Maths Paper 1 2012

1 a f_1 - not a function, for each x-value there may be several y values, the function is not 1-2-1 and will fail vertical line test does not determine a single person for each day

f_2 - is a function for each x-value (person) there is only one y-value (day)

b $f(x) = -2+\sqrt{x-3}$ domain: $[3, \infty)$
range = $[-2, \infty)$

$f^{-1} =$ $y = -2+\sqrt{x-3}$
$x = -2+\sqrt{y-3}$
$x+2 = \sqrt{y-3}$
$(x+2)^2 = y-3$ $y = (x+2)^2+3$ dom range $[-2, \infty]$
$f^{-1}(x) = (x^2+2)^2+3$ range $[3, \infty]$

c $\lim_{x\to 3^+} \sqrt{3x-9} = 0$

$0 < |f(x)-L| < \epsilon$ if $0<|x-a|<\delta$

$0 < |\sqrt{3x-9} - 0| < \epsilon$ if $0<|x-3|<\delta$

$0 < |3x-9| < \epsilon^2$

$0 < 3|x-3| < \epsilon^2$

$0 < |x-3| \frac{\epsilon^2}{3}$ if $0 < |x-3| < \delta$

$\frac{\epsilon^2}{3} = \delta$

d $f(x) = \begin{cases} \frac{\sin 7x}{x} & x\neq 0 \\ k(k+6) & x=0 \end{cases}$

$\lim_{x\to 0} \frac{\sin 7x}{x} \cdot \lim_{x\to 0} \frac{7}{7}$

$\lim_{x\to 0} \frac{\sin 7x}{7x} \cdot \lim_{x\to 0} 7$

$1 \cdot 7 = 7$

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need $\lim_{x \to 0} f(x) = f(0)$

$f(0) = k(k+6)$

$k(k+6) = 7$

$k^2 + 6k - 7 = 0$

$(k-1)(k+7) = 0$

$k = 1$ or $k = -7$

2 a The function $f'(x)$ defined by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

is called the derivative in respect to x

The domain of $f'(x)$ consists of all $x \in D(f)$ for which the limit above exists

b $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \Rightarrow \lim_{h \to 0} \frac{|x+h| - |x|}{h}$

$x=0$ $f'(0) = \lim_{h \to 0} \frac{|h|}{h}$ $\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$

$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$

The two sided limit does not exist, $f(x)$ is not differentiable at $x=0$.

c i $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ $\frac{x^2+3x-1}{x+5}$ u v

$\frac{(x+5)(2x+3) - (x^2+3x-1)(1)}{(x+5)^2}$

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2a) $y = \frac{d\left(\cos^2\left((x^2+7)^2\right)\right)^{\frac{1}{2}}}{d\cos^2\left((x^2+7)^2\right)} \quad \frac{d\cos^2\left((x^2+7)^2\right)}{d\cos\left((x^2+7)^2\right)} \quad \frac{d\cos(x^2+7)^2}{d(x^2+7)^2} \quad \frac{d(x^2+7)^2}{d\,x^2+7} \quad \frac{d\,x^2+7}{dx}$

$\frac{1}{2}\left(\cos^2\left((x^2+7)^2\right)\right)^{-3/2}(2)\cos\left((x^2+7)^2\right) - \sin(x^2+7)^2\,(2)(x^2+7)(2x)$

2aii) $x^3y^2 - 5x^2y + x = 1$

$x^3\,2y\frac{dy}{dx} + y^2\,3x^2 \quad -5x^2\frac{dy}{dx} + y\,10x + 1 = 0$

$\frac{dy}{dx}\left(x^3 2y - 5x^2\right) = -10xy\ 3y^2x^2 - 1$

$\frac{dy}{dx} = \frac{-10xy - 3y^2x^2 - 1}{x^3 2y - 5x^2}$

2d $f(x) = \sqrt[4]{x}$ $\quad x = 17$, $x_0 = 16$ $\quad f(x) \approx f(x_0) + f'(x_0)(x - x_0)$

$f'(x) = \frac{1}{4(x)^{3/4}}$

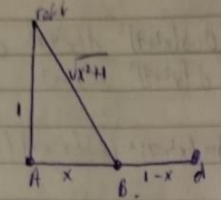
$\sqrt{x} \approx \sqrt[4]{x_0} + \frac{1}{4(x)^{3/4}}(x - x_0)$

$\sqrt[4]{16} + \frac{1}{4(16)^{3/4}}(17 - 16)$

$2 + \frac{1}{32} = \frac{65}{32} = 2.03125$

- If a function f is differentiable at x_0 then a sufficiently magnified portion of the graph of f centred at point $P(x_0, f(x_0))$ takes on the appearance of a straight line segment
- For this reason a function is differentiable at x_0 is sometimes said to be locally linear at x_0.

3a



distance rock to b.

$y^2 = x^2 + 1^2$

$y = \sqrt{x^2+1}$

length b to d = 1-x

Speed in water = 3 km/h

land = 5/h $\quad$ time = $3(\sqrt{x^2+1}) + 5(1-x)$

$0 \leq x \leq 1$

$\frac{dy}{dx} = 3\left(\frac{1}{2}\right)(x^2+1)^{-1/2}\, 2x \quad + 5(-1)$

$\frac{3x}{\sqrt{x^2+1}} - 5 = 0$

$\frac{3x}{\sqrt{x^2+1}} = 5$

$3x = 5(\sqrt{x^2+1})$

$9x^2 = 25(x^2+25)$

$16x^2 + 25 = 0$

$16x^2 = -25$

$x^2 = \frac{-25}{16}$

$x = \pm\sqrt{5/4}$

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3b $y = 3x^4 - 12x^2$

Critical point

Critical point are at the stationary point

$f'(x) = 12x^3 - 24x = 0$

$12x(x^2-2) = 0$

$12x = 0 \mid x^2 - 2 = 0$

$x = 0 \mid x^2 = 2$

$x = \pm\sqrt{2}$

$f''(x) = 36x^2 - 24$

$x = 0$ – relative max

$x = +\sqrt{2}$ + $-\sqrt{2}$ + + + 0 - - - $\sqrt{2}$ + + + +

min

$x = -\sqrt{2}$ – max

$x = 0$ = max

$x = +\sqrt{2}$ = min

$x = -\sqrt{2}$ = max.

Intervals increasing/decreasing

- - - $-\sqrt{2}$ + + + 0 - - - $+\sqrt{2}$ +

decreasing on $(-\infty, -\sqrt{2}]$ and $[0, +\sqrt{2}]$

Increasing on $[-\sqrt{2}, 0]$ and $[+\sqrt{2}, \infty)$

Concave up/down

$f''(x) = 36x^2 - 24 = 0$ + + + + $-\frac{\sqrt{2}}{3}$ - - - - $\sqrt{\frac{2}{3}}$ + + + +

$3x^2 - 2 = 0$

$x^2 = \frac{2}{3}$ concave ~~up~~ down on $\left(-\sqrt{\frac{2}{3}}, +\sqrt{\frac{2}{3}}\right)$

$x = \pm\sqrt{\frac{2}{3}}$ concave up $\left(-\infty, -\sqrt{\frac{2}{3}}\right)$ and $\left(\sqrt{\frac{2}{3}}, \infty\right)$

Inflection points

$f''(x) = 0$ $36x^2 - 24 = 0$ $x = +\sqrt{\frac{2}{3}}$ $y = -\frac{5}{4}$

$x = \pm\sqrt{\frac{2}{3}}$ $x = -\sqrt{\frac{2}{3}}$ $y = \frac{28}{3}$

4a $\frac{dy}{dx} = x^2 (x^4)^{1/2}$ $x^2 \times \sqrt{x}$

$x^2 x^4 x^{1/2}$

$u = x^2$ $du = \cancel{x^{3/2}} = x\sqrt{x}$ $\int x^{3.5}\,dx$

$du = \cancel{2x\,dx}$ $= \frac{x^{4.5}}{4.5} + c$

$\frac{x^{4.5}}{4.5} + c = 0$ $x = 0 \Rightarrow \frac{0^{4.5}}{4.5} + c = 0$

$c = 0$

$\frac{x^{4.5}}{4.5}$

b) $\int x^3 \sqrt{5+x^4}\,dx$ $u = 5+x^4$ $du = 4x^3\,dx$

$\int \frac{\sqrt{u}}{4}\,du$

$= \frac{1}{4}\int \sqrt{u}\,du$

$= \frac{1}{4}\int u^{1/2}$

$= \frac{1}{4}\frac{u^{3/2}}{3/2} = \frac{2u^{3/2}}{12} = \frac{u^{3/2}}{6} = \frac{(5+x^4)^{3/2}}{6}$

ii. $\int \frac{\cos 4x}{(1+2\sin 4x)^3}$ $u = 1+2\sin 4x$

$dx = 2\cos 4x\,(4)$

$= 8\cos 4x\,du$

$\frac{1}{8}\int \frac{1}{u^3}\,du$

$= \frac{1}{8}\int u^{-3}\,du = \frac{1}{8}\int\left(\frac{u^{-2}}{-2}\right) = \frac{1}{-16u^2} = \frac{1}{-16(1+2\sin 4x)^2}$

c ~~If the function f is continuous on [a,b] and if $f(x) \geq 0$ for all x in [a,b], then the area A under the curve $y = f(x)$ over interval [a,b] is defined by~~

~~$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$ $\Delta x = \frac{b-a}{n}$~~

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4c If f is continuous on a finite interval $[a,b]$ and the limit

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and does not depend on choice of Δx_k or x_k^*. Then in this case we write the limit as

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

where this which is the definite integral of f from a to b

d Average value $= \frac{1}{b-a}\int_a^b f(x)dx$

$$\frac{1}{4-2}\int_2^4 2x^2-3x+7$$

$$\frac{1}{2}\left[\frac{2x^3}{3}-\frac{3x^2}{2}+7x\right]_2^4$$

$$\frac{1}{2}\left[\frac{128}{3}-24+28\right]-\left[\frac{16}{3}-6+14\right]$$

$$\frac{1}{2}\left[\frac{140}{3}-\frac{40}{3}\right]$$

$$=\frac{1}{2}\left[\frac{100}{3}\right] \quad = \frac{50}{3}$$

E. If f is continuous on an interval, then f has an antiderivative on that interval. In particular if a is any point in the interval, then the function F defined by

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of f; that is $F'(x) = f(x)$ for each x in the interval, or in alternate notation

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

By Second part of fundamental theorem of calculus:

$$\frac{d}{dx}\left(\int_0^x \frac{t^7}{(1+\cos t \sin t)^{13}}\,dt\right) = \frac{x^7}{(1+\cos x \sin x)^{13}}$$