MA1E01: Chapter 5 Summary

Integration

Definitions

- **Antiderivatives**: A function F is called an antiderivative of a function f on an interval I if
$$F'(x) = f(x) \quad \forall\, x \in I.$$

- **Indefinite Integral**: Finding an antiderivative is known as antidifferentiation or integration,
$$\frac{dF(x)}{dx} = f(x) \qquad \Longleftrightarrow \qquad F(x) = \int f(x)dx + C.$$

- **Area with a Regular Partition**: To compute the area under a non-negative continuous curve $y = f(x)$ over the interval $[a, b]$, we can divide the interval into n equal sub intervals (this called a regular partition), each of width
$$\Delta x = \frac{b-a}{n}.$$
We can approximate the area of the k^{th} rectangle by
$$A_k \approx f(x_k^*)\Delta x$$
where x_k^* is an arbitrary point in the k^{th} interval $[x_{k-1}, x_k]$. Adding each of these n rectangles gives an approximation for the area under the curve
$$A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x.$$
Clearly, the larger the number of sub-intervals (i.e. the larger the value of n), the better this approximation will be. Taking $n \to \infty$ gives the exact area
$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$
The result does not depend on the choice of x_k^*.
If f is allowed to take on both positive and negative values, then A defined above gives the *net-signed area.*

- **Natural Choices for x_k^***: Although the limit above is independent of the choice of x_k^*, there are three natural choices when it comes to computing the area:
  - Left end-point: $x_k^* = x_{k-1} = a + (k-1)\Delta x$

- Right end-point: $x_k^* = x_k = a + k\Delta x$
- Midpoint: $x_k^* = \frac{1}{2}(x_{k-1} + x_k) = a + (k - \frac{1}{2})\Delta x$.

- **Area with an Irregular Partition**: We now consider the net-signed area for an irregular partition, where the interval is divided up into n subintervals not necessarily of equal length. The partition of the interval $[a, b]$ is defined by the points
$$a = x_0 < x_1 < x_2 \cdots < x_{n-1} < x_n = b.$$
The width of the k^{th} interval is therefore
$$\Delta x_k = x_k - x_{k-1},$$
and an approximation to the net-nigned area is given by summing the approximate area of the n rectangles defined by the partition, which gives
$$A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x_k,$$
where, as before, x_k^* is an arbitrary point in the k^{th} interval. In this case, since the intervals are not equally spaced, each of the Δx_k need not tend to zero as $n \to \infty$. Instead we take the limit $\max \Delta x_k \to 0$, which guarantees that each subinterval will shrink to zero. So the exact net-signed area is
$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$
This is known as a *Riemann Sum.*

- **The Definite Integral**: A function is said to be integrable on $[a, b]$ if the limit
$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$
does not depend on the choice of x_k^* or on the choice of partition. We then denote this limit by
$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$
This is the *Definite Integral* or *Riemann Integral.*

- **Boundedness**: A function is bounded on an interval I if there exists a positive M such that
$$-M \le f(x) \le M \qquad \forall\, x \in I.$$

- **Total Area**: The total area between $y = f(x)$ and $[a, b]$ is given by

$$\text{total area} = \int_a^b |f(x)|dx.$$

- **Average of a Function**: The average value of a function over $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)dx.$$

- **Displacement/ Distance**: If $v(t)$ is the velocity function of a particle, then the displacement over the time interval $[t_1, t_2]$ is

$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)dt.$$

  The distance over this same time interval is given by

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|dt.$$

Theorems

- **Intervals of Integration**: If f is integrable on a closed interval containing the points a, b, c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

  regardless of how the points are ordered.

- **Integral Inequalities**: If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$, then

$$\int_a^b f(x)dx \geq 0.$$

  If f, g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

- **Integrability Criteria**: Let f be a function on $[a, b]$.

  (a) If f has finitely many discontinuities and is bounded on $[a, b]$, then f is integrable.

  (b) If f is not bounded on $[a, b]$, then f is not integrable on $[a, b]$.

- **Fundamental Theorem of Calculus: Part I**: If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a) = \Big[F(x)\Big]_a^b.$$

- **Mean Value Theorem for Integrals**: If f is continuous on $[a, b]$, there exists at least one point $x^* \in [a, b]$ such that

$$\int_a^b f(x)dx = f(x^*)(b - a).$$

- **Fundamental Theorem of Calculus: Part II**: If f is continuous on an interval I, then f has a antiderivative on I. In particular, if a is any point in I, then the function F defined by

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of f on I, i.e.,

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x).$$

Miscellaneous Results

- **Initial Value Problems**: Given a differential equation of the form

$$\frac{dy}{dx} = f(x),$$

all solutions are antiderivatives of $f(x)$,

$$y = \int f(x)\,dx + C.$$

The constant C is determined by an initial condition of the form

$$y(x_0) = y_0.$$

- **Integration by Substitution**: To compute an integral of the form

$$\int f(g(x))g'(x)dx,$$

we make the substitution

$$u = g(x) \quad \Longrightarrow \quad du = g'(x)dx,$$

and hence the integral reduces to

$$\int f(g(x))g'(x)dx = \int f(u)\,du = F(u) + C$$

where F is an antiderivative of f.

For definite integrals by substitution, we note that the integration limits are with respect to x not u. We can either convert back to x and use the limits in terms of x or we can change the limits to their corresponding u values and perform the definite integral completely in terms of u. In the latter case, we use

$$\begin{aligned} x = a \quad &\implies \quad u = g(a) \\ x = b \quad &\implies \quad u = g(b), \end{aligned}$$

and hence

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)\,du = \Big[F(u)\Big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a)).$$