ST3451: Problem set 2

October, 2015

Monday 2nd november 5 o'clock

Problem 4 is due at class on Wednesday 28th October.

1. For a simple linear regression model with

$$\widehat{\epsilon}_i = Y_i - \widehat{Y}_i$$

show that

(a) $\sum_{i=1}^{n} \widehat{\epsilon}_i = 0$

(b) $\sum_{i=1}^{n} \widehat{\epsilon}_i X_i = 0$

2. Show that the expected value of MS(Reg) for the standard SLR model is

$$\sigma^2 + \beta_1^2 S_{XX}.$$

3. A linear regression model may be written either

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i \qquad i = 1, \ldots, n$$

or

$$Y_i = \alpha_0 + \alpha_1 (X_i - \bar{X}) + \epsilon_i \qquad i = 1, \ldots, n.$$

(a) Find the relationship between the α's and the β's.

(b) Use the method of least squares to estimate α_0 and α_1.

4. A random variable Y is distributed with expectation depending linearly on another variable X and with constant variance σ^2. Observations are denoted by $(X_i, Y_i), i = 1, \ldots, n$. If we write

$$\mathrm{E}\{Y_i\} = \alpha_0 + \alpha_1 (X_i - \bar{X})$$

(a) Find the expected values and variances of the least squares estimators $\widehat{\alpha}_0$ and $\widehat{\alpha}_1$.

(b) Show that

$$\mathrm{Cov}\{Y_i, \widehat{\alpha}_1\} = \frac{\sigma^2 (X_i - \bar{X})}{\sum (X_i - \bar{X})^2}$$

(c) Hence or otherwise prove that $\widehat{\alpha}_0$ and $\widehat{\alpha}_1$ are uncorrelated and interpret this property.

5. A random variable Y has mean $\alpha + \beta X$ and variance σ^2. At each of n values of X, a value of Y is observed giving n pairs of independent observations $(X_1, Y_1), \ldots, (X_n, Y_n)$. Suppose that the observations are made on two different days, n_1 on the first day and $n - n_1$ on the second. Suppose that there are practical reasons for assuming that conditions vary from day to day in such a way as to affect α but not β or σ^2. Show that the least squares estimator of the common β derived from the two samples is given by

$$\widehat{\beta} = \frac{\omega_1 \widehat{\beta}_1 + \omega_2 \widehat{\beta}_2}{\omega_1 + \omega_2}$$

where $\widehat{\beta}_1$ and $\widehat{\beta}_2$ are the least squares estimators of β from the samples on the first and second days respectively, and ω_1, ω_2 are the corresponding sums of squares of the X's about their means. Find $\mathrm{Var}\{\widehat{\beta}\}$.

6. Given a random sample of n data pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$ a data analyst decides to draw a line on the scatterplot by joining the first and last points. Write out a formula for the slope of this line which will be denoted $\tilde{\beta}$. Is $\tilde{\beta}$ a linear estimator? Assuming a linear regression model

$$E(Y) = \alpha + \beta X$$

is $\tilde{\beta}$ an unbiased estimator of β? How does the variance of $\tilde{\beta}$ compare with the variance of the least squares estimator $\hat{\beta}$?

7. (*) Consider the simple linear regression model

$$Y_i = \theta X_i + \epsilon_i \qquad i = 1, \ldots, n$$

where the X_i are fixed constants and the ϵ_i are uncorrelated normal variables with zero means and the same variance σ^2. Show how a confidence interval for θ may be obtained based on the Student-t variable

$$t_1 = \frac{\hat{\theta} - \theta}{(s^2 / \sum X_i^2)^{1/2}}$$

where $\hat{\theta}$ is the least squares unbiased estimator of θ and s^2 is the usual unbiased estimator of σ^2.

Show that this confidence interval is always of smaller length than the alternative interval based on the variable

$$t_2 = \frac{\bar{Y} - \theta \bar{X}}{(s^2/n)^{1/2}}$$

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Q 1.(a) $\hat{\varepsilon}_i = y_i - \hat{y}_i$

$$
\begin{aligned}
\Sigma \hat{e}_i &= \Sigma (y_i - \hat{y}_i \\
&= \Sigma y_i - \Sigma (b_1 x_i + b_0) \\
&= \Sigma y_i - \Sigma (b_1 x_i + \bar{y} - b_1 \bar{x}) \\
&= \Sigma y_i - [b_1 \Sigma x_i + n\bar{y} - b_1 n\bar{x}] \quad (\Sigma x_i = n\bar{x}) \\
&= \Sigma y_i - [\cancel{b_1 n\bar{x}} + \Sigma y_i - \cancel{b_1 n\bar{x}}] \\
&= \Sigma y_i - \Sigma y_i \\
&= 0
\end{aligned}
$$

(B) Prove $\Sigma \hat{\varepsilon}_i x_i = 0$

$$
\begin{aligned}
\Sigma \hat{\varepsilon}_i x_i &= \Sigma x_i (y_i - \hat{y}_i) \\
&= \Sigma x_i y_i - \Sigma x_i (\hat{y}_i) \\
&= \Sigma x_i y_i - \Sigma x_i (b_1 x_i + b_0) \\
&= \Sigma x_i y_i - \Sigma x_i (b_1 x_i + \bar{y} - b_1 \bar{x}) \\
&= \Sigma x_i y_i - \Sigma x_i [b_1 \Sigma x_i + n\bar{y} - b_1 n\bar{x}] \\
&= \Sigma x_i y_i - \Sigma x_i [b_1 n\bar{x} + \Sigma y_i - b_1 n\bar{x}] \\
&= \cancel{\Sigma x_i y_i} - \cancel{b_1 n\bar{x} \Sigma x_i} - \cancel{\Sigma x_i y_i} + \cancel{b_1 n\bar{x} \Sigma x_i} \\
&= 0.
\end{aligned}
$$

(2) Show $E[MS(reg)] = \sigma^2 + \beta_1^2 S_{xx}$

$$E\left[\Sigma(\hat{y} - \bar{y})^2\right] \qquad \hat{y} = b_0 + b_1\bar{x}$$

$$SSR = \Sigma(b_0 - b_1 x_i - b_0 - b_1\bar{x})^2$$
$$= b_1^2 \, \Sigma(x_i - \bar{x})^2$$

$$E(SSR) = E(b_1)^2 \; \Sigma(x_i - \bar{x})^2$$
$$= [Var(b_1) + (E b_1)^2] \; \Sigma(x_i - \bar{x})^2$$

$$= \left[\frac{\sigma^2}{\Sigma(x_i - \bar{x})^2} + \beta_1^2\right] \Sigma(x_i - \bar{x})^2$$

$$= \sigma^2 + \beta_1^2 \, \Sigma(x_i - \bar{x})^2$$
$$= \sigma^2 + \beta_1^2 S_{xx}$$

3 (A) In the second model we are using a predictor of $(x_i - \bar{x})$ which is a centered predictor with mean 0

(B) Model $= (y_i - \alpha_0 + \alpha_1 x_i - \alpha_1 \bar{x})$

$$\frac{d}{d\alpha_0}(y_i - \alpha_0 + \alpha_1 x_i - \alpha_1\bar{x})^2$$

$$= (2)(-1)\Sigma(y_i - \alpha_0 + \alpha_1 x_i - \alpha_1\bar{x}) = 0$$
$$\Sigma y_i - n\alpha_0 + \alpha_1 \Sigma x_i - \alpha_1 n\bar{x} = 0$$
$$n\alpha_0 = \Sigma y_i + \alpha_1 \Sigma x_i - \alpha_1 n \frac{\Sigma x_i}{n}$$
$$\alpha_0 = \frac{\Sigma y_i}{n} + \frac{\alpha_1 \Sigma x_i}{n} - \frac{\alpha_1 \Sigma x_i}{n}$$
$$\alpha_0 = \bar{y} \quad \text{or} \quad \Sigma y_i / n$$

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Q3 b. $\frac{d\,(y_i - \alpha_0 - \alpha_1(x_i - \bar{x}))^2}{d\alpha_1}$ $\alpha_0 = \bar{y}$

$(-2)\Sigma\left[(x_i-\bar{x})(y_i - \alpha_0 - \alpha_1 x_i + \alpha_1\bar{x})\right] = 0$

$\Sigma\left[x_i y_i - \alpha_0 x_i - \alpha_1 x_i^2 + \alpha_1\bar{x}x_i - \bar{x}y_i + \alpha_0\bar{x} + \alpha_1\bar{x}x_i - \alpha_1\bar{x}^2\right]$

$\Sigma x_i y_i - \alpha_0\Sigma x_i - \alpha_1\Sigma x_i^2 + \alpha_1\bar{x}\Sigma x_i - \bar{x}\Sigma y_i + n\alpha_0\bar{x} + \alpha_1\bar{x}\Sigma x_i - n\alpha_1\bar{x}^2 = 0$

$- \frac{\Sigma x_i \Sigma y_i}{n} + n\frac{\Sigma y_i}{n}\frac{\Sigma x_i}{n} = 0$

(5) and (6) cancel

$\alpha_1\frac{\Sigma x_i \Sigma x_i}{n} - n\alpha_1\frac{\Sigma x_i}{n}\frac{\Sigma x_i}{n} = 0$

(4) and (8) cancel

$\Sigma x_i y_i - \bar{y}\Sigma x_i - \alpha_1\Sigma x_i^2 + \alpha_1\bar{x}\Sigma x_i = 0$

$\alpha_1(\bar{x}\Sigma x_i - \Sigma x_i^2) - \bar{y}\Sigma x_i + \Sigma x_i y_i = 0$

$\alpha_1 = \frac{\bar{y}\Sigma x_i - \Sigma x_i y_i}{(\bar{x}\Sigma x_i - \Sigma x_i^2}$

$\alpha_1 = \frac{\Sigma x_i(\bar{y} - y_i)}{(\frac{\Sigma x_i \Sigma x_i}{n} - \Sigma x_i^2)}$

minus sign from here → flip around

$= \alpha_1 = \frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{\Sigma(x_i-\bar{x})^2}$

$=$

4 (A) $\alpha_0 = \bar{y}$ or $\frac{\Sigma y_i}{n}$

$\mathbb{E}[b_0] = \mathbb{E}[\bar{y}] = \frac{1}{n}\mathbb{E}[y_1] + \dots + \frac{1}{n}\mathbb{E}[y_n]$

$= \frac{1}{n}\mu_y + \dots + \frac{1}{n}\mu_y$

$= \mu_y$

$Var[b_0] = Var[\bar{y}] = Var\left[\frac{1}{n}y_1 + \dots + \frac{1}{n}y_n\right]$

$= Var\left[\frac{1}{n}y_1\right] + \dots + Var\left[\frac{1}{n}y_n\right] + Cov\left[\frac{1}{n}y_1, \frac{1}{n}y_2\right] + \dots + Cov\left[\frac{1}{n}y_{n-1}, \frac{1}{n}y_n\right]$

$= \frac{1}{n^2}Var[y_1] + \dots + \frac{1}{n^2}Var[y_n] + 2\sum_{i=1}^{n-1}\frac{1}{n^2}Cov[y_i, y_j]$

↳ Cov = 0 by independence of y_i's and y_j's

$= \frac{1}{n^2}\sigma^2 + \ldots + \frac{1}{n^2}\sigma^2$

$= \frac{n}{n^2}\sigma^2 = \sigma^2/n$ $= Var[\alpha_0]$

$$\mathbb{E}[b_1] = \mathbb{E}\left[\frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{\Sigma(x_i-\bar{x})^2}\right]$$

$= \frac{1}{\Sigma(x_i-\bar{x})^2}\mathbb{E}[\Sigma(x_i-\bar{x})(y_i-\bar{y})]$ (assumed x_i's are fixed/known)

$= \text{"} \ \mathbb{E}[(x_1-\bar{x})(y_1-\bar{y}) + \ldots + (x_n-\bar{x})(y_n-\bar{y})]$

$= \text{"} \ \mathbb{E}[(x_1-\bar{x})\mathbb{E}[y_1-\bar{y}] + \ldots + (x_n-\bar{x})\mathbb{E}[y_n-\bar{y}]]$

$= \frac{1}{\Sigma(x_i-\bar{x})^2}\Sigma(x_i-\bar{x}) \ \mathbb{E}[y_i-\bar{y}]$

$= \frac{1}{S_{xy}}\Sigma(x_i-\bar{x})[\mathbb{E}[y_i]-\mathbb{E}[\bar{y}]]$ and $\mathbb{E}[y_i] = \beta_0+\beta_1(x_i-\bar{x})$

also $\mathbb{E}[\bar{y}] = \frac{1}{n}\Sigma[\beta_0+\beta_1(x_i-\bar{x})]$

$= \frac{1}{S_{xx}}\Sigma(x_i-\bar{x})[\beta_0+\beta_1(x_i-\bar{x}) - (\beta_0-\beta_1\bar{x}\Sigma x_i)]$

$= \frac{1}{S_{xx}}\Sigma(x_i-\bar{x}) \ \beta_1(x_i-\bar{x}-\Sigma x_i)$

$= \beta_1 \frac{1}{S_{xy}}\Sigma(x_i-\bar{x})^2 = \beta_1\frac{S_{xx}}{S_{xx}} = \beta_1$

you've swapped notation

$\alpha_1 = \beta_1$,

$\alpha_0 = \beta_0$,

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4 (A) $Var[\hat{\alpha}_1] = Var\left[\frac{S_{xy}}{S_{xx}}\right] = \frac{1}{S_{xx}^2} Var[S_{xy}]$

$= \frac{1}{S_{xx}^2} Var\left[\Sigma (x_i - \bar{x}) y_i\right]$

$= \frac{1}{S_{xx}^2} Var\left[(x_1 - \bar{x})(y_1) + \ldots + (x_n - \bar{x})(y_n)\right]$

$= \frac{1}{S_{xx}^2}\Big[Var[(x_1-\bar{x})(y_1)] + \ldots + Var[(x_n-\bar{x})(y_n)] + \cancel{Cov[(x_n-\bar{x}), y_n]} \ldots Cov$

$+ Cov[(x_1-\bar{x})(y_1), (x_2-\bar{x})(y_2)] + \ldots + Cov[(x_{n-1}-\bar{x})(y_{n-1}), (x_n-\bar{x})(y_n)]\Big]$

$= \frac{1}{S_{xx}^2}\left[\Sigma Var[(x_i-\bar{x})(y_i)] + 2\sum_{i=1}^{n-1}\sum_{j>i} Cov[(x_i-\bar{x})(y_i), (x_j-\bar{x})(y_j)]\right]$

$= \frac{1}{S_{xx}^2}\left[\Sigma (x_i-\bar{x})^2 Var[y_i] + 2\sum_{i=1}^{n-1}\sum_{j>i} (x_i-\bar{x})(x_j-\bar{x})\, Cov[y_i, y_j]\right]$

$cov = 0 \Rightarrow$ independence

$= \frac{1}{S_{xx}^2}\left[\Sigma (x_i - \bar{x})^2 \sigma^2 + 0\right]$

$= \sigma^2 S_{xx} / S_{xx}^2 = \sigma^2 / S_{xx} = Var[\hat{\alpha}_1]$

4 (B) Show $Cov[y_i, \hat{\alpha}_1] = \frac{\sigma^2 (x_i - \bar{x})}{\Sigma (x_i - \bar{x})^2}$ [illegible]

$Cov\left[y_i, \frac{\Sigma (x_i - \bar{x})(y_i - \bar{y})}{\Sigma (x_i - \bar{x})^2}\right]$

$= \frac{1}{S_{xx}} Cov\left[y_i, \Sigma (x_i - \bar{x})(y_i - \bar{y})\right]$ will only pick up i^{th} term

$= \frac{1}{S_{xx}} Cov\left[y_i, (x_i - \bar{x})(y_i)\right]$

$= \frac{(x_i - \bar{x})}{S_{xx}} Cov[y_i, y_i] = \frac{\sigma^2 (x_i - \bar{x})}{\Sigma (x_i - \bar{x})^2}$

4(c) Cov - calc and check if 0 betw $\hat{a}_0$ and $\hat{a}_1$

$$Cov[\hat{a}_0, \hat{a}_1] = Cov\left[\bar{y}, \frac{S_{xy}}{S_{xx}}\right]$$

$$= \frac{1}{N S_{xx}} Cov\left[\Sigma y_i, S_{xy}\right]$$

$$= \frac{1}{N S_{xx}} Cov\left[y_1 + y_2 + \dots + y_n, S_{xy}\right]$$

$$= \frac{1}{N S_{xx}} \left[Cov[y_1, S_{xy}] + Cov[y_2, S_{xy}] + \dots Cov[y_n, S_{xy}]\right]$$

$$= \frac{1}{N S_{xx}} \left[\Sigma Cov[y_i, S_{xy}]\right]$$

$$= \frac{1}{N S_{xx}} \Sigma Cov\left[y_i, \sum_j (x_j - \bar{x})(y_j)\right]$$

$$= \frac{1}{N S_{xx}} \Sigma Cov\left[y_i, (x_1 - \bar{x})y_1 + \dots + (x_i - \bar{x})(y_i) + \dots + (x_n - \bar{x})(y_n)\right]$$

will only have cov for y_i with y_i because of independence

$$= \frac{1}{N S_{xx}} \Sigma Cov\left[y_i, (x_i - \bar{x}) y_i\right]$$

$$= \frac{1}{N S_{xx}} (x_i - \bar{x}) Cov[y_i, y_i]$$

$$= \frac{\sigma^2}{n} \frac{\Sigma (x_i - \bar{x})}{S_{xx}} = 0$$

- Interpretation ⇒ no relationship between $\hat{a}_0$ and $\hat{a}_1$, i.e no relationship between mean $(\bar{y})$ and estimator.
- Does not guarantee independence of variables
- Variables do NOT change together.

04/11/15 ALSM 1

PROBLEM SHEET 2 ANSWERS

Q1 $\hat{\varepsilon}_i = y_i - \hat{y}_i$

A. $\Sigma \hat{\varepsilon}_i = 0 \Rightarrow \Sigma(y_i - \hat{y}_i) = \Sigma(y_i - \beta_0 - \beta_1 x_i)$

$= \Sigma(y_i - (\bar{y} - \hat{\beta}_1\bar{x}) - \beta_1 x_i)$

$= \Sigma(y_i - \bar{y} - \hat{\beta}_1(x_i - \bar{x}))$

$= \Sigma(y_i - \bar{y}) - \hat{\beta}_1 \Sigma(x_i - \bar{x})$

$= 0 - \hat{\beta}_1(0) = 0$

B $\Sigma \hat{\varepsilon}_i x_i = 0$ $\Rightarrow$ Show

$= \Sigma(y_i - \bar{y} - \hat{\beta}_1(x_i - \bar{x}))x_i$

$= \Sigma(y_i - \bar{y})x_i - \hat{\beta}_1\Sigma(x_i - \bar{x})x_i$

$= Sxy - \hat{\beta}_1 Sxx$ Aside: $\hat{\beta}_1 = \frac{Sxy}{Sxx}$

$= Sxy - \frac{Sxy}{Sxx} Sxx$

$= 0$

Q2 Show $\mathbb{E}[MS(Reg)] = \sigma^2 + \beta_1^2 Sxx$

$MS(Reg) = SS(Reg)/1$

$SS(Reg) = \Sigma(\hat{y}_i - \bar{y})^2 = \hat{\beta}_1^2 Sxx$ Shown in notes

$= \mathbb{E}[\hat{\beta}_1^2 Sxx] = Sxx\,\mathbb{E}[\hat{\beta}_1^2]$ aside: $Var[R] = \mathbb{E}[R^2] - (\mathbb{E}[R])^2$ Use this

$= Sxx\,[Var[\hat{\beta}_1] + \mathbb{E}[\hat{\beta}_1]^2]$

$= Sxx\,[\frac{\sigma^2}{Sxx} + \hat{\beta}_1^2]$ unbiased estimator - from least squares

$= \sigma^2 + \hat{\beta}_1^2 Sxx$

Q3 $y_i = \alpha_0 + \alpha_1(x_i - \bar{x}) + \varepsilon_i$ usually $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$

A. Relationship between α and β

$y_i = \alpha_0 + \alpha_1 x_i - \alpha_1\bar{x} + \varepsilon_i$

$= (\alpha_0 - \alpha_1\bar{x}) + \alpha_1 x_i + \varepsilon_i$

B Least Squares Criterion for a_0 a_1

$$\sum \epsilon_i^2 = \sum (y_i - \hat{a}_0 - \hat{a}_1 \underbrace{(x_i - \bar{x})}_{c_i})^2$$

$$\frac{dQ}{da_0} = -2\sum(y_i - \hat{a}_0 - \hat{a}_1 c_i) = 0 \quad (1)$$

$$\frac{dQ}{da_1} = -2\sum(c_i)(y_i - \hat{a}_0 - \hat{a}_1 c_i) = 0 \quad (2)$$

$(1) \Rightarrow \sum y_i - n\hat{a}_0 - \hat{a}_1 \sum c_i = 0$

$\sum c_i = \sum(x_i - \bar{x}) = 0$

$\Rightarrow \sum y_i - n\hat{a}_0 = 0 \qquad \hat{a}_0 = \bar{y}$

$(2) \Rightarrow \sum c_i y_i - \hat{a}_0 \overbrace{\sum c_i}^{=0} - \hat{a}_1 \sum c_i^2 = 0$

$$\hat{a}_1 = \frac{\sum c_i y_i}{\sum c_i^2} = \frac{\sum (x_i - \bar{x}) y_i}{\sum (x_i - \bar{x})^2} = \frac{S_{xy}}{S_{xx}} = \hat{\beta}_1$$

Q4 A Same Model as Q3 $\hat{a}_0 = \bar{y}$ $\hat{a}_1 = S_{xy}/S_{xx}$

$$\mathbb{E}[\hat{a}_0] = \mathbb{E}[\bar{y}] = \mathbb{E}\left[\tfrac{1}{n}\sum y_i\right]$$
$$= \tfrac{1}{n}\sum \mathbb{E}[y_i] = \tfrac{1}{n}\sum[a_0 + a_1(x_i - \bar{x})]$$
$$= \tfrac{1}{n}[na_0 + a_1 \overbrace{\sum(x_i - \bar{x})}^{=0}]$$
$$= a_0 \Rightarrow \text{unbiased}$$

$$Var[\hat{a}_0] = Var[\bar{y}] = Var\left[\tfrac{1}{n}\sum y_i\right]$$
$$= \tfrac{1}{n^2} Var\left[\sum y_i\right]$$
$$= \tfrac{1}{n^2}\left[\sum Var[y_i] + 2\sum_{i=1}^{n-1}\sum_{j>i} Cov[y_i, y_j]\right] \quad Cov = 0 \text{ by independence}$$
$$= 1/n^2 \sum \sigma^2 = n\sigma^2/n^2 = \sigma^2/n$$

$$\mathbb{E}[\hat{a}_1] = \mathbb{E}\left[\tfrac{S_{xy}}{S_{xx}}\right] = \tfrac{1}{S_{xx}} \mathbb{E}[S_{xy}]$$
$$= \tfrac{1}{S_{xx}} \mathbb{E}\left[\sum (x_i - \bar{x}) y_i\right]$$
$$= \tfrac{1}{S_{xx}} \sum (x_i - \bar{x}) \mathbb{E}[y_i]$$
$$= \text{" "} [a_0 + a_1(x_i - \bar{x})]$$
$$= \tfrac{1}{S_{xx}}\left[a_0 \sum (x_i - \bar{x}) + a_1 \sum (x_i - \bar{x})^2\right]$$

24/11/15 ALSM1

$= \frac{1}{S_{xx}} [a_1 S_{xx}] = a_1$

$Var[\hat{a}_1] = Var[\frac{S_{xy}}{S_{xx}}] = \frac{1}{S_{xx}^2} Var[S_{xy}]$

$= \frac{1}{S_{xx}^2} Var[\Sigma (x_i - \bar{x}) y_i]$

$= \frac{1}{S_{xx}^2} [\Sigma Var[(x_i-\bar{x})y_i] + 2 \sum_{i=1}^{n} \sum_{j>i} Cov[(x_i-\bar{x})(y_i), (x_j-\bar{x})(y_j)]]$

$= \text{"} \quad [\Sigma (x_i-\bar{x})^2 Var[y_i] + 2 \Sigma\Sigma (x_i-\bar{x})(x_j-\bar{x}) Cov[y_i, y_j]]$ y's independent $\Rightarrow 0$

$= \frac{1}{S_{xx}^2} [\Sigma (x_i-\bar{x})^2 \sigma^2]$

$= \frac{S_{xx}}{S_{xx}^2} \sigma^2 = \sigma^2 / S_{xx}$

B $Cov[y_i, \hat{a}_1] = Cov[y_i, \frac{S_{xy}}{S_{xx}}]$

$= \frac{1}{S_{xx}} Cov[y_i, \Sigma (x_j-\bar{x}) y_j]$

$= \text{"} \quad Cov[y_i, ((x_1-\bar{x})y_1 + \dots + (x_n-\bar{x})y_n)]$

$= \frac{1}{S_{xx}} \Sigma (x_j-\bar{x}) Cov[y_i, y_j]$ $\quad Cov[y_i, y_j] = \begin{cases} \sigma^2 & i=j \\ 0 & \text{otherwise} \end{cases}$

$= \frac{1}{S_{xx}} (x_i-\bar{x}) \sigma^2$

$= \frac{\sigma^2 (x_i-\bar{x})}{S_{xx}}$

C Prove $Cov[\hat{a}_0, \hat{a}_1] = 0$

$Cov[\bar{y}, \hat{a}_1] = Cov[\frac{1}{n}\Sigma y_i, \hat{a}_1] = \frac{1}{n} \Sigma Cov[y_i, \hat{a}_1]$

$= \frac{1}{n} \Sigma \frac{\sigma^2 (x_i-\bar{x})}{S_{xx}}$ (from previous part)

$= \frac{\sigma^2}{n S_{xx}} \underbrace{\Sigma (x_i-\bar{x})}_{0} = 0$

- If you center covariates, you're making joint covariance matrix of $\hat{\beta}_0$ and $\hat{\beta}_1$ into a diagonal matrix $\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$
- AKA orthogonalisation

Q5 $(x_1, y_1) \dots (x_{n_1}, y_{n_1})$ Day 1

$(x_{n_1+1}, y_{n_1+1}) \dots (x_n, y_n)$ Day 2

$\alpha + \beta x$ but α can vary day to day but β doesn't

Day 1: $y_i = \alpha_1 + \beta x_i \quad i = 1, \dots, n_1$

Day 2: $y_i = \alpha_2 + \beta x_i \quad i = n_1+1, \dots, n$

B Least Squares Criterion for a_0 a_1

$$\sum \epsilon_i^2 = \sum (y_i - \hat{a}_0 - \hat{a}_1 \underbrace{(x_i - \bar{x})}_{c_i})^2$$

$$\frac{dQ}{da_0} = -2 \sum (y_i - \hat{a}_0 - \hat{a}_1 c_i) = 0 \quad (1)$$

$$\frac{dQ}{da_1} = -2 \sum (c_i)(y_i - \hat{a}_0 - \hat{a}_1 c_i) = 0 \quad (2)$$

$(1) \Rightarrow \sum y_i - n\hat{a}_0 - \hat{a}_1 \sum c_i = 0$

$\sum c_i = \sum (x_i - \bar{x}) = 0$

$\Rightarrow \sum y_i - n\hat{a}_0 = 0 \qquad \hat{a}_0 = \bar{y}$

$(2) \Rightarrow \sum c_i y_i - \hat{a}_0 \overbrace{\sum c_i}^{=0} - \hat{a}_1 \sum c_i^2 = 0$

$$\hat{a}_1 = \frac{\sum c_i y_i}{\sum c_i^2} = \frac{\sum (x_i - \bar{x}) y_i}{\sum (x_i - \bar{x})^2} = \frac{Sxy}{Sxx} = \hat{\beta}_1$$

Q4 A Same Model as Q3 $\hat{a}_0 = \bar{y}$ $\hat{a}_1 = {}^{Sxy}/_{Sxx}$

$\mathbb{E}[\hat{a}_0] = \mathbb{E}[\bar{y}] = \mathbb{E}[\frac{1}{n} \sum y_i]$

$= \frac{1}{n} \sum \mathbb{E}[y_i] = \frac{1}{n} \sum [a_0 + a_1 (x_i - \bar{x})]$

$= \frac{1}{n} [n a_0 + a_1 \overbrace{\sum (x_i - \bar{x})}^{=0}]$

$= a_0 \Rightarrow$ unbiased

$Var[\hat{a}_0] = Var[\bar{y}] = Var[\frac{1}{n} \sum y_i]$

$= \frac{1}{n^2} Var[\sum y_i]$

$= \frac{1}{n^2} [\sum Var[y_i] + 2 \sum_{i=1}^{n-1} \sum_{j>i} Cov[y_i, y_j]]$ Cov = 0 by independence

$= 1/n^2 \sum \sigma^2 = n\sigma^2/n^2 = \sigma^2/n$

$\mathbb{E}[\hat{a}_1] = \mathbb{E}[\frac{Sxy}{Sxx}] = \frac{1}{Sxx} \mathbb{E}[Sxy]$

$= \frac{1}{Sxx} \mathbb{E}[\sum (x_i - \bar{x}) y_i]$

$= \frac{1}{Sxx} \sum (x_i - \bar{x}) \mathbb{E}[y_i]$

$=$ " " $[a_0 + a_1 (x_i - \bar{x})]$

$= \frac{1}{Sxx} [a_0 \sum (x_i - \bar{x}) + a_1 \sum (x_i - \bar{x})^2]$

LS criterion show $\beta = \frac{w_1\hat{\beta} + w_2\hat{\beta}}{w_1 + w_2}$

$Q(\hat{\alpha}_1, \hat{\alpha}_2, \beta) = \sum \varepsilon_i^2 = \sum_{i=1}^{n_1} \varepsilon_i^2 + \sum_{n_1+1}^{n} \varepsilon_i^2$

$= \sum_{i=1}^{n_1} (y_i - \hat{\alpha}_1 - \beta x_i)^2 + \sum_{i=n_1+1}^{n} (y_i - \hat{\alpha}_2 - \beta x_i)^2$

$\frac{dQ}{d\hat{\alpha}_1} = -2 \sum_{i=1}^{n_1} (y_i - \hat{\alpha}_1 - \hat{\beta} x_i)$ (1)

$\frac{dQ}{d\hat{\alpha}_2} = -2 \sum_{i=n_1+1}^{n} (y_i - \hat{\alpha}_2 - \hat{\beta} x)$ (2)

$\frac{dQ}{d\beta} = -2 \sum_{i=1}^{n_1} x_i (y_i - \hat{\alpha}_1 - \hat{\beta} x_i) - 2 \sum_{i=n_1+1}^{n} x_i (y_i - \hat{\alpha}_2 - \hat{\beta} x_i)$ (3)

(1) $\sum y_i - n_1 \hat{\alpha}_1 - \hat{\beta} \sum x_i = 0$

$\hat{\alpha}_1 = \frac{\sum y_i}{n_1} - \hat{\beta} \frac{\sum x_i}{n_1}$

$= \bar{y}_1 - \hat{\beta} \bar{x}_1$

(2) $\Rightarrow \bar{y}_2 - \hat{\beta} \bar{x}_2$

(3) Plug (1) and (2) into (3)

$\sum_{i=1}^{n_1} x_i (y_i - (\bar{y}_1 - \hat{\beta} x_1) - \hat{\beta} x_i) + \sum_{i=n_1+1}^{n} x_i (y_i - (\bar{y}_2 - \hat{\beta} \bar{x}_2) - \hat{\beta} x_i) = 0$

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Tutorial Question Continued.

(3) $\sum_{(1)} x_i y_i - \hat{a}_1 \sum_{(1)} x_i - \hat{\beta} \sum_{(1)} x_i^2 + \sum_{(2)} x_i y_i - \hat{a}_2 \sum_{(2)} x_i - \hat{\beta} \sum_{(2)} x_i^2 = 0$

$$\sum_{(1)} x_i y_i - (\bar{y}_1 - \hat{\beta}\bar{x}_1) \sum_{(1)} x_i - \hat{\beta} \sum_{(1)} x_i^2 + \sum_{(2)} x_i y_i - (\bar{y}_2 - \hat{\beta}\bar{x}_2) \sum_{(2)} x_i - \hat{\beta} \sum_{(2)} x_i^2 = 0$$

$$\sum_{(1)} x_i y_i - n_1 \bar{x}_1 \bar{y}_1 - \hat{\beta}\left(\sum_{(1)} x_i^2 - n_1 \bar{x}_1^2\right) + \sum_{(2)} x_i y_i - n_2 \bar{x}_2 \bar{y}_2 - \hat{\beta}\left(\sum_{(2)} x_i^2 - n_2 \bar{x}_2^2\right) = 0$$

$$S_{xy}^{(1)} - \hat{\beta} S_{xx}^{(1)} + S_{xy}^{(2)} - \hat{\beta} S_{xx}^{(2)} = 0$$

$$\hat{\beta}\left(S_{xx}^{(1)} + S_{xx}^{(2)}\right) = S_{xy}^{(1)} + S_{xy}^{(2)}$$

$$\hat{\beta} = \frac{S_{xy}^{(1)} + S_{xy}^{(2)}}{S_{xx}^{(1)} + S_{xx}^{(2)}} = \frac{w_1 \hat{\beta}_1 + w_2 \hat{\beta}_2}{w_1 + w_2}$$

$$= \frac{S_{xx}^{(1)}\left(\frac{S_{xy}^{(1)}}{S_{xx}^{(1)}}\right) + S_{xx}^{(2)}\left(\frac{S_{xy}^{(2)}}{S_{xx}^{(2)}}\right)}{S_{xx}^{(1)} + S_{xx}^{2}}$$

$$w_1 = S_{xx}^{(1)} \qquad w_2 = S_{xx}^{(2)}$$

$$Var[\hat{\beta}] = Var\left[\frac{w_1 \hat{\beta}_1 + w_2 \hat{\beta}_2}{w_1 + w_2}\right]$$

$$= \frac{1}{(w_1 + w_2)^2} Var[w_1 \hat{\beta}_1 + w_2 \hat{\beta}_2]$$

$$= \frac{1}{(w_1 + w_2)^2}\left[w_1^2 Var[\hat{\beta}_1] + w_2^2 Var[\hat{\beta}_2] + 2 w_1 w_2 Cov[\hat{\beta}_1, \hat{\beta}_2]\right]$$

$$Var[\hat{\beta}_1] = \sigma^2 / S_{xx}^{(1)} \qquad Var[\hat{\beta}_2] = \sigma^2 / S_{xx}^{(2)}$$

$Cov[\hat{\beta}_1, \hat{\beta}_2] = Cov\left[\sum_{(1)} c_i y_i, \sum_{(2)} d_i y_i\right]$ all cov will be 0 $\Rightarrow$ Different Days

$$Var[\hat{\beta}] = \frac{1}{(w_1 + w_2)^2}\left[\frac{[w_1]^2 \sigma^2}{S_{xx}^{(1)}} + \frac{[w_2]^2 \sigma^2}{S_{xx}^{(2)}}\right]$$