2011 MANG. SCI PAPER 1 Q6 DAVID WEITBRECHT.

18/20

6a Show that the following integral below is independent of path:

$$\int_{(-\pi,0)}^{(\pi/2,\pi)} (-3x^2+\pi^3\cos(x)+2y^2)\,dx - (3\pi^3\sin(y)-4xy)\,dy$$

If it is conservative, $\frac{df}{dy} = \frac{dg}{dx}$

$\frac{d(-3x^2+\pi^3\cos(x)+2y^2)}{dy} = 4y$ $\qquad \frac{d[-(3\pi^3\sin(y)-4xy)]}{dx} = +4y$

$\frac{df}{dy} = \frac{dg}{dx} \Rightarrow$ independent of path and conservative 2

B. Find a potential function $\phi(x,y)$

As F is a conservative, potential function exists.

We have $\frac{d\phi}{dx} = (-3x^2+\pi^3\cos(x)+2y^2)$ $\qquad \frac{d\phi}{dy} = (-3\pi^3\sin(y)+4xy)$

Integrate first eq^n with respect to x.

$$\int \frac{d\phi}{dx}\,dx = \int(-3x^2+\pi^3\cos(x)+2y^2)\,dx$$

$$\phi = -x^3+\pi^3\sin(x)+2y^2x+C_1(y)$$

(where $C_1(y)$ depends on y only and is therefore a constant with respect to integration in x

Differentiate this expression for ϕ with respect to y to equal it to our 2nd original eq^n.

$$\frac{d}{dy}(-x^3+\pi^3\sin(x)+2y^2x+C_1(y)) = 4yx+\frac{d(C_1(y))}{dy} = -3\pi^3\sin y+4xy$$

2

$$\frac{d(C)}{dy} = -3\pi^3 \sin y$$

integrate $\Rightarrow$ $3\pi^3 \cos y + C$ where C is any constant

$\Rightarrow \Phi = -x^3 + \pi^3 \sin(x) + 2y^2 x + 3\pi^3 \cos(y) + C$ ✓   10

C. Use the fundamental theorem of line integrals to find the value of the integral

When $F(x,y) = \nabla\phi(x,y)$

$$\int_C F(x,y)\,dr = \int_C \nabla\phi \cdot dr = \phi(x_1,y_1) - \phi(x_0,y_0)$$

$(x_1,y_1) = (\pi/2, \pi)$ $(x_0,y_0) = (-\pi, 0)$

$$-\left(\frac{\pi}{2}\right)^3 + \pi^3 \sin\left(\frac{\pi}{2}\right) + 2(\pi^2)\left(\frac{\pi}{2}\right) + 3\pi^3 \cos(\pi) - \left[-(-\pi)^3 + \pi^3 \sin(-\pi) + 2(0)^2(-\pi) + 3\pi^3 \cos(0)\right]$$

$$-1\left[\pi^3 - \pi^3 + 0 + 3\pi^3\right]$$

$$-\frac{\pi^3}{8} + \pi^3 + \pi^3 \qquad - [3\pi^3$$

$$= \frac{15\pi^3}{8} \qquad - 3\pi^3 = -\frac{9}{8}\pi^3$$

$$-\frac{9\pi^2}{8} \quad - \quad 4\pi^3 = -\frac{41\pi^3}{8}.$$

1

D Choose the integration path C between the points $(-\pi, 0)$ and $(\pi/2, \pi)$ to be a curve formed from two line segments C_1 and C_2, where C_1 is joining $(-\pi, 0)$ and $(\pi/2, 0)$ and C_2 is joining $(\pi/2, 0)$ and $(\pi/2, \pi)$.

i. Plot the integration path C and show its orientation on the plot.

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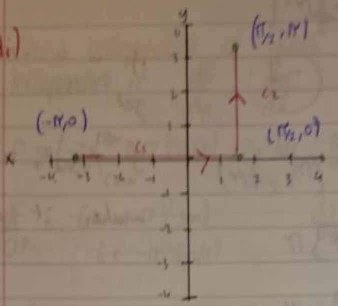
6(d i)

6 d i



6 D ii. Parameterise C_1 and C_2 and evaluate

$\int_C (-3x^2 + \pi^3\cos(x) + 2y^2)\,dx - (3\pi^3\sin(y) - 4xy)\,dy$

r_0 to $r_1 \Rightarrow (1-t)r_0 + t(r_1)$ for $0 \le t \le 1$

$C_1\,(-\pi, 0) \to (\pi/2, 0) = (1-t)(-\pi, 0) + t\left(\frac{\pi}{2}, 0\right) = \left(-\pi + \pi t + \frac{t\pi}{2}, 0\right)$

$\left(-\pi + \frac{3t\pi}{2}, 0\right)$

$C_2\,(\pi/2, 0) \to (\pi/2, \pi) = (1-t)(\pi/2, 0) + t\left(\frac{\pi}{2}, \pi\right) = \left(\frac{\pi}{2} - \frac{t\pi}{2} + \frac{t\pi}{2}, t\pi\right)$

$\left(\frac{\pi}{2}, t\pi\right)$

Parameterise C_1 $\frac{dx}{dt} = \frac{3\pi}{2}$ $\frac{dy}{dt} = 0$ dont need second part

$\int_0^1 \left[-3\left(-\pi + \frac{3t\pi}{2}\right)^2 + \pi^3\cos\left(-\pi + \frac{3t\pi}{2}\right) + 2(0)^2\right]\frac{3\pi}{2}\,dt$

$\left[-3\left(\pi^2 + \frac{9t^2\pi^2}{4} - \frac{6t\pi^2}{2}\right) + \pi^3\cos\left(-\pi + \frac{3t\pi}{2}\right)\right]\frac{3\pi}{2}$

watch the sign.

$\frac{3\pi}{2}\int_0^1 -3\pi^2 - \frac{27t^2\pi^2}{4} + 9t\pi^2 + \pi^3\cos\left(-\pi + \frac{3t\pi}{2}\right)$

$u = \frac{3\pi t}{2}$ $du = \frac{3\pi}{2}\,dt$

$= \frac{-2}{3\pi}\int \cos(u)\,du$

$= \frac{2\sin(u)}{3\pi}$

Sub back in u

this is only for u substitution

$= \frac{3\pi}{2}\left[-3\pi^2 t - \frac{9}{4}t^3\pi^2 + \frac{9t^2\pi^2}{2} + \pi^3\left(-\frac{2\sin\left(\frac{3\pi t}{2}\right)}{3\pi}\right)\right]_{t=0}^{1}$

Sorry!

$\frac{3\pi}{2}\left[-3\pi^2 - \frac{9}{4}\pi^2 + \frac{9\pi^2}{2} + -\pi^3\left(\frac{2}{3\pi}\right)(1)\right]$

$\frac{3\pi}{2}\left[-3\pi^2 - \frac{9\pi^2}{4} + \frac{9\pi^2}{2} + \frac{2\pi^2}{3}\right]$

$-\frac{55}{8}\pi^3 \quad -\frac{\pi^3}{8}$ -7

if $u = -\pi + \frac{3\pi t}{2}$

get

$\int_{u(0)}^{u(1)} (-3u^2 + \pi^3 \cos u)\,du$

makes it easier!

C_2: $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = \pi$. $(\pi/2, t\pi)$

$[-3\pi^3 \sin(t\pi) + 4(\frac{\pi}{2})(t\pi)]\pi$

$\pi \int_0^1 [-3\pi^3 \sin(t\pi) + 2t\pi^2]\,dt$

$\pi\left[-3\pi^3\left(-\frac{\cos(\pi t)}{\pi}\right) + t^2\pi^2\right]\Big|_{t=0}^{t=1}$

$\pi\left[\left[\frac{-3\pi^3}{\pi} + \pi^2\right] - \left[\frac{3\pi^3}{\pi}\right]\right]$

$[-2\pi^2 - 3\pi^2]\pi$

$-5\pi^3$ ✓

Add both results together.

$-5\pi^3 - \frac{55}{8}\pi^3 = \frac{-15\pi^3}{8}$

4.

2011 MANG SCI PAPER Q 6 CORRECTION

6 a Show independent of path

Independent if $\frac{df}{dy} = \frac{dg}{dx}$

$F = (-3x^2 + \pi^3 \cos(x) + 2y^2)dx \quad g(3\pi^3 \sin(y) - 4xy)dy$

$\frac{df}{dx} = -6x + \pi^3(-\sin) \qquad \frac{dg}{dy} = -3\pi^3 \cos$

$-6x - \pi^3 \sin(x)$

$\frac{df}{dy} = 4y \qquad \frac{dg}{dx} = +4y$

$\frac{df}{dy} = \frac{dg}{dx} \Rightarrow$ conservative $\Rightarrow$ path independent

$d\cos = -\sin$
$d\sin = \cos$
$\int \sin = -\cos$
$\int \cos = \sin$

B Find potential function

$\phi(x,y)$

$\frac{d\phi}{dx} = (-3x^2 + \pi^3 \cos(x) + 2y^2) \qquad \frac{d\phi}{dy} = (-3\pi^3 \sin(y) + 4xy)$

$\phi = \int(-3x^2 + \pi^3 \cos(x) + 2y^2)dx$

$\phi = -x^3 + \pi^3(\sin(x) + 2xy^2 + G(y)$

where $G(y)$ depends on y only and is therefore a constant with respect to integration in x.

$\frac{d\phi}{dy} = 4xy + \frac{dG(y)}{dy} = -3\pi^3 \sin(y) + 4xy$

$\frac{d[G(y)]}{dy} = -3\pi^3 \sin(y)$

$G(y) = 3\pi^3 \cos(y) + C$ where C is any value

$\phi = \cancel{4xy + 3\pi^3 \cos(y)} + C$

$= -x^3 + \pi^3 \sin(x) + 2xy^2 + 3\pi^3 \cos(y) + C$

C Use Fundamental theorem of line integrals to find value of integral

when $F(x,y) = \nabla\phi(x,y)$

$$\int_C F(x,y)\cdot dr = \int_C \nabla\phi(\cdot dr) = \phi(x_1,y_1) - \phi(x_0,y_0)$$

$(-\pi, 0) \quad (\pi/2, \pi)$

$$-(-\pi)^3 + \pi^3 \underbrace{\sin(-\pi)}_{0} + 2\underbrace{(0^2)}_{0}(-\pi) + 3\pi^3\cos(0) - \left[-\left(\tfrac{\pi}{2}\right)^3 + \pi^3\sin\left(\tfrac{\pi}{2}\right) + 2(\pi^2)\left(\tfrac{\pi}{2}\right) + 3\pi^3\cos(\pi)\right]$$

$$\left[\pi^3 + 3\pi^3\right] - \left[\frac{-\pi^3}{8} + \pi^3 + \pi^3 - 3\pi^3\right]$$

$$\cancel{4\pi^3 - \left[-\tfrac{7}{8}\pi^3\right]}$$

$$4\pi^3 + \frac{7}{8}\pi^3$$

$$\frac{39\pi^3}{8}$$

$$-\left(\frac{\pi}{2}\right)^3 + \pi^3\sin\left(\frac{\pi}{2}\right) + 2(\pi^2)\left(\frac{\pi}{2}\right) + 3\pi^3\cos(\pi)$$

$$\frac{-\pi^3}{8} + \pi^3 + \pi^3 - 3\pi^3 = -\frac{9}{8}\pi^3$$

$$-\frac{9}{8}\pi^3 \quad -4\pi^3 = \frac{-41\pi^3}{8}.$$

D r_0 to $r_1 = (1-t)r_0 + t(r_1)$

$C_1\ (-\pi,0) \Rightarrow (\pi/2, 0) : (1-t)(-\pi,0) + t\left(\frac{\pi}{2},0\right) \cdot \left(-\pi + \pi t + \frac{t\pi}{2}, 0\right)$

$C_2 : (\pi/2, 0) \Rightarrow (\pi/2, \pi) = (1-t)(\pi/2, 0) + t(\pi/2, \pi) = (\pi/2, t\pi)$

$C_1\ \frac{dx}{dt} = \pi + \frac{\pi}{2} = \frac{3\pi}{2} \qquad \frac{dy}{dt} = 0$ no need for second part $\left[-\pi + \pi t + \frac{t\pi}{2}, 0\right]$

$$\left[-3\left(-\pi + \frac{3t\pi}{2}\right)^2 + \pi^3\cos\left(-\pi + \frac{3t\pi}{2}\right) + 2(0)^2\right]\frac{3\pi}{2}$$

$$\left[-3\left(\pi^2 + \frac{9t^2\pi^2}{4} - 3t\pi^2\right) + \pi^3\cos\left(-\pi + \frac{3t\pi}{2}\right)\right]\frac{3\pi}{2}$$

$$\left[-3\pi^2 - \frac{27t^2\pi^2}{4} + 9t\pi^2 + \pi^3\cos\left(-\pi + \frac{7t\pi}{2}\right)\right]\frac{3\pi}{2}$$

$$\frac{3\pi}{2}\left[-3\pi^2 t - \frac{27\pi^2}{4}\frac{t^3}{3} + \frac{9t^2\pi^2}{2} + \left(-\frac{\sin\left(\frac{3\pi}{2}\right)}{2}\right)3\pi\right] \quad u = \frac{3\pi t}{2} \quad du = \frac{3\pi}{2}\,dt$$

$$\frac{3\pi}{2}\int \cos(u)\,du$$

$$\frac{3\pi}{2}\sin\left(\frac{3\pi}{2}\right)$$

$$\frac{3\pi}{2}\left[-3\pi^2 t - \frac{27\pi^2}{4}\frac{t^3}{3} + \frac{9t^2\pi^2}{2} - \frac{3\pi}{2}\sin\left(\frac{3\pi}{2}\right)\right|_{t=0}^{1}$$

$$\frac{3\pi}{2}\left[\left[-3\pi^2 - \frac{27\pi^2}{12} + \frac{9\pi^2}{2} + \frac{3\pi}{2}\right] - \left[\frac{3\pi}{2}\right]\right]$$

$$\frac{3\pi}{2}\left[-\frac{3}{4}\pi^2 + \frac{3\pi}{2}\right] - \frac{9\pi^2}{4}$$

$$\cancel{= -\frac{3}{4}\pi^2}$$

$$\frac{3\pi}{2}\left[-3\pi^2 - \frac{9\pi^2}{4} + \frac{9\pi^2}{2} + \pi^3\left(\frac{2}{3\pi}\right)\right] -$$

$$\frac{3\pi}{2} \quad \cancel{\frac{\pi}{12}} \quad -\frac{1}{12} = -\frac{1}{8}\pi^3$$

$$C_2: \frac{dx}{dt} = 0 \quad \frac{dy}{dt} = \pi \quad (\pi/2,\ t\pi)$$

$$-3\pi^3\sin(t\pi) - 4\left(\frac{\pi}{2}\right)(t\pi) \quad dt$$

$$\int_0^1 -3\pi^3 \sin t\pi - 2t\pi^2 \quad dt$$

$$-3\pi^3\left(-\frac{\cos(t\pi)}{\pi}\right) + \frac{t^2\pi^2}{1}\Big|_{t=0}^{1}$$

$$\pi \cdot \left[\left[\frac{-3\pi^3}{\pi} + \pi^2\right] - \left[\frac{3\pi^3}{\pi}\right]\right]$$

$$\left[-2\pi^2 - 3\pi^2\right]\pi$$

$$= -5\pi^3$$

$$-5\pi^3 - \frac{\pi^3}{6} = -\frac{41}{8}\pi^3$$