2011 Maths 4 Exam Paper

1 A $u = (2, -1, 0, 3)$

$x = 2t$
$y = -t$
$z = 0$
$w = 3t$

B $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = k_1 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

$2k_1 + k_2 = x$
$-k_1 = y$
$3k_1 + 3k_2 = z$

$k_2 = x + 2y$

$-3y + 3x + 3y = z$

$3x = z$

$z = 3(-y) + 3(x + 2y)$

$3x + 3y - z = 0$ = eq^n of plane

2a $k_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + k_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + k_4 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$k_2 + k_3 + k_4 = 0$
$k_1 + k_3 + k_4 = 0$
$k_1 + k_2 + k_4 = 0.$

$k_3 + k_4 = -k_2 = -k_1$

$k_1 = k_2.$

$\Rightarrow 2k_2 + k_4 = 0 \quad k_2 = \frac{-k_4}{2}$

$k_1 + k_3 + 2k_4 = 0$

$k_3 = k_1$

$k_1 = k_3 = k_2 = \frac{-k_4}{2}$ There is a non zero solution $\therefore$ vectors are linearly dependent

b $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = k_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + k_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + k_4 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$k_2 + k_3 + k_4 = a$
$k_1 + k_3 + k_4 = b$
$k_1 + k_2 + k_4 = c$

$k_3 + k_4 = a - k_2$

$a - k_2 = b - k_1$

$a - b = -k_1 + k_2$

There is always a solution $\Rightarrow$ vectors span $\mathbb{R}^3$

C Basis? The vectors are linearly dependent because any set of n vectors with $n > 3$ is linearly dependent in $\mathbb{R}^3$, hence they do not form a basis

3A $\begin{pmatrix} 0 & 2 & -4 & -2 \\ 0 & -3 & 6 & 3 \end{pmatrix}$ $r_2 \div 2$ $\begin{pmatrix} 0 & 1 & -2 & -1 \\ 0 & -3 & 6 & 3 \end{pmatrix}$ r_2+3r_1 $\begin{pmatrix} 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Rank $(A) = \dim \text{Row}(A) = \text{col} \dim(A) = 1$
Null Space $= 4-1 = 3$

Basis for row space $= \text{Span}\{(0,1,-3,-1)\}$
Basis for column space $= \text{Span}\{\begin{pmatrix} 2 \\ -3 \end{pmatrix}\}$

Basis for null space $\begin{pmatrix} 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $y - 2z - w = 0$.
$y = w + 2z$
$\overset{s_1}{w}, \overset{s_2}{z}, \overset{s_3}{x}$ free
$x = s_1$ $z = s_3$ $w = s_4$.

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} s_1 \\ s_4 + 2s_3 \\ s_3 \\ s_4 \end{pmatrix} \quad s_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s_3\begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + s_4\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Basis for null space $= \text{Span}\{(1,0,0,0), (0,2,1,0), (0,0,0,1)\}$

B $\begin{pmatrix} 0 & 1 \\ 0 & -2 \\ -2 & 1 \end{pmatrix}$ $r_3 \div 2$ $\begin{pmatrix} -1 & 1/2 \\ 0 & 1 \\ 0 & -2 \end{pmatrix}$ $r_3 + 2r_1$ $\begin{pmatrix} -1 & 1/2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & -1/2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

Rank $(A) = \dim \text{Row}(A) = \dim \text{Col}(A) = 2$
Null Space $= 2-2 = 0$
Basis for row space $= \text{Sp}\{(1,-1/2), (0,1)\}$
Basis for column space $= \text{Span}\{\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\}$

Basis for null space $= \begin{pmatrix} 1 & -1/2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $x - 1/2 y = 0$
$y = 0 \rightarrow x = 0$
Null of $(A) = \text{Span}\{0, 0\}$

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Q4

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad AA^T = BA^T$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$5x - 2y = 1$

$-2x + 2y = -1$

$3x = 0 \quad x = 0 \Rightarrow 2y = -1 \quad y = -\tfrac{1}{2}$

$\Rightarrow (x,y) = (0, -\tfrac{1}{2})$

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Q1 a $t(2,-1,0,3) = (2t, -t, 0\ 3t)$

$x = 2t$
$y = -t$
$z = 0$
$w = 3t$

b $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = k_1 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

$2k_1 + k_2 = x \quad k_2 = x - 2k_1$
$-k_1 = y$
$3k_1 + 3k_2 = z$

$3(-y) + 3(x - 2(-y)) = z$
$-3y + 3x + 6y = z$
$3y + 3x - z = 0 = eq^n$

2 a $k_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + k_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + k_4 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$

$k_2 + k_3 + k_4 = 0 \quad k_3 + k_4 = -k_2$
$k_1 + k_3 + k_4 = 0 \quad k_1 - k_2 = 0 \quad k_1 = k_2$
$k_1 + k_2 + k_4 = 0 \rightarrow 2k_1 = -k_4 \quad k_1 = \frac{1}{2}k_4$
$k_1 + k_3 - 2k_1 = 0$
$k_1 = k_3$

$k_1 = k_2 = k_3 = -\frac{1}{2}k_4$ There is a non-zero solution
∴ vectors are linearly dependent

b Span $\mathbb{R}^3$?

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = k_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + k_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + k_4 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} k_2 + k_3 + k_4 \\ k_1 + k_3 + k_4 \\ k_1 + k_2 + k_4 \end{pmatrix}$ There is always a solution
∴ the vectors span $\mathbb{R}^3$

c Basis of $\mathbb{R}^3$?
The vectors are linearly dependent because any set of n vectors with $n > 3$ is linearly dependent in $\mathbb{R}^3$, hence they do not form a basis

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2

3a $\begin{pmatrix} 0 & 2 & -4 & -2 \\ 0 & -3 & 6 & 3 \end{pmatrix} \xrightarrow{r_1 \div 2} \begin{pmatrix} 0 & 1 & -2 & -1 \\ 0 & -3 & 6 & 3 \end{pmatrix} \xrightarrow{r_2 + 3r_1} = \begin{pmatrix} 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

row dim(A) = colum dim(A) = 1
null space dimension = $\cancel{3}4 - 1 = 3$

Basis of row space = Span $\{(0,1,-2,1)\}$
Basis of colum space = $\left\{\begin{pmatrix} 2 \\ -3 \end{pmatrix}\right\}$

$A\bar{x} = \bar{0}$ $\quad x = \text{free} \quad x = s_1 \quad w = s_{\cancel{3}4} \quad z = s_2$

$\begin{pmatrix} 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad y - 2z - w = 0$

$y = w + 2z$

$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = s_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s_3 \begin{pmatrix} 0 \\ +2 \\ 0 \\ 0 \end{pmatrix} + s_4 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

null space = $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ +2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}$

B $\begin{pmatrix} 0 & 1 \\ 0 & -2 \\ -2 & 1 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ -2 & 1 \end{pmatrix} \xrightarrow{r_3 \div 2} \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

Dim row = Dim colum = 2
Null space = 2 - 2 = 0

Basis of row = Span $\{(1, -\frac{1}{2}), (0,1)\}$
Basis of colum space = $\left\{\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right\}$

$A\bar{x} = 0$

$\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ $\quad x - \frac{1}{2}y = 0$

$y = 0 \Rightarrow x = 0$

basis of null space = $\{(0,0)\}$

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Q4 $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$A A^T \vec{x} = b A^T$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} +1 \\ -1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$

$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$

$r_2 = 2$

~~$\begin{pmatrix} 5 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ $r_1 + 2r_1$~~

~~$\begin{pmatrix} 3 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $3x = 1$ $x = 1/2$~~

~~$-x+y=-1$ $y = -1 + 1/3$~~ $= -2/3$

$(1/2, -2/3)$

$\begin{pmatrix} 5 & -2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $5x - 2y = \phi$

$3x = 0$ $x = 0$ $-2y = 1$ $y = -1/2$

$(x, y) \Rightarrow (0, -1/2)$

5. $\begin{vmatrix} -2 & 0 & 0 \\ -1 & -2 & 3 \\ 1 & 2 & -3 \end{vmatrix}$ $P(\lambda) = \det(\lambda I - A)$

$\det \begin{pmatrix} \lambda+2 & 0 & 0 \\ 1 & \lambda+2 & -3 \\ -1 & -2 & \lambda+3 \end{pmatrix}$

$(\lambda+2)[(\lambda+2)(\lambda+3) - (-3)(-2))$

$\lambda+2 \; [\lambda^2 + 3\lambda + 2\lambda + 6 - 6]$

$\lambda+2 \; [\lambda^2 + 5\lambda$

$\lambda(\lambda+2)(\lambda+5)$

Roots $\lambda = 0,$ $\lambda = -2,$ $\lambda = -5$

$$\lambda = 0 \quad \begin{pmatrix} -2 & 0 & 0 \\ -1 & -2 & 3 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & 0 \\ -1 & -2 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad -2x = 0 \quad x = 0 \quad -x - 2y + 3z = 0 \quad -2y + 3z = 0 \quad x = 0 \quad 3z = 2y \quad y = \tfrac{3}{2}z \quad \bar{v}_1 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

$$\lambda_2 = -2 \quad \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 3 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{0} \quad -x + 3z = 0 \quad 2y + 2z = 0 \quad x = 3z \quad y = -z \quad v_2 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$\lambda_3 = -5 \quad \begin{pmatrix} 3 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{0}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{0} \quad x = 0 \quad y = 0 \quad 2y + 2z = 0 \quad x = 0,\ y = 0,\ z = 0 \quad \bar{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

5b $$P = \begin{pmatrix} 0 & 3 & 0 \\ 3 & -1 & 0 \\ 2 & 1 & 0 \end{pmatrix} \quad D = P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

Q6 $$f(x) \begin{cases} 1 & \text{if } -\pi \le x < 0 \\ -x & \text{if } 0 \le x \le \pi \end{cases}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{0} 1\,dx + \int_0^{\pi} x\,dx$$
$$\tfrac{1}{2\pi}\left[\left(x\Big|_{-\pi}^{0}\right) + \left(x^2\Big|_0^{\pi}\right)\right]$$
$$= \frac{1}{2} + \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} \cos nx\,dx + \int_0^{\pi} x\cos nx\,dx$$
$$\tfrac{1}{\pi}\left[\frac{\sin nx}{n}\Big|_{-\pi}^{0} + \frac{x\sin nx}{n} \quad \frac{\cos nx}{n^2}\Big|_0^{\pi}\right]$$
$$\tfrac{1}{\pi}\left[(0) + (0 + \frac{(-1)^n}{n^2}) - 0 + \frac{1}{n^2}\right]$$
$$= \frac{(-1)^n - 1}{\pi n^2}$$

$$b_n = \tfrac{1}{\pi}\int_{-\pi}^{0} \sin nx\,dx + \int_0^{\pi} x\sin nx\,dx$$
$$\tfrac{1}{\pi}\left[\left(-\frac{\cos nx}{n}\right)\Big|_{-\pi}^{0} + \left(-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right)\Big|_0^{\pi}\right]$$
$$\tfrac{1}{\pi}\left[\left(-\frac{1}{n}\right) - \left(-\frac{(-1)^n}{n}\right) + \left(-\frac{\pi(-1)^n}{2}\right)\right]$$
$$= \frac{(-1)^n - 1}{\pi n} - \frac{(-1)^n}{n}$$

6B $$f(x) = \frac{1}{2} + \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{(-1)^n - 1}{\pi n^2}\right)\cos nx + \left(\frac{(-1)^n - 1}{\pi n} - \frac{(-1)^n}{n}\right)\sin nx\right]$$