1 2012 May SCI PAPER Q5 DAVID WEMBRECHT

Q5 A Express rectangular coordinates in terms of cylindrical coordinates

rectangular = x, y, z.

In cylindrical = $x = \rho \cos\theta$ $y = \rho \sin\theta$ $z = z$ jacobian ρ.

2.

example $z = x^2 + y^2 - 4$

cylindrical => $z = \rho^2 \cos^2\theta + \rho^2 \sin^2\theta - 4$

$= z = \rho^2 - 9$

B. Consider the Solid G which is bounded above by the Surface $x^2 + y^2 + 2z = 15$ and below by the Surface $z = \sqrt{x^2+y^2}$

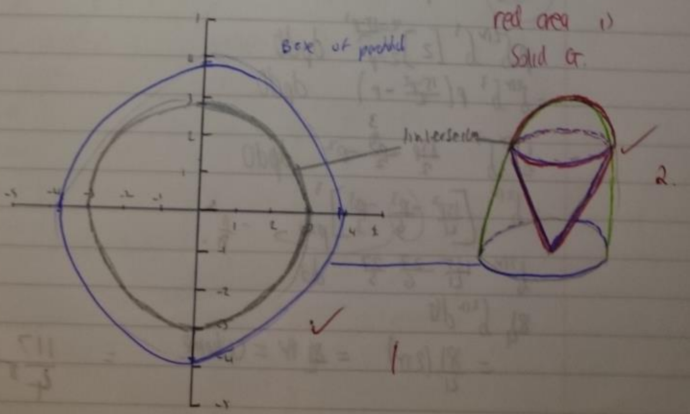
B i What is the Surface $x^2 + y^2 + 2z = 15$?

It is an infinite paraboloid 1

B ii What is the Surface $z = \sqrt{x^2+y^2}$?

This is an infinite cone with origin at (0,0,0) 2.

Biii



V Use triple Integral and cylindrical coordinates to compute the volume V of the Solid G

$x = p\cos\theta \quad y = p\sin\theta \quad z = z.$

* $0 \le \theta \le 2\pi.$

$\sqrt{x^2+y^2} \le z \le \frac{15-x^2-y^2}{2}$

$\sqrt{p^2} \le z \le \frac{15-p^2\cos^2\theta-p^2\sin^2\theta}{2}$

* $p \le z \le \frac{15-p^2}{2}$

For p, equate $\sqrt{x^2+y^2} = \frac{15-x^2-y^2}{2}$ $\Rightarrow x^2+y^2=t^2$

$t = \frac{15-t^2}{2}$

$2t = 15-t^2$

$t^2+2t-15=0$

$t=-5$ or $t=+3$

$t=3 \Rightarrow$ * $0 \le p \le 3$

* jacobian is p.

$V = \int_0^{2\pi}\int_0^3\int_p^{\frac{15-p^2}{2}} p \, dz\,dp\,d\theta$

$p\int_0^{2\pi}\int_0^3 [z]_{z=p}^{z=\frac{15-p^2}{2}} \, dp\,d\theta$

$= \int_0^{2\pi}\int_0^3 p\left(\frac{15-p^2}{2} - p\right) dp\,d\theta$

$\int_0^{2\pi}\int_0^3 \frac{15p}{2} - \frac{p^3}{2} - p^2 \, dp\,d\theta$

$= \int_0^{2\pi}\left[\frac{15p^2}{4} - \frac{p^3}{6} - \frac{p^3}{3}\right]_{p=0}^{3} \rightarrow -\frac{p^4}{8}.$

$\int_0^{2\pi} \frac{135}{4} - \frac{27}{6} - \frac{27}{3} \, d\theta$

$\frac{81}{4}\int_0^{2\pi} d\theta$

$= \frac{81}{4}(2\pi) = \frac{81}{2}\pi = \text{volume} = \frac{117}{4}\pi$ 4

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Vi. Use triple integral and cylindrical coordinates to find the mass M of the solid G if its density is

$$\delta(x,y,z) = \frac{e^{-\sqrt{x^2+y^2}}}{x^2+y^2+5\sqrt{x^2+y^2}}$$

Change density to cylindrical $x = \rho\cos\theta$ $y = \rho\sin\theta$ $z = z$

$$\frac{e^{-\sqrt{\rho^2}}}{\rho^2+5\sqrt{\rho^2}} = \frac{e^{-\rho}}{\rho^2+5\rho} = \frac{1}{\rho^2e^\rho+5\rho e^\rho}$$

2

6

$$\int_0^{2\pi}\int_0^3\int_\rho^{\frac{15-\rho^2}{2}} \frac{1}{\rho^2e^\rho+5\rho e^\rho} \cdot \rho \; dz\,d\rho\,d\theta$$

$$= \frac{e^{-\rho}\rho\left(\frac{15-r^2}{2}-r\right)}{r^2+5r}$$

$$= \frac{1}{2}e^{-\rho}(r-2)$$

can integrate by parts.

$$= \int_0^{2\pi}\int_0^3\int_\rho^{\frac{15-\rho^2}{2}} \frac{1}{\rho e^\rho} + \frac{1}{5e^\rho} \; dz\,d\rho\,d\theta$$

cannot integrate $\frac{1}{\rho e^\rho}$ or $\frac{1}{5e^\rho}$

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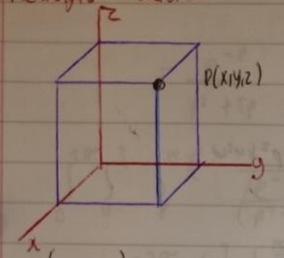
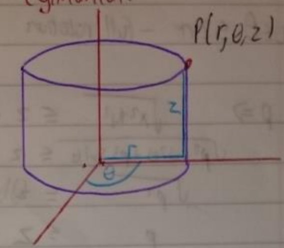
2012 MANG SCI PAPER 1 Q 5 CORRECTION

Q5 A. Rectangular Cordinates



(x, y, z)

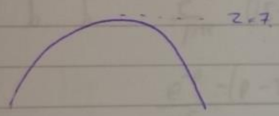
Cylindrical.



$x = \rho\cos\theta \quad y = \rho\sin\theta \quad z = z$

Jacobian = r.

ii. Surface $x^2 + y^2 + 2z = 15$ is an infinite parabloid ~~with origin at (0,0,0)~~ open downwards starting at $z = 7.5$



iii. Surface $z = \sqrt{x^2+y^2}$ is an infinite cone with origin 0,0,0 extending upwards and downwards infinitely

iv. Intersect at $z = \frac{15 - x^2 - y^2}{2}$

$\sqrt{x^2+y^2} = \frac{15 - x^2 - y^2}{2}$ $\qquad x^2 + y^2 = t^2$

$t = \frac{15 - t^2}{2}$

$2t = 15 - t^2$

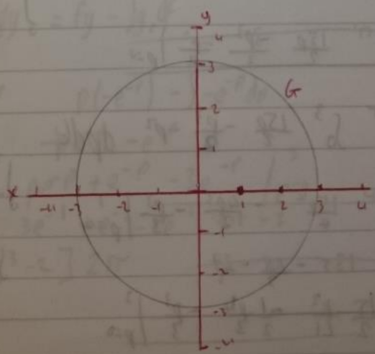
$t^2 + 2t - 15 = 0$

$(t - 3)(t + 5)$

$t = 3$ or $t = -5$

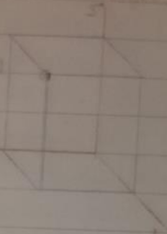
$\boxed{t = 3}$

$x^2 + y^2 = 3^2$



V. Cylindrical (ρ, θ, z) $x = \rho\cos\theta$ $y = \rho\sin\theta$ $z = z$. Jacobian $= \rho$.

$0 \le \theta \le 2\pi$ - full rotation



$\rho \Rightarrow$ $\sqrt{x^2+y^2} \le z \le \frac{15-x^2-y^2}{2}$

$\sqrt{\rho^2\cos^2\theta + \rho^2\sin^2\theta} \le z \le \frac{15 - \rho^2\cos^2\theta - \rho^2\sin^2\theta}{2}$

$\sqrt{\rho^2} \le z \quad \frac{15-\rho^2}{2}$

$\rho \le z \quad \frac{15-\rho^2}{2}$

$0 \le \rho \le 3$ equate $\sqrt{x^2+y^2} \le z \le \frac{15-x^2-y^2}{2}$ $x^2+y^2 = t^2$

$t \le z \le \frac{15-t^2}{2}$

$t = \frac{15-t^2}{2}$

$t^2 - 2t - 15 = 0$

$(t-3)(t+5) = 0$

$t = 3$ or $t = -5$.

$$\int_0^{2\pi}\int_0^3\int_\rho^{\frac{15-\rho^2}{2}} \rho \, dz\, d\rho\, d\theta$$

$$\int_0^{2\pi}\int_0^3 \left[z\right]_{z=\rho}^{\frac{15-\rho^2}{2}} \rho \, d\rho\, d\theta$$

$$\int_0^{2\pi}\int_0^3 \frac{15-\rho^2}{2} - \rho$$

$$\int_0^{2\pi}\int_0^3 \left[\frac{15}{2} - \frac{\rho^2}{2} - \rho\right]\rho \, d\rho\, d\theta$$

$$\cancel{\int_0^{2\pi} \frac{15\rho}{2} \frac{-3\rho^3}{2} \frac{-\rho^2}{2}\Big|_{\rho=0}^{3}}$$

$$\int_0^{2\pi}\int_0^3 \frac{15\rho}{2} - \frac{\rho^3}{2} - \rho^2 \, d\rho\, d\theta$$

$$\cancel{\int_0^{2\pi} \frac{15\rho^2}{4} - \frac{4\rho^3}{2} - \frac{\rho^3}{3}\Big|_{\rho=0}^{3}}$$

$$\cancel{135 - \frac{108}{3} - \frac{27}{3}}$$

$$\cancel{\frac{15}{2}\frac{\rho^2}{4} \frac{-1}{2}\frac{\rho^4}{3} - \frac{\rho^3}{3}}\Big|_{\rho=0}^{3}$$

$$\int_0^{2\pi}\int_0^3 \frac{15}{2}\rho - \frac{\rho^3}{2} - \rho^2 \, d\rho\, d\theta$$

$$\frac{15}{2}\frac{\rho^2}{2} \frac{-1}{2}\frac{\rho^4}{4} - \frac{\rho^3}{3}\Big|_{\rho=0}^{3}$$

$$\int_0^{2\pi} \frac{15}{2}\left[\frac{9}{2}\right] - \frac{81}{8} - \frac{27}{3} \, d\theta$$

$$\int_0^{2\pi} \frac{117}{8} \, d\theta$$

$$= \frac{117}{8}(2\pi) = \frac{117}{4}\pi = \text{volume}$$

$d(x,y,z) = \delta(\rho,\theta,z) = \dfrac{e^{-\sqrt{\rho^2}}}{\rho^2+5\rho}$

$= \dfrac{e^{-\rho}}{\rho^2+5\rho}$

$\int_0^{2\pi}\int_0^3\int_\rho^{\frac{15-\rho^2}{2}} \left(\dfrac{e^{-\rho}}{\rho^2+5\rho}\right)(\rho)\, dz\, d\rho\, d\theta$

$= \int_0^{2\pi}\int_0^3\int_\rho^{\frac{15-\rho^2}{2}} \dfrac{e^{-\rho}}{\rho+5}\, dz\, d\rho\, d\theta$

$\int_0^{2\pi}\int_0^3 \left(\dfrac{e^{-\rho}}{\rho+5}\right)\left(\dfrac{15-\rho^2}{2} - \rho\right)$

$\int_0^{2\pi}\int_0^3 \dfrac{e^{-\rho}}{\rho+5}(15-\rho^2-2\rho)$

$\dfrac{e^{-\rho} - (\rho-3)(\rho+5)}{\rho+5}$

$-e^{-\rho}(\rho-3)$

$-\rho e^{-\rho} + 3e^{-\rho}$

$3e^{-\rho} \qquad -\int \rho e^{-\rho} + \int 3e^{-\rho}$

$= \qquad\qquad + (-3e^{-\rho})$

Integrate by parts $\int f\, dy = fg - \int g\, df$

$f = \rho \qquad df = d\rho$

$dy = e^{-\rho} \qquad g = -e^{-\rho} \qquad \rho(-e^{-\rho}) - \int -e^{-\rho} d\rho$

$-\rho e^{-\rho} - e^{-\rho}$

$\rho e^{-\rho} + e^{-\rho} - 3e^{-\rho} \Big|_{\rho=0}^{3}$

$3e^{-3} + e^{-3} - 3e^{-3} + 1 - 3$

$[e^{-3} - 2]\, 2\pi$

Inno.

Inmelluh $\frac{15p - p^3}{2} - p^2$

4 yre) $\frac{1}{2} \quad 15p - p^3 - 2p^2$

hdroril $\frac{1}{2}\left[\frac{15p^2}{2} - \frac{p^4}{4} - \frac{2p^3}{3}\right]$

CPP GPP $\frac{135}{2} - \frac{81}{4} - 18$

Replouant coll / coll cunen $0.5\left[\frac{117}{4}\right]$

far vde /nor rallin $\frac{117}{8}$

$\times 2\pi = \frac{117}{4}\pi.$

$\frac{15p}{2} - \frac{p^3}{3} - p^2$ $\frac{117}{8}$

$\frac{15}{2}\frac{p^2}{2} - \frac{1}{2}\frac{p^4}{4} - 1\frac{p^3}{2}$

$\int f\,dg = f \cdot g \int g\,df.$

$\frac{15}{4}p^2 - \frac{p^4}{12} - \frac{p^3}{3}$

$f = $ " $df = $ "dy

$\frac{15(9)}{4} - \frac{81}{18} - \frac{27}{3}$

$dg = $ ~~~ $g =$

$\cancel{\frac{405}{4}}\frac{135}{4} - \cancel{\frac{27}{4}}\frac{81}{8} - 9$

$\frac{171}{2} \times$ ~ 18

$\frac{135}{4} - \frac{27}{4} - \frac{27}{3}$