example: Find the critical point of $f(x,y) = xy - x^3 - y^2$ and determine if they are maxima, minima or saddle points

Solution: Set $f_x(x,y) = f_y(x,y) = 0$

$\Rightarrow \quad y - 3x^2 = 0 \qquad = x - 2y$

From the second equation, $y = \frac{x}{2}$ and the first equation becomes

$\frac{x}{2} - 3x^2 = 0$

$x(x - \frac{1}{6}) = 0 \qquad x = 0$ or $x = \frac{1}{6} \qquad (0,0) \quad (\frac{1}{6}, \frac{1}{12})$

The y values are $y = 0 \qquad y = \frac{1}{12}$

The second order derivatives are $f_{xx}(x,y) = -6x$

$f_{yy}(x,y) = -2 \qquad f_{xy}(x,y) = 1$

$\Rightarrow \quad D = (-6x)(-2) - (1)^2 = 12x - 1$

At $(0,0)$ $D = -1$, < 0, $(0,0)$ is saddle point

At $(\frac{1}{6}, \frac{1}{12})$ $D = 12(\frac{1}{6}) - 1 = 1$, > 0, could be min or max, check $f_{xx}(\frac{1}{6}, \frac{1}{12})$

$= -6(\frac{1}{6}) = -1$, < 0

so $(\frac{1}{6}, \frac{1}{12})$ is a global maximum

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2 PARTIAL DERIVATIVES

Functions of Several variables:

Before $z = f(x)$ Now $z = f(x,y)$ $w = f(x,y,z)$

The domain of $z = f(x,y)$ is all (x,y) for which f is defined

$D(w)$ is all (x,y,z) for which f is defined.

Example: ~~Find~~ $f(x,y,z)$

$= x^2 + xy + y^2 - \sqrt{z}$ ← letter

Find its value at $(1,3,4)$ we get:

$f(1,3,4) = 1^2 + (1)(3) + 3^2 - \sqrt{4} = 11$

To find the domain: the first three terms are defined for all ~~[illegible]~~ x, y $\sqrt{z}$ is defined for $z \geq 0$

$\Rightarrow f(x,y,z)$ is defined for $z \geq 0$

$f(x,y) = \sqrt{x^2+y^2-4}$ → This is a circle of radius 4.

The argument of $\sqrt{\ }$ must be positive so,

$0 < x^2 + y^2 - 4$

$\Rightarrow x^2 + y^2 \geq 4$

f is all points on or outside the circle of radius 4

Graphing functions of several variables $z = f(x,y)$

Find points (x,y) and plot them

Lets Sketch $z = \sqrt{1-x^2+y^2}$

This is a hemisphere of radius 1 $x^2+y^2+z^2=1$

Hemisphere because $z \geq 0$ due to $\sqrt{\ }$