1.

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Random number generators (RNG's) ideally yield values Y that are uniformly distributed on (0,1) and are independent. They are "inputs" to systems that transform them.

A. Use sketches and formulae for the pdf and the cdf to explain what is meant by saying Y is uniformly distributed on (0,1)

Pdf

If the distribution is described as uniform on (0,1), the probability density function associated with this experiment should be constant on (0,1).

A continuous random variable has a uniform distribution on the interval (a,b) if its probability density function f(x) is given by $f(x)=0$, if x is not in $[a,b]$ and $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$

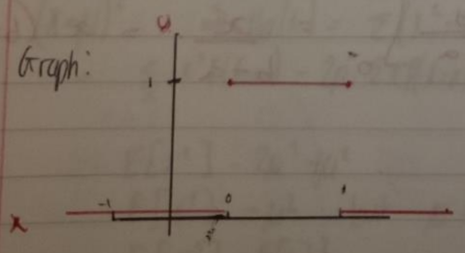
general case $\begin{cases} 0 & x \leq 0 \\ X & a \leq x \leq b \\ 0 & x \geq 1 \end{cases}$

We denote this distribution $U(a,b)$

In our case: $a=0$, $b=1$

$f(x) = \frac{1}{1-0} = 1/1 = 1$ Or are $\begin{cases} 0 & x \leq 0 \\ 1 & 0 \leq x \leq 1 \\ 0 & x \geq 1 \end{cases}$

Graph:



We need pdf to find probability of a continuous random variable lying with some interval (a,b). This lead to cdf.

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3ai Show how calculus can be used to compute $E[y]$, $E[y^2]$ and $var[y]$ explaining the meanings of these terms

E [y]

Expected value of y, the long run average value

$$E(y) = \int_{-\infty}^{\infty} y\, f(y)\, dy$$
$$= \int_a^b y \left(\frac{1}{b-a}\right) dy$$
$$= \frac{1}{b-a} \int_a^b y\, dy$$
$$= \frac{1}{b-a} \left(\frac{1}{2}\right)(y^2)\Big|_{y=a}^{b}$$
$$= \frac{1}{b-a}\left(\frac{1}{2}\right)(b^2-a^2) = \frac{1}{2} \cdot \frac{\cancel{(b-a)}(b+a)}{\cancel{(b-a)}} = \frac{b+a}{2} \Rightarrow \frac{1}{2}$$

$a=0$ $b=1$

$$E(y^2) = \frac{1}{b-a} \int_a^b y^2\, dy$$ long run average of y^2

$$= \frac{1}{b-a}\left[\frac{y^3}{3}\right]\Big|_{y=a}^{b}$$
$$= \frac{1}{b-a}\left[\frac{b^3}{3} - \frac{a^3}{3}\right] = \frac{1}{3}\left[\frac{b^3-a^3}{b-a}\right] = \frac{1}{3}\left[\frac{1^3-0^3}{1-0}\right] = \frac{1}{3}$$

Variance.

Variance is a measure of the degree of dispersion of x about its expectation $E(x)$ ($E(x)$ is mean of x)

$$(x-\mu)^2 \Rightarrow Var(x) = \sum (x^2 - 2\mu x + \mu^2) P(X=x)$$
$$= \sum_x x^2 P(X=x) - 2\mu \sum x P(X=x) + \mu^2 \sum p(X=x)$$

* $E(x) = \sum x P(X=x)$ and $\sum P(X=x) = 1$

$$E[x^2] - 2\mu^2 + \mu^2$$
$$E[x^2] - \mu^2 \quad \text{but } \mu = E(x)$$
$$E[x^2] - E[x]^2$$
$$\Rightarrow \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

4

3(A ii) Discuss independence in the context of the pseudo-RNG's commonly used in software

A truly random number is independent of any other random numbers, or anything at that

Pseudo random numbers are deterministic, they are NOT independent

Pseudo random numbers are produced using random number before, and inputting it into a complex algorithm

Easily reproduced when we know algorithm

For our purposes, pseudo RNG are "statistically" random for any of our experiment

3(B) The transformation $X = -\ln(1-y)$ yields value for a variable X that is said to have an Exponential Distribution with expected value of 1. Use an event identity to determine algebraically the cdf for X and hence it's pdf. Explain with sketches how these can each be used to compute $Pr(X \leq 1)$

We want our function in terms of X

$$x = -\ln(1-y)$$

$$-x = \ln(1-y)$$

$$e^{-x} = 1-y$$

$$y = 1 - e^{-x}$$

this is our cdf.

$$\text{cdf} \begin{cases} 0 & x \leq 0 \\ 1-e^{-x} & 0 \leq x \leq \infty \\ \cancel{1} & \cancel{x \geq 1} \end{cases}$$

e.g. $F_X(2) = 1-e^{-2} \neq 1$

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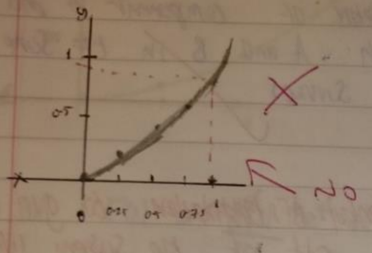
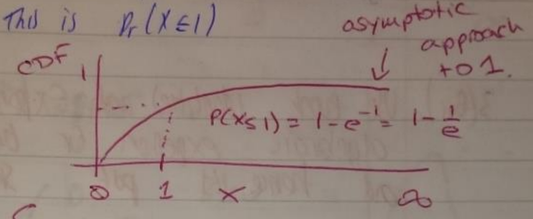
Graph of cdf.

0.22
·39
·63
·86

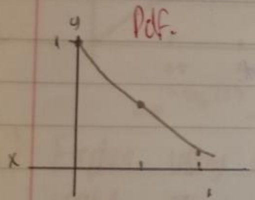


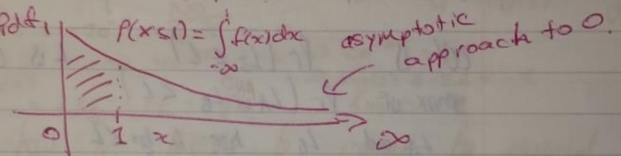
$Pr(X\leq 1)$ from cdf.
Use $x=1$, bring line vertical until it hits curve bring line horizontal to meet y axis
This is $Pr(X \leq 1)$



Pdf is cdf differentiated

$$\frac{d(1-e^{-x})}{dx} = f(x) = e^{-x} \begin{cases} 0\leq x \\ 0 \quad (o/w) \end{cases}$$





1 A System has 2 components that run in parallel. Providing that atleast one is working, the System works! The lifetimes are exponentially distributed with expected value 1 and are taken as independent. Explain using spread sheet ideas of rows and columns how to simulate system life time

	lifetime A	lifetime B	Max of A or B
1	-ln (1-RAND())	-ln(1-RAND())	= Max ("A1:B1")
2	-ln (1-RAND())	-ln(1-RAND())	= Max ("A2:B2")
:	:	:	:
n	-ln(1-RAND())	-ln(1-RAND())	=MAX ("AN:BN")

Pick n suitably large and simulate as above.
To find average system lifetime, add all values in

in "C" column and divide by count.

- Column A and B represent the lifetime of component A and B.
- Column C is max of both A and B in the same row.
- System survives if A or B survives

3(Bii) Use event identities and Exponential cdf's ~~distribution~~ to give an algebraic expression for the cdf of the System lifetime and hence its pdf. Sketch both functions

L_S = lifetime system L_A = lifetime A L_B = lifetime B

cdf is $Pr(L_S \leq L)$ L is a constant.

max of $Pr(L_A \overset{AND}{or} L_B < L)$

L_A and L_B have to be $< L$

$Pr(L_A \leq L)$ $Pr(L_B \leq L)$

lifetime of A and B are $-\ln(1-Y)$ No. $P(L_A \leq L) = 1 - e^{-L}$

$\Rightarrow P(-\ln(1-Y_A) \leq L)$ and $P(-\ln(1-Y_B) \leq L)$

$\Rightarrow$ manipulate like earlier   No.

$Pr(Y_A \leq 1-e^{-L})$ and $Pr(Y_B \leq 1-e^{-L})$

$Pr(Y_A, Y_B) = (1-e^{-L})*(1-e^{-L})$

$1-e^{-L}-e^{-L}+e^{-L}e^{-L}$

$Pr(Y_A, Y_B) \leq (1-2e^{-L}+e^{-2L})$

$P(L_S \leq l)$
$= P(L_A \& L_B \leq l)$
$= P(L_A \leq l)\,P(L_B \leq l)$ as independent
$= (1-e^{-l})(1-e^{-l})$
$= 1-2e^{-l}+e^{-2l}$ $l \geq 0$
$[0$ if $l \leq 0]$

Y_A and Y_B are random numbers between 0 and 1.

$$\text{cdf} \Rightarrow \begin{cases} 0 & L \leq 1 \\ 1-2e^{-L}+e^{-2L} & 0 \leq L \leq 1 \\ 1 & L \geq 1 \end{cases}$$

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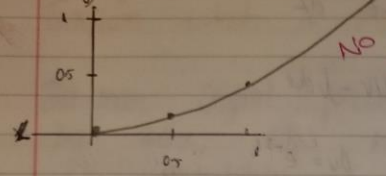
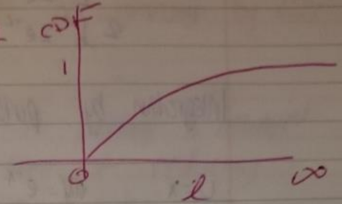
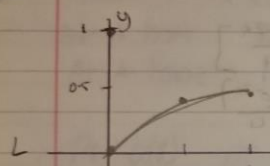
Sketch of cdf: $1-2e^{-L}+e^{-2L}$



No asymptotic approach to 1

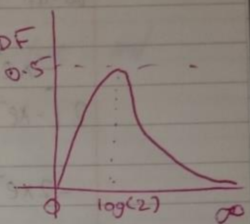


$$\text{Pdf} = F'(x) = \frac{d\left(1-2e^{-L}+e^{-2L}\right)}{dL} = \cancel{1}+2e^{-L}-2e^{-2L}$$



$$\text{pdf}\begin{cases} 0 & L \le 0 \\ 2e^{-L}-2e^{-2L} & 0 \le L \le 1 \\ 0 & L \geq 1 \end{cases}$$

No



$$\frac{df(x)}{dx} = -2e^{-l}+4e^{-2l} = 0$$

$$\Rightarrow 4e^{-2l} = 2e^{-l} \Rightarrow 2e^{-2l} = 4e^{-l}$$

$$\Rightarrow \log(2) \cancel{=} -2l = -l$$

$$\log(2) = l$$

$$2e^{-\log(2)} - 2e^{-2\log(2)} = 2\times\frac{1}{2} - 2\times\frac{1}{4} = 1-\frac{1}{2} = \frac{1}{2}$$

3(B iii) Explain using calculus how in principle to compute the expected value of system lifetime. Will this be greater, or less than 1, the expected lifetime of a component? Why?

To calculate expected value we would integrate pdf.

$$= \int_{-\infty}^{\infty} L\left(2e^{-L}-2e^{-2L}\right)\, dL \qquad \int_0^{\infty} L\left(2e^{-L}-2e^{-2L}\right) dL$$

$$\int_{-\infty}^{\infty} l\, f(l)\, dl = \int_0^{\infty} l \times \left(2e^{-l}-2e^{-2l}\right) dl$$

The expected value of variable L with param $\lambda = 1$ is

$$E[L] = \frac{1}{\lambda} = \frac{1}{1} = 1$$

NOT SURE HOW TO DO THIS

$$2\int_0^{\infty} L e^{-L}\, dL - 2\int_0^{\infty} L e^{-2L}\, dL$$

$$3/2$$

$$2e^{-L} \qquad -e^{-L}$$

$t = u \quad e^{-t}dt = dv$

$du = dt \quad v = -e^{-t}$

$-e^{-t}t \quad + e^{-t}dt$

$2(-e^{-t}t) + 2e^{-t} \cancel{-2e^{-1}} + 2e^{-1}$

$L\,2e^{-L}$ $\quad -L2e^{-2L}$

$2\int_0^{\infty} Le^{-L}\,dL \quad -2\int_0^{\infty} Le^{-2L}\,dL$

Integration by parts: $\int u\,dv = uv - \int v\,du$

$u = x \quad dv = e^{-x}$

$du = dx \quad v = -e^{-x}$

$-xe^{-x} - \int -e^{-x}\,dx$

$2[-xe^{-x} - e^{-x}]$

$u = x \quad dv = e^{-2x}$

$du = dx \quad v = \int e^{-2x} = \frac{-e^{-2x}}{2}$

$2\left[-\frac{xe^{-2x}}{2} - \int \frac{-e^{-2x}}{2}\,dx\right]$

$- \quad 2\left[-\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4}\right]$

$-2xe^{-x} - 2e^{-x} \quad - \quad \frac{(-2xe^{-2x})}{2} - \frac{(-2e^{-2x})}{4}$

$-2xe^{-x} - 2e^{-x} + xe^{-2x} + \frac{2e^{-2x}}{2} \Big|_{x=0}^{\infty}$

$-2(0)e^{0} \quad -2e^{0} \quad +(0)e^{0} \quad + \frac{2e^{0}}{2}$ at $x=0$.

$-2 + \frac{1}{2} \quad = \frac{-3}{2}$

at $x=\infty$ $\quad e^{-x} \to 0 \quad \Rightarrow \quad = 0$

$0 \; - \; -\frac{3}{2} \; = \; 3/2.$

L is > 1 it is $3/2$

✓

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Q4 A. Playing at home 'increases the odds of winning' by 20% Explain what it means

This means original probability of say A beating B increases by 20%, this has the effect of reducing B's probability of winning away by 20%

For example if only two outcomes, win or lose and
$P_{A win} = P_{B win} = 0.5$

At A home $P_{A win} = 0.5 \times 1.2 = 0.6$

At A home $P_{B win} = 0.5 \times 0.8 = 0.4$

NO.

The odds are now in favour of A at home in the ratio of 3:2 A Home.

$\frac{P(A\ win)}{P(B\ win)} = 1 \times 1.2 = 1.2$

$\overset{=p}{P(A\ win)} = 1.2\ P(B\ win)$ $(1-p)$

$p = 1.2(1-p) = 1.2 - 1.2p$ $2.2p = 1.2$ $p = \frac{1.2}{2.2} = 0.55$

4 b It is known that A beats B Explain how to compute the probability that this was a Home Game Explain both the probability and odds form of Bayes Rule

Here we want $P(A_{won}^{home} | A_{home}^{win})$

Using bayes rule we get $P(A_{win} | A_{home}) = \frac{P(A_{won}^{home} \text{ and } A_{home}^{win})}{P(A_{home}^{win})}$

This is a conditional probability which reflects our knowledge of the occurrence of A at home

The usual probability form of bayes rule is
$P(A|B) = \frac{P(AB)}{P(B)}$ if $p(B) > 0$

$P(A\ Home | A win) = \frac{P(A win | A\ Home) P(A\ Home)}{P(A\ win)} = \frac{0.55 \times 0.5}{(0.55 \times 0.5) + 0.45 \times 0.5)}$

$= 0.55$

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2

Bayes rule specifies how probabilities must be updated in the light of new information

Before knowing an event happened we assign prior probabilities $P(H)$ and $P(\bar{H}) = 1 - P(H)$, where H and $\bar{H}$ are mutually exclusive

How do the prior probabilities change once evidence in the form of the knowledge that the event E has occurred becomes available?

The updated value of the probability that the hypothesis is true given the fact that E occurred is denoted $P(H|E)$

To calculate posterior probability $P(H|E)$ we use Bayes's rule

$$\frac{P(A\ Home|A\ win)}{P(A\ Away|A\ win)} = \frac{\cancel{P(A\ Home)}}{\cancel{P(A\ Away)}} \cdot \frac{P(A\ win|A\ Home)}{P(A\ win|A\ Away)}$$

We use odds.

$$= \frac{P(A\ win|A\ Home)}{P(A\ win|A\ Away)} = \frac{0.55}{0.45} = 1.222$$

Posterior probability $P(H|E)$ satisfies: $\frac{P(H|E)}{P(\bar{H}|E)} = \frac{P(H)}{P(\bar{H})} \frac{P(E|H)}{P(E|\bar{H})}$

In words, Bayes rule in odds forms states: posterior odds = prior odds x likelihood ratio

$$\frac{p}{1-p} = 1.222$$

$$p = 1.222 - 1.222p$$

$$2.222p = 1.222$$

$$p = \frac{1.222}{2.222} = 0.55$$

Factor $\frac{P(H)}{P(\bar{H})}$ gives prior odds in favour of H occurring

The second fraction represents the effect E will have on the belief H happened

C i. In these circumstances the number of games N won by any team follows, marginally a binomial distribution. Explain what this means and the key assumption that it is based on. What is the variance of the number of games won by each team?

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3 2013 MANG SCI PAPER 1 Q4 DAVID LEITERECHT.

4c i. Binomial $Y \sim B(n,p)$

Basic Assumptions:

1. Y are positive integers: $0, 1, 2 \ldots n$
2. Finite number of trials n
3. Only two outcomes "success" and "failure."
4. Independent trials. → with same probability of success in each trial.

- Our Situation Satisfies these conditions, y is 0, 1 ~~or~~ 2, 3 or 4.
- $N = 4$ = number of games
- Each game independent of each other.
- P Success for each trial is the Same

$$\Pr(Y=y) = \binom{n}{y} p^y (1-p)^{n-y}$$
$$\Pr(Y=0) = \binom{4}{0}(0.5)^0(0.5)^4 = 1/16$$
$$\Pr(Y=1) = \binom{4}{1}(0.5)^1(0.5)^3 = 1/4$$
$$\Pr(Y=2) = \binom{4}{2}(0.5)^2(0.5)^2 = 3/8$$
$$\Pr(Y=3) = \binom{4}{3}(0.5)^3(0.5)^1 = 1/4$$
$$\Pr(Y=4) = \binom{4}{4}(0.5)^4(0.5)^0 = 1/16$$
$$= 1$$

Variance calculated by $np(1-p)$ for binomial dist

$$4(0.5)(1-0.5) = 1.$$

4 cii. Suppose probability of a draw in every game is 0.1. Win attract 3 points, draws 1 point, and losses no points. Explain why Binomial distribution no longer applies to the distribution of the number of points won by each of the teams. Show it is simple to compute the variance of this random variable

4

4c (ii) Basic assumption has been breached with introduction of third probability, we have 3 outcomes not 2.
⇒ Binomial no longer applies
No longer Bernoulli trials

	pr	value (x)	x^2
win	0.45	3	9
lose	0.45	0	0
draw	0.1	1	1

$E[x] = 0.45 \times 3 + 0.45 \times 0 + 0.1 \times 1 = 1.45$ points
$E[x^2] = 0.45 \times 9 + 0.45 \times 0 + 0.1 \times 1 = 4.15$ points

Variance $= E[x^2] - E[x]^2$
$4.15 - (1.45)^2 = 2.0475$ points this is for 1 game.

For 4 games it is 4×2.0475

iii. If home advantage is 20% as above, and draws are as described, what is the variance of the number of points won by each of the teams

home
Pr win 0.54
lose 0.36
draw 0.1

away
Pr win 0.36
lose 0.54
draw 0.1

SEE SHEET

$\boxed{6}$ →

Var(home)
$E[x^2] - E[x]^2$
$[0.54 \times 9 + 0.1 \times 1] - [0.54 \times 3 + 0.1]^2$
$= 2.0016$

away
$E(x) = 1.18$ $E(x^2) = 3.34$

var $= 1.9476$

+

$= 3.9492$

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5. 2013 MANG SCI PAPER 1 Q4 DAVID WEITBRECHT

d Understanding uncertainty site showed that for the premier league with 20 teams and 38 matches, theoretical variance of number of points gained is 61; computed in a way similar to (iii), in 2007 computed variance across 20 teams was 239. The site concludes "about half the spread of points is due to chance alone".

Assuming all teams were of equal standard, so that each match resulted in a home win with probability p_H, draw with p_D and away win with p_A where $p_H + p_A + p_D = 1$

X_H is number of points a team wins in a home match, variance of X_H is

$$E(X_H) = 3p_H + p_D$$

$$E(X_H^2) = 9p_H + p_D$$

$$Var = E(X_H^2) - E[X_H]^2$$

$$= 9p_H + p_D - \cancel{8}\ 9p_H^2 + p_D^2 + 6p_H p_D$$

$$= 9p_H + p_D + E(X_H)^2$$

Same for A.

N/2 home games, N/2 away games.
Total points T is sum of all points in the individual matches

Expected points $m_T = E(T) = \frac{N}{2}(m_H + m_A) = \frac{N}{2}(3(p_H + p_A) + 2p_D)$

$$= \frac{N}{2}(3 - p_D)$$

Variance is $V(T) = \frac{N}{2}(v_H + v_A) = \frac{N}{2}(9(p_H + p_A) + 2p_D - m_H^2 - m_A^2)$

$$= \frac{N}{2}(9 - 7p_D - m_H^2 - m_A^2).$$

6

This is how variance = 61.

The variance of the actual league points at the end of season was 234 compared with theoretical variance of 61 were all teams all of equal quality and the results of matchs were due to chance.

half in a!

Since $61/234 = 26$, we conclude that 26% of the variance in the premier league points is due to chance

$SD = \sqrt{61} \approx 7.8 \leftarrow$ spread of points according to model

$SD = \sqrt{239} \approx 15.5 \leftarrow$ total spread of points (only 7.8 explained by model).

4 (iii)

	home		away	
win	0.54	0.9×0.55	0.36	0.9×0.45
lose	0.36	0.9×0.45	0.54	0.9×0.55
draw	0.1	0.1	0.1	0.1

$E(x) = 0.54 \times 3 + 0.1$

1.72

$E(x^2) = 0.54 \times 9 + 0.1$

4.96

$Var = 4.96 - (1.72)^2$

2.0016

$E(x) = 0.36 \times 3 + 0.1$

1.18

$E(x^2) = 0.36 \times 9 + 0.1$

$= 3.34$

$Var = 3.34 - (1.18)^2$

1.9476

otherwise fine

Variance total points $n=4$ is $\frac{n}{2}(V_A + V_H)$

$\frac{4}{2}(2.0016 + 1.9476)$

$= 7.8984$ points

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2013 MANG COMP PAPER 1 Q1 SECTION A

2013 MANGSCI PAPER 1 Q3 CORRECTION

3ai. $E[y]$

Expected value of y the long Run average vale

$E[y] = \int_{-\infty}^{\infty} y f(y) dy$

$= \int_0^b y \left(\frac{1}{b-a}\right) dy$

$\frac{1}{b-a} \frac{y^2}{2}\Big|_0^b = \frac{1}{2}\frac{1}{b-a}(b^2-a^2) = \frac{1}{2}\frac{1}{\cancel{b-a}}(\cancel{b-a})\, b+a \quad \frac{b+a}{2} = \frac{1}{2}$

$E[y^2]$ long run average of y^2.

$\frac{1}{b-a}\int_0^b y^2 dy$

$\frac{1}{b-a}\left[\frac{y^3}{3}\right]\Big|_{y=0}^{b} = \frac{1}{b-a}\left[\frac{b^3}{3}-\frac{a^3}{3}\right] = \frac{1}{3}\left[\frac{b^3-a^3}{b-a}\right] = \frac{1}{3}\left[\frac{1^3-0^3}{1-0}\right] = \frac{1}{3}$

Variance

Variance is a measure of the degree of dispersion of X about its expetation $E[x]$ $E[x]$ is mean of x

$(x-\mu)^2 \Rightarrow Var(x) = \Sigma(x^2-2\mu x+\mu^2)p(X=x)$

$= \Sigma x^2 p(X=x) - 2\mu\Sigma x p(X=x) + \mu^2 \Sigma p(X=x)$

$E[x^2] \quad - 2\mu E[x] + \mu^2(1) \quad \mu = E(x)$

$E[x^2] \quad - E[x]^2$

$\frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

A ii A truely random number is important of any other random number or anything of that.

Pseudo random numbers are deterministic, they are not independ.

Pseudo random numbers are produced using random numbers before and imputting it into a complex algorithm

DAVID LIEMBRECHT

2

They are easily reproduced when we know the algorithm

For our purposes, pseudo RNG are "statistically" random for any of our experiments

b i. We want our function in terms of y = "

$x = -\ln(1-y)$

$-x = \ln(1-y)$

$e^{-x} = 1-y$

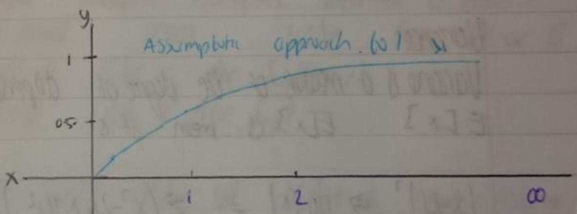
$e^{-x}-1 = -y$

$y = 1-e^{-x}$ this is our cdf

$$\begin{cases} 0 & 0 \le x \\ \cancel{\quad} \\ 1-e^{-x} & 0 < x \le \infty \end{cases}$$

Graph of cdf.

x	y
0.25	0.22
0.5	0.39
1	0.63
2	0.86



$Pr(X \le 1)$ from cdf $= 1-e^{-1} = 1-\frac{1}{e} = 0.63$

or bring line vertical until it hits the curve and bring it horizontal and read off value

Pdf is cdf differentiated

$$\frac{d(1-e^{-x})}{dx} = e^{-x}$$

$$\begin{cases} \cancel{\quad} \\ e^{x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

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2012 Mana Science Paper 1 Q1 SECTION A

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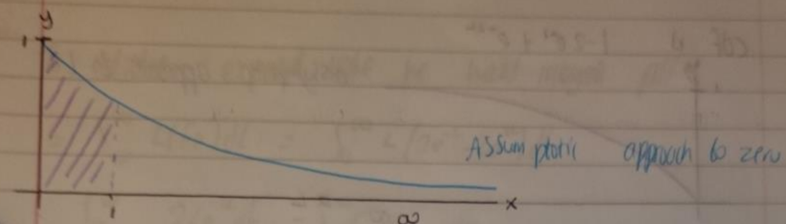


$P_r(x \le 1)$ is area to left of one or $e^{-1} = \frac{1}{e} = 0.368$

bi.

Lifetime A	Lifetime B	Max of A and B
$-\ln(1-RAND())$	$-\ln(1-RAND())$	=Max ("A1:B1")
⋮	⋮	
$-\ln(1-RAND())$	$-\ln(1-RAND())$	= Max ("AN:BN")

- Use Rand() to generate random numbers between 0 and 1.
- In column C pr record max value from A or B as if one component is alive the system is
- Choose N suitably large (1000 or more say)
- To find average system lifetime, add all value in third column and divide by count.
- System survive if A or B survive

bii. L_S - lifetime System L_A - lifetime A L_B lifetime B

cdf is $P_r(L_S \le L)$ L is a constant

max of $P_r(L_A \text{ and } L_B \le L)$

L_A and L_B must be $\le L$

$P_r(L_A \le L)$ and $P_r(L_B \le L)$

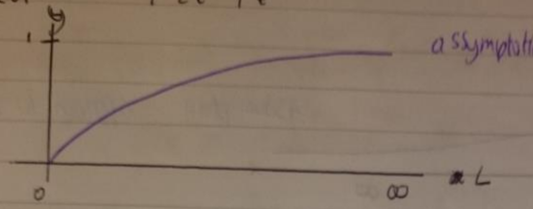
$= (1-e^{-L})(1-e^{-L})$

$1 - e^{-L} - e^{-L} + e^{-2L} \quad = 1 - 2e^{-2L} + e^{-2L}$ for $L > 0$

0 otherwise

4

cdf is $1-2e^{-1}+e^{-2L}$



Pdf is $\frac{d\left(1-2e^{-1}+e^{-2L}\right)}{dL} = 2e^{-1} - 2e^{-2L}$

at max/min $\frac{d''}{dL} = 0 \quad -2e^{-L}+4e^{-2L}=0$

$$4e^{-2L} = 2e^{-L}$$

$$2e^{-2L} = e^{-L}$$

$$\cancel{2\ln(-2L) = \ln(-L)}$$

$$\ln(2) - 2L = -L$$

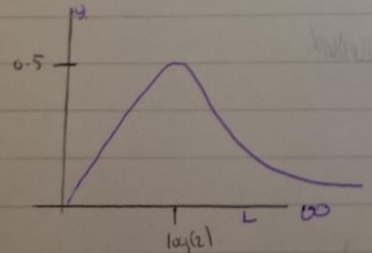
$$\ln(2) = L \text{ at max/min}$$

into pdf. $2e^{-\ln(2)} - 2e^{-2\ln(2)}$

$$2\left(\tfrac{1}{2}\right) - 2\left(\tfrac{1}{4}\right) = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

at $\log(2)$ max value of $\frac{1}{2}$

$$\text{pdf} = \begin{cases} 0 & \text{otherwise} \\ 2e^{-L} - 2e^{-2L} & L \geq 0 \end{cases}$$



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8iii.

To calculate expected value we would integrate pdf.

$$= \int_{-\infty}^{\infty} L(f_L)\,dL = \int_0^{\infty} L(2e^{-L} - 2e^{-2L})\,dL$$

$$\int_0^{\infty} 2Le^{-L}\,dL - \int_0^{\infty} 2Le^{-2L}\,dL$$

Integration by parts $\int u\,dv = u \cdot v - \int v\,du$

$u = L \qquad du = dL$	$u = L \qquad du = dL$
$dv = e^{-L} \qquad v = -e^{-L}$	$dv = e^{-2L} \qquad v = \frac{-e^{-2L}}{2}$
$= -Le^{-L} - \int -e^{-L}\,dL$	$\frac{-Le^{-2L}}{2} - \int \frac{-e^{-2L}}{2}\,dL$
$= -Le^{-L} - e^{-L}$	$-\frac{Le^{-2}}{2} - \frac{e^{-2L}}{4}$

$$\left[-2Le^{-L} - 2e^{-L} + \frac{2Le^{-L}}{2} + \frac{2e^{-2L}}{4}\right]_{L=0}^{\infty}$$

$$-2(\infty)e^{-\infty} - 2e^{-\infty} + \frac{2Le^{-\infty}}{2} + \frac{2e^{-2L}}{4}$$

$$[0 - 0 + 0 + 0] - [0 - 2(1) + 0 + \tfrac{2}{4}]$$

$$= \frac{\cancel{1}3}{2} = \frac{3}{2} = 1.5$$

L is > 1 it is $3/2$

1 2013 MANG SCI PAPER Q4 CORRECTION

Q4A. $P(A\ win) = 1 \times 1.2 = 1.2$

$P(A\ win) = 1.2\, P(B\ win)$

$p = 1.2(1-p) = (1.2 - 1.2p)$

$1.2(1k - 1.2p)$ $\quad 1.2 - 2.2p = 0$

$2.2p = 1.2$ $\quad 2.2p = 1.2$

$p = \frac{1.2}{2.2} = 0.55$ $\quad p = 0.55$

Probably A win = 0.55

Probably B win) = 0.45

B. Here we want $P(A\ home \mid A\ win)$

Using bayes rule we get

$$P(A\ home \mid A\ win) = \frac{P(A\ win \mid A\ home)\,P(A\ home)}{P(A\ win)}$$

$$= \frac{0.55 \times 0.5}{0.55 \times 0.5 + 0.45 \times 0.5} = 0.55 \text{ chance}$$

By manipulation $P(A|B) = \frac{P(AB)}{PB}$

$P(AB) = P(A|B)P(B)$

Bayes rule specified how probabilities must be updated in light of new information

Before knowing an event happened we assign prior probabilities $P(H)$ and $P(\bar{H}) = 1 - P(H)$ where H and $\bar{H}$ are mutually exclusive

The updated value of the probability that C4

2

hypothesis is true given the fact that E occurs is denoted $P(H|E)$.

To calculate posterior probability $P(A_{home}|A_{win})$ we use bayes rule

$$\frac{P(A\ home|A\ wins)}{P(A\ away|A\ wins)} = \frac{P(A\ home)}{P(A\ away)} \cdot \frac{P(A\ wins|A\ home)}{P(A\ win|A\ away)}$$

$$= \frac{0.55}{0.45} = 1.222$$

$$\frac{p}{1-p} = 1.222$$

$p = 1.222 - 1.222p$

$2.222p = 1.222$

$p = \frac{1.222}{2.222} = 0.55.$

4a Binomial $Y \sim B(n,p)$

Basic assumptions

- y are positive integers $0,1 \ldots N$
- finite number of trials n
- Only two outcomes "success" or failure
- Independent trials with same probability of success in each trial

- Our situation satisfies those conditions, $y = 0,1,2,3$ or 4
- $N = 4$ = number of games
- Each game independent of each other
- p success for each trial is the same

$Pr(Y=y) = \binom{n}{y} p^y (1-p)^{n-y}$

$Pr(Y=0) = \binom{4}{0} 0.5^0 (0.5)^4 = 1/16$

$Pr(Y=1) = \binom{4}{1} 0.5^1 (0.5)^3 = 1/4$

$Pr(Y=2) = \binom{4}{2} 0.5^2 (0.5)^2 = 3/8$

$Pr(Y=3) = \binom{4}{3} 0.5^3 (0.5)^1_0 = 1/4$

$Pr(Y=4) = \binom{4}{4} 0.5^4 (0.5) = 1/16 = 1$

Variance calculated by $np(1-p) =$

$4(0.5)(1-0.5) = 1$

4ii. Basic assumptions have been breached with introduction of third probability, we have three outcomes not 2 as stated in our assumptions
=> Binomial no longer applies as they are no longer bernoulli trials

	pr	value X	x^2
win	0.45	3	9
lose	0.45	0	0
draw	0.1	1	1

$E[X] = 0.45(3) + 0.45(0) + 0.1(1) = 1.45$ points per game
$E[X^2] = 0.45(9) + 0.45(0) + 0.1(1) = 4.15$ points

Variance $= E[X^2] - E[X]^2$
$= 4.15 - (1.45)^2 = 2.0475$ points for 1 game
4×2.0475 is 8.19 for 4 matches

(iii.

	home		away
win	0.9×0.55.	0.495	0.405
lose		0.405	0.495
draw	0.1	0.1	1

$E[X] = 0.495(3) + 0.1(1)$
1.585
$E[X^2] = 4.564$
var $= 2.051775$

$E[X] = 0.405(3) + 1(0.1)$
$= 1.315$
$E[X^2] = 3.745$.
var $= 2.015775$.

$\frac{4}{2}(2.051775 + 2.015775)$

$= 8.1351$ points in 4 matches
2 home 2 away

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Exam Paper 1 Q1 SECTION A

12 Variance of actual league points at the end of season was 239 compared with theoretical variance of 61, were all teams of equal quality and the result of matches were due to chance

SD $= \sqrt{61} \approx 7.8$ ← Spread of points according to model

$\sqrt{239} \approx 15.5$ ← total spread of points

Only 7.8 of this is explained by model which is around half.