[illegible] Exam Paper

Q4 $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $AA^T = BA^T$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$5x - 2y = 1$

$-2x + 2y = -1$

$3x = 0 \quad x = 0 \Rightarrow 2y = -1 \quad y = -\frac{1}{2}$

$\Rightarrow (x, y) = \quad (0, -\frac{1}{2})$

2

Laplace Transform Notes

1. The given ODE is transformed into an algebraic equation called the subsidiary equation

2. Subsidiary eqn is solved by purely algebraic ~~ex~~ manipulations

3. Solution in step 2 is transformed back, resulting in the solution of the given problem

Laplace Transform. Linearity. First Shifting Theorem

If $f(t)$ is a function defined for all $t \geq 0$, its Laplace transform is the integral of $f(t)$ times e^{-st} for $t=0$ to ∞. It is a function of s, say, $F(s)$, and is denoted by $\mathcal{L}(f)$:

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$$

It is an integral transform:

$$F(s) = \int_0^\infty k(s,t)\, f(t)\, dt.$$

with kernel $k(s,t) = e^{-st}$.

Note that the Laplace transform is called an integral transform because it transforms (changes) a function in one space to a function in another space by a process of integration that involves a kernel

The kernel or kernel function is a function of the variables in the two spaces and defines the integral transform

Furthermore the given function $f(t)$ earlier is called the inverse transform of $F(s)$ and is denoted by $\mathcal{L}^{-1}(F)$:

$$f(t) = \mathcal{L}^{-1}(F)$$

$\mathcal{L}^{-1}(\mathcal{L}(f)) = f$ and $\mathcal{L}(\mathcal{L}^{-1}(F)) = F$

Original function depend on t and their transforms on s.

$F(s)$ denoted transform of $f(t)$

$Y(s)$ denotes transform of $y(t)$

Example:

$f(t) = 1 \quad t \geq 0$. Find $F(s)$

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty (1)e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

Example: Laplace transform $\mathcal{L}(e^{at})$ of exponential function e^{at}

$f(t) = e^{at} \quad t \geq 0$ Find $\mathcal{L}(f)$.

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st}e^{at} = \frac{1}{a-s}e^{-(s-a)t}\Big|_0^\infty$$

hence when $s-a > 0$ $\quad \mathcal{L}(e^{at}) = \frac{1}{s-a}$

Linearity of the Laplace Transform

Laplace transform is a linear operation, for any function $f(t)$ and $g(t)$ whose transforms exist and any constants a and b the transform of $af(t)$ and $bg(t)$ exists and:

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$$

This is true because integration is a linear operation

Example:

Find transform of $\cosh(at) = \frac{1}{2}(e^{at} + e^{-at})$

$$\mathcal{L}(\cosh(at)) = \frac{1}{2}\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at}) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2-a^2}$$

3

Laplace Transform Notes

S-Shifting : Replacing s by s-a in the transform

L.T has useful property that if we know the transform of $f(t)$ we can immediately get that of $e^{at} f(t)$ as follows

Theorem:

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at} f(t)$ has the transform $F(s-a)$ (where $s-a > k$):

$$\mathcal{L}\left[e^{at} f(t)\right] = F(s-a)$$

or if we take inverse of both sides:

$$e^{at} f(t) = \mathcal{L}^{-1}\left[F(s-a)\right]$$

Key point

if $\mathcal{L}(f(t)) = F(s)$ then $(e^{-at} f(t)) = F(s+a)$

Example $\mathcal{L}(t^3 u(t)) = \frac{6}{s^4}$

and by F.S.T $\Rightarrow \mathcal{L}(e^{-2t} t^3 u(t)) = \frac{6}{(s+2)^4}$

$\mathcal{L}(1) = 1/s = F(s)$

Thus $\mathcal{L}(e^{at}) = F(s-a) = 1/(s-a)$ for $s > a$

Example $\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2 + 2^2}\right] = e^{-t} \cos 2t$

Transforms of Derivatives and Integrals. ODEs.

Laplace transform is a method of solving ODEs and initial value problems

Crucial idea is that operations of calculus on functions are replaced by operations of algebra on transforms

Laplace transform of derivatives

$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$

$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$

holds for all $t \geq 0$

Prove $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$

$\Rightarrow \mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\,dt = [e^{-st} f(t)]\big|_0^\infty + s\int_0^\infty e^{-st}(f(t))\,dt.$

$= (0) - (1)y(0) + s\mathcal{L}(y) = s\mathcal{L}(y) - y(0)$

$\mathcal{L}(y'') = s\mathcal{L}((y')') = s\mathcal{L}(y') - y'(0)$

which gives us

$\mathcal{L}(y'') = s\mathcal{L}(y') - y'(0) = s[s\mathcal{L}(y) - y(0)] - y'(0)$

$= s^2\mathcal{L}(y) - sy(0) - y'(0)$

Differential Equations, Initial Value Problems

$y'' + ay' + by = r(t) \qquad y(0) = k_0 \qquad y'(0) = k_1$

- where a and b are constant
- $r(t)$ is the given input (driving force)
- $y(t)$ is output (response to the input)

Setting up the Subsidiary equation:

This is an algebraic eq^n for the transform $Y = \mathcal{L}(y)$:

$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s).$

where $R(s) = \mathcal{L}(r)$

Collecting the Y-terms we have subsidiary eq^n:

$(s^2 + as + b)Y = (s+a)y(0) + y'(0) + R(s)$

Laplace Transforms Notes

Example:

Solve $y'' - y = t \qquad y(0) = 1, \quad y'(0) = 1$

$$s^2 Y - s\,y(0) - y'(0) - Y = 1/s^2$$

$$(s^2 - 1)\,Y = s + 1 + 1/s^2$$

$$Y = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)}.$$

Simplification of first fraction and expansion of latter fraction gives

$$Y = \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}$$

$$y(t) = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^t + \sinh t - t$$

Example:

Solve: $y'' + y' + 9y = 0 \qquad y(0) = 0.16 \quad y'(0) = 0.$

$$s^2 Y - 0.16s + sY - 0.16 + 9Y = 0$$

$$(s^2 + s + 9)\,Y = 0.16(s+1)$$

$$Y = \frac{0.16(s+1)}{(s^2+s+9)}$$

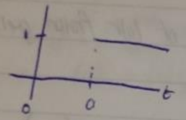
$$= \frac{0.16\left(s + \frac{1}{2}\right) + 0.08}{\left(s + \frac{1}{2}\right)^2 + 35/4}$$

$$y(t) = e^{-t/2}\left(0.16 \cos\sqrt{\tfrac{35}{4}}\,t + \frac{0.08}{\frac{1}{2}\sqrt{35}} \sin\sqrt{\tfrac{35}{4}}\,t\right)$$

UNIT STEP FUNCTION (Heaviside function) SECOND SHIFTING THEORM (t-shifting)

Unit step function (Heaviside function) $u(t-a)$
is $u(t-a)$ is 0 for $t<a$, has a jump of size 1 at $t=a$ and is one for $t>a$

$$u(t-a) = \begin{cases} 0 & \text{if } t<a \\ 1 & \text{if } t>a \end{cases} \qquad a \geq 0.$$



$u(t-a)$

$$\mathcal{L}[u(t-a)] = \frac{e^{-as}}{s} \qquad s>0$$

Second Shifting theorm

If $f(t)$ has the transform $F(s)$, then the "shifted function"

$$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t<a \\ f(t-a) & \text{if } t>a \end{cases}$$

has the transform $e^{-as}F(s)$. That is, if $\mathcal{L}(f(t)) = F(s)$, then

$$\mathcal{L}(f(t-a)u(t-a)) = e^{-as}F(s)$$

Or taking inverse

$$f(t-a)u(t-a) = \mathcal{L}^{-1}[e^{-as}F(s)]$$

Example:

Find $\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+s}\right)$

Since this function of s contains an exponential e^{-as} we know the inverse LT must involve the second shifting theorm

2

Laplace Transform Notes

Namely we have $\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+s}\right) = \mathcal{L}^{-1}\left(e^{-s}\frac{1}{s^2+1}\right)$

So by the second shift theorem backwards we have

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+s}\right) = u(t-1)\,\mathcal{L}^{-1}\left(\frac{1}{s^2+s}\right)\Big|_{(t-1)}$$

In order to continue we need partial fractions

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{(s+1)}\right) = 1 - e^{-t}$$

Combining these results we have:

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+s}\right) = u(t-1)\left[1 - e^{-(t-1)}\right]$$

Example:

Find $\mathcal{L}^{-1}\left(\frac{se^{-5s}}{s^2+4}\right)$ Since function of s contains e^{-as} we know inverse LT involves second shift theorem.

$$\mathcal{L}^{-1}\left(\frac{se^{-5s}}{s^2+4}\right) = \mathcal{L}^{-1}\left(e^{-5s}\frac{s}{s^2+4}\right) = u(t-5)\cos(2(t-5))$$

Example:

Solve $y'' - y = 2u(t-1)$ $\quad y(0) = 0 \quad y'(0) = 0$

$$\mathcal{L}(y'') - \mathcal{L}(y) = \frac{2e^{-s}}{s}$$

$$s^2 Y - Y = \frac{2e^{-s}}{s}$$

$$Y = \frac{2e^{-s}}{s(s^2-1)} \qquad Y = \frac{2e^{-s}}{s(s-1)(s+1)}$$

So to find y we need to take the inverse LT requiring partial fractions

$$\frac{2}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{s+1}$$

$A = -2$
$B = 1$
$C = 1$

8.

$$y = \mathcal{L}^{-1}(y) = \mathcal{L}^{-1}\left(e^{-s}\frac{-2}{s}\right) + \mathcal{L}^{-1}\left(e^{-s}\frac{1}{(s-1)}\right) + \mathcal{L}\left(e^{-s}\frac{1}{(s+1)}\right)$$
$$= u(t-1)\left[e^{-(t-1)} + e^{-(t-1)} - 2\right]$$

Dirac Delta Function

Defined by
$$\delta(t-a) = \begin{cases} \infty & \text{if } t=a \\ 0 & \text{otherwise} \end{cases} \quad (29)$$

and $\int_0^\infty \delta(t-a)\,dt = 1$

Because of property 29, when inserted in an integral it has effect of picking out the function at $t=a$ i.e.

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

The Laplace transform is $\mathcal{L}[\delta(t-a)] = \int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-sa}$

hence $\mathcal{L}^{-1}(e^{-sa}) = \delta(t-a)$

Example

Solve $y'' - 5y' + 6y = \delta(t-2)$ $\quad y'(0)=0 \quad y(0)=0$

$$[s^2 y - s(0) - (0)] - 5[s\mathcal{L}(y) - y(0)] + 6\mathcal{L}(y) = e^{-2s}$$

$$y = \frac{e^{-2s}}{s^2 - 5s + 6}$$

$$y = \mathcal{L}^{-1}\ e^{-2s}\left[\frac{1}{s-3} - \frac{1}{s-2}\right]$$
$$= u(t-2)\left[\left(e^{3t-6} - e^{2t-4}\right)\right]$$

9

Laplace Transform Notes

Convolution

If $f(t)$ and $g(t)$ are causal functions then the convolution is defined by

$$(f * g)(t) = \int_0^t f(t-x)g(x)\,dx$$

Example

Use convolution to solve $H(s) = \frac{1}{(s^2+a^2)^2}$

Write as a product

$$\left(\frac{1}{s^2+a^2}\right)\left(\frac{1}{s^2+a^2}\right)$$

So in this case we have $F(s) = G(s) = \frac{1}{s^2+a^2}$

$$\Rightarrow f(t) = g(t) = \frac{1}{a}\sin(at).$$

Example

$\mathcal{L}\{f(t)\} = F(s)$

$\mathcal{L}\{g(t)\} = G(s)$

$\mathcal{L}[(f * g)(t)] = F(s)G(s) \Leftrightarrow (f*g)(t) = \mathcal{L}^{-1}[F(s)G(s)]$

$$H(s) = \frac{2s}{(s^2+1)^2}$$

We want $\mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}\left[\frac{2s}{(s^2+1)^2}\right]$

Try to write as product of 2 L.T we know

$$\frac{2s}{(s^2+1)^2} = 2 \cdot \underbrace{\frac{1}{(s^2+1)}}_{F(s)\ \to\ \sin(t)} \cdot \underbrace{\frac{s}{(s^2+1)}}_{G(s)\ \to\ \cos(t)}$$

2

$f(t) = \sin t \qquad g(t) = \cos t$

$$\mathcal{L}^{-1}\left[F(s)\,G(s)\right] \quad F($$

$$\uparrow \frac{2}{s^2+1} \qquad \uparrow \frac{s}{s^2+1}$$

$$= \mathcal{L}^{-1}\left[\frac{2}{s^2+1}\right] * \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]$$

$$2\sin t * \cos t$$

$$\mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = 2\sin t * \cos t = \int_0^t 2\sin(t-\tau)\cos(\tau)\,d\tau$$

$$f * g = \int_0^t f(t-\tau)g(\tau)\,d\tau \qquad 2\int_0^t \sin(t-\tau)\cos(\tau)\,d\tau$$

$$2 \times = \frac{1}{2} t \sin(t)$$

$$\mathcal{L}^{-1}\left[\frac{2s}{(s^2+1)^2}\right] = t\sin(t)$$