Introduction to Probability and Statistics
Fourteenth Edition

Chapter 4
Probability and Probability Distributions

4.1 EVENTS AND SAMPLE SPACE

WHAT IS PROBABILITY?

- In Chapters 2 and 3, we used graphs and numerical measures to describe data sets which were usually **samples.**
- We measured "how often" using

Relative frequency = *f/n*

Sample → Population

And "How often"
= Relative frequency → Probability

Ch 1, 2, 3	Ch 4, 5, 6
Sample	Population
Data (measurements of a variable): qualitative, quantitative (discrete, continuous)	Random variable: quantitative (discrete, continuous)
$x_1, \ldots, x_n$	X
Relative frequency	Probability
Graphs (histogram, boxplot, dot plot, stemplot)	Probability mass function (pmf), probability density function (pdf)

Basic Concepts

- An **experiment** is the process by which an observation (or measurement) is obtained.
- **Experiment: Record an age**
- **Experiment: Toss a die**
- **Experiment: Record an opinion (yes, no)**
- **Experiment: Toss two coins**

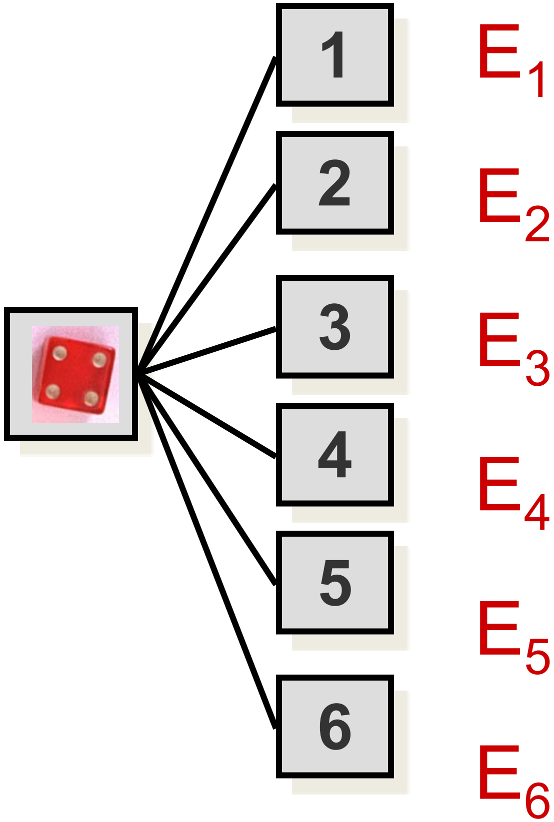
Basic Concepts

- A **simple event** is the outcome that is observed on a single repetition of the experiment.
  - The basic element to which probability is applied.
  - One and only one simple event can occur when the experiment is performed.

- A **simple event** is denoted by E with a subscript.

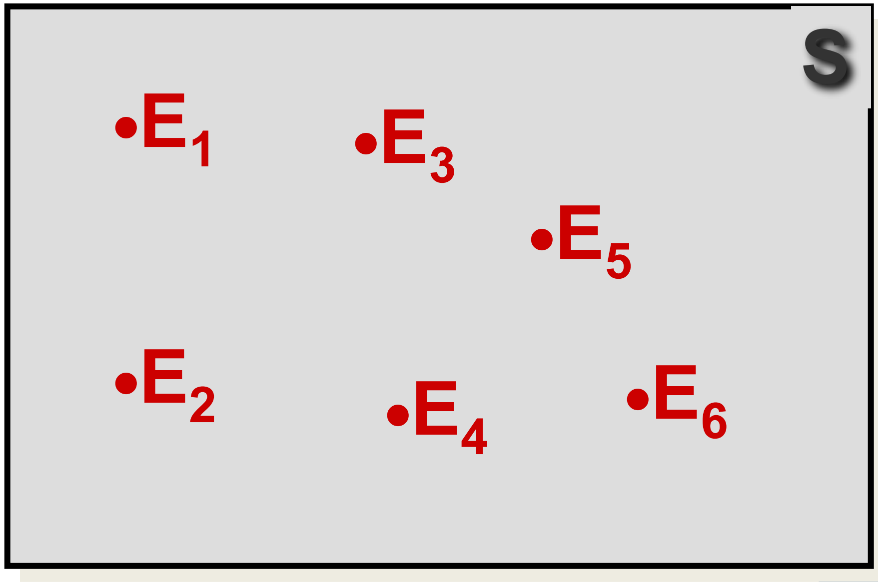
Example

- The die toss:
- Simple events:



Sample space:

$$S = \{E_1, E_2, E_3, E_4, E_5, E_6\}$$

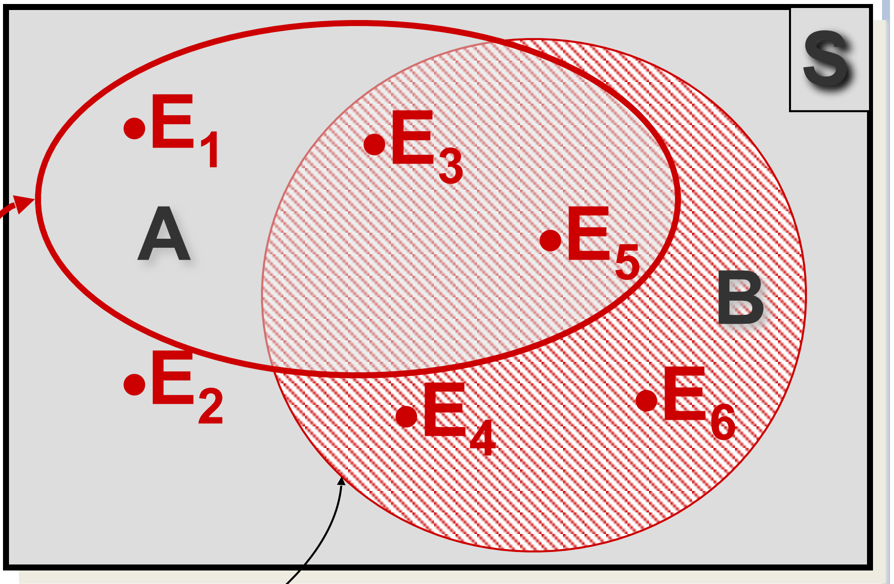


Basic Concepts

- Each simple event will be assigned a probability, measuring “how often” it occurs.
- The set of all simple events of an experiment is called the **sample space, S.**
- The set of no events of an experiment is called the **empty set, ϕ.**

Basic Concepts

- An **event** is a collection of one or more simple events.



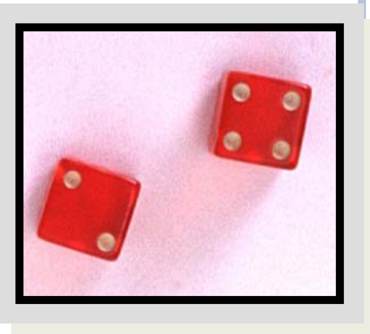
- **The die toss:**
  - A: an odd number
  - B: a number > 2

$A = \{E_1, E_3, E_5\}$

$B = \{E_3, E_4, E_5, E_6\}$

Basic Concepts

- Two events are **mutually exclusive** if, when one event occurs, the other cannot, and vice versa.



- **Experiment: Toss a die**
  - –A: observe an odd number
  - –B: observe a number greater than 2
  - –C: observe a 6
  - –D: observe a 3

$$\Omega = \{1,2,3,4,5,6\}$$
$$A = \{1,3,5\}$$
$$B = \{3,4,5,6\}$$
$$C = \{6\}$$
$$D = \{3\}$$

B and C?
C and D?

$B \cap C = \{6\}$. Not Mutually Exclusive

$C \cap D = \phi$. Mutually Exclusive

Some experiments can be generated in stages, and the sample space can be displayed in a **tree diagram**. Each successive level of branching on the tree corresponds to a step required to generate the final outcome.

EXAMPLE 4.4 A medical technician records a person's blood type and Rh factor. List the simple events in the experiment.

Solution For each person, a two-stage procedure is needed to record the two variables of interest. The tree diagram is shown in Figure 4.2. The eight simple events in the tree diagram form the sample space, $S = \{A+, A-, B+, B-, AB+, AB-, O+, O-\}$.

Figure 4.2
Tree diagram for Example 4.4

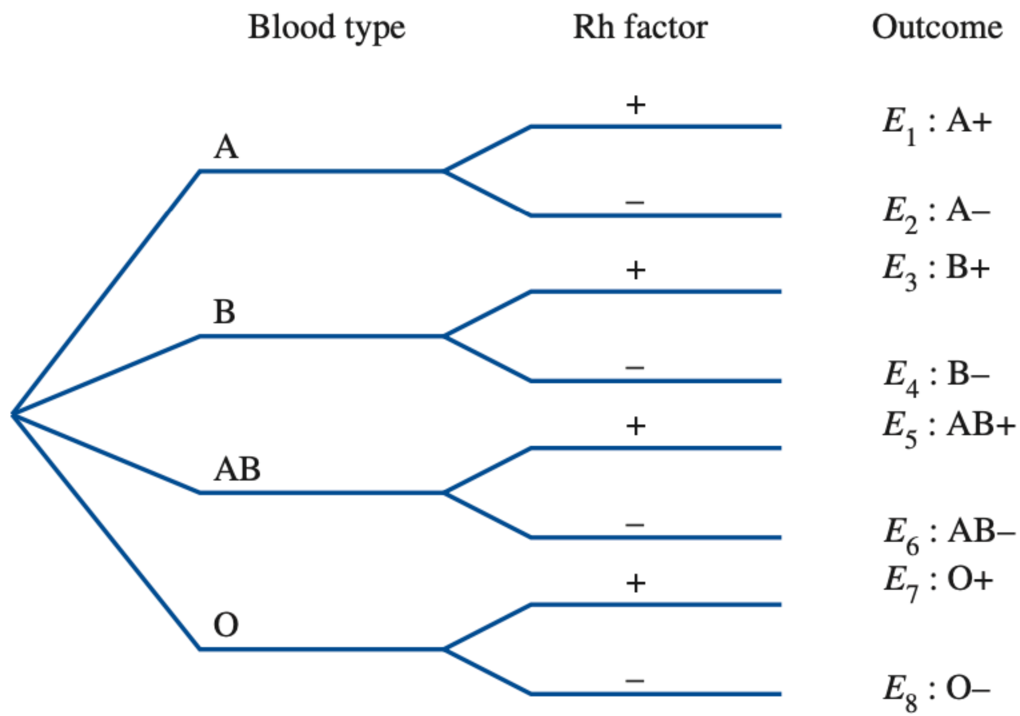
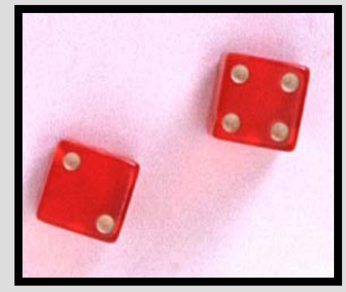


Table 4.1 Table of Outcomes for Example 4.4

	Blood Type			
Rh Factor	**A**	**B**	**AB**	**O**
Negative	A−	B−	AB−	O−
Positive	A+	B+	AB+	O+

4.2 Calculating probabilities using simple events

The Probability of an Event



- The probability of a simple event E measures "how often" we think A will occur. We write $P(E)$.
- Suppose that an experiment is performed n times. The relative frequency for a simple event E is

$$\frac{\text{Number of times A occurs}}{n} = \frac{f}{n}$$

• If we let n get infinitely large,

$$\boxed{P(E) = \lim_{n\to\infty} \frac{f}{n}}$$

Difficult to calculate in practice!

The Probability of an Event: Math Axioms

1. $P(A)$ must be between 0 and 1.

$$0 \leq P(A) \leq 1 \quad \leftarrow$$ Relative frequency!

2. The sum of the probabilities for all simple events in S equals 1.

$$P(S) = 1$$

3. The probability of an event A is found by adding the probabilities of all the simple events contained in A:

$$P(A) = P(E_1, \ldots, E_k) = P(E_1) + \cdots + P(E_k)$$

S.O.P to find a probability

1. Write down the sample space and simple events:
$$S = \{E_1, E_2, \ldots, E_N\}$$
2. Identify the event A using simple events to find the number of simple events in A, n_A
3. Then, $P(A) = \dfrac{n_A}{N}$

Note

- We denote an empty set by { } or ϕ.
- For an emtpy set, ϕ, $P(\phi) = 1$
- If event A can never occur, P(A) = 0.
- If event A always occurs when the experiment is performed, P(A) =1.

Counting Rules

Axiom 3: The **probability of an event** A is found by adding the probabilities of all the simple events contained in A.

If the simple events in an experiment are **equally likely**, you can calculate

$$P(A) = \frac{n_A}{N} = \frac{\text{number of simple events in A}}{\text{total number of simple events}}$$

You can use **counting rules in Ch 4.3** to find n_A and N.

Finding Probabilities

- Probabilities can be found using
  - Estimates from empirical studies
  - Common sense estimates based on equally likely events

- **Examples:**
  - Toss a fair coin.

    P(Head) = 1/2
  - 10% of the U.S. population has red hair. Select a person at random.

    P(Red hair) = .10

EXAMPLE

- Toss a fair coin twice. What is the probability of observing at least one head?

1st Coin	2nd Coin	E_i	$P(E_i)$
H	H	HH	1/4
	T	HT	1/4
T	H	TH	1/4
	T	TT	1/4

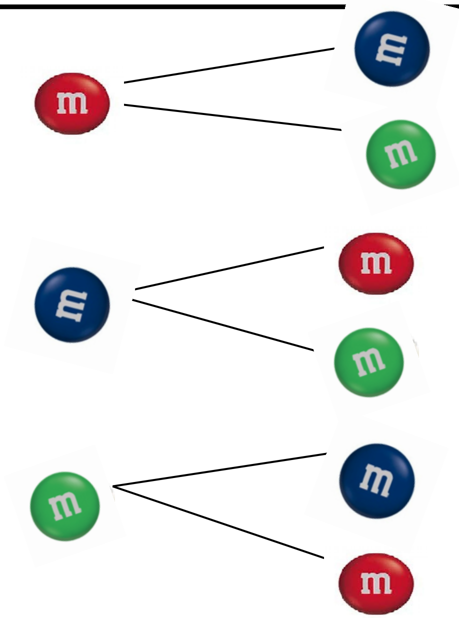
P(at least 1 head)

$= P(E_1) + P(E_2) + P(E_3)$

$= 1/4 + 1/4 + 1/4 = 3/4$

Example

- A bowl contains three M&Ms®, one red, one blue and one green. A child selects two M&Ms at random. What is the probability that at least one is red?



1st M&M	2nd M&M	E_i	$P(E_i)$
		RB	1/6
		RG	1/6
		BR	1/6
		BG	1/6
		GB	1/6
		GR	1/6

P(at least 1 red)

= P(RB) + P(BR)+ P(RG) + P(GR)

= 4/6 = 2/3

Example 4.6

The proportions of blood types A, B, AB, and O in the population of all Caucasians in the United States are reported as .40, .11, .04, and .45, respectively.[1] If a single Caucasian is chosen randomly from the population, what is the probability that he or she will have either type A or type AB blood?

Solution The four simple events, A, B, AB, and O, do *not* have equally likely probabilities. Their probabilities are found using the relative frequency concept as

$$P(\mathrm{A}) = .40 \quad P(\mathrm{B}) = .11 \quad P(\mathrm{AB}) = .04 \quad P(\mathrm{O}) = .45$$

The event of interest consists of two simple events, so

$$\begin{aligned} P(\text{person is either type A or type AB}) &= P(\mathrm{A}) + P(\mathrm{AB}) \\ &= .40 + .04 = .44 \end{aligned}$$

4.3 Useful Counting Rules

Why counting rules?

- Assume simple events in an experiment are **equally likely:**

$$P(A) = \frac{n_A}{N} = \frac{\text{number of simple events in A}}{\text{total number of simple events}}$$

S.O.P

1. List the Sample Space and simple events
   $\Omega = \{E_1, E_2, \ldots, E_N\}$
2. Find A as a collection of $E_{(1)}, E_{(2)}, \ldots, E_{(n_A)}$.

You can use **counting rules** to find n_A and N.

1. The *mn* Rule

- If an experiment is performed in two stages, with m ways to accomplish the first stage and n ways to accomplish the second stage, then there are mn ways to accomplish the experiment.
- This rule is easily extended to k stages, with the number of ways equal to

$$n_1 n_2 n_3 \ldots n_k$$

Example: Toss two coins. The total number of simple events is:

$$2 \times 2 = 4$$

EXAMPLES

Example: Toss three coins. The total number of simple events is:

$2 \times 2 \times 2 = 8$

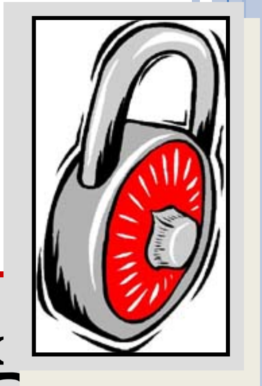
Example: Toss two dice. The total number of simple events is:

$6 \times 6 = 36$

Example: Two M&Ms are drawn from a dish containing two red and two blue candies. The total number of simple events is:

$4 \times 3 = 12$

2. PERMUTATIONS



- The number of ways you can arrange ***n*** distinct objects, taking them ***r*** at a time is $P_r^n = \dfrac{n!}{(n-r)!}$

  where $n! = n(n-1)(n-2)\ldots(2)(1)$ and $0! \equiv 1$.

Why?

Taking *r* items

$$P_r^n = n \times (n-1) \cdots \times (n-r+1)$$

$$= n \times (n-1) \cdots \times (n-r+1) \frac{(n-r)(n-r-1) \times \cdots 1}{(n-r)(n-r-1) \times \cdots 1}$$

$$= \frac{n!}{(n-r)!}$$

EXAMPLE

How many 3-digit lock combinations can we make from the numbers 1, 2, 3, and 4?

The order of the choice is important!

$$P_3^4 = \frac{4!}{1!} = 4(3)(2) = 24$$

EXAMPLES

Example: A lock consists of five parts and can be assembled in any order. A quality control engineer wants to test each order for efficiency of assembly. How many orders are there?

The order of the choice is important!

$$P_5^5 = \frac{5!}{0!} = 5(4)(3)(2)(1) = 120$$

3. COMBINATIONS

- The number of distinct combinations of ***n*** distinct objects that can be formed, taking them ***r*** at a time is

$$C_r^n = \frac{n!}{r!(n-r)!}$$

Why?

$$C_r^n = \frac{P_r^n}{r!} = \frac{n!}{(n-r)!r!}$$

Example

Three members of a 5-person committee must be chosen to form a subcommittee. How many different subcommittees could be formed?

The order of the choice is not important!

$$C_3^5 = \frac{5!}{3!(5-3)!} = \frac{5(4)(3)(2)1}{3(2)(1)(2)1} = \frac{5(4)}{(2)1} = 10$$

Example

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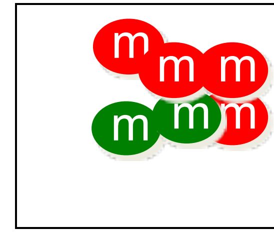


- A box contains six M&Ms®, four red
- and two green. A child selects two M&Ms at random. What is the probability that exactly one is red?

The order of the choice is not important!

$$C_2^6 = \frac{6!}{2!4!} = \frac{6(5)}{2(1)} = 15$$

ways to choose 2 M & Ms.

$$C_1^2 = \frac{2!}{1!1!} = 2$$

ways to choose 1 green M & M.

$$C_1^4 = \frac{4!}{1!3!} = 4$$

ways to choose 1 red M & M.

4 × 2 =8 ways to choose 1 red and 1 green M&M.

P(exactly one red) = 8/15

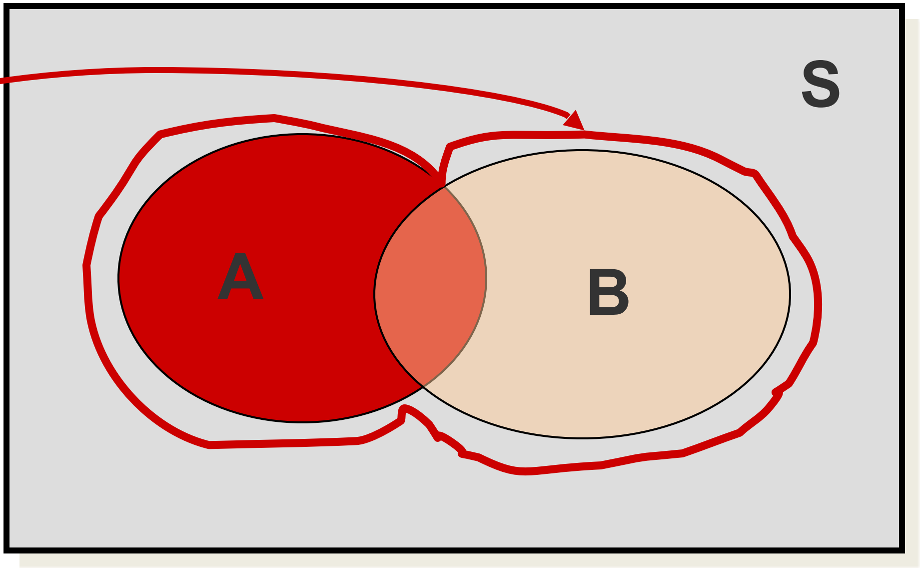
4.4 Rules for Calculating Probabilities

1. Union of two events

- The **union** of two events, A and B, is the event that either A **or** B **or both** occur when the experiment is performed. We write

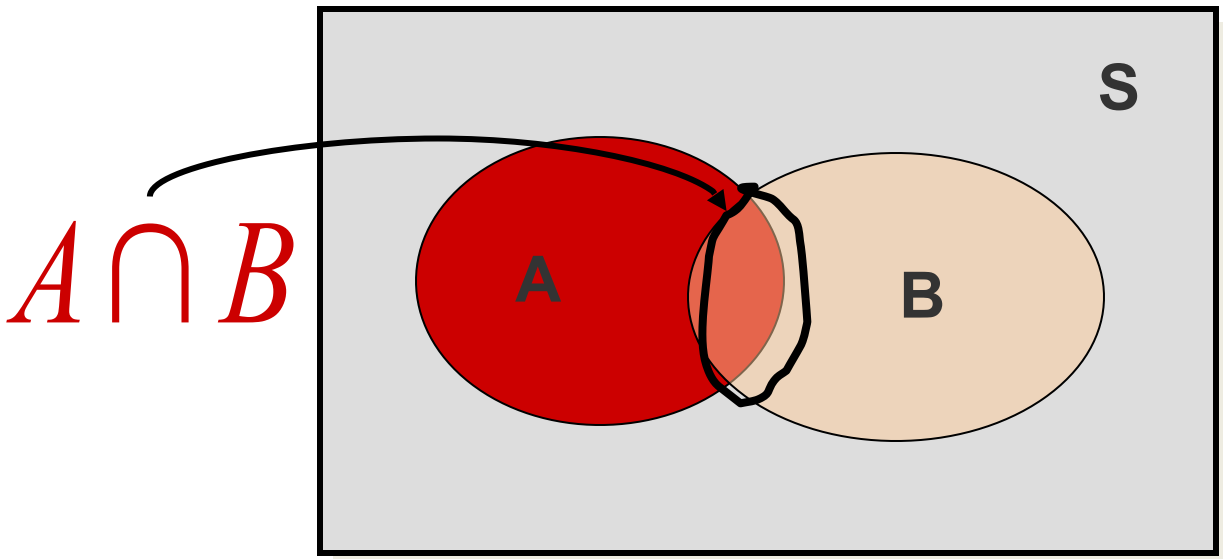
$$A \cup B$$

$A \cup B$



2. Intersection of two events

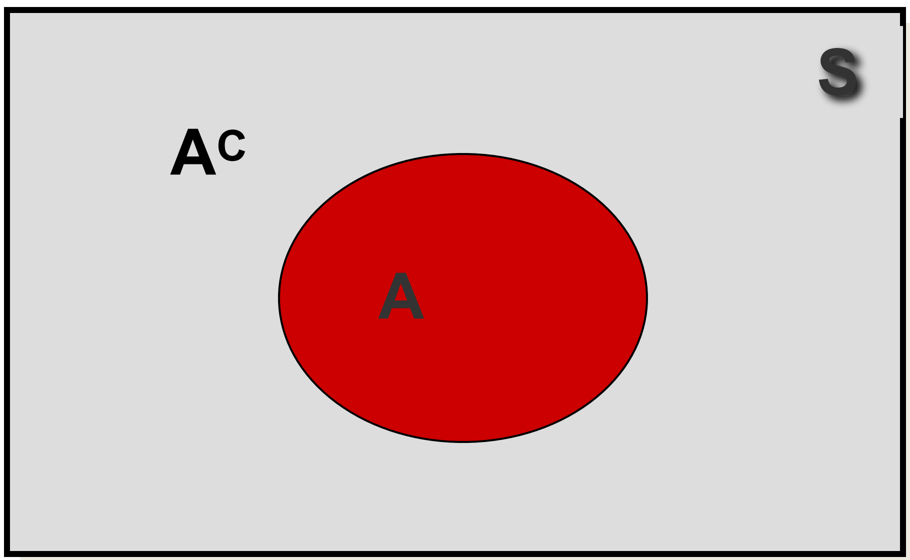
- The **intersection** of two events, **A** and **B**, is the event that both A **and** B occur when the experiment is performed. We write **A ∩ B**.



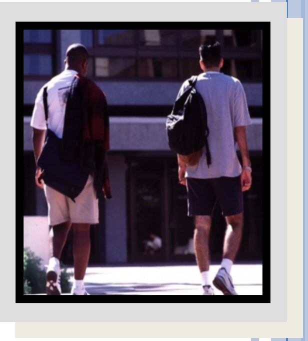
- If two events A and B are **mutually exclusive, we write** $A \cap B = \phi$, then P(**A** ∩ **B) = 0.**

3. Compliment of an event

- The **complement** of an event **A** consists of all outcomes of the experiment that do not result in event A. We write $\mathbf{A}^{C}$.



Example



- Select a student from the classroom and record his/her **hair color** and **gender**.
  - **A:** student has brown hair
  - **B:** student is female
  - **C:** student is male

Mutually exclusive; B = C^C

- What is the relationship between events **B** and **C**?
- $\mathbf{A^C}$:

Student does not have brown hair

- **B∩C:**

Student is both male and female: $B \cap C = \phi$

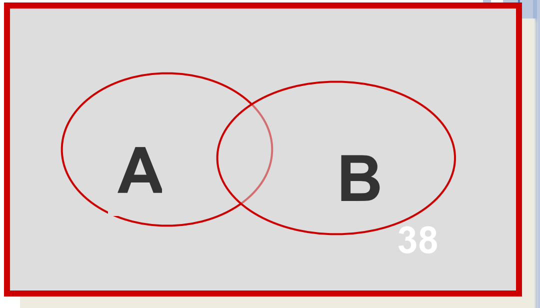
- **B∪C:**

Student is either male and female = all students: $B \cup C = S$

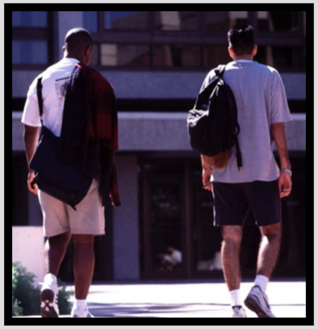
1. The Additive Rule for Unions

- There are special rules that will allow you to calculate probabilities for composite events.
- **The Additive Rule for Unions:**
- For any two events, **A** and **B**, the probability of their union, **P(A ∪ B)**, is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Example: Additive Rule



Example: Suppose that there were students in the classroom, and that they could be classified as follows:

A: brown hair

P(A) = 50/120

B: female

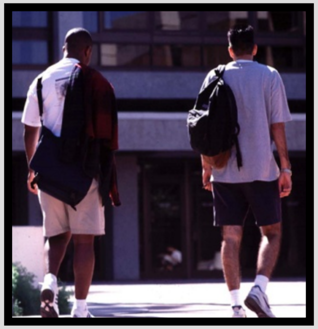
P(B) = 60/120

	Brown	Not Brown
Male	20	40
Female	30	30

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 50/120 + 60/120 - 30/120$
$= 80/120 = 2/3$

Check: $P(A \cup B)$
$= (20 + 30 + 30)/120$

A Special Case



When two events A and B are **mutually exclusive,** $A \cap B = \phi$, P(A∩B) = 0 and **P(A∪B) = P(A) + P(B).**

	Brown	Not Brown
Male	20	40
Female	30	30

A: male with brown hair
P(A) = 20/120
B: female with brown hair
P(B) = 30/120

A and B are mutually exclusive, so that

P(A∪B) = P(A) + P(B)
= 20/120 + 30/120
= 50/120

2. The Multiplicative Rule for Intersections

- In the previous example, we found $P(A \cap B)$ directly from the table. Sometimes this is impractical or impossible.
- The rule for calculating $P(A \cap B)$ depends on the idea of **conditional probabilities.**

CONDITIONAL PROBABILITIES

- The probability that A occurs, given that event B has occurred is called the **conditional probability** of A given B and is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) \neq 0$$

"given"

Original Definition of Independence

- Two events, **A** and **B**, are said to be **independent** *if and only if* the probability that event **A** occurs does not change, depending on whether or not event **B** has occurred. And vice versa.

1. $P(A|B)=P(A)$
2. $P(B|A)=P(B)$

EXAMPLE

- Toss a fair coin twice. Define
  - A: head on second toss
  - B: head on first toss

P(A|B) = $\frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2}$ = ½

P(A|not B) $= \frac{P(A \cap B^c)}{P(B^c)} = \frac{\frac{1}{4}}{\frac{1}{2}} =$ = ½

HH	1/4
HT	1/4
TH	1/4
TT	1/4

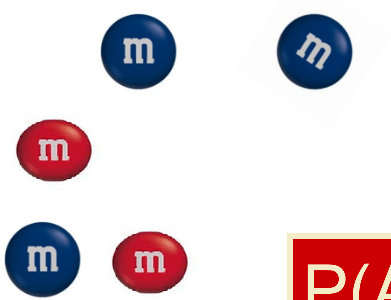
P(A) does not change, whether B happens or not… → A and B are independent!

Example

Skipped!

- A bowl contains five M&Ms®, two red and three blue. Randomly select two candies, and define
  - A: second candy is red.

  P(A) = P(BR)+P(RR) =3x2/6x5+2x1/6x5 = 8/30 = 4/15
  - B: first candy is blue.



$P(A|B) = P(2^{nd} \text{ red}|1^{st} \text{ blue}) = 2/4 = 1/2$

$P(A|\text{not } B) = P(2^{nd} \text{ red}|1^{st} \text{ red}) = 1/4$

P(A) does change, depending on whether B happens or not…

A and B are dependent (not independent)!

The Multiplicative Rule for Intersections

- Recall $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$
- For any two events, **A** and **B**, the probability that both **A** and **B** occur is

$$\boxed{P(A \cap B) = P(A)P(B|A)}$$

Example

Suppose we have additional information in the previous example. We know that only 49% of the population are female. Also, of the female patients, 8% are high risk. A single person is selected at random. What is the probability that it is a high risk female?

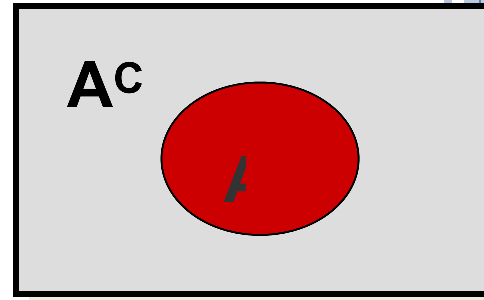
Define H: high risk F: female

From the example, $P(F) = .49$ and $P(H|F) = .08$. Use the Multiplicative Rule:

$$P(\text{high risk female}) = P(H \cap F)$$

$$= P(F)P(H|F) = .49(.08) = .0392$$

3. Probabilities for Complements



- We know that for any event **A:**

$$A \cap A^c = \phi, \ P(A \cap A^C) = 0$$

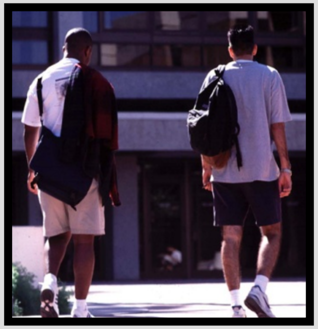
- Since either **A** or **A^C** must occur,

$$A \cup A^c = S, \ P(A \cup A^C) = 1$$

- Herefore, **$P(A \cup A^C) = P(A) + P(A^C) = 1$**

$$\boxed{P(A^C) = 1 - P(A)}$$

Example



Select a student at random from the classroom. Define:

A: male

$P(A) = 60/120$

B: female

	Brown	Not Brown
Male	20	40
Female	30	30

A and B are complementary, so that

$P(B)$
$= 1 - P(A)$
$= 1 - 60/120 = 60/120$

Defining Independence

- We can redefine independence in terms of conditional probabilities:

Two events A and B are **independent** if and only if

$$P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B)$$

Otherwise, they are **dependent**.

DEFINITION OF INDEPENDENCE

1. If $P(A|B) = P(A)$, then $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = P(A)$

Thus, $P(A \cap B) = P(A)P(B)$.

2. If $P(B|A) = P(B)$, then $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = P(B)$

Thus, $P(B \cap A) = P(A)P(B)$.

DEFINITION OF INDEPENDENCE

- A and B are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

- A_1, A_2, A_3 are mutually independent if and only if

$$P(A_1 \cap A_2) = P(A_1)P(A_2)$$

$$P(A_1 \cap A_3) = P(A_1)P(A_3)$$

$$P(A_2 \cap A_3) = P(A_2)P(A_3)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

The Multiplicative Rule for Intersections: A special case

If the events **A** and **B** are independent, then the probability that both **A** and **B** occur is

$$\boxed{P(A \cap B) = P(A)\,P(B)}$$

EXAMPLE 1

In a certain population, 10% of the people can be classified as being high risk for a heart attack. Three people are randomly selected from this population. What is the probability that exactly one of the three are high risk? Assume mutually independence.

Define H: high risk N: not high risk

P(exactly one high risk)

=P(HNN, NHN, NNH)

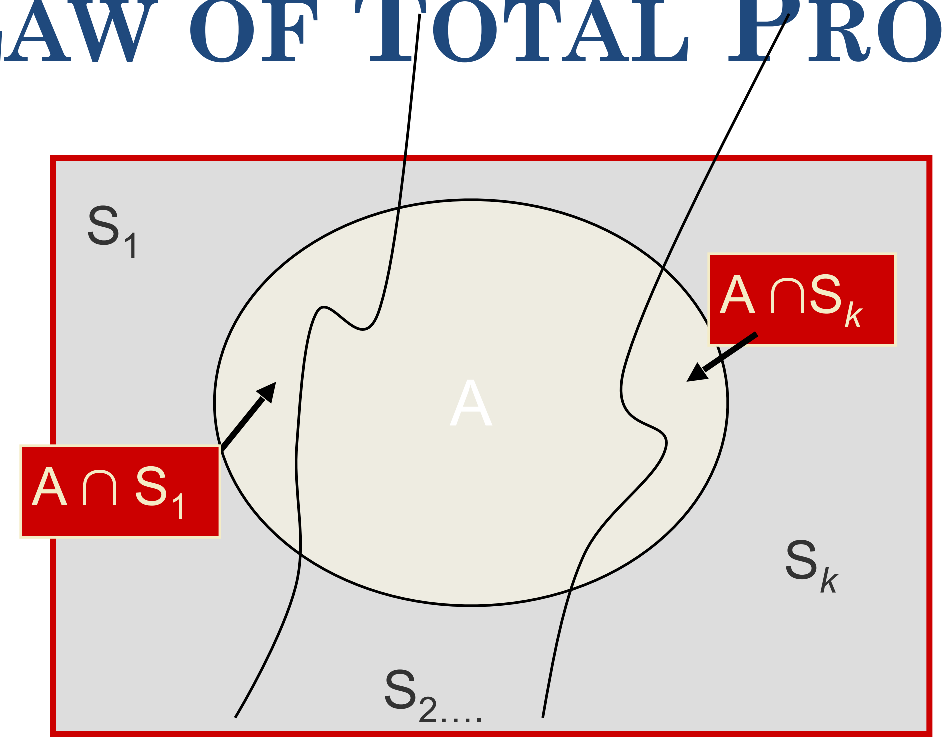
= P(HNN) + P(NHN) + P(NNH)

= P(H)P(N)P(N) + P(N)P(H)P(N) + P(N)P(N)P(H)

= $(.1)(.9)(.9) + (.9)(.1)(.9) + (.9)(.9)(.1) = 3(.1)(.9)^2 = .243$

4.5 Bayes Rules

The Law of Total Probability



- Let S_1 , S_2 , S_3 ,...., S_k be mutually exclusive and exhaustive events (that is, one and only one must happen).

The Law of Total Probability

- Then the probability of another event A can be written as

$$
\begin{aligned}
& P(A) \\
&= P(A \cap S_1) + P(A \cap S_2) + \cdots + P(A \cap S_k) \\
&= P(S_1)P(A \mid S_1) + P(S_2)P(A \mid S_2) + \cdots + P(S_k)P(A \mid S_k) \\
&= \sum P(S_i)P(A \mid S_k)
\end{aligned}
$$

BAYES' RULE

- Let S_1 , S_2 , S_3 ,…., S_k be mutually exclusive and exhaustive events with prior probabilities $P(S_1)$, $P(S_2)$,…,$P(S_k)$.
- If an event A occurs, the posterior probability of S_i, given that A occurred is
- $$P(S_i \mid A) = \frac{P(S_i)P(A \mid S_i)}{\sum_{j=1}^{k} P(S_j)P(A \mid S_j)}, \quad \text{for} \quad i = 1,\ldots,k$$

EXAMPLE

From a previous example, we know that 49% of the population are female. Of the female patients, 8% are high risk for heart attack, while 12% of the male patients are high risk. A single person is selected at random and found to be high risk. What is the probability that it is a male?

Define {H: high risk L: Low Risk}, {F: female M: male}

We know:

P(F) =	.49
P(M) =	.51
P(H|F) =	.08
P(H|M) =	.12

$$P(M \mid H) = \frac{P(M)P(H \mid M)}{P(M)P(H \mid M) + P(F)P(H \mid F)}$$

$$= \frac{.51(.12)}{.51(.12) + .49(.08)} = .61$$

Key Concepts

I. Experiments and the Sample Space

1. Experiments, events, mutually exclusive events, simple events
2. The sample space
3. Venn diagrams, tree diagrams, probability tables

II. Probabilities

1. Relative frequency definition of probability
2. Properties of probabilities
    a. Each probability lies between 0 and 1.
    b. Sum of all simple-event probabilities equals 1.
3. $P(A)$, the sum of the probabilities for all simple events in A

Key Concepts

III. Counting Rules

1. *mn* Rule; extended *mn* Rule
2. Permutations: $P_r^n = \dfrac{n!}{(n-r)!}$
3. Combinations: $C_r^n = \dfrac{n!}{r!(n-r)!}$

IV. Event Relations

1. Unions and intersections
2. Events
   a. Disjoint or mutually exclusive: $P(A \cap B) = 0$
   b. Complementary: $P(A) = 1 - P(A^C)$

Key Concepts

3. Conditional probability: $\boxed{P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}}$
4. Independent and dependent events
5. Additive Rule of Probability:

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

6. Multiplicative Rule of Probability:

$$\boxed{P(A \cap B) = P(A)P(B \mid A)}$$

7. Law of Total Probability
8. Bayes' Rule

KEY CONCEPTS

V. Discrete Random Variables and Probability Distributions

1. Random variables, discrete and continuous
2. Properties of probability distributions

$$0 \le p(x) \le 1 \text{ and } \sum p(x) = 1$$

3. Mean or expected value of a discrete random variable: $\text{Mean}: \mu = \sum xp(x)$
4. Variance and standard deviation of a discrete random v

$$\text{Variance}: \sigma^2 = \sum (x-\mu)^2 p(x)$$

$$\text{Standard deviation}: \sigma = \sqrt{\sigma^2}$$