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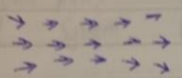
5 VECTOR CALCULUS AND LINE INTEGRALS

Vector field

In two or three dimensional space we define a vector field to be a function that defines a unique $\vec{F}(P)$ for every point P

It can be written as $\vec{F}(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k}$



a vector field how it behaves in space
drop a point in and see where it flows
like dropping leaf in stream

Acting on a vector field: Gradient, Divergence and Curl

Recall that the gradient of a function gives a vector at every point in space

$$\vec{\nabla} F = \frac{df}{dx}\vec{i} + \frac{df}{dy}\vec{j} + \frac{df}{dz}\vec{k}$$

In other words this is a vector field

From now on we will use ϕ as the function since $f(x,y,z)$ appears in $\vec{F}(x,y,z)$

$$\Rightarrow \vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{d\phi}{dy}\vec{j} + \frac{d\phi}{dz}\vec{k}$$

This is called the gradient field of ϕ. By definition, it points in the direction in ϕ increases most. Vector fields defined in this way are special

A vector field $\vec{F}$ is conservative if a ϕ exists such that $\vec{F} = \vec{\nabla}\phi$

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Vector field of this type have special properties

Example: Check if the potential function
$\phi(x,y,z) = x^2 - 3y^2 + 4z^3$ gives rise to the vector field $\vec{F}(x,y,z) = 2x\vec{i} - 6y\vec{j} + 12z^2\vec{k}$

Solution

$\vec{F}(x,y,z) = \vec{\nabla}(x^2 - 3y^2 + 4z^3)$

$\frac{d}{dx}(x^2 - 3y^2 + 4z^3)\vec{i} + \frac{d}{dy}(x^2 - 3y^2 + 4z^3)\vec{j} + \frac{d(x^2 - 3y^2 + 4z^3)}{dz}\vec{k}$

$= 2x\vec{i} - 6y\vec{j} + 12z^2\vec{k}$

We now define two operations that act on vector fields

The divergence of a vector field

$\vec{F}(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k}$

$\text{div}\,\vec{F} = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}$ This is a scaler (a number), not a vector.

The second operation is the curl of vector field $\vec{F}(x,y,z)$ (which is the same as before) given by

$\text{curl}\,\vec{F}\ \left(\frac{dh}{dy} - \frac{dg}{dz}\right)\vec{i} + \left(\frac{df}{dz} - \frac{dh}{dx}\right)\vec{j} + \left(\frac{dg}{dx} - \frac{df}{dy}\right)\vec{k}$

The curl itself is a vector field

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The curl of a vector field $F(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k}$ is

$$\text{curl } \vec{F} = \left(\frac{dh}{dy} - \frac{dg}{dz}\right)\vec{i} + \left(\frac{df}{dz} - \frac{dh}{dx}\right)\vec{j} + \left(\frac{dg}{dx} - \frac{df}{dy}\right)\vec{k}$$

→ this is a vector field

Note that the value of $\text{div}\vec{F}$ and $\text{curl}\vec{F}$ depend on the point they are evaluated at since f, g, h depend on this point, and should give a result in any coord system (note that the exact form might change → see definition of $\vec{\nabla}$ later)

The curl may be symbolically represented by a determinant

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ f & g & h \end{vmatrix}$$

where we interpret $\left(\frac{d}{dx}\right)(g)$ or $(g)\left(\frac{d}{dx}\right)$ as $\frac{dg}{dx}$

Example: Find the divergence and curl of the vector field

$$\vec{F} = xyz\,\vec{i} + y^2z\,\vec{j} + z\,\vec{k}$$

Solution: $\text{div }\vec{F} = \frac{d}{dx}(xyz) + \frac{d(y^2z)}{dy} + \frac{d(z)}{dz}$

$= yz + 2yz + 1 = 3yz + 1$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ xyz & y^2z & z \end{vmatrix}$$

$$= \left[\frac{d(z)}{dy} + \frac{d(y^2z)}{dz}\right]\vec{i} + \left[\frac{d(xyz)}{dz} - \frac{dz}{dx}\right]\vec{j} + \left[\frac{d(y^2z)}{dx} - \frac{d(xyz)}{dy}\right]\vec{k}$$

$$= [0 - y^2]\vec{i} + [xy - 0]\vec{j} + [0 - xz]\vec{k}$$

$$= -y^2\vec{i} + xy\vec{j} + -xz\vec{k}$$

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Up until now, we treated $\vec{\nabla}$ as an abstract object, however, in the terminology here, the natural interpretation is

$$\vec{\nabla} = \frac{d}{dx}\vec{i} + \frac{d}{dy}\vec{j} + \frac{d}{dz}\vec{k}$$

Apply this to $\phi(x,y,z)$

$$\vec{\nabla}\phi = \frac{d\phi}{dx}\vec{i} + \frac{d\phi}{dy}\vec{j} + \frac{d\phi}{dz}\vec{k}$$

It also allows us to write div and curl in a new way.

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}$$
$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \left(\frac{dh}{dy} - \frac{dg}{dz}\right)\vec{i} + \left(\frac{df}{dz} - \frac{dh}{dx}\right)\vec{j} + \left(\frac{dg}{dx} - \frac{df}{dy}\right)\vec{k}$$

Div gives the tendency of the "fluid" represented by $\vec{F}$ to "flow". It's a number that represents how much "stuff" moves past the point

The curl represents the rotation around the vector field ie how it "curls" around

We can also consider taking the dot product of $\vec{\nabla}$ with itself.

$$\vec{\nabla}^2 = \vec{\nabla}\cdot\vec{\nabla} = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$$

This is called the Laplace operator and applied to $\phi(x,y,z)$ it generates Laplace's equation:

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0$$ which can be written $\vec{\nabla}^2\phi = 0$

Appears in heatflow and fluid flow for example

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Example: Prove that: $\phi = x^2 z - \frac{z^3}{3}$ satisfies Laplace equation

Solution: $\frac{d\phi}{dx} = 2xz \Rightarrow \frac{d^2\phi}{dx^2} = 2z$

$\frac{d\phi}{dy} = 0 \Rightarrow \frac{d^2\phi}{dy^2} = 0$

$\frac{d\phi}{dz} = x^2 - z^2 \Rightarrow \frac{d^2\phi}{dz^2} = -2z$

$\Rightarrow \vec{\nabla}^2\phi = 2z + 0 + (-2z) = 0$

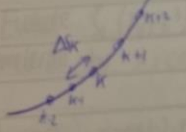
So Laplace's equation is satisfied

LINE INTEGRALS

By a line integral, we mean integrating a function along a curve C. We denote this by

$= \int_C f(x,y,z)\,ds$

where s is the arc length parameter along the curve. We break the curve up into segments such that k^{th} segment has arc length Δs_k.



The integral is written as a sum over these segments:

$$\int_C f(x,y,z)\,ds = \lim_{\max \Delta s_k \to 0} \sum_{k=0}^{n} f(x_k, y_k, z_k)\,\Delta s_k$$

The length Δs_k is an arc length

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \|\vec{r}\,'(t)\|\,dt = \|\vec{r}\,'(t_k^*)\|\,\Delta t_k$$

some value of t in range

Using the mean value theorem for integral
with $\Delta t_k = t_k - t_{k-1}$

As the length of these segments goes to 0, in the limit we get a continuous variable

If C is smoothly parameterised by the following
$\vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k}$

Then the line integral along c is given by

$$\int_C f(x,y,z)\,dl = \int_a^b f(x,y,z)\,\|\vec{r}\,'(t)\|\,dt$$

NOTE: If we want to integrate with respect to the x-direction, we would simply replace $\vec{r}\,'$ with x' and dl with dx

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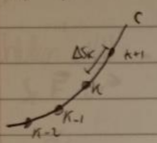
Lines Integrals

$\int_C f(x,y,z)\,ds = \int_a^b f(x,y,z)\,\|\vec{r}'(t)\|\,dt$



parameterised by $\vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k}$

Example: Evaluate $\int_C (\underbrace{1+xy^2}_{f(x,y)})\,ds$ along the curve $\vec{r}(t) = t\vec{i} + 2t\vec{j}$ for $0 \le t \le 1$

Solution: $\vec{r}'(t) = \vec{i} + 2\vec{j}$

$\|\vec{r}'\| = \sqrt{1^2+2^2} = \sqrt{5}$

Looking at the parameterisation, $x = t$, $y = 2t$

$\int_C (1+xy^2)\,ds = \int_0^1 (1+(t)(2t)^2)\sqrt{5}\,dt$

$\sqrt{5}\int_0^1 1+4t^3\,dt$

$\sqrt{5}\left[t+\frac{4t^4}{4}\right]_{t=0}^{1}$

$\Rightarrow = \sqrt{5}(1+1)$

$= 2\sqrt{5}$

Example: Evaluate $\int_C (xy+z^3)\,ds$ from $(1,0,0)$ to $(-1,0,\pi)$ along the helix C, parameterised by $x = \cos t$, $y = \sin t$, $z = t$.

Solution $t=0$, gives
$$\left.\begin{aligned} x &= \cos(0) = 1 \\ y &= \sin(0) = 0 \\ z &= 0 \end{aligned}\right\} (1,0,0) \text{ for } t=0$$

$(-1,0,\pi) \Rightarrow$ $x = \cos t = -1 \Rightarrow t = \cos^{-1}(-1) = \pi, 3\pi$ etc.

$z = t = \pi$

$y = \sin \pi = 0$

$t = \pi$

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$\Rightarrow 0 \le t \le \pi$

$\vec{r}t = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$

$\vec{r}t' = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$

$\|\vec{r}t'\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}$

$= \sqrt{2}$

We sub in $x = \cos t$, $y = \sin t$, $z = t$

$\rightarrow \int_c (xy+z^3)\,ds = \int_0^{\pi} \left((\cos t)(\sin t) + t^3\right)\sqrt{2}\,dt$

$= \sqrt{2}\int_0^{\pi} (-\cos t)\,d(\cos t) + \sqrt{2}\,\frac{t^4}{4}\Big|_0^{\pi}$

$= \sqrt{2}\left(-\frac{\cos^2 t}{2}\right)\Big|_0^{\pi} + \sqrt{2}\,\frac{\pi^4}{4} = \sqrt{2}\left[-\left(\frac{-1}{2}\right) - \frac{(+1)}{2} + \sqrt{2}\,\frac{\pi^4}{4}\right]$

$= \sqrt{2}\left[-\left(\cancel{\frac{1}{2} - \frac{1}{2}}\right) + \sqrt{2}\,\frac{\pi^4}{4}\right] = \sqrt{2}\,\frac{\pi^4}{4}$

different way: $\int_0^{\pi} \cos t \sin t\,dt \qquad u = \cos t \Rightarrow du = -\sin t\,dt$

$= -\int_{u(0)}^{u(\pi)} -u\,du = -\frac{u^2}{2}\Big|_{u(0)}^{u(\pi)}$

$= -\frac{\cos^2 t}{2}\Big|_0^{\pi}$

Consider a vector field

$\vec{F}(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k}$ integrated along a curve c

If c is parameterised by $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

then $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$, a small change along the curve

We define the line integral as $\int \vec{F} \cdot d\vec{r} = \int_c \left[f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz\right]$

In analogy to the simpler case

$\int_c \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$

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Example: Evaluate $\vec{F} = \cos t\,\vec{i} + \sin t\,\vec{j}$ along the curve parameterized by $\vec{r} = t\vec{i} + t^2\vec{j}$ in the interval $-\frac{\pi}{2} \le t \le \pi$

Solution $r'(t) = \vec{i} + 2t\vec{j}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{-\frac{\pi}{2}}^{\pi} (\cos t\,\vec{i} + \sin t\,\vec{j})\cdot(\vec{i} + 2t\vec{j})\,dt$$

$$= \int_{-\frac{\pi}{2}}^{\pi} (\cos t + 2t\sin t)\,dt$$

$$\int_{-\frac{\pi}{2}}^{\pi} 2t\sin t\,dt = 2t(-\cos t)\Big|_{-\frac{\pi}{2}}^{\pi} - \int_{-\frac{\pi}{2}}^{\pi} 2(-\cos t)\,dt = -2\pi\underbrace{\cos\pi}_{=-1} + 2\left(-\frac{\pi}{2}\right)\underbrace{\cos\frac{\pi}{2}}_{=0} + 2\sin t\Big|_{-\frac{\pi}{2}}^{\pi}$$

$$= 2\pi + 2(0) - 2(-1) = 2\pi + 2$$

$$\int_{-\frac{\pi}{2}}^{\pi} \cos t\,dt = \sin t\Big|_{-\frac{\pi}{2}}^{\pi} \qquad \underbrace{\sin\pi}_{=0} - \sin\left(-\frac{\pi}{2}\right)$$

$$-(-1) = 1$$

$$= \int_{-\frac{\pi}{2}}^{\pi} (\cos t + 2t\sin t)\,dt$$

$$= 1 + (2\pi + 2) = 2\pi + 3$$

If we parameterize with respect to arc length parameter s, then $\vec{T} = \vec{r}\,'(s)$

$\vec{T}$ is unit tangent vector along the curve C, and

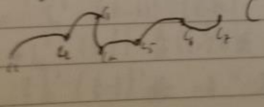
$$\int_C \vec{F}\cdot\vec{T}\,ds \quad \text{where } \vec{F}\cdot\vec{T} = \|\vec{F}\|\,\|\vec{T}\|\cos\theta = \|\vec{F}\|\cos\theta$$ ← angle between vectors

Where θ is angle between $\vec{F}$ and $\vec{T}$ and $\|\vec{T}\| = 1$ (unit vector)

Line Integrals along piecewise smooth curves

Consider a line integral taken along a series of smooth curves joined end to end. At the joints, we loose the smoothness but the sections are called <u>piecewise smooth</u>

If we have smooth curves $C_1, C_2, \ldots C_n$, then C is the curve made by joining these



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The line integral along C is

$$\int_C \vec{F}\,d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} \ldots \int_{C_n} \vec{F}\,dr$$

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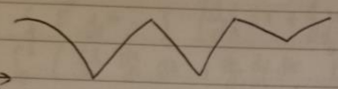
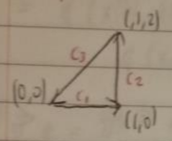


$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \dots \int_{C_n} \vec{F}\cdot d\vec{r}$$

Example: Evaluate $\int_C (x^2 y\,dx + x\,dy)$ along the path



Solution: Line from $\vec{r_0}$ to $\vec{r_1}$

$\vec{r} = (1-t)\vec{r_0} + t\vec{r_1} \qquad 0 \le t \le 1$ general form of line

$C_1: \vec{r}(t) = (1-t)(0,0) + t(1,0) = (t,0)$

$C_2: \vec{r}(t) = (1-t)(1,0) + t(1,2) = (1,2t)$

$C_3: \vec{r}(t) = (1-t)(1,2) + t(0,0) = (1-t, 2-2t)$

On C_1 $\quad y=0 \Rightarrow dy=0$

$\Rightarrow \int_{C_1} (x^2 y\,dx + x\,dy)$

$\Rightarrow \int_{C_1} (x^2(0)\,dx + x(0)) = 0$

On C_2 $\quad dx = 0, \quad dy = 2$

$\Rightarrow \int_{C_2} (x^2 y\,dx + x\,dy) \Rightarrow \int_{C_2} x\,dy = \int_0^1 (1)\frac{dy}{dt}\,dt$

$= \int_0^1 \frac{d\,2t}{dt}\,dt = \int_0^1 2\,dt = 2.$

On C_3 $\quad \frac{dx}{dt} = -1 \qquad \frac{dy}{dt} = -2$

$\Rightarrow \int_{C_3} (x^2 y\,dx + x\,dy)$

$= \int_0^1 (1-t)^2 (2-2t) \frac{d(1-t)}{dt}\,dt$

$+ \int_0^1 (1-t) \frac{d}{dt}(2-2t)\,dt$

$= 2\int_0^1 (t-1)^3\,dt + 2\int_0^1 (t-1)\,dt$

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Proof: $\int_C \vec{F}\cdot d\vec{r} = \int_C \left(\frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy + \frac{d\phi}{dz}dz\right)$

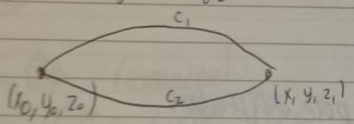
$= \int_a^b \left(\frac{d\phi}{dx}\frac{dx}{dt}dt + \frac{d\phi}{dy}\frac{dy}{dt}dt + \frac{d\phi}{dz}\frac{dz}{dt}dt\right)$

$= \int_a^b \frac{d}{dt}\left[\phi((x(t), y(t), z(t))\right]\cdot dt$

$= \phi(x(b), y(b), z(b)) - \phi(x(a), y(a), z(a))$

$= \phi(x_1, y_1, z_1) - \phi(x_0, y_0, z_0)$

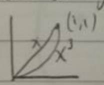
$\vec{\nabla}\phi = \frac{d\phi}{dx}\vec{i} + \frac{d\phi}{dy}\vec{j} + \frac{d\phi}{dz}\vec{k}$



$\int_{(x_0,y_0,z_0)}^{(x_1,y_1,z_1)} \vec{F}\cdot d\vec{r} = \int_{(x_0,y_0,z_0)}^{(x_1,y_1,z_1)} \vec{\nabla}\phi\cdot d\vec{r}$ — ie $\vec{F}$ conservative

$= \phi(x_1, y_1, z_1) - \phi(x_0, y_0, z_0)$

Example: $\vec{F} = y\vec{i} + x\vec{j}$ evaluated on $y=x$ and $y=x^3$ from $(0,0)$ to $(1,1)$
Prove path independence



Solution: $C_1: y=x$ $\quad x=t, \Rightarrow y=t.$

$d\vec{r} = dx\vec{i} + dy\vec{j}$ (2D)

$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} (y\vec{i} + x\vec{j}).(dx\vec{i} + dy\vec{j})$

$= \int_{C_1} (y\,dx + x\,dy)$ ← dot product

$= \int_0^1 \left(y(t)\frac{dx}{dt} + x(t)\frac{dy}{dt}\right)dt$

$= \int_0^1 ((t)(1) + (t)(1))\,dt$

$= \int_0^1 2t\,dt = t^2\big|_0^1 = 1$

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$C_2: \quad x = t \Rightarrow y = t^3$

$\Rightarrow \frac{dy}{dt} = 3t^2$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} (y\vec{i} + x\vec{j}) \cdot (dx\vec{i} + dy\vec{j})$$

$$= \int_{C_2} (y\,dx + x\,dy)$$

$$= \int_0^1 \left(y(t) \frac{dx}{dt} + x(t) \frac{dy}{dt} \right) dt$$

$$= \int_0^1 \left((t^3)(1) + (t)(3t^2) \right) dt$$

$$= \int_0^1 (4t^3)\, dt$$

$$= t^4 \Big|_0^1 = 1$$

$\int_{C_1} = \int_{C_2} \Rightarrow$ path independence

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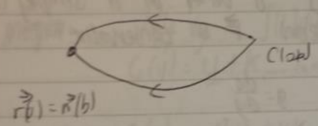
$\vec{F} = y\vec{i} + x\vec{j}$ Try $\phi = xy \Rightarrow \vec{F} = \vec{\nabla}\phi = \frac{d}{dx}(xy)\vec{i} + \frac{d}{dy}(xy)\vec{j} = y\vec{i} + x\vec{j}$

$\vec{F}$ conservative $\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = \phi(1,1) - \phi(0,0) = (1)(1) - (0)(0) = 1$

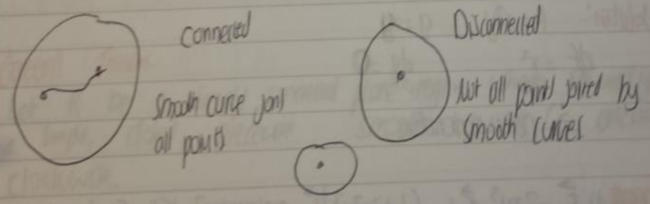
Closed path:

Let $\vec{r}(a) = \vec{r}(b)$ ie. $(x_1, y_1, z_1) = (x_0, y_0, z_0)$



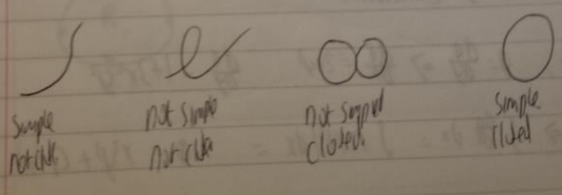
If $\vec{F} = \vec{\nabla}\phi$ (conservative)

$$\int_C \vec{F}\cdot d\vec{r} = \phi(\vec{r}(b)) - \phi(\vec{r}(a))$$
$$= \phi(x_1, y_1, z_1) - \phi(x_0, y_0, z_0) = 0$$



Theorem: For an open connected set

- $\vec{F}(x,y) = f(x,y)\vec{i} + g(x,y)\vec{j}$ is conservative
- $\int_C \vec{F}\cdot d\vec{r} = 0$ for C piecewise closed smooth curve
- $\int_C \vec{F}\cdot d\vec{r}$ path independent are all equivalent



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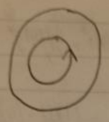
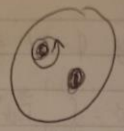


Simply connected
no holes



multiply connected
hole

If $\vec{F}(x,y) = f(x,y)\vec{i} + g(x,y)\vec{j}$ is defined on a simply connected region, then if $\frac{df}{dy} = \frac{dg}{dx}$ means $\vec{F}$ is conservative

Note if $\vec{F} = \vec{\nabla}\phi$, $f = \frac{d\phi}{dx}$, $g = \frac{d\phi}{dy}$

$$\frac{df}{dy} = \frac{d^2\phi}{dy\,dx} \qquad \frac{dg}{dx} = \frac{d^2\phi}{dx\,dy}$$

Example. Is $\vec{F} = x^2y\vec{i} + y\vec{j}$ conservative? If it is find ϕ

Solution: $f = x^2y \quad g = y$

$\frac{df}{dy} = x^2 \qquad \frac{dg}{dx} = 0$

$\Rightarrow$ not conservative

Example: Is $\vec{F} = 2xy^3\vec{i} + (1+3x^2y^2)\vec{j}$ conservative? If so find ϕ

Solution: $f = 2xy^3 \qquad g = 1+3x^2y^2$

$\frac{df}{dy} = 6xy^2 \qquad \frac{dg}{dx} = 6xy^2$

$\frac{df}{dy} = \frac{dg}{dx} \Rightarrow$ conservative

Note $f = \frac{d\phi}{dx}$, $g = \frac{d\phi}{dy} \Rightarrow \frac{d\phi}{dx} = 2xy^3 \qquad \frac{d\phi}{dy} = 1+3x^2y^2$

Integrate $\frac{d\phi}{dx} \Rightarrow \int \frac{d\phi}{dx}\,dx = \int (2xy^3)\,dx = \quad \phi = x^2y^3 + C_1(y)$

15/11/13 $\overset{3}{\text{Minimal}}$

$$\frac{d\phi}{dy} = \frac{d}{dy}(x^2y^3 + G(y)) = 3x^2y^3 + \frac{dG}{dy}$$

Also $\frac{d\phi}{dy} = g = 1 + 3x^2y^2$

$\Rightarrow \frac{dG}{dy} = 1$

Integrate $\int \frac{dG}{dy}\,dy = \int (1)\,dy$

$G(y) = y + k$ — constant

$\phi = x^2y^3 + G(y) = x^2y^3 + y + k$

Check: $\vec{F} = \vec{\nabla}\phi = \frac{d}{dx}(x^2y^3 + y + k)\vec{i} + \frac{d}{dy}(x^2y^3 + y + k)\vec{j}$

$= 2xy^3\vec{i} + (3x^2y^2 + 1)\vec{j}$

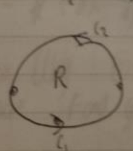
Since $\frac{dk}{dx} = 0 = \frac{dk}{dy}$

Green's Theorem:

Let R be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve C orientated anti-clockwise.

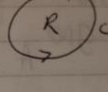
If $f(x,y)$ and $g(x,y)$ are continuous and have continuous first partial derivatives on some open set containing R, then

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{dg}{dx} - \frac{df}{dy}\right) dA$$

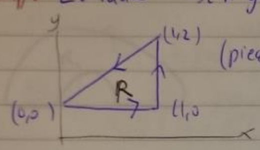


18/11/13 Green's Theorem Mum!

$$\int_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{dg}{dx} - \frac{df}{dy}\right) dA$$



Example: Evaluate $\int_C x^2y\,dx + x\,dy$ around



(piecewise smooth)

Solution: $f(x,y) = x^2y$

$g(x,y) = x$

The line from $(0,0)$ to $(1,2)$ is given by equation $y=2x$. We now apply green's theorem

$$\int_C x^2y\,dx + x\,dy = \iint_R \left[\frac{d}{dx}(x) - \frac{d}{dy}(x^2y)\right] dA = \int_0^1 \int_0^{2x} (1-x^2)\,dy\,dx$$

$$= \int_0^1 \left[(y - x^2y)\Big|_{y=0}^{2x} \right] dx$$

$$= \int_0^1 (2x - 2x^3)\,dx$$

$$= \left[x^2 - \frac{x^4}{2}\right]_{x=0}^{1} = 1 - \frac{1}{2} = \frac{1}{2}$$

$\oint f(x,y)\,dx$ denotes an integral around a closed curve. Green's theorem can be written $\oint f(x,y)\,dx + g(x,y)\,dy$

$$= \iint_R \left(\frac{dg}{dx} - \frac{df}{dy}\right) dA$$

Area Using Green's Theorem

Choose f and g such that $\frac{dg}{dx} - \frac{df}{dy} = 1$

gives an area integral, e.g. choose CORRECT $g(x,y) = x$ gives

$$A = \iint_R dA = \oint x\,dy$$

Also $g(x,y) = 0$ $f(x,y) = -y$

gives
$$A = -\iint_R dA = \oint y\,dx$$

Combining these gives
$$A = \frac{1}{2} \oint -y\,dx + x\,dy$$

$\oint -y\,dx$
$\uparrow$
f

$$= \iint_R \frac{d(0)}{dx} - \frac{d(-y)}{dy}\,dA$$

$$= \iint_R (0+1)\,dA = \iint_R dA$$

Example: Find the area of the disk bounded by the circle $x^2+y^2=a^2$

Solution $x = a\cos t$ $y = a\sin t$ $(0 \le t \le 2\pi)$
$$A = \frac{1}{2} \oint -y\,dx + x\,dy$$

Note: $\frac{dx}{dt} = -a\sin t \Rightarrow dx = -a\sin t\,dt$

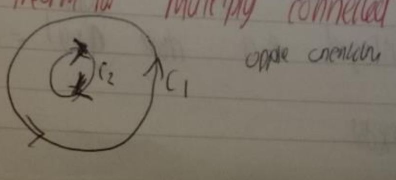
$\frac{dy}{dt} = a\cos t \Rightarrow dy = a\cos t\,dt$

$$\Rightarrow A = \frac{1}{2}\int_0^{2\pi} [-\underbrace{(a\sin t)}_{y}\underbrace{(-a\sin t)}_{dx} + \underbrace{(a\cos t)}_{x}\underbrace{(a\cos t)}_{dy}]\,dt$$

$$= \frac{a^2}{2}\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

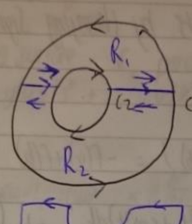
$$= \frac{a^2}{2}\int_0^{2\pi} dt = \frac{a^2}{2}(2\pi) = \pi a^2$$

Green's Theorem for multiply connected regions

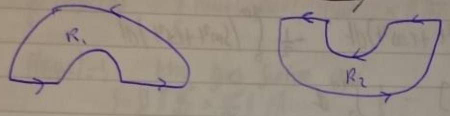


18/11/13 Maths 3

Cut region in half



DIFFERENT ORIENTATIONS



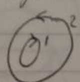
$$\iint_R \left(\frac{dg}{dx} - \frac{df}{dy}\right) dA = \iint_{R_1} \left(\frac{dg}{dx} - \frac{df}{dy}\right) dA + \iint_{R_1} \left(\frac{dg}{dx} - \frac{df}{dy}\right) dA$$

$$= \oint_{\text{bondry of } R_1} f(x,y)dx + g(x,y)dy + \oint_{\text{bondry of } R_2} f(x,y)dx + g(x,y)dy$$

Contributions from common boundaries cancel (we twice, but in opposite directions) $\Rightarrow$

$$\iint_R \left(\frac{dg}{dx} - \frac{df}{dy}\right) dA = \oint_{C_1} f(x,y)dx + g(x,y)dy + \oint_{C_2} f(x,y)dx + g(x,y)dy$$

Example: Find the area of the region bounded by a circle of radius 1 and a circle of radius 2



C_1 : $x = 2\cos t$ $y = 2\sin t$ $(0 \le t \le 2\pi)$

C_2 : $x = \cos t$ $y = \sin t$ $\cancel{(0 \le t \le 2\pi)}$ $(2\pi \le t \le 0)$

OR $\boxed{x = \cos t, \; y = -\sin t \quad (0 \le t \le 2\pi)}$

$$A = \frac{1}{2}\oint_{C_1} (-y\,dx + x\,dy) + \frac{1}{2}\oint_{C_2} (-y\,dx + x\,dy)\, dA$$

C_1 $dx = -2\sin t\, dt$
$dy = 2\cos t\, dt$

C_2 $dx = -\sin t\, dt$
$dy = \cos t\, dt$

$$= \frac{1}{2}\int_0^{2\pi} \left(-(2\sin t)(-2\sin t) + (2\cos t)(2\cos t)\right) dt$$
$$+ \int_{2\pi}^{0} \left(-(\sin t)(-\sin t) + (\cos t)(\cos t)\right) dt$$

4

Can switch limits by changing sign   in front of integral

$\int_a^b = -\int_b^a$

$\int_a^b = f(b) - f(a)$

$-\int_b^a = -(f(a) - f(b)) = -f(a) + f(b)$

$= \frac{1}{2} \, 4 \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt \quad -\frac{1}{2} \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt$

$= 2\left[\int_0^{2\pi} dt \right] - \frac{1}{2} \int_0^{2\pi} dt$

$= \underbrace{4\pi}_{\text{area disc } r=2} - \underbrace{\pi}_{\text{area disc } r=1} = 3\pi$