MA1E01: Chapter 3 Summary

Differentiation

Definitions

- **The Derivative Function**: The function $f'(x)$ defined by

$$f'(x) = \lim_{h \to 0} \frac{(f(x+h) - f(x)}{h}$$

is called the derivative of f with respect to x. The domain of the derivative function consists of all x in the domain of f for which the limit above exists.

- **Differentiability**: A function f is said to be differentiable at x_0 if the limit

$$f'(x_0 = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

exists.

- **Equation of tangent**: The equation of the tangent line to a curve $y = f(x)$ at the point $x = x_0$ is

$$y = f(x_0) + f'(x_0)(x - x_0).$$

- **Higher derivatives**: The n^{th} derivative function is obtained by differentiating the $(n-1)^{\text{th}}$ derivative function,

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = \lim_{h \to 0} \frac{f^{(n-1)}(x+h) - f^{(n-1)}(x)}{h}.$$

- **Local Linear Approximation**: The local linear approximation to the curve $y = f(x)$ in the vicinity of the point x_0 is given by the equation of the tangent line to the curve at that point, i.e.,

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

- **Differentials**: The differential dy gives the change in the tangent line of $f(x)$ at x as x runs over the interval from x to $x + dx$. This is different to Δy which gives the change in the value of the function $f(x)$ over this interval.

- **Measurement/Propagated Error**: If x_0 is the exact value of a certain quantity and x is the measured value, then the measurement error is

$$dx = \Delta x = x - x_0.$$

Any quantity that depends on this measurement will have an associated propagated error. If $y_0 = f(x_0)$ is the exact value of a quantity being computed, then the value based on the measurement x is $y = f(x)$. Hence the propagated error is

$$\Delta y = y - y_0 = f(x) - f(x_0).$$

If the measurement error of x_0 is small, then we can approximate the propagated error by the differential

$$\Delta y \approx dy = f'(x)dx.$$

Relative/Percentage Error: The relative error associated with measuring a quantity is $\Delta y/y$. Again, if $\Delta x = dx$ is small enough, we can approximate this by the differential

$$\frac{\Delta y}{y} \approx \frac{dy}{y} = \frac{f'(x)dx}{f(x)}.$$

Theorems

- **Differentiability and continuity**: If a function f is differentiable at x_0, then f is continuous at x_0. The converse is false, differentiability does not necessarily imply continuity, for example, the absolute value function. As a corollary, if a function f is not continuous at x_0, then it is not differentiable at x_0.

Miscellaneous Results

- **Techniques of Differentiation**:
  1. Product Rule:
  $$\frac{d}{dx}(f \cdot g) = \frac{df}{dx}g + f\frac{dg}{dx}.$$
  2. Quotient Rule:
  $$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}.$$
  3. Chain Rule: For compositions $y = f(g(x))$, we set $u = g(x)$, which implies $y = f(u)$, and then
  $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{df(u)}{du}\frac{dg(x)}{dx}.$$
  4. Implicit Differentiation: For functions defined implicitly by some equation, we can differentiate both sides of the equation using the rules above and solve for dy/dx in terms of x and y.

- **Related Rates**: For quantities related by a particular formula, then the rate of change at a particular time requires knowledge of the quantities and their derivatives at that time. For example, the volume of a cone is $V = \frac{\pi}{3}r^2h$, then the rate of change of the volume as r and h are allowed to vary is
$$\frac{dV}{dt} = \frac{\pi}{3}\left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right].$$