Spherical Kerr-Schild Coordinates

Yitian Chen (yc2377@cornell.edu)

July 12, 2021

1 Introduction

This article contains all useful formulae to construct a spinning black-hole (BH) in spherical Kerr-Schild (SphKS) coordinates. Since They are closely related to Kerr-Schild (KS) coordinates, we follow the symbol convention of Kerr-Schild (KS) in the file *KerrSchildCoords.tex* in the same folder. We first describe a rotating BH with spin along z axis in SphKS, and later generalize the spin to any direction. As KS coordinates use symbol $\{t, x, y, z\}$, we denote SphKS coordinates as $\{t, \bar{x}, \bar{y}, \bar{z}\}$.

2 Spin in the z direction

2.1 Transformation

In KS, the Boyer-Lindquist radius r at a point satisfies an equation of spheroid

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1, \tag{1}$$

or equivalently,

$$r^2 = \frac{1}{2}(x^2 + y^2 + z^2 - a^2) + \left(\frac{1}{4}(x^2 + y^2 + z^2 - a^2)^2 + a^2 z^2\right)^{1/2}. \tag{2}$$

$\vec{\bar{x}} = \bar{x}^i = \bar{x}_i = (\bar{x}, \bar{y}, \bar{z})$ is related to $\vec{x} = x^i = x_i = (x, y, z)$ by

$$\rho^2 \equiv r^2 + a^2, \tag{3}$$

$$\left(\frac{\bar{x}}{r}, \frac{\bar{y}}{r}, \frac{\bar{z}}{r}\right) \equiv \left(\frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{r}\right), \tag{4}$$

or more compactly,

$$x^i = P^i{}_j \bar{x}^j, \tag{5}$$

$$\bar{x}^i = Q^i{}_j x^j, \tag{6}$$

$$P^i{}_j \equiv \text{Diagonal}\left(\frac{\rho}{r}, \frac{\rho}{r}, 1\right), \tag{7}$$

$$Q^i{}_j \equiv (P^{-1})^i{}_j = \text{Diagonal}\left(\frac{r}{\rho}, \frac{r}{\rho}, 1\right). \tag{8}$$

Thus, r satisfies the equation of sphere in SphKS

$$r^2 = \vec{\bar{x}} \cdot \vec{\bar{x}} = \bar{x}^2 + \bar{y}^2 + \bar{z}^2. \tag{9}$$

In other words, Euclidean radius coincides with Boyer-Lindquist radius, in SphKS. The Jacobian $T^i{}_j$ is

$$dx^i = T^i{}_j d\bar{x}^j, \tag{10}$$

$$F^i{}_k \equiv -\frac{a^2}{\rho r^3} \cdot \text{Diagonal}(1, 1, 0), \tag{11}$$

$$T^i{}_j = P^i{}_j + F^i{}_k \bar{x}^k \bar{x}_j. \tag{12}$$

Its inverse $S^i{}_j = (T^{-1})^i{}_j$ is

$$(G_1)^i{}_m \equiv -\frac{r^2}{\rho} F^i{}_m = \frac{a^2}{\rho^2 r} \cdot \text{Diagonal}(1, 1, 0), \tag{13}$$

$$s \equiv r^2 + \frac{a^2 z^2}{r^2}, \tag{14}$$

$$(G_2)^n{}_j \equiv \frac{\rho^2}{sr} Q^n{}_j = \frac{\rho}{s} \cdot \text{Diagonal}\left(1, 1, \frac{\rho}{r}\right),, \tag{15}$$

$$S^i{}_j = Q^i{}_j + (G_1)^i{}_m \bar{x}^m \bar{x}_n (G_2)^n{}_j. \tag{16}$$

2.2   Metric

The spatial metric is

$$g_{ij} = \bar{\eta}_{ij} + 2H \bar{l}_i \bar{l}_j, \tag{17}$$

$$\bar{\eta}_{ij} = \eta_{mn} T^m{}_i T^n{}_j, \tag{18}$$

$$H = \frac{r^3}{r^4 + a^2 z^2} = \frac{r}{s}, \tag{19}$$

$$l_i = l^i = \frac{r\vec{x} - \vec{a} \times \vec{x} + \frac{(\vec{a} \cdot \vec{x})\vec{a}}{r}}{\rho^2}, \tag{20}$$

$$\bar{l}_i = T^m{}_i l_m, \tag{21}$$

$$\bar{l}^i = S^i{}_m l^m, \tag{22}$$

and the spacetime metric is

$$\psi_{\mu\nu} = \bar{\eta}_{\mu\nu} + 2H \bar{l}_\mu \bar{l}_\nu, \tag{23}$$

$$\bar{l}_\mu = (1, \bar{l}_i), \tag{24}$$

$$\bar{l}^\mu = (-1, \bar{l}^i), \tag{25}$$

$$\bar{\eta}_{\mu\nu} = (-1) \otimes \bar{\eta}_{ij}. \tag{26}$$

Lapse and shift are

$$\beta^i = \frac{2H \bar{l}^i}{1 + 2H} = 2H\alpha^2 \bar{l}^i, \tag{27}$$

$$\beta_i = 2H \bar{l}_i, \tag{28}$$

$$\alpha = (1 + 2H)^{-1/2}. \tag{29}$$

2.3 Derivatives

In the following, symbol ∂_i always refers to the derivative relative to $\bar{x}^i$. Derivatives of the Jacobian and its inverse are

$$D^i{}_j \equiv \frac{a^2}{\rho^3 r} \cdot \text{Diagonal}(1,1,0), \tag{30}$$

$$C^i{}_m \equiv D^i{}_m - 3F^i{}_m = \frac{a^2}{\rho r}\left(\frac{1}{\rho^2} + \frac{3}{r^2}\right) \cdot \text{Diagonal}(1,1,0), \tag{31}$$

$$\partial_k T^i{}_j = F^i{}_j \bar{x}_k + F^i{}_k \bar{x}_j + F^i{}_m \bar{x}^m \delta_{jk} + C^i{}_m \frac{\bar{x}_k \bar{x}^m \bar{x}_j}{r^2}, \tag{32}$$

$$(E_1)^i{}_m \equiv -\frac{a^2}{\rho^2}\left(\frac{1}{r^2} + \frac{2}{\rho^2}\right) \cdot \text{Diagonal}(1,1,0), \tag{33}$$

$$(E_2)^n{}_j \equiv \left[-\frac{a^2}{\rho^2 r} - \frac{2}{s}\left(r - \frac{a^2 \bar{z}^2}{r^3}\right)\right] \cdot (G_2)^n{}_j + \frac{1}{s}P^n{}_j, \tag{34}$$

$$\partial_k S^i{}_j = D^i{}_j \bar{x}_k + (G_1)^i{}_k \bar{x}_n (G_2)^n{}_j + (G_1)^i{}_m \bar{x}^m (G_2)_{kj}$$
$$+ (E_1)^i{}_m \frac{\bar{x}_k \bar{x}^m \bar{x}_n}{r}(G_2)^n{}_j + (G_1)^i{}_m \frac{\bar{x}_k \bar{x}^m \bar{x}_n}{r}(E_2)^n{}_j - (G_1)^i{}_m \bar{x}^m \bar{x}_n (G_2)^n{}_j \frac{2a^2 \bar{z}}{sr^2}\delta_{k\bar{z}}. \tag{35}$$

where $(G_2)_{kj} \equiv (G_2)^k{}_j$ and $\delta_{k\bar{z}}$ is 1 if $k = \bar{z}$ but 0 otherwise. Other important derivatives are

$$\frac{\partial r}{\partial x^i} = \frac{r^2 x_i + (\vec{a} \cdot \vec{x})a_i}{rs}, \tag{36}$$

$$\partial_i H = H T^m{}_i \left[\frac{3}{r}\frac{\partial r}{\partial x^m} - \frac{4r^3 \frac{\partial r}{\partial x^m} + 2(\vec{a} \cdot \vec{x})a_m}{r^4 + (\vec{a} \cdot \vec{x})^2}\right], \tag{37}$$

$$\partial_j \bar{l}_i = T^k{}_i T^m{}_j \frac{1}{\rho^2}\left[\left(x_k - 2rl_k - \frac{(\vec{a} \cdot \vec{x})a_k}{r^2}\right)\frac{\partial r}{\partial x^m} + r\delta_{km} + \frac{a_k a_m}{r} - \epsilon^{kmn}a_n\right] + l_k \partial_j T^k{}_i, \tag{38}$$

$$\partial_k g_{ij} = 2\bar{l}_i \bar{l}_j \partial_k H + 4H\bar{l}_{(i}\partial_k \bar{l}_{j)} + T^m{}_j \partial_k T^m{}_i + T^m{}_i \partial_k T^m{}_j, \tag{39}$$

$$\partial_k \alpha = -(1 + 2H)^{-3/2}\partial_k H = -\alpha^3 \partial_k H, \tag{40}$$

$$\partial_k \beta^i = 2\alpha^2 \left[\bar{l}^i \partial_k H + H(S^i{}_j S^m{}_j \partial_k \bar{l}_m + S^i{}_j \bar{l}_m \partial_k S^m{}_j + S^m{}_j \bar{l}_m \partial_k S^i{}_j)\right] - 4H\bar{l}^i \alpha^4 \partial_k H, \tag{41}$$

where ϵ^{kmn} is the antisymetric symbol and $\epsilon^{xyz} = +1$.

3 Spin in an arbitrary direction

3.1 Transformation

Now, we generalize the spin to arbitrary direction. The spin vector is $\vec{a}$ and the unit vector along its direction is $\hat{a}$ (meaningful only if spin is nonzero). In KS, r satisfies

$$r^2 = \frac{1}{2}(\vec{x} \cdot \vec{x} - a^2) + \left(\frac{1}{4}(\vec{x} \cdot \vec{x} - a^2)^2 + (\vec{a} \cdot \vec{x})^2\right)^{1/2}, \tag{42}$$

$$\rho^2 \equiv r^2 + a^2, \tag{43}$$

and we define $\vec{\bar{x}} = \bar{x}^i = \bar{x}_i = (\bar{x}, \bar{y}, \bar{z})$ in terms of $\vec{x} = x^i = x_i = (x, y, z)$ as

$$Q^i{}_j \equiv \frac{r}{\rho}\delta^i{}_j + \frac{1}{(\rho + r)\rho}a^i a_j, \tag{44}$$

$$P^i{}_j \equiv \frac{\rho}{r}\delta^i{}_j - \frac{1}{(\rho + r)r}a^i a_j, \tag{45}$$

$$\vec{\bar{x}} \equiv Q^i{}_j x^j = Q\vec{x}, \tag{46}$$

$$\vec{x} = P^i{}_j \bar{x}^j = P\vec{\bar{x}}. \tag{47}$$

Note that

- as $a \to 0$, both P and Q tend to the identity. If a is nonzero, P and Q can be written in projection matrices:

$$Q^i{}_j = \frac{r}{\rho}(P_\perp)^i{}_j + (P_{//})^i{}_j, \tag{48}$$

$$P^i{}_j = \frac{\rho}{r}(P_\perp)^i{}_j + (P_{//})^i{}_j, \tag{49}$$

$$(P_\perp)^i{}_j \equiv \delta^i{}_j - \hat{a}^i\hat{a}_j, \tag{50}$$

$$(P_{//})^i{}_j \equiv \hat{a}^i\hat{a}_j. \tag{51}$$

One should be careful about the difference between $\hat{a}^i$ and a^i in the above formulae.

- r still satisfies

$$r^2 = \vec{\bar{x}} \cdot \vec{\bar{x}} = \bar{x}^2 + \bar{y}^2 + \bar{z}^2. \tag{52}$$

- $\vec{a} \cdot \vec{x}$ and $\vec{a} \cdot \vec{\bar{x}}$ give the same result, i.e.

$$\vec{a} \cdot \vec{x} = \vec{a} \cdot \vec{\bar{x}}. \tag{53}$$

The Jacobian and its inverse are

$$dx^i = T^i{}_j d\bar{x}^j, \tag{54}$$

$$F^i{}_k \equiv -\frac{1}{\rho r^3}(a^2\delta^i{}_j - a^i a_j), \tag{55}$$

$$T^i{}_j = P^i{}_j + F^i{}_k \bar{x}^k \bar{x}_j, \tag{56}$$

$$(G_1)^i{}_m \equiv \frac{1}{\rho^2 r}(a^2\delta^i{}_m - a^i a_m), \tag{57}$$

$$s \equiv r^2 + \frac{(\vec{a} \cdot \vec{x})^2}{r^2}, \tag{58}$$

$$(G_2)^n{}_j \equiv \frac{\rho^2}{sr}Q^n{}_j, \tag{59}$$

$$S^i{}_j = (T^{-1})^i{}_j = Q^i{}_j + (G_1)^i{}_m \bar{x}^m \bar{x}_n (G_2)^n{}_j. \tag{60}$$

3.2 Metric

Metrics, lapse and shift formulae are unchanged, but we copy them here for convenience.

$$g_{ij} = \bar{\eta}_{ij} + 2H\bar{l}_i\bar{l}_j, \tag{61}$$

$$\bar{\eta}_{ij} = \eta_{mn}T^m{}_iT^n{}_j, \tag{62}$$

$$H = \frac{r^3}{r^4 + (\vec{a}\cdot\vec{x})^2}, \tag{63}$$

$$l_i = l^i = \frac{r\vec{x} - \vec{a}\times\vec{x} + \frac{(\vec{a}\cdot\vec{x})\vec{a}}{r}}{\rho^2}, \tag{64}$$

$$\bar{l}_i = T^m{}_i l_m, \tag{65}$$

$$\bar{l}^i = S^i{}_m l^m, \tag{66}$$

$$\psi_{\mu\nu} = \bar{\eta}_{\mu\nu} + 2H\bar{l}_\mu\bar{l}_\nu, \tag{67}$$

$$\bar{l}_\mu = (1, \bar{l}_i), \tag{68}$$

$$\bar{l}^\mu = (-1, \bar{l}^i), \tag{69}$$

$$\bar{\eta}_{\mu\nu} = (-1) \otimes \bar{\eta}_{ij}, \tag{70}$$

$$\beta^i = \frac{2H\bar{l}^i}{1 + 2H} = 2H\alpha^2\bar{l}^i, \tag{71}$$

$$\beta_i = 2H\bar{l}_i, \tag{72}$$

$$\alpha = (1 + 2H)^{-1/2}. \tag{73}$$

3.3 Derivatives

Important derivatives are

$$D^i{}_j \equiv \frac{1}{\rho^3 r}(a^2\delta^i{}_m - a^i a_m), \tag{74}$$

$$C^i{}_m \equiv D^i{}_m - 3F^i{}_m = \frac{1}{\rho r}\left(\frac{1}{\rho^2} + \frac{3}{r^2}\right)(a^2\delta^i{}_m - a^i a_m), \tag{75}$$

$$\partial_k T^i{}_j = F^i{}_j\bar{x}_k + F^i{}_k\bar{x}_j + F^i{}_m\bar{x}^m\delta_{jk} + C^i{}_m\frac{\bar{x}_k\bar{x}^m\bar{x}_j}{r^2}, \tag{76}$$

$$(E_1)^i{}_m \equiv -\frac{1}{\rho^2}\left(\frac{1}{r^2} + \frac{2}{\rho^2}\right)(a^2\delta^i{}_m - a^i a_m), \tag{77}$$

$$(E_2)^n{}_j \equiv \left[-\frac{a^2}{\rho^2 r} - \frac{2}{s}\left(r - \frac{(\vec{a}\cdot\vec{x})^2}{r^3}\right)\right]\cdot(G_2)^n{}_j + \frac{1}{s}P^n{}_j, \tag{78}$$

$$\partial_k S^i{}_j = D^i{}_j\bar{x}_k + (G_1)^i{}_k\bar{x}_n(G_2)^n{}_j + (G_1)^i{}_m\bar{x}^m(G_2)_{kj}$$
$$+ (E_1)^i{}_m\frac{\bar{x}_k\bar{x}^m\bar{x}_n}{r}(G_2)^n{}_j + (G_1)^i{}_m\frac{\bar{x}_k\bar{x}^m\bar{x}_n}{r}(E_2)^n{}_j - (G_1)^i{}_m\bar{x}^m\bar{x}_n(G_2)^n{}_j\frac{2\vec{a}\cdot\vec{x}}{sr^2}a_k, \tag{79}$$

$$\frac{\partial r}{\partial x^i} = \frac{r^2 x_i + (\vec{a}\cdot\vec{x})a_i}{rs}, \tag{80}$$

$$\partial_i H = HT^m{}_i\left[\frac{3}{r}\frac{\partial r}{\partial x^m} - \frac{4r^3\frac{\partial r}{\partial x^m} + 2(\vec{a}\cdot\vec{x})a_m}{r^4 + (\vec{a}\cdot\vec{x})^2}\right], \tag{81}$$

$$\partial_j\bar{l}_i = T^k{}_i T^m{}_j\frac{1}{\rho^2}\left[\left(x_k - 2rl_k - \frac{(\vec{a}\cdot\vec{x})a_k}{r^2}\right)\frac{\partial r}{\partial x^m} + r\delta_{km} + \frac{a_k a_m}{r} - \epsilon^{kmn}a_n\right] + l_k\partial_j T^k{}_i, \tag{82}$$

$$\partial_k g_{ij} = 2\bar{l}_i\bar{l}_j\partial_k H + 4H\bar{l}_{(i}\partial_k\bar{l}_{j)} + T^m{}_j\partial_k T^m{}_i + T^m{}_i\partial_k T^m{}_j, \tag{83}$$

$$\partial_k\alpha = -(1 + 2H)^{-3/2}\partial_k H = -\alpha^3\partial_k H, \tag{84}$$

$$\partial_k\beta^i = 2\alpha^2\left[\bar{l}^i\partial_k H + H(S^i{}_j S^m{}_j\partial_k\bar{l}_m + S^i{}_j\bar{l}_m\partial_k S^m{}_j + S^m{}_j\bar{l}_m\partial_k S^i{}_j)\right] - 4H\bar{l}^i\alpha^4\partial_k H. \tag{85}$$

5