Maths 2. 2011

Q1. a. $y = \sin^{-1}(x)$

$\sin y = x$

$-\cos y \frac{dy}{dx} = x$

$\frac{dy}{dx} = \frac{x}{-\cos(\sin^{-1}(x))}$

b. i. $\lim_{x \to \pi^{+}} \frac{\sin x}{x - \pi}$ $\frac{0}{0}$ $\frac{-\cos x}{1} = 1$

ii. $\lim_{x \to +\infty} x e^{-x}$ $\infty . 0$

$\frac{e^{-x}}{1/x} = \frac{-e^{-x}}{\frac{-1}{x^2}} = \frac{x^2 e^{-x}}{1}$ $\infty . 0$

$\frac{e^{-x}}{\frac{1}{x}} = x^{?} e^{-x} = \infty . 0$

$\lim_{x \to +\infty} \frac{x}{e^x} = \frac{\infty}{\infty}$ $\lim_{x \to \infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$

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Q 1 d ii $\int \sin(\sqrt{x})$ $\quad u=\sqrt{x} \quad du = \frac{1}{2\sqrt{x}}$

$x^{1/2}$

$\frac{1 \sin u}{2\sqrt{u}} du$

$\cancel{\frac{1}{2}\int \frac{\sin u}{\sqrt{x}}}$

$\frac{1}{2}\ u \sin u \qquad f = u \qquad dg = \sin(u)\,du$

$df = du \qquad g = -\cos u$

$= 2\int \cos(u)\,du - 2u\cos(u)$

$= 2\sin(u) - 2u\cos(u)$

$= 2\sin(\sqrt{x}) - 2\sqrt{x}\cos\sqrt{x}$

2 i a $\int \cos^3 x \sin^2 x\,dx$ $\quad \cancel{u = \cos^3 x \quad du = 3\cos^2 x(\sin x)\,dx}$

$u = \sin x \quad du = \cos x\,dx$

$\int \frac{\cos^3 x}{\cos x} u^2$

$\int \cos^2 x\, u^2$

$\int (1-u^2)u^2$

$\int u^2 - u^4$

$\frac{u^3}{3} - \frac{u^5}{5}$

$\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}$

iii $\cosh = \text{even} = \frac{e^x + e^{-x}}{2}$ $\sinh = \text{odd} = \frac{e^x - e^{-x}}{2}$

$d\cosh = \frac{1}{2}(e^x + (-e^{-x}))$

$= \frac{e^x - e^{-x}}{2} = \sinh(x)$

A hyperbolic function is a group of functions of an angle expressed as the relationship between the distance from any point on the hyperbola to its origin and the coordinate axes

di. $\int x \sin 2x\,dx$ $u = x$ $dv = \sin 2x\,dx$ $u = 2x$

$\frac{1}{2}\int \frac{u}{2}\sin u\,dx$

$\frac{1}{4}\left[u(-\cos u) - \int -\cos u\,du\right]$

$\frac{1}{4}\left[u(-\cos u) + \sin u\right]$

$\frac{1}{4}\left[-2x\cos 2x + \sin 2x\right]$

ii. $\int \sin(\ln x)\,dx$ $u = \ln x$ $du = \frac{1}{x}dx$ Sunny eqn

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$\int \frac{\sin u}{x}\,dx$

$\int \frac{1}{x}\sin u$ $u = \frac{1}{x}$ $dv = \sin u\,du$

$v = -\cos u$

$\frac{1}{x}(-\cos u) - \int -\cos u \; -\frac{1}{x^2}\,dx$

$-\frac{\cos u}{x}$

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2ib $\int \frac{1}{x^2+7x+12}$ $\frac{1}{(x+4)(x+3)}$

$\frac{A}{x+4} \cdot \frac{B}{x+3}$

$\frac{Ax+3A+Bx+4B}{(x+4)(x+3)}$

$A+B=0$ $\quad 3A+4B=1$

$B=-A$ $\quad 3A+-4A=1$

$-A=1$

$A=-1$

$B=1$

$\int \frac{-1}{x+4} - \frac{1}{x+3}$

$\ln|x+4| - \ln|x+3|$

2ic $\int \frac{1}{(x^4-1)}$ $\frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x-1)(x+1)(x^2+1)}$

$1 = Ax^4 + (C+D)x^3 + (A+B-C+D)x^2 + (C+D)x + (D-B-C)$

$A=0$ | $C+D=0$, $C=-D$ | $D-B-C=1$, $2D-B=1$, $2D=1+B$ | $A+B-C+D=0$, $B-C+D=0$, $B+2D=0$, $2B+1=0$

$\Rightarrow \frac{1}{2}\int \frac{1}{x^2+1}\,dx - \frac{1}{4}\int \frac{1}{x+1} + \frac{1}{4}\int \frac{1}{x-1}$ $D=\frac{1}{4}$ $C=\frac{1}{4}$

$= -\frac{1}{2}\tan^{-1}x - \frac{1}{4}\ln|x+1| + \frac{1}{4}|x-1| + C$

2id $\int \frac{2x+3}{x^2+2x+3}$ $\quad U = x^2+2x+3 \quad du = 2x+2\,du$

$\frac{\frac{2x+3}{u}}{\frac{2x+2du}{1}} = \int \frac{1}{U} = \int U^{-1} = \int$

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3b i. $\sum_{n=0}^{\infty} (-1)^n$ ratio test $a_n = (-1)^n$

$a_{n+1} = (-1)^{n+1}$

$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}}{(-1)^n} = -1 < 1$ ~~conver~~ diverge

ii. $\sum_{n=1}^{\infty} \frac{1}{p^{3/2}}$ a_n $\frac{1}{n^{3/2}}$ p-series

a_{n+1} $\frac{1}{(n+1)^{3/2}}$ $p = 3/2$ hence diverge

$\frac{1}{(n+1)^{3/2}}$ $\frac{n^{3/2}}{1}$ $\frac{n^{3/2}}{(n+1)^{3/2}}$

$\int x^{-3/2}$

$\int_1^{\infty} -\frac{3}{2\sqrt{x}}$ -1.5

$-\frac{3}{2} - [0]$

$= -1.5$

iii. $\sum_{n=1}^{\infty} \frac{1}{n+3}$ $\frac{ar}{1-r}$ 1

$\frac{1}{n+3}$ $\frac{n+1+3}{1}$ $\frac{n+4}{n+3} = 1$ as $n \to \infty$ both conv or diver

$\int \frac{1}{n+3}$ $u = n+3$

$du = dx$

$\int \frac{1}{u} dx$

$= \left[\ln(u)\right]_4^{\infty}$

$\ln 4 - \ln(\infty)$ negative hence diverge d i $\int u\,dv = uv - \int v\,du$

$\int x \sin 2x\,dx$ — $u = x$ $\quad dv = \sin 2x\,dx$

$\frac{du}{dx} = 1$ $\quad v = \int dv = \int \sin 2x$

— $du = dx$ $\quad V = -\frac{1}{2}\cos(2x)$

$\int x \sin 2x\,dx = x\left(-\frac{1}{2}\cos(2x)\right) - \int -\frac{1}{2}\cos(2x)\,dx$

$-\frac{x}{2}\cos(2x) - \frac{1}{2}\int -\cos(2x)$ $\quad \cos(2x) = u = 2x$

$\frac{du}{dx} = 2$

$\cancel{-\frac{x}{2}\cos(2x) + \frac{1}{2}\sin(x)\cos(x)}$

$du = 2dx$

$\frac{1}{4}\int \cos(u)\,du$

$= \frac{\sin(u)}{4}$

$= \frac{\sin 2x}{4} - \frac{1}{2}x\cos 2x$

$= \frac{1}{4}\left(\sin(2x) = 2x\cos(2x)\right)$

ii $\int \sin(\ln x)\,dx$ $\quad u = \ln x$ $\quad u = \sin(x)$

$\frac{du}{dt} = \frac{1}{x}$ $\quad du = \cos$

$\int \frac{\sin(u)}{x}\,du$ $\quad du = \frac{1dx}{x}$

$\int \overset{u}{\frac{1}{x}} \overset{dv}{\sin(u)}\,du$

$u = \cancel{x}^{-1}$ $\quad \frac{du}{dx} = \frac{-1}{x^2}$ $\quad dv = \sin(u)$

$v = \int \sin(u)$

$du = \frac{-1}{x^2}dx$ $\quad = -\cos(u)$

$\overset{uv}{\int \frac{1}{x}\sin(u)} = \frac{1}{x}\left(-\cos(u)\right) - \int +\cos(u)\,\frac{+1}{x^2}\,dx$

$\frac{-\cos(u)}{x}$ $\quad -\int \frac{1}{x^2}\cos u$

$\frac{\sin(u)}{x^2} - \int \sin(u)\left(-\frac{1}{x^3}\right)dx$

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Q. 1d iii. $\sin(\ln x)dx$ $u = \ln x$

$du = \frac{1}{x} dx.$

$= \int e^u \sin(u) dx \qquad (e^{\ln x} = \frac{1}{x})$

rule $\int \exp(\alpha u) \sin(\beta u) du = \frac{\exp(\alpha u)(\alpha \sin \beta u) - \beta \cos(\beta u)}{\alpha^2 + \beta^2}$

$\frac{1}{2} e^u \sin(u) - \frac{1}{2} e^u \cos(u)$

$u = \ln x \Rightarrow \frac{1}{2} x \sin(\ln x) - \frac{1}{2} x \cos(\ln x) + c$

$= -\frac{1}{2} (x (\cos(\ln x)) - \sin(\ln x))) + c$

iii. $\int \sin(\sqrt{x}) dx$ $u = x^{\frac{1}{2}}$

$\frac{du}{\sqrt{x}} = \frac{1}{\sqrt{x}}$ $du = \frac{1}{2\sqrt{x}} dx$

$\int \frac{\sin(u)}{\sqrt{x}} dx$

$\int \frac{1}{2(u)} \sin(u) dx$

$\frac{1}{2} \int \frac{\sin(u)}{u}$

$\frac{1}{2} \int u^{-1} \sin(u)$

$\frac{1}{2} \left(\frac{u^{-2}}{2} \cos(u) \right)$

$-2u^{-1}$

$2 \sin(u) - 2u \cos u$

1v $\sum_{n=0}^{\infty} \frac{(2n)!}{4n}$ $a = \frac{(2n)!}{4n}$.

a_{n+1} $\frac{2(n+1)!}{4(n+1)}$

$$\frac{2(n+1)!}{4\cancel{(n+1)}^{n+1}} \frac{\cancel{4}^n}{(2n)!}$$

$$\frac{(2n+2)!}{(n+1)(2n)!} \qquad \frac{(\cancel{\frac{2n+2}{2n}})!\,(2n+2)(2n+1)}{4} \quad \to \infty \text{ as } n \to \infty$$

3c $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3n}$

$$\frac{(x-2)^{n+1}}{3^{\cancel{n+1}}} \quad \frac{3^{\cancel{n}}}{(x-2)^n} \qquad \frac{(x-2)}{3}$$

Conv when <1 $x<5$ $x \neq 5$

divg >1 $x>5$

iii. $\int \sin(\sqrt{x})dx$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} dx$

$= \int \frac{\sin(u)dx}{\frac{1}{2\sqrt{x}} dx}$

$= 2\int \sqrt{x} \sin u$

$= 2\int u \sin u$

$= 2\int$

By Parts

$f = u$

$\frac{df}{du} = 1$

$df = du$

$dg = \sin u \, du$

$\int \sin u$

$g = -\cos(u)$

$= u(-\cos(u)) - \int -\cos(u)$

$= u(-\cos u) + \int \cos(u)$

$= -2u \cos u + 2\sin(u)$

$= 2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x} + c$

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2i. a $\int \cos^3 x \sin^2 x$

$u = \cos^3 x$

$du = 3\cos^2$

$\int \cancel{\cos^3 x (1-\cos^2 x)}$

$\int \cancel{\cos^3} x - \cos^5 x$

$\int \cancel{\cos^{-2}} x$

$= \cancel{\frac{\cos^{-1} x}{-1}}$

$u = \sin x \qquad du = \cos x \, dx$

$\int \cos^2 x \; u^2 \, du$

$\int (1-u^2) u^2$

$\int u^2 - u^4 \, du$

$= \frac{u^3}{3} - \frac{u^5}{5} + C$

$u = \sin x \qquad \frac{\sin x^3 x}{3} - \frac{\sin x^5}{5}$

b $\int \frac{1}{(x^2+7x+12)} \qquad \frac{1}{(x+4)(x+3)} \qquad \frac{a}{x+4} + \frac{b}{x+3}$

$ax + 3a + bx + 4b$

$(a+b)x + 3a + 4b$

$a+b=0 \qquad 3a+4b=1$

$b = -a \qquad 3a + 4(-a) = 1$

$a = -1$

$b = 1$

$\int \frac{1}{x+4} - \frac{1}{x+3}$

$\ln|x+4| - \ln|x+3|$

$= \ln\left|\frac{x+4}{x+3}\right|$

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1a $y = \sin^{-1} x$

$\sin y = x$

$\frac{dy(x)}{dx} = \frac{d(\sin y)}{dx}$

$1 = \cos y \frac{dy}{dx}$

$\frac{1}{\cos(\sin^{-1} x)} = \frac{dy}{dx}$

bi $\lim_{x \to \pi^+} \frac{\sin x}{x - \pi}$ $\frac{\sin \pi}{0} = \frac{0}{0}$

$f(x) = \sin x$ $\frac{\cos x}{1} = \cos(x)$ $\cos(\pi) = -1$

$g(x) = x - \pi$

ii $\lim_{x \to +\infty} x e^{-x}$ $\infty e^{-\infty} = \infty \cdot 0$

$\cancel{\frac{e^{-x}}{1/x} \quad f(x) = e^{-x} \quad g(x) = 1/x \quad f'x = -e^{-x} \quad g(x) = -1/x^2 \quad \frac{-e^{-\infty}}{-1/\infty}} = \frac{0}{0}$

$\cancel{\frac{e^{-x}}{1/x^3} \quad \frac{e^{-\infty}}{1/\infty} \quad \frac{0}{0}}$

$\frac{x}{e^x} = \frac{1}{e^x} = \frac{1}{\infty} = \infty$

c $\cosh x = \frac{e^x + e^{-x}}{2}$ $\sinh x = \frac{e^x - e^{-x}}{2}$

$\frac{d(\cosh x)}{dx} = \frac{d \frac{e^x + e^{-x}}{2}}{dx} = \frac{1}{2} d\, e^x + e^{-x}$

$= \frac{1}{2}(e^x + (-1)e^{-x})$

Any of a group of functions of an angle expressed as a relationship between the $= \frac{(e^x - e^{-x})}{2} = \sinh(x)$ distances of a point on a hyperbola to the origin and to the coordinate axes, as sinh or cosh c $\int \frac{1}{(x^4-1)}$ or $\frac{1}{(x^2-1)(x^2+1)}$ $\frac{Ax+b}{x^2-1} + \frac{cx+d}{x^2+1}$

$Ax+b(x^2+1) + cx+d(x^2-1)$

$Ax^3 + Ax + bx^2 + b \quad + cx^3 - cx + dx^2 - d$

$x^3(A+c) + x^2(b+d) + (A-c)x + (b-d)$

$b-d=1$

$b=1+d$

$A+c=0$ $\quad b+d=0$ $\quad A-c=0$

$A=-c$

$1+d+d=0$

$2d=1$

$\boxed{d=\frac{1}{2}}$ $\quad \boxed{b=\frac{-1}{2}}$

$A+c=0$

$A-c=0$

$b+\frac{1}{2}=0$

$A=0$

$\frac{1}{x^4-1} = \frac{Ax+B}{x^2-1} + \frac{C}{x+1} + \frac{D}{x-1}$

$1 = Ax^4 + (C+D)x^3 + (A+B-C+D)x^2 + (C+D)x + (D-B-C)$

$A=0$ $\quad \Big| \quad C+D=0$, $C=-D$ $\quad \Big| \rightarrow D-B-C=1$, $2D-B=1$, $2D=1+B$ $\quad \Big| \quad A+B-C+D=0$, $B-C+D=0$, $B+2D=0$, $2B+1=0$

$B=\frac{1}{2}$ $\quad D=\frac{1}{4}$ $\quad C=\frac{-1}{4}$

$\Rightarrow \frac{1}{2}\int \frac{1}{x^2+1}dx - \frac{1}{4}\int\frac{1}{x+1} + \frac{1}{4}\int\frac{1}{x-1}$

$= -\frac{1}{2}\tan^{-1}x - \frac{1}{4}\ln|x+1| + \frac{1}{4}|x-1| + C$

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Q2d $\int \frac{2x+3}{x^2+2x+3}$ $\qquad \frac{2x+3}{(x-1)(x+3)}$ $\qquad \frac{x^2+2x+3-x^2}{x^2+2x+3}$

$u = x^2+2x+3 \qquad du = 2x+2dx$

$\int \frac{2x+2 \quad +1}{x^2+2x+3} \qquad \frac{2x+2+1}{u} du = \int \frac{1}{u}$

e $\int_0^1 \frac{1}{\sqrt{x}} dx$ unbounded on [0, 1] therefore an improper integral and must be evaluated as a limit

$\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{\sqrt{x}} dx = \lim_{\varepsilon \to 0^+} (2\sqrt{x})\Big|_\varepsilon^0$

The right hand side is $\lim_{\varepsilon \to 0^+} (2 - 2\sqrt{\varepsilon}) = 2$

2iii a $x\frac{dy}{dx} - y^2 = 1 \qquad \cancel{\int -y^2 dx = -\frac{y^3}{3}}$

$\frac{dy}{dx} - y^2 = \frac{1}{x} \qquad \int -1 = -x$

$-x\frac{dy}{dx} - (-x)y^2 = \frac{-x}{x}$

$= -1$