Mohid Sender 1 2012 JF

CHAPTER 3

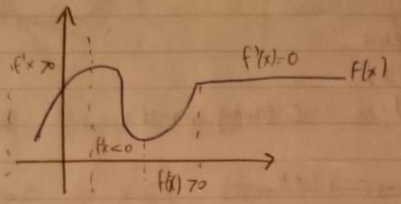
THE DERIVATIVE IN GRAPHING AND APPLICATIONS

12/11/12 Maths Week 8 Chapter 4 Derivative in graphing and Applications

maths.tcd.ie/~playfr

Definitely be asked in exam

Analysis of Functions



For f a continuous on $[a,b]$ and differentiable on (a,b)

- $f'(x) > 0$ $\quad$ for all $\forall x \in (a,b)$ (for all)

  $\Rightarrow f$ is increasing on $[a,b]$

- $f'(x) < 0$ $\quad$ $\forall x \in (a,b)$

  $\Rightarrow f$ is decreasing on $[a,b]$

Also true for intervals such as $[a,\infty)$, $(-\infty,b]$, $(-\infty,\infty)$

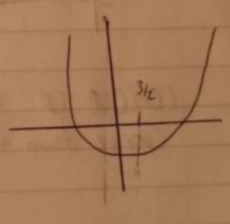
Example: Find the intervals on which $f(x) = x^2 - 3x + 8$ is increasing and decreasing.

$f'(x) = 2x - 3$

$\Rightarrow f'(x) > 0$ when $2x - 3 > 0$

$\Rightarrow x > \frac{3}{2}$

Similarly $f'(x) < 0$ when $x < \frac{3}{2}$



f is increasing on $[3/2, \infty)$

f is decreasing on $(-\infty, 3/2]$

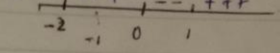
12/11/12 2.

Example: Find the intervals on which $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ is increasing and decreasing

$f'(x) = 12x^3 + 12x^2 - 24x$

$12x(x^2 + x - 2)$

$12x((x-1)(x+2))$ 3 turning points



$f'(-1) = 12(-1)^3 + 12(-1)^2 - 24(-1)$

> 0

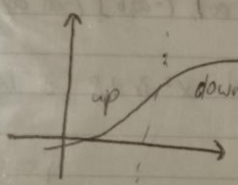
F is increasing on $[-2, 0]$ and $[1, \infty)$

f is decreasing on $(-\infty, -2]$ and $[0, 1]$

Concavity

Concave up - graph "holds water" ∪

Concave down - graph ∩



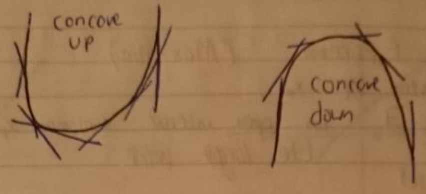
concave up ⇒ slope of tangent line is increasing i.e. f'x is increasing

Concave down ⇒ " " decreasing " f'x is decreasing

3.

12/11/17 Maths

Slope of tangent line is everywhere increasing



Slope of tangent lines is everywhere decreasing

Let f be twice differentiable on an open interval I

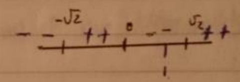
if $f''(x) > 0 \ \forall x \in I$, then f is concave up on I
always open

if $f''(x) < 0, \ \forall x \in I$, then f is concave down on I

Points where f changes direction of concavity are called points of inflection. (usually given by $f''(x) = 0$)

Example: Consider the function $f(x) = 3x^4 - 12x^2$
Where is the function increasing, decreasing, concave up, concave down?

$$f'(x) = 12x^3 - 24x$$
$$= 12x(x^2 - 2)$$



Turning points are $0, \pm\sqrt{2}$

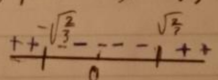
$f(1) = 12(1)(1^2) - 2) < 0$

* exam just 9 year

f is increasing on $[-\sqrt{2}, 0]$ and $[\sqrt{2}, \infty)$
f is decreasing on $(-\infty, -\sqrt{2}]$ and $[0, \sqrt{2}]$

$$f''(x) = 36x^2 - 24$$
$$36(x^2 - \tfrac{2}{3})$$

Turning pnt is $x = \pm\sqrt{\tfrac{2}{3}}$



f is concave up on $(-\infty, -\sqrt{\tfrac{2}{3}})$ and $(\sqrt{\tfrac{2}{3}}, \infty)$
f is concave down on $(-\sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{2}{3}})$

12/11/12 4

Relative Extrema (Max/Min)

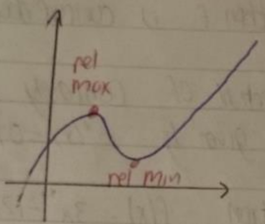
Relative max at x_0

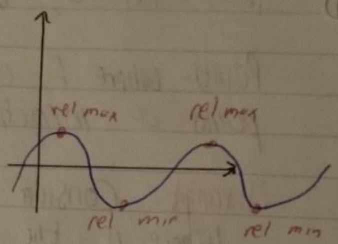
$\Rightarrow$ $\exists$ an open interval containing x_0 which $f(x_0)$ is the largest value

if there exists

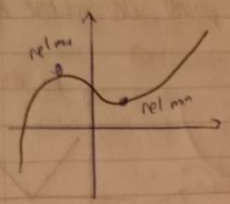
Relative Min at x_0

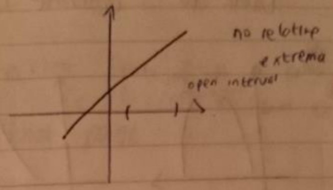
$\Rightarrow$ $\exists$ an open interval containing x_0. which $f(x_0)$ is the smallest value

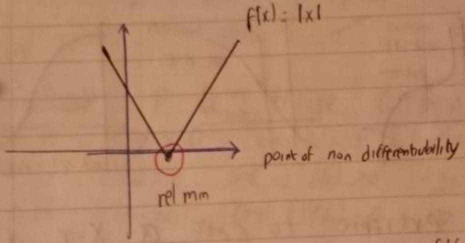




13/11/12 Week 8 Maths







Critical points - are points at which $f'(x_0) = 0$ or f is NOT differentiable.

If $f'(x_0) = 0 \Rightarrow x = x_0$ is a stationary point.

Example: Find all critical points of $f(x) = 2x + 3x^{2/3}$

$f'(x) = 2 + 3(\frac{2}{3})(x^{-1/3}$

$2 + \frac{2}{x^{1/3}}$ $x = 0$ is a point of non differentiability

Stationary point $f'(x) = 0$

$2 + \frac{2}{x^{1/3}} = 0$

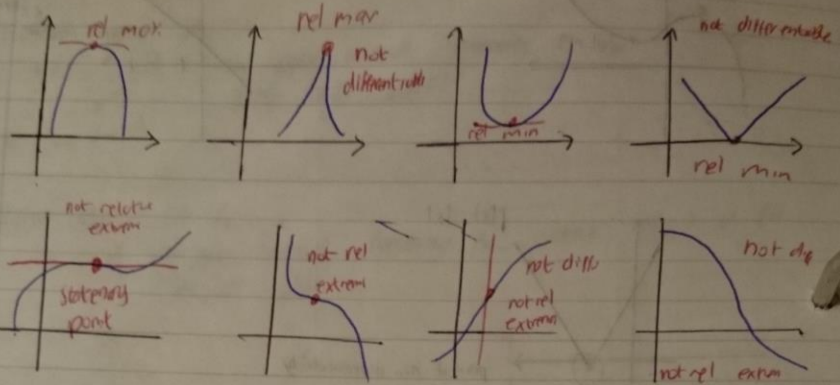
$x^{1/3} = -1$ $x = \sqrt[3]{-1}$

$x = -1$ Critical points are $x = 0$, $x = -1$

First Derivative Test.

Used to determine which critical points are relative extrema

Signs of $\frac{dy}{dx}$ change



For a relative extreme to exist at $x=x_0$ $f'(x)$ must have opposite signs on either side of x.

If $f'(x)>0$ on the left of $x=x_0$ and $f'(x)<0$ on the right of $x=x_0$, then $x=x_0$ is a relative max. Vice-versa (positive then negative)

If $f'(x)$ has the same sign on either side of $x=x_0$ then there is no rel. extrema at x_0

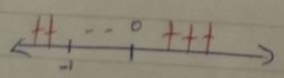
Example Returning to $f(x)=2x+3x^{2/3}$

$f'(x)$ $2+\frac{2}{x^{1/3}}$ with critical points at $x=0$, $x=-1$.

For $x<-1$ $f'(x)>0$

For $-1<x<0$ $f'(x)<0$

For $x>0$, $f'(x)>0$



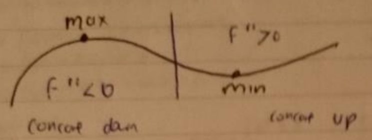
$x=-1$ and $x=0$ are relative extrema

③

13/11/12 Week 8 maths tuesday

Second derivative test:

For stationary points it is clear that $x = x_0$ is a rel. max or rel. min if x_0 is concave down or concave up, respectively at that point



Suppose f is twice differentiable at x_0,

a. If $f'(x_0) = 0$ and $f''(x_0) < 0$ (negative) $\Rightarrow$ rel. Max at x_0

b. If $f'(x_0) = 0$ and $f''(x_0) > 0$ (positive) $\Rightarrow$ rel min at x_0

c. If $f'(x_0) = 0$ and $f''(x_0) = 0$ $\Rightarrow$ inconclusive

Example: Find and classify the relative extrema of

$$f(x) = 4x^4 - 16x^2 + 17$$

$f'(x) = 16x^3 - 32x$

Stationary points are given by

$f'(x) = 0 \Rightarrow 16x^3 - 32x = 0$

$16x(x^2 - 2) = 0$

$x = 0, \quad x = \pm\sqrt{2}$

$f''(x) = 48x^2 - 32$

$\Rightarrow f''(0) = -32 < 0 \Rightarrow x = 0$ is a relative MAX

$\Rightarrow f''(\sqrt{2}) = 48(2) - 32 > 0 \Rightarrow x = \sqrt{2}$ is relative MIN

$\Rightarrow f''(-\sqrt{2}) = 48(2) - 32 > 0 \Rightarrow x = -\sqrt{2}$ is rel MIN

Multiplicity of Roots:

A root $x=r$ of a polynomial $p(x)$ is multiplicity m, if $(x-r)^m$ divides $p(x)$ but $(x-r)^{m+1}$ does not divide

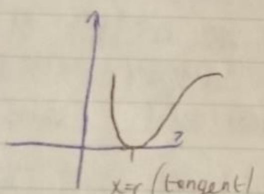
M is highest power divisible m.

Example: $p(x) = x^3+3x^2$
$= x^2(x+3)$

roots = $\underbrace{x=0,\ x=0}_{\text{multiplicity}-2}$, $x=-3$

Theorem: Suppose $p(x)$ is a polynomial with a root of multiplicity m at $x=r$

(a) If M is even $-y=p(x)$ is tangent to the x-axis at $x=r$, does not cross the x-axis at $x=r$ and doesn't have an inflection point there.

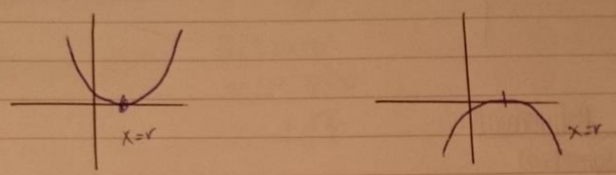


week 8 Thursday

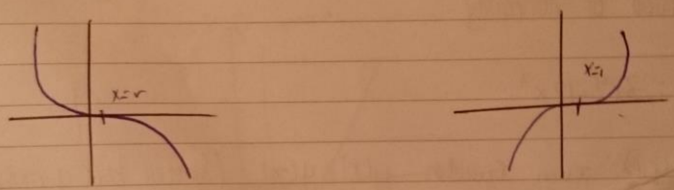
15/11/12 Maths week 8

New assignment

(a) If m is even y = p(x) is tangent to the x-axis at x=r, doesnt cross the x-axis and doesn't have an inflection point

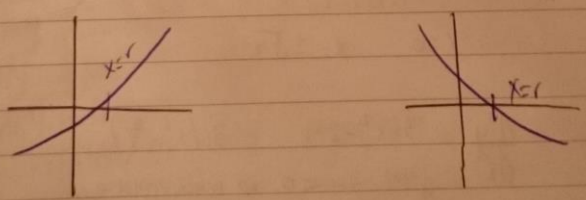


(b) If m is odd, and greater than 1 then the graph is tangent to the x-axis at x=r, crosses the x-axis and has an inflection point. th



(c) If m=1 (simple root) - graph isnt tangent at x=r.

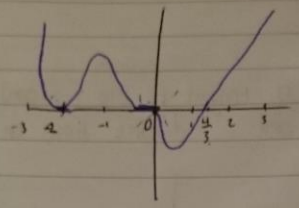
- crosses x axis at x=r
- may or may not have an inflection point



Example: $y = x^3(3x-4)(x+2)^2$

root at (3) $x=0$ (1) $x=\frac{4}{3}$ (2) $x=-2$

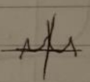
$m=3$ $m=1$ $m=2$



Sketching Polynomials:

1. Identify symmetries.
2. Find x, y intercepts.
3. Find critical points
4. Intervals of increase/decrease
5. Interval of concavity

Example: $y = 3x^4 - 12x^2$



1. Symetries even function - $y(x) = y(-x)$ (Symetric bout y-axis)
2. ~~x, y~~ x intercept, $y = 0$ $3x^4 - 12x^2 = 0$

$3x^2(x^2-4) = 0$ $x = 0$

y intercept, $x = 0$, $y = 0 \sim (0, 0)$ $x = \pm 2$

3. Critical points $\frac{dy}{dx} = 12x^3 - 24x = 0$

$12x(x^2-2) = 0$

$x = 0$ $x = \pm\sqrt{2}$

$\frac{d^2y}{dx^2} = 36x^2 - 24 = 36(x^2 - \frac{2}{3})$

$y''(0) = -24 < 0 \Rightarrow$ max relative

$y''(\pm\sqrt{2}) = > 0 \Rightarrow$ rel. min

4. Interval of increase/decrease: $\frac{dy}{dx} = 12x^3 - 24x$

$- - \quad + + + \quad - - - - \quad + +$

$-\sqrt{2}$ 0 $\sqrt{2}$

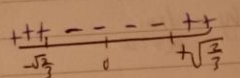
week 8 Thursday

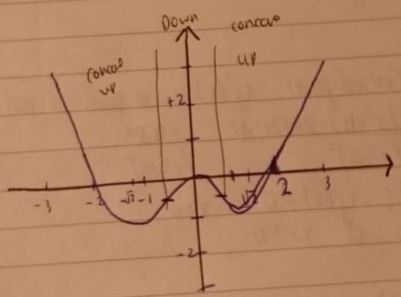
15/11/12 3

$y(x)$ is increasing on $[-\sqrt{2}, 0]$ $[\sqrt{2}, \infty)$
decreasing on $(-\infty, \sqrt{2}]$ $[0, \sqrt{2})$

5. Interval of concavity: $\frac{d^2y}{dx^2} = 36x^2 - 24$

Inflection point: $36x^2 - 24 = 0$
$36(x^2 - \frac{2}{3}) = 0$
$x = \pm\sqrt{\frac{2}{3}}$





Sketching rational Functions $f(x) = \frac{p(x)}{q(x)}$

1. Identify Symetries
2. Find x, y intercepts
3. Find vertical/horizontal asymptotes
4. Find critical points
5. Intervals of increase/decrease.
6. Intervals of concavity

4. Skeching graph on Summer Test ☺ wish wink

Example: Sketch the graph of $y = \frac{2x^2-8}{x^2-16}$

1. Identify Symetries: $y(x) = y(-x)$ - even function (Symetric bout y-axis)

2. x intercept, $y=0$ $\frac{2x^2-8}{x^2-16} = 0$ $2x^2-8=0 \Rightarrow x^2=4$ $x = \pm 2$
   y intercept $x=0$ $y = 1/2$

3. Vertical asymptotes $x^2-16=0 \Rightarrow x = \pm 4$
   Horizontal asymptotes given by $\lim_{x \to \pm\infty}$

   $\lim_{x \to \pm\infty} \frac{2x^2-8}{x^2-16}$ $\frac{2 - \frac{8}{x^2}}{1 - \frac{16}{x^2}} = 2$ $y=2$ is horizontal asymptote

4. ~~Intervals of increase~~ Critical points: $y'(x)=0$ or points of non differentiability
   $y'(x) = \frac{4x(x^2-16) - 2x(2x^2-8)}{(x^2-16)^2}$ $\frac{-48x}{(x^2-16)^2}$
   Stationary points $y'(x)=0$ $x=0$
   Non differentiable at $x = \pm 4$

5. Intervals increase/decrease $y'(x) = \frac{-48x}{(x^2-16)^2}$

   $x=0$ rel max by the first derivative test

   +++ ∞ ++++ 0 - - - - ∞ - - - -
   -4 0 +4

   $y(x)$ is increasing on $(-\infty, -4)$ $[-4, 0]$ decreasing on $[0, \cancel{4}) (4, \infty]$

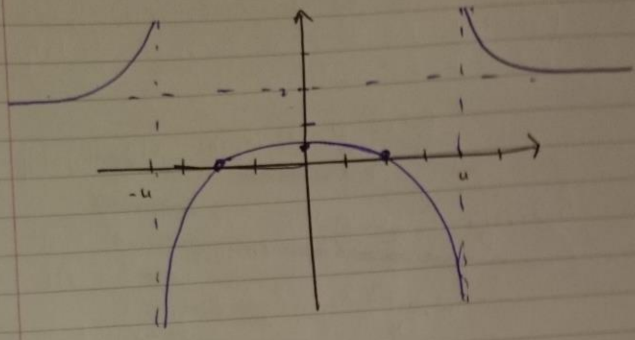
6. Intervals of Concavity:
   $y''(x) = \frac{48(16+3x^2)}{(x^2-16)^3}$ + + + ∞ - - - - - ∞ +++++
   -4 0 4

   Concave up on $(-\infty, -4)$ $(4, \infty)$
   concave down on $(-4, 4)$

Maths week 8

15/11/10

5.



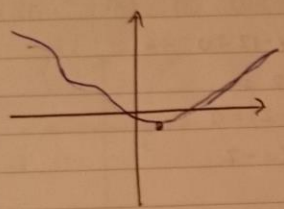
9/11/12 Math1 Week 9

Absolute max/min

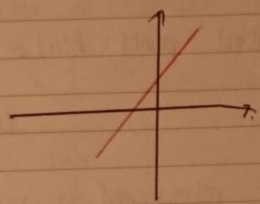
Def^n Let I be an interval on the domain of f, then

- there is an absolute max at x_0 if $f(x) \le f(x_0)$ for all $x \in I$.

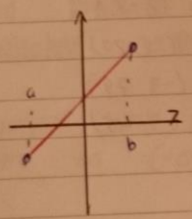
- there is an absolute min at x_0 if $f(x) \ge f(x_0)$ for all $x \in I$.



absoulte min on $(-\infty, \infty)$



no absolut max or min on $(-\infty, \infty)$



abs max/min on $[a, b]$

Extreme-Value Theorm. If a function f is continous in a finite closed interval, $[a, b]$ then f has both absolute max and absolute min on $[a, b]$

Theorm: if f has an absolute extrem on an open interval (a, b) (exclude end points) then it must occur at a critical point of f.

19/11/12 2. week 9

Finding the absolute extrema on a Finite closed interval $[a,b]$

1. Find the critical points of f in (a,b)
2. Evaluate f at all critical points **AND** at the end points a and b.
3. The largest of the values in step 2 is absolute max and the smallest the absolute min

Example:

Find the absolute extrema of $f(x) = 2x^3+3x^2-12x$ on $[-3,3]$

1. critical points: $f'(x) = 6x^2+6x-12=0 \quad \div 6$

$$x^2+x-2=0$$
$$(x-1)(x+2)=0$$
$$x=1 \quad \text{or} \quad x=-2$$

rel extrema at $x=1$, $x=-2$

2. $f(1) = 2(1)^3+3(1)^2-12(1) = -7 \quad (1,-7)$

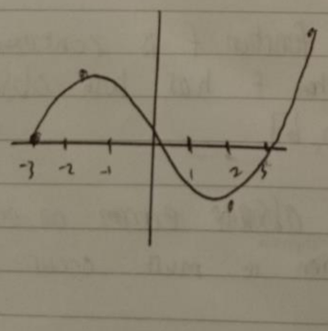
$f(-2) = 2(-2)^3+3(-2)^2-12(-2) = 20 \quad (-2,20)$

$f(-3) = 2(-3)^3+3(-3)^2-12(-3) = 79 \quad (-3,79)$

$f(3) = 2(3)^3+3(3)^2-12(3) = 45 \quad (3,45)$

3. Absolute min occurs at $x=1$

Absolute max occurs at $x=3$



8. Maths Week 9

19/11/12.

Absolute extrema on infinite intervals.

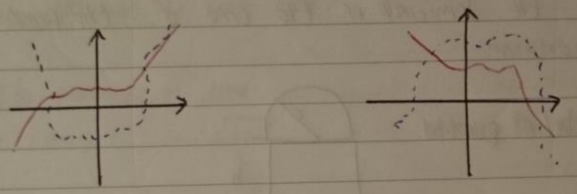
For a continous function f, if we have

$\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to -\infty} f(x) = -\infty$

or

$\lim_{x \to \infty} f(x) = -\infty$ and $\lim_{x \to -\infty} f(x) = \infty$

then f has no absolute extrema on $(-\infty, \infty)$



If end behaviour of f has the same sign, we do have an absolute extreme

This implies only polynomials of even degree have absolute extrema on $(-\infty, \infty)$

Absolute Extrema of functions with one rel extrema:

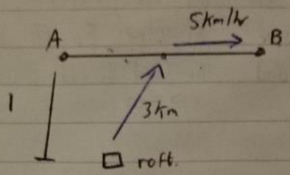
If a continuous function has only one turning point, then this point has to be an absolute extrema.

If f has exactly one rel. extrema on an open interval I at x_0 the $f(x)$ is an absolute extrema

** Applied max/min problem on test.

Applied max/min Problems:

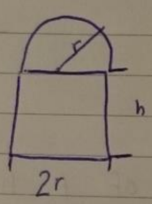
Example:



Example:

A church window consisting of a rectangle topped by a semi circle is to have a perimeter of 6m. Find the radius of the semicircle if the area of the window is to be a maximum

1. Draw and label all quantities:



2. Find a formula for the quantity to be extremised;

Area $A = 2hr + \frac{1}{2}\pi r^2$

Perimeter $P = 2h + 2r + \pi r = 6$

3. Use constraint equation to eliminate a variable:

Perimeter $= 2h + 2r + \pi r = 6$

$2h = 6 - 2r - \pi r$

$\Rightarrow h = 3 - r - \frac{\pi r}{2}$

$\downarrow$

$A = 2r\left(3 - r - \frac{\pi r}{2}\right) + \frac{1}{2}\pi r^2$

$= 6r - 2r^2 - \frac{1}{2}\pi r^2$

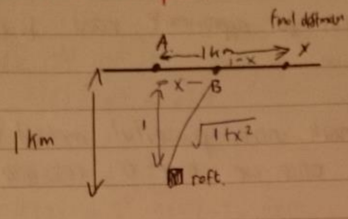
4. Identify the interval of possible values for the independent variable:

$r > 0, \quad r(2+\pi) \leq 6 \qquad r \in \left(0, \frac{6}{2+\pi}\right]$

①.

20/11/12 Maths Week 9

2a2 max/min problem



Floats at 3km/h.
Walk at 5km/h

To what point B on the bank should he go in order to reach his final destination in least time?

Want to minimize the time

Time floating $= \frac{D}{S} = \frac{\sqrt{1+x^2}}{3}$

Time walking $= \frac{D}{S} = \frac{1-x}{5}$

Time function

$T(x) = \frac{\sqrt{1+x^2}}{3} + \frac{1-x}{5}$

at max/min

$\frac{dT}{dx} = \frac{\frac{1}{2}(1+x^2)^{-\frac{1}{2}} 2x}{3} - \frac{1}{5} = \frac{x}{3\sqrt{1+x^2}} - \frac{1}{5} = 0$

$\Rightarrow 5x = 3\sqrt{1+x^2}$

$25x^2 = 9(1+x^2)$

$= 16x^2 = 9$

$x^2 = \sqrt{\frac{9}{16}}$

$x = \pm\frac{3}{4}$

Since $x \in [0,1]$ we ignore the negative solution
so the relative extrema is $x = \frac{3}{4}$

$T(3/4) = 0.46666'$

$T(0) = 0.5333'$

$T(1) = 0.471405$

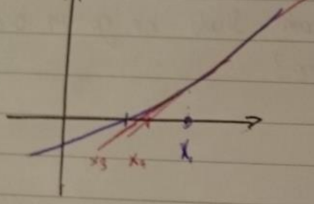
Hence $x = \frac{3}{4}$ is the global minimum we should land at 0.75km out of A

2.

Newton Rapson Method

- Method for approximating roots
- Previously we used the IVT to approximate roots but this method is not very efficient

- Newton rapson method gives a much more powerful method for approximating roots of equation, once we have a resonable initial guess.



tangent line cutting x axis gets closer to root

If our initial guess is x_1 (where $f(x) \neq 0$) then a better guess is ~~obtained~~ obtained by taking the x-intercept of the tangent line at x.

The tangent line at x_1 is $y - f(x_1) = f'(x_1)(x - x_1)$

The intercept the x axis at $y=0$

$$\Rightarrow 0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

A better approximation again in the x-intercept of the tangent line at x_2

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \quad \text{etc}$$

The n^{th} approximation is given by $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

③ Week 9 maths

Example: Use the newton-raphson method to approximate the root of $x^5 - 5x^3 - 2 = 0$ on $(1, \infty)$ to 5 dp

1. Use the IVT to choose an initial guess
$f(x) = x^5 - 5x^3 - 2$
$f(1) = 1 - 5 - 2 = -6 < 0$
$f(2) = 2^5 - 5(2^3) - 2 = < 0$
$f(3) = 3^5 - 5(3^3) - 2 = > 0$

By the IVT there is a root in the interval $(2, 3)$
Take our initial guess to x $x_1 = 2.5$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad f(x) = x^5 - 5x^3 - 2 \quad f'(x) = 5x^4 - 15x^2$$

$$= 2.5 - \frac{f(2.5)}{f'(2.5)}$$
$$= 2.3273846154$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.2776776776$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.2738140346$$

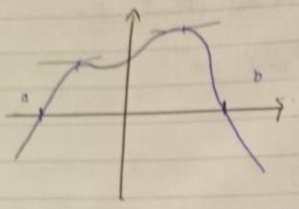
$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 2.2737917331$$

$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = 2.2737917324$$

5 decimal places we have $r = 2.27379$

④

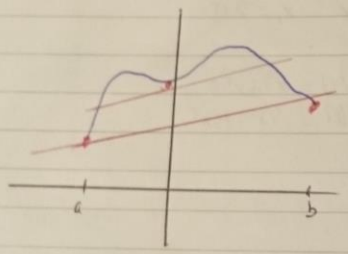
Rolf's theorm



let f be continuous on [a,b] and differentiable on (a,b)

If f(a)=0 = f(b) then there is at least one point $c \in (a,b)$ such that $f'(c)=0$

Mean Value theorm:



must be a point that has same slope as a, b

f is a continuous function on [a,b] and differentiable on (a,b) then there is at least one point $c \in (a,b)$ such that

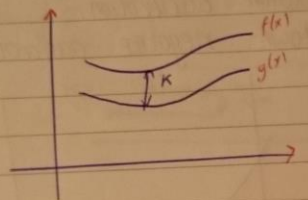
$$f'(c) = \frac{f(b)-f(a)}{b-a} = \text{slope of } AB$$

1/12 Week 9 maths

Constant Difference Theorm

If f and g are differentiable in an interval I where $f'(x) = g'(x)$ for all $x \in I$ then $f - g$ is a constant on I

i.e. $f(x) = g(x) + k$



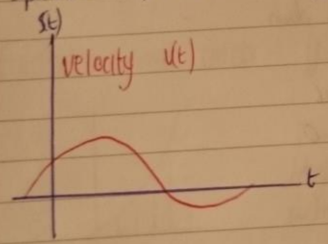
Linear Motion

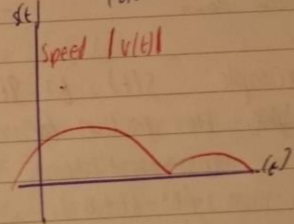
$s(t)$ - (t-time) position function

$s(t_2) - s(t_1)$ - displacement over interval $[t_1, t_2]$

$v(t) = s'(t)$ - velocity function (rate of change of position with respect to time)

Speed = $|v(t)|$ - always positive = $\left|\frac{ds}{dt}\right|$





Acceleration = $a(t) = v'(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ (rate of change of velocity with respect to time)

Speeding up: If $a(t)$ and $v(t)$ have the same sign, we say a particle is speeding up

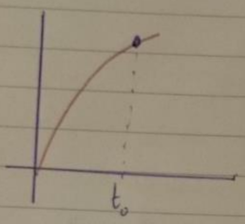
Slowing down: If $a(t)$ and $v(t)$ have opposite sign we say the particle is slowing down

2.

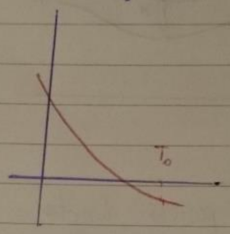
Analysing Position verses time Curves

- $s(t) > 0$ particle on positive side of s-axis
- $s(t) < 0$ " " negative " "
- slope of curve at any point gives instantaneous velocity
- $v(t) > 0$ (positive slope), moving in positive s-direction
- $v(t) < 0$ neg slope) moving in neg s-direction

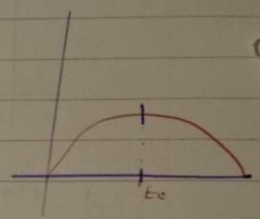
- curve is concave up, positive acceleration.
- Curve is concave down- negative acceleration.



- positive side of origin
- moving in positive direction
- velocity is decreasing
- concave down



- on neg side of s-axis
- moving in neg direction
- velocity is increasing
- slowing down



- postive side of origin
- momentarily stopped at t_0
- velocity decreasing
-

Example: $s(t) = t^3 - 9t^2 + 24t$

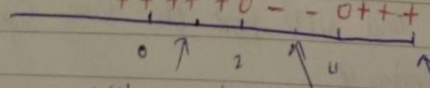
Analyse the motion for $t \geq 0$

$v(t) = 3t^2 - 18t + 24$

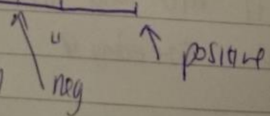
$= 3(t^2 - 6t + 8)$

$3((t-2)(t-4))$

velocity is zero at $t=2$, or $t=4$ Sub into original eq



moving in positive direction



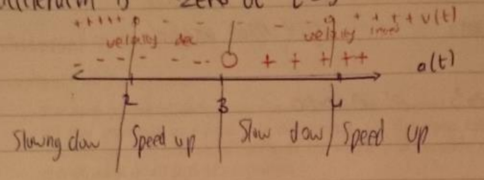
22/11/12

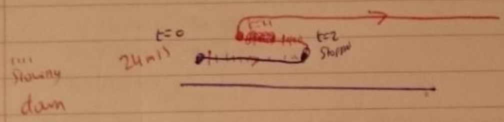
3.

Meth.

$a(t) = v'(t) = 6t - 18$

acceleration is zero at $t = 3$

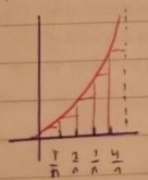




5 Integration

Area Problems: Given a continuous function that is non-negative on $[a,b]$, find the area between the graph of f and the interval $[a,b]$ on the x-axis.

Example: Area under $y = x^2$ on $[0,1]$



Form rectangles using the right end points
Height of each rectangle $= f'(x_n)$
where x_0 we are the right end point.

heights are $\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2$ etc

Width of each rectangle $= \frac{1}{n}$

An approximation of the area under the curve (A_n), is

$A \approx A_n = \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \left(\frac{3}{n}\right)^2 \ldots + 1^2\right]$

for example, $n=10$ $\quad A_{10} = \frac{1}{10}\left[\left(\frac{1}{10}\right)^2 + \left(\frac{2}{10}\right)^2 \ldots + 1^2\right] = 0.385000$

4

As n gets larger our approximation gets better

n 10, 100 1000 10 000 100 000

0.385 0.3850 ... get more accurate

Seems that $A_n \to \frac{1}{3}$ as $n \to \infty$

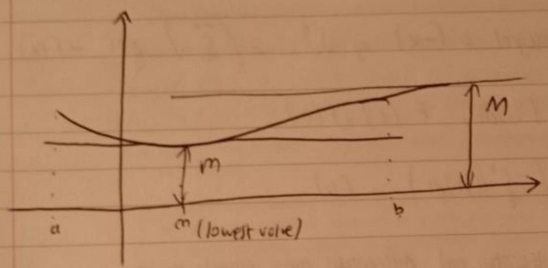
|12|12. Maths Week 11

Total area $= \int_a^b |f(x)|\,dx$

Total distance $= \int_a^b |v(t)|\,dt$

Displacement $\int_{t_0}^{t_1} v(t)\,dt$

MEAN VALUE THEORM FOR INTEGRALS



Area under $y = f(x)$ is bigger then $m.(b-a)$ and smaller then $M.(b-a)$

THEORM: If f is continuous on $[a,b]$ there is atleast one point x^* in $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(x^*)(b-a)$$

average height by width

The average value of a function $f(x)$ over $[a,b]$ is

$$f_{average} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

THE FUNDAMENTAL THEORM OF CALCULUS (II)

Theorm: if f is continuous on an interval I, then f has an antiderivative on I. In particular if a is any point in I, then the function F defined by

$$F(x) = \int_a^x f(t)\,dt$$

2

is an anti-derivative of f on I i.e. $F'(x) = f(x)$

$$\frac{d}{dx}\left(\int_0^x f(t)dt\right) = f(x)$$

(genuine function)

EXAMPLE:

$$\frac{d}{dx}\left(\int_0^x \frac{\sin^7(4t)\cos^6(10t)}{\sqrt{t^2-4t+9}}\,dt\right) = \frac{\sin^7(4x)\cos^6(10x)}{\sqrt{x^6-4x+9}}$$

reversing integral $= (-x)$ eg $\int_x^9 = \left(-\int_9^x\right)$ get $-f(x)$.

Part 1 $\int_0^x F'(t)dt = F(x) - F(0)$

Part 2 $\frac{d}{dx}\left(\int_0^x f(t)dt\right) = f(x)$

i.e. integration and differentiation are inverse processes

DEFINITE INTEGRALS

Example: Evaluate $\int_0^{\pi/8} \sin^5(2x)\cos(2x)\,dx$

Use $u = \sin(2x)$ → The corresponding indefinite integral is $\int u^5 \frac{du}{2}$

$du = 2\cos(2x)dx$

$\cos(2x)dx = \frac{1}{2}du$

Method 1: Evaluate indefinite integral and then sub in for $(u(x))$

$$\int u^5 \frac{du}{2} = \frac{u^6}{12} = \left.\frac{\sin^6(2x)}{12}\right|_0^{\pi/8} = \left(\frac{\sin^6(\pi/4)}{12} - 0\right) = \frac{1}{96}$$

Method 2: Write the limits in terms of ~~x~~ and evaluate the definite integral completely in terms of u

when $x = 0$, $u = 0$

when $x = \pi/8$ $u = \frac{1}{\sqrt{2}}$

$\int_0^{\pi/8} \sin^5(2x)\cos(2x)dx = \frac{u^5}{2}du$