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Ordinary Differential Equations

Laplace Transform of Derivatives

The transforms of derivatives are $\mathcal{L}(y') = s\,\mathcal{L}(y) - y(0)$

$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0)$

Proof:

Integration by parts.

$\mathcal{L}(y') = \int_0^\infty e^{-st} y'(t)\, dt$

$= e^{-st} y(t) \Big|_0^\infty - (-s)\int_0^\infty e^{-st} (y(t))\, dt$

$= \underbrace{e^{-\infty} y(\infty)}_{=0} - \underbrace{e^{0}}_{=1} y(0) + \underbrace{s\int_0^\infty e^{-st} y(t)\, dt}_{= \mathcal{L}(y)}$

$= (0) - (1)y(0) + s\,\mathcal{L}(y)$

$= s\,\mathcal{L}(y) - y(0)$

Note $\mathcal{L}(y'') = \mathcal{L}((y')')$

$= s\,\mathcal{L}(y') - y'(0)$

$\Rightarrow \mathcal{L}(y'') = s\mathcal{L}(y') - y'(0) = s\,[s\mathcal{L}(y) - y(0)] - y'(0)$

$= s^2\mathcal{L}(y) - s(y(0)) - y'(0)$

First and Second order ODE's

The most basic type is a first order differential equation this means that we have a first derivative, y' only

$y' + ay = 0$

is the general form of a homogeneous first order differential equation

This means that the right hand side is zero

An inhomogeneous first order ODE looks like $y' + ay = r(t)$ inhomogeneous part

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Example: Find the transform of $f(t) = 1$

Solution: $\int_0^\infty e^{-st}(1)\,dt = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = 0 - \left(-\frac{1}{s}(1)\right) = \frac{1}{s}$

(Since $e^{-\infty} = 0$, $e^0 = 1$)

LAPLACE TRANSFORM $\Rightarrow$ $F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt.$

$\int e^{at} = \frac{1}{a}e^{at} + c$

Example: Find the Laplace Transform of $f(t) = e^{at}$

Solution: $\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\,dt$

$= \int_0^\infty e^{-(s-a)t}\,dt$

$= \frac{1}{s-a} e^{-(s-a)t} \Big|_0^\infty$

$= 0 - \left(-\frac{1}{s-a}(1)\right) = \frac{1}{s-a}$

Linearity Property:

$\mathcal{L}(af(t) + bg(t))$

$= a\mathcal{L}(f) + b\mathcal{L}(g) = aF(s) + bG(s)$

Capital letter refers to Laplace transform of function [illegible] $G(s) = \mathcal{L}(g)$

First Shifting Theorm

From $\mathcal{L}(f)$ we can find $\mathcal{L}(e^{at}f(t))$ using the first shifting theorm.

$\mathcal{L}(e^{at}f(t)) = F(s-a)$

Conversely

$e^{at}f(t) = \mathcal{L}^{-1}[F(s-a)]$

Proof: $F(s-a) = \int_0^{\infty} e^{-(s-a)t} f(t)\,dt$

$$= \int_0^{\infty} e^{-st}\,[e^{at} f(t)]\,dt \quad = \mathcal{L}[e^{at} f(t)]$$

Example: Find the Laplace Transform of $y(t) = 2e^{t} + 3e^{2t}$

Solution

$f(t)$	$F(t)$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$

$$\mathcal{L}(y) = \mathcal{L}(2e^{t} + 3e^{2t})$$
$$= 2\mathcal{L}(e^{t}) + 3\mathcal{L}e^{2t}$$
$$= \frac{2}{s-1} + \frac{3}{s-2} = \frac{5s-7}{(s-1)(s-2)}$$

Example: Find the Laplace Transform of $y(t) = e^{at} t^n$

Solution: $\mathcal{L}(y) = \mathcal{L}[e^{at} t^n] = F(s-a)$

Now $F(s) = \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

$\Rightarrow \mathcal{L}(y) = F(s-a) = \frac{n!}{(s-a)^{n+1}}$

Example: Find $\mathcal{L}^{-1}\left(\frac{1}{s^2-4s+5}\right)$

Solution: $\frac{1}{s^2-4s+5} = \frac{1}{(s-2)^2-4+5} = \frac{1}{(s-2)^2+1}$

Since $\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$ calling $F(s) = \frac{1}{s^2+1}$ and $f(t) = \sin t$ we then shift by $a=2$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s^2-4s+5}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s-2)^2+1}\right) = e^{2t}\sin t$$

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Laplace Transforms

We can define second order differential equations which include y''. Again, there are two types:

Homogeneous: $y'' + ay' + by = 0$

Inhomogeneous: $y'' + ay' + by = r(t)$

To solve ODE's we need a general solution for the homogeneous equation, and a particular solution for the inhomogeneous equation

- We also need initial conditions
- For a first order equation we need $y(0) = k_0$
- For a second order equation we need $y(0) = k_0$ + $y'(0) = k_1$
- An ODE with necessary conditions is called an initial value problem

Solving ODEs

Recall that we found the Laplace transforms of y' and y''

$L(y') = sL(y) - y(0)$
$L(y'') = s^2 L(y) - sy(0) - y'(0)$

Laplace transforms convert differential equations to algebraic equations including polynomials etc

Also, an inhomogeneous equation can be solved directly without first solving the homogeneous one

Finally, the initial conditions are included automatically

Solving first order ODE's

Example: Solve the first order equation $y'(t) = y(t)$ with initial condition $y(0) = y_0$

Solution: Take the Laplace Transform of both sides

$$\mathcal{L}(y') = \mathcal{L}(y)$$
$$\Rightarrow s\mathcal{L}(y) - y(0) = \mathcal{L}(y)$$
$$\Rightarrow s\mathcal{L}(y) - y_0 = \mathcal{L}(y)$$
$$\Rightarrow (s-1)\mathcal{L}(y) = y_0$$
$$\Rightarrow \mathcal{L}(y) = \frac{y_0}{s-1}$$

We now use an inverse transform

$$y(t) = \mathcal{L}^{-1}\left(\frac{y_0}{s-1}\right) = y_0 \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = y_0 e^t$$

Check the solution satisfies the original equation

$$y'(t) = y_0 \frac{d}{dt}(e^t) = y_0 e^t = y(t)$$
$$y'(t) = y(t)$$

i.e. the original equation is satisfied

Example. Solve the equation $y'(t) = 2y(t) = 4$ subject to the condition $y(0) = 1$ using Laplace Transform

Solution:

$$\mathcal{L}(y) - 2\mathcal{L}(y) = \mathcal{L}(4)$$
$$\Rightarrow s\mathcal{L}(y) - y(0) - 2\mathcal{L}(y) = \frac{4}{s}$$
$$\Rightarrow s\mathcal{L}(y) - 1 - 2\mathcal{L}(y) = 4/s$$
$$(s-2)\mathcal{L}(y) = 1 + 4/s$$
$$\mathcal{L}(y) = \frac{1}{s+2} + \frac{4}{s(s-2)}$$

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Notes

We use partial fraction decomposition on the second term

$$\frac{4}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

$$= \frac{A(s-2) + Bs}{s(s-2)}$$

$$\Rightarrow 4 = A(s-2) + Bs$$

$$4 = (A+B)s - 2A$$

$$\Rightarrow \begin{rcases} A+B=0 \\ -2A=4 \end{rcases} \quad A=-2,\ B=2$$

$$\Rightarrow \frac{4}{s(s-2)} = -\frac{2}{s} + \frac{2}{s-2}$$

$$\Rightarrow \mathcal{L}(y) = \frac{1}{s-2} - \frac{2}{s} + \frac{2}{s-2}$$

$$= \frac{3}{s-2} - \frac{2}{s}$$

We now take the inverse transform

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left(\frac{3}{s-2}\right) - \mathcal{L}^{-1}\left(\frac{2}{s}\right)$$

$$= 3e^{2t} - 2$$

Check it solves the original equation

$$y'(t) = 6e^{2t}$$

$$y'(t) - 2y(t) = 6e^{2t} - 2(3e^{2t} - 2)$$

$$= 6e^{2t} - 6e^{2t} + 4 = 4$$

as required

Solving Second Order ODEs:

Consider the Laplace transform of the general equation

$$y'' + ay' + by = r(t)$$

We can view $r(t)$ as the input or driving force and $y(t)$ as the output

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Writing $Y(s) = \mathcal{L}(y)$ and $R(s) = \mathcal{L}(r(t))$

$$\mathcal{L}(y'') + a\mathcal{L}(y') + b\mathcal{L}(y) = \mathcal{L}(r)$$
$$\Rightarrow [s^2Y(s) - sy(0) - y'(0)] + a[sY - y(0)] + bY(s) = R(s)$$

This is called the Subsidiary equation

We want to solve for $Y(s)$,
$$(s^2 + as + b)Y(s) = (s+a)y_0 + y'(0) + R(s)$$

Introducing the transfer function
$$Q(s) = \frac{1}{s^2+as+b} = \frac{1}{(s+\frac{a}{2})^2 + b - \frac{1}{4}a^2}$$

We write
$$Y(s) = [(s+a)y(0) + y'(0)]Q(s) + R(s)Q(s)$$

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Second Order ODEs

General equation: $y'' + ay' + by = r(t) \Rightarrow \mathcal{L}(y'') + a\mathcal{L}(y') + b\mathcal{L}(y) = \mathcal{L}(r)$

$\Rightarrow [s^2 Y(s) - sy(0) - y'(0)] + a[sY(s) - y(0)] + bY(s) = R(s)$

$(Y(s) = \mathcal{L}(y),\quad R(s) = \mathcal{L}(r))$

$\Rightarrow$ Subsidiary equation

$(s^2 + as + b) = (s+a)y(0) + y'(0) + R(s)$

Define transform function

$$Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{(s + \frac{a}{2})^2 + b - \frac{1}{4}a^2}$$

$\Rightarrow Y(s) = [(s+a)y(0) + y'(0)]Q(s) + R(s)Q(s)$

Take inverse transform to find $y(t)$ using Y

Example: Solve the initial value problem

$y''(t) + 4y(t) = 2e^t \qquad y(0) = 1,\ y'(0) = 0$

Solution: $\mathcal{L}(y'') + 4\mathcal{L}(y) = 2\mathcal{L}(e^t)$

$\Rightarrow [s^2\mathcal{L}(y) - sy(0) - y'(0)] + 4\mathcal{L}(y) = 2\mathcal{L}(e^t)$

$\Rightarrow [s^2\mathcal{L}(y) - s(1) - 0] + 4\mathcal{L}(y) = \frac{2}{s-1}$

$\Rightarrow (s^2 + 4)\mathcal{L}(y) = s + \frac{2}{s-1}$

$\overset{Y(s)}{\mathcal{L}(y)} = \frac{s}{s^2+4} + \frac{2}{(s^2+4)(s-1)}$

decompose into fraction

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$\frac{A}{s-1} + \frac{Bs+C}{s^2+4}$

$\Rightarrow \frac{2}{(s-1)(s^2+4)} = \frac{A(s^2+4)+(Bs+C)(s-1)}{(s-1)(s^2+4)}$

$2 = A(s^2+4) + (Bs+C)(s-1)$ this should hold for any value of s

sub in $s=1 \Rightarrow$

$2 = A(1+4) + (B(1)+C)(1-1)$

$2 = 5A \Rightarrow A = \frac{2}{5}$

$s=0 \Rightarrow 2 = A(0^2+4) + (B(0)+C)(0-1) \Rightarrow 2 = 4A - C$

$\Rightarrow 2 = 4(\frac{2}{5}) - C \Rightarrow C = \frac{8}{5} - 2 = \frac{-2}{5}$

$s = -1 \Rightarrow A((-1)^2+4) + (B(-1)+C)(-1-1) = 2$

$5A + 2B - 2C = 2$

$5(\frac{2}{5}) + 2B - 2(\frac{-2}{5}) = 2$

$\Rightarrow 2 + 2B - \frac{4}{5} = 2$

$B = -2/5$

Therefore, our decomposition is

$\frac{2}{(s-1)(s^2+4)} = \frac{2}{5}\frac{1}{s-1} - \frac{2}{5}\frac{s}{s^2+4} - \frac{2}{5}\frac{1}{s^2+4}$

$\Rightarrow \mathcal{L}(y) = \frac{s}{s^2+4} + \frac{2}{5}\frac{1}{s-1} - \frac{2}{5}\frac{s}{s^2+4} - \frac{2}{5}\frac{1}{s^2+4}$ $\quad -\frac{1}{5}\frac{2}{s^2+4}$

$y(t) = \cos(2t) + \frac{2}{5}e^t - \frac{2}{5}\cos 2t - \frac{1}{5}\cos t 2t$

$= \frac{3}{5}\cos(2t) + \frac{2}{5}e^t - \frac{1}{5}\sin(2t)$

Returning to the original equation $y''(t) + 4y(t) = 2e^t$

$y'(t) = -\frac{6}{5}\sin(2t) + \frac{2}{5}e^t + \frac{-2}{5}\cos(2t)$

$\Rightarrow y''t = -12/5 \cos(2t) + 2/5 e^t - 12/5 \sin(2t)$

$(t) \to [\frac{-12}{5}\cos 2t + \frac{2}{5}e^t - \frac{12}{5}\sin(2t)] + 4[\frac{3}{5}\cos(2t) + \frac{2}{5}e^t + \frac{3}{5}\sin(2t)]) = 2e^t$

$\Rightarrow -\frac{12}{5}\cos 2t + 2/5 e^t - 12/5 \sin 2t + 12/5 \cos(2t) + 8/5 e^t + 12/5 \sin(2t) = 2e^t$

$10/5 e^t = 2e^t \qquad 2e^t = 2e^t$

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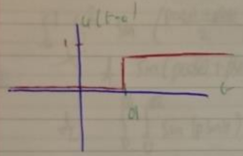
Heaviside Function / Unit Step Function

We denote the Heaviside function/unit step function by $u(t-a)$:

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$



The laplace transform is: $\mathcal{L}(u(t-a)) = \int_0^{\infty} e^{-st} u(t-a)\,dt$

$$= \int_0^a e^{-st}(0) + \int_a^{\infty} e^{-st}(1)\,dt$$

$$= \int_a^{\infty} e^{-st}\,dt = -\frac{1}{s} e^{-st} \Big|_{t=a}^{\infty}$$

$$= -\frac{1}{s} e^{-\infty} + \frac{1}{s} e^{-as} = -\frac{1}{s}(0) + \frac{1}{s}(e^{-as})$$

$$= \frac{e^{-as}}{s}$$

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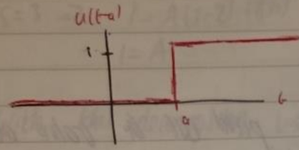
Unit Step Function

$u(t-a) = \begin{cases} 0 & \text{if } t<a \\ 1 & \text{if } t>a \end{cases}$



With Laplace transform $L[u(t-a)]\ \frac{e^{-as}}{s}$

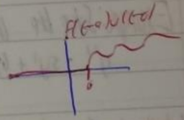
The Second Shifting theorem

Says that if $L[f(t)] = F(s)$ then

$L[f(t-a)u(t-a)] = e^{-as}F(s)$

Conversely $f(t-a)u(t-a) = L^{-1}[e^{-as}F(s)]$

Note: $f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t>a \\ f(t-a) & \text{if } t>a \end{cases}$



Anytime we have something like $\frac{e^{-as}}{s^2}$ we apply the second shifting theorem

Consider $e^{-s}F(s)$ where $F(s) = \frac{2}{s^2}$, ie $e^{-s}\frac{2}{s^2}$

$\Rightarrow L^{-1}[e^{-s}\frac{2}{s^2}] = L^{-1}[e^{-s}F(s)] = f(t-1)u(t-1)]$

Since $F(s) = \frac{2}{s^2} \xrightarrow{\text{transform}} f(t) = 2t \xrightarrow{\text{sub in}} f(t-1)$

$\Rightarrow L^{-1}[\frac{2e^{-s}}{s^2}] = 2(t-1)u(t-1)$

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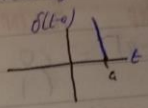
Dirac Delta Function

The Dirac delta function is defined by:

$$\delta(t-a) = \begin{cases} \infty & \text{if } t=a \\ 0 & \text{otherwise} \end{cases}$$



and $\int_0^\infty \delta(t-a)\,dt = 1$

When inserted in any integral it picks out the value at $t=a$

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

$\Rightarrow$ Laplace Transform is

$$\mathcal{L}[\delta(t-a)] = \int_0^\infty e^{-st}\,\delta(t-a)\,dt = e^{-sa}$$

and $\mathcal{L}^{-1}(e^{-sa}) = \delta(t-a)$

Example: Solve the initial value problem

$$y'' - 5y' + 6y = \delta(t-2) \qquad y'(0)=0, \qquad y(0)=0$$

Solution: The laplace transform gives

$$[s^2\mathcal{L}(y) - sy(0) - y'(0)] - 5[s\mathcal{L}(y) - y(0)] + 6\mathcal{L}(y) = e^{-2s}$$

$$\Rightarrow [s^2Y - s(0) - 0] - 5[sY - 0] + 6Y = e^{-2s}$$

$$\Rightarrow [s^2 - 5s + 6]Y = e^{-2s} \Rightarrow Y = \frac{e^{-2s}}{s^2-5s+6}$$

$$= \frac{e^{-2s}}{(s-3)(s-2)} \qquad \text{But } \frac{1}{(s-3)(s-2)} = \frac{A}{s-3} + \frac{B}{s-2} = \frac{A(s-2)+B(s-3)}{(s-3)(s-2)}$$

$$\Rightarrow 1 = A(s-2) + B(s-3)$$

$$\Rightarrow 1 = (A+B)s - (2A+3B)$$

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$\Rightarrow s=2 \Rightarrow 1 = A(0) + B(2-3)$

$B = -1$

$s=3 \Rightarrow 1 = A(3-2) + B(0)$

$1 = A$

With this decomposition, we find

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-5s+6}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-3} - \frac{1}{s-2}\right)$$

$$= e^{3t} - e^{2t} = f(t)$$

We now have $Y = e^{-2s} \cancel{F(s)}\, F(s)$

where $F(s) = \dfrac{1}{s^2-5s+6}$

Hence $y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\dfrac{e^{-2s}}{s^2+5s+6}\right)$

$= u(t-2)\left(e^{3(t-2)} - e^{2(t-2)}\right)$

$= u(t-2)\left(e^{3t-6} - e^{2t-4}\right)$

with $f(t) = e^{3t} - e^{2t}$

Example: Solve the equation:

$y''(t) + y(t) = \delta(t-1) + u(t-2)$ subject to the conditions

$y'(0) = 1 \qquad y(0) = 0$

Solution: $[s^2 Y - s\overbrace{y(0)}^{=0} - \overbrace{y'(0)}^{=1}] + Y = \dfrac{e^{-s}}{1} + \dfrac{e^{-2s}}{s}$

$\Rightarrow (s^2+1)Y = 1 + e^{-s} + \dfrac{e^{-2s}}{s}$

$Y = \dfrac{1}{s^2+1} + \dfrac{e^{-s}}{s^2+1} + \dfrac{e^{-2s}}{s(s^2+1)}$

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$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \Rightarrow 1 = A(s^2+1) + (Bs+C)s$$

$$\Rightarrow 1 = (A+B)s^2 + Cs + A$$ comparing coefficients $A = 1$

$B = -A = -1$

$C = 0$

$$\Rightarrow Y = \frac{1}{s^2+1} + \frac{e^{-s}}{s+1} + \frac{e^{-2s}}{s} \quad \frac{-se^{-2s}}{s^2+1}$$

$$\Rightarrow y(t) = \sin(t) \quad + u(t-1)\sin(t-1) + u(t-2) - u(t-2)\cos(t-2)$$

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Solution:

The Laplace transform gives

$$[s^2L(y) - sy(0) - y'(0)] - 5[sL(y) - y(0)] + 6L(y) = e^{-2s}$$

Note $L[\delta(t-a)] = e^{-as}$

$$\Rightarrow [s^2Y - y(0) - (0)] - 5[sY - (0)] + 6Y = e^{-2s}$$

$$(s^2 - 5s + 6)Y = e^{-2s}$$

$$\Rightarrow Y = \frac{1}{s^2-5s+6}e^{-2s} = Q(s)R(s)$$

Where $Q(s) = \frac{1}{s^2-5s+6}$

$R(s) = e^{-2s}$

$$\frac{1}{s^2-5s+6} = \frac{1}{(s-3)(s-2)} = \frac{A}{s-3} + \frac{B}{s-2}$$

$1 = A(s-2) + B(s-3)$

$s=2 \Rightarrow 1 = A(0) + B(2-3)$

$\Rightarrow B = -1$

$s = 3 \Rightarrow 1 = A(3-2) = A$

$\Rightarrow A = 1$

$$\Rightarrow L^{-1}\left(\frac{1}{s^2+5s+6}\right) = L^{-1}\left(\frac{1}{s-3} - \frac{1}{s-2}\right) = q(t)$$

$q(t) = e^{3t} - e^{2t}$

Also $r(t) = \delta(t-2)$

Since $Y(s) = Q(s) \cdot R(s)$ we apply convolution theorem to give

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$y(t) = \int_0^t$

$y(t) = \int_0^t$

Recall $\int_0^\infty f(t)$

There are

1. $t<2$
the integr
i.e.

2. $t>2$
non-
Since
take
i.e
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In

Th
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$$y(t) = \int_0^t r(\tau)\, g(t-\tau)\, d\tau$$

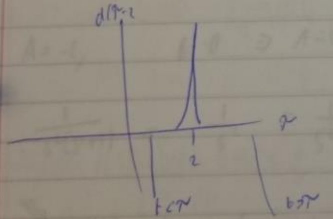
$$y(t) = \int_0^t \delta(\tau-2)\left(e^{3(t-\tau)} - e^{2(t-\tau)}\right) d\tau$$

Recall $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$

There are two possibilities in this integral

1. $t<2$ The $\delta(t-2)=0$ for all τ since t is the upper limit of the integral. Therefore the integral is 0.
   i.e. for $t<2$ $y(t)=0$

2. $t>2$ then at $\tau=2$ the ~~value of the~~ delta function is non-zero, but everywhere else it is 0.
   Since $t>2$, all $\tau>t$ gives $\delta(\tau-2)=0$. Therefore if we take $t\to\infty$, the integral does not change
   i.e. $y(t) = \int_0^\infty \delta(\tau-2)\, e^{3(t-\tau)} - e^{2(t-\tau)}\, d\tau$
   $= e^{3(t-2)} - e^{2(t-2)} = e^{3t-6} - e^{2t-4}$



In summary $y(t) \begin{cases} e^{3t-6} - e^{2t-4} & \text{if } t>2 \\ 0 & \text{if } t<2 \end{cases}$

This is the behaviour of a step function
$\Rightarrow y(t) = u(t-2)\left(e^{3t-6} - e^{2t-4}\right)$ as before