1

2011 MANG SCI PAPER 1 Q 5 DAVID WEITBRECHT.

A. Express rectangular coordinates in terms of spherical coordinates.
Rectangular coordinates, x, y, z.

11/20

Spherical coordinates

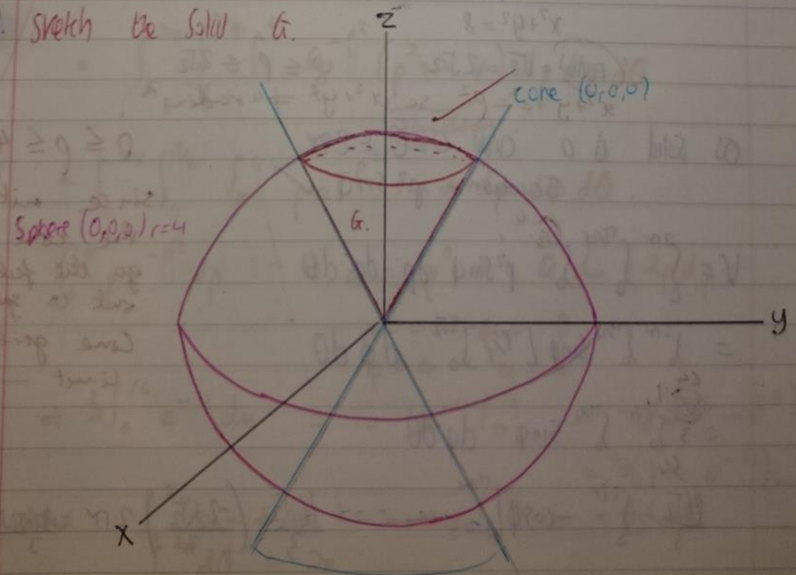
$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \cos\theta \qquad z = \rho\cos\phi$ 1

B. Consider the solid G bounded above by surface $x^2+y^2+z^2=16$ and below by surface $z=x^2+y^2$

i. What is surface $x^2+y^2+z^2=16$?
This is a sphere with centre (0,0,0) and radius 4. 1

ii. What is surface $z=\sqrt{x^2+y^2}$?
This is a cone originating at (0,0,0) and extending upwards and downwards infinitely 2

iii. Sketch the solid G.



3

iv. Use triple integral and spherical coordinates to compute the volume V of solid G.

$$V = \iiint_G dV$$

$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho\cos\phi$

$\sqrt{x^2+y^2} < z < \sqrt{16-x^2-y^2}$ (bound above and below)

$\rho = 4$

$\sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} < \rho\cos\phi < \sqrt{16-\rho^2\sin^2\phi\cos^2\theta - \rho^2\sin^2\theta\sin^2\phi}$

$\sqrt{16\sin^2\phi(\cos^2\theta+\sin^2\theta)} < \rho\cos\phi < \sqrt{16-16\sin^2\phi(\cos^2\theta-\sin^2\theta)}$

$\sqrt{16\sin^2\phi} < 4\cos\phi < \sqrt{16-16\sin^2\phi}$

$4\sin\phi < 4\cos\phi < \sqrt{(4-4\sin\phi)(4+4\sin\phi)}$

$4\sin\phi < 4\cos\phi$

$\sin\phi = \cos\phi$

$\phi = \pi/4 \qquad 0 \le \phi \le \pi/4$ ✓

For ρ (radius) $\sqrt{x^2+y^2} = \sqrt{16-x^2-y^2}$

$x^2+y^2 = 16-x^2-y^2$

$2x^2+2y^2 = 16$

$x^2+y^2 = 8$

radius $= \sqrt{8} = 2\sqrt{2} \qquad 0 \le \rho \le 2\sqrt{2}$

$x^2+y^2+z^2 = \rho^2$ so $x^2+y^2 \neq$ radius^2.

as solid is a circle $0 \le \theta \le 2\pi$

Jacobian $= \rho^2 \sin\phi$ ✓

$0 \le \rho \le 4$

Since within the cone we go the full way out to sphere. Cone gives ϕ limit

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cancel{2\sqrt{2}}\ 4} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3}\right]_0^{2\sqrt{2}} d\phi \, d\theta$$

$$= \frac{\cancel{16\sqrt{2}}\ 64}{3}\int_0^{2\pi}\int_0^{\pi/4} \sin\phi \, d\phi \, d\theta$$

$$\frac{\cancel{16\sqrt{2}}\ 64}{3}\int_0^{2\pi} -\cos\phi\Big|_{\phi=0}^{\pi/4} = \frac{16\sqrt{2}}{3}\left(\frac{-2+\sqrt{2}}{2}\right)2\pi = \frac{32-32\sqrt{2}}{3}\pi$$

64π ...

3

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V. Use triple integral and spherical coordinates to find the mass M of the solid G if its density is

$$\delta(x,y,z) = \frac{e^{-(x^2+y^2+z^2)}}{\sqrt{x^2+y^2+z^2}}$$

Change density using spherical coords

$$x^2+y^2+z^2 = \rho^2 \qquad \frac{e^{-\rho^2}}{\rho}\rho^2 \sin\phi \qquad 1$$

$$\sqrt{x^2+y^2+z^2} = \rho \qquad \iiint \rho e^{-\rho^2} \sin\phi \quad ?$$

Please could you write a more detailed procedure method, the last one was unclear thanks.

$$M = \iiint_G \delta(x,y,z)\,dV \quad ; \quad \delta(x,y,z) = \delta(\rho,\theta,\phi) = \frac{e^{-\rho^2}}{\rho}$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^4 \frac{e^{-\rho^2}}{\rho}\underbrace{(\rho^2 \sin\phi)}_{\text{Jacobian}}\,d\rho\,d\phi\,d\theta.$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^4 \rho e^{-\rho^2} \sin\phi \,d\rho\,d\phi\,d\theta.$$

$$\int_0^4 \rho e^{-\rho^2} d\rho : \quad u = \rho^2 \Rightarrow du = 2\rho\,d\rho.$$

$$\rho = 0 \Rightarrow u = 0$$

$$\rho = 4 \Rightarrow u = 16$$

$$\Rightarrow \frac{1}{2}\int_0^{16} e^{-u}\,du = -\frac{1}{2}e^{-u}\Big|_0^{16} = -\frac{1}{2}(e^{-16} - e^{-0})$$

$$= \frac{1}{2}(1 - e^{-16}).$$

Also $\int_0^{\pi/4} \sin\phi\,d\phi = -\cos\phi\Big|_0^{\pi/4} = -\frac{1}{\sqrt{2}} + 1.$

$$\int_0^{2\pi} d\theta = 2\pi$$

$$\Rightarrow M = \pi\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 - \frac{1}{e^{16}}\right)$$

2011 MANG SCI PAPER 1 Q1 Correction

$V = \iiint_G dV$

$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$

$\sqrt{x^2+y^2} < z < \sqrt{16-x^2-y^2}$

$\cancel{\rho^2\sin^2\phi + \rho}$

$\sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} < z < \sqrt{16-\rho^2\sin^2\phi\cos^2\theta - \rho^2\sin^2\phi\sin^2\theta}$

$\sqrt{\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)} < z < \sqrt{16-\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)}$

$\sqrt{16\sin^2\phi} < z < \sqrt{16-16\sin^2\phi}$

$4\sin\phi < \cancel{\phi}\cos\phi < \cancel{4}4\sin\phi \quad \sqrt{16(1-\sin^2\phi)}$

$4\sqrt{\cos^2\phi}$

$4\sin\phi < 4\cos\phi \qquad 4\cos\phi < 4\cos\phi$

$\sin\phi < \cos\phi$

$\phi = \pi/4 \qquad 0 \leq \phi \leq \pi/4$

$x^2+y^2+z^2 = \rho^2$ so and $x^2+y^2 \neq$ radius^2

hence $0 \leq \rho \leq 4 \qquad 0 \leq \theta \leq 2\pi$.

Jacobian is $\rho^2\sin\phi$.

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^4 \rho^2\sin\phi \; d\rho\, d\phi\, d\theta$$

$$\int_0^{2\pi}\int_0^{\pi/4} \sin\phi \left(\frac{\rho^3}{3}\right)\Big|_{\rho=0}^{4} d\phi\, d\theta$$

$$\frac{64}{3}\int_0^{2\pi}\int_0^{\pi/4} \sin\phi \; d\phi\, d\theta$$

$$\frac{64}{3}\int_0^{2\pi} \cancel{\sin\phi} -\cos\phi\Big|_{\phi=0}^{\phi=\pi/4} d\theta$$

$$\frac{64}{3}\left(\frac{2-\sqrt{2}}{2}\right)\int_0^{2\pi} d\theta$$

$$= 2\pi\left(\frac{64}{3}\right)\left(\frac{2-\sqrt{2}}{2}\right) = \frac{64}{3}\pi(2-\sqrt{2})$$

$$d(x,y,z) = \frac{e^{-(x^2+y^2+z^2)}}{\sqrt{x^2+y^2+z^2}}$$

$$M = \iiint_G \delta(x,y,z)\,dV \qquad \delta(x,y,z) = d(\rho,\theta,\phi) = \frac{e^{-\rho^2}}{\rho}$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \int_0^4 \frac{e^{-\rho^2}}{\rho} (\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \int_0^4 \rho e^{-\rho^2} \sin\phi \, d\rho\, d\phi\, d\theta$$

Aside $\Rightarrow \int_0^4 \rho e^{-\rho^2} \cancel{\sin\phi}$ $\qquad u = \rho^2 \quad du = 2\rho\, d\rho$

$\rho = 0 \Rightarrow u = 0$

$\rho = 4 \Rightarrow u = 16$

$$\int_0^4 \frac{\cancel{\rho} e^{-\rho^2}\, \cancel{d\rho}}{2\cancel{\rho}\, \cancel{d\rho}}$$

$$\frac{1}{2} \int_0^{16} e^{-u}\, du = \frac{1}{2}\left[-e^{-u} \Big|_{u=0}^{16}\right] = -\frac{1}{2}\left[e^{-16} - e^{0}\right]$$
$$= \frac{1}{2}\left[1 - e^{-16}\right]$$

$$\int_0^{\pi/4} \sin\phi \, d\phi$$
$$-\cos\phi \Big|_0^{\pi/4} = \frac{2-\sqrt{2}}{2}$$

$$\int_0^{2\pi} d\theta = 2\pi \qquad 2\pi\left(\frac{2-\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\left(1 - e^{-16}\right)$$

$$M = \pi\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 - e^{-16}\right)$$