1

Markov chain

Distinguished by fact that the current state of a system depends only on the immediately preceding state of the system

The transition probabilities are dependent only on the current state of the system

The transition probabilities are constant over time

The transition probabilities of moving to alternative states in the next time period, given a state in the current time period, must sum to 1.

Example 1.

3 Supermarkets A, B and C. Each family in town visits one supermarket once a week.

- 70% who shop in A will do so next week
  10% will shop at B and 20% at C
- 70% who shop in B will do so next week.
  10% will shop at A and 20% at C.
- 90% who shop in C will do so next week.
  5% will shop at A and 5% at B

At present, 50% at A 30% at B and 20% at C

Solution - transition Matrix

$$
\begin{array}{cc|ccc}
 & & & \text{To} & \\
 & & A & B & C \\
\text{From} & A & 0.7 & 0.1 & 0.2 \\
 & B & 0.1 & 0.7 & 0.2 \\
 & C & 0.05 & 0.05 & 0.9
\end{array}
$$

2 Solution - State vectors (1)

Initial state vector $(0.5, 0.3, 0.2)$

State vector for week 2 - A will have

$(0.7)(0.5) + (0.1)(0.3) + (0.05)(0.2) = 0.39$

B will have

$(0.1)(0.5) + (0.7)(0.3) + (0.05)(0.2) = 0.27$

C will have

$(0.2)(0.5) + (0.2)(0.3) + (0.9)(0.2) = 0.34$

Solution - State vectors (2)

The week 2 state vector $(0.39, 0.27, 0.34)$ can be found by multiplying the state vector for week 1 by the transitional matrix, what we did

The state vector for week 3 $(0.317, 0.245, 0.438)$ can be found by multiplying the state vector for week 2 by the transition matrix

Solution - State vectors (3)

Week 1 $(0.5, 0.3, 0.2)$

Week 2 $(0.39, 0.27, 0.34)$

Week 3 $(0.317, 0.245, 0.438)$

Week 4 $(0.2683, 0.2251, 0.5066)$

- Eventually in week n we will reach a steady state vector - in this case it is $(0.1667, 0.1667, 0.6667)$

Proof (1)

Denote transition matrix by P and the initial state vector S_0 and let S_i be the subsequent state vectors then:

$S_1 = S_0 P$
$S_2 = S_1 P$
$S_3 = S_2 P$
$S_n = S_{n-1}$

From above we get:

$S_2 = S_0 P(P) = S_0 P^2$
$S_3 = S_0 P^2(P) = S_0 P^3$
$S_n = S_0 P^{n-1}(P) = S_0 P^n$

A markov process will reach its steady state eventually regardless of the initial state

In other words it is "memoryless"

Example 2
If it rains today, it will rain tomorrow with probability 0.7.
If it doesn't rain it will rain tomorrow with probability 0.4

Transition matrix
Let 0 = rain and 1 = No rain

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0.7 & 0.3 \\ 1 & 0.4 & 0.6 \end{array}$$

Rain in 4 days time
Calculate p^2 and then p^4 to get probability

$$p^4 = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix}$$

Rain in four days $= 0.5749$

4

Steady State - Different method

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0.7 & 0.3 \\ 1 & 0.4 & 0.6 \end{array}$$

Steady State vector is (x, y)

We know, steady state vector multiplied by transition matrix is equal to the steady state vector

$$(x\ y)\begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = (x, y)$$

$x + y = 1 \rightarrow y = 1 - x$

$(x, y) \rightarrow (x, 1-x)$

Solution

$$(x, 1-x)\begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = (x, 1-x)$$

Therefore: $0.7x + 0.4(1-x) = x$

$0.3x + 0.6(1-x) = 1-x$

- Solving gives $x = 0.5714$ $y = 0.4286$
- In long run it will rain 0.5714 time.

3 State Process

- For 3 state process, let steady state vector be (x, y, z)
- From this we get $\Rightarrow$ $x + y + z = 1$
- Therefore $z = 1 - x - y$
- Giving the steady state vector as $(x, y, 1-x-y)$
- Proceed as before and solve eq^ns