ESE 2180 Project Two: Least-Squares Image Classification

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1 Project Information

This project will explore least squares methods for classification. The project is due on **November 10**. You may work in groups of up to **two**. You will turn in a lab report and source code.

2 Least Squares Classification

Classification is one of the fundamental problems in data science. The goal of classification is to correctly identify the "class" of a particular object. This could include identifying objects in an image or video frame, identifying the speaker of a particular audio file, or categorizing the type of music in an audio recording.

Mathematically, the classification problem can be stated as follows. Let $\mathcal{X}$ denote the set of possible inputs, and let $\mathcal{C}$ denote the set of classes. Let $f : \mathcal{X} \rightarrow \mathcal{C}$ be a function that maps each input to its class. The goal of classification is to construct a function $\tilde{f} : \mathcal{X} \rightarrow \mathcal{C}$ such that $\tilde{f}$ is "close" to f in some sense based on a set of training data $\{(x_1, y_1), \ldots, (x_N, y_N)\}$, where $x_i \in \mathcal{X}$ and $y_i = f(x_i)$.

In this project, we will explore a simple type of classifier, namely, the least squares classifier. Our discussion follows that in [1, Ch. 14]. For simplicity, we assume that $\mathcal{C} = \{-1, 1\}$, i.e., there are two possible labels. The idea of the least squares classifier is to choose

$$\tilde{f}(x) = \mathrm{sign}(\theta_1 f_1(x) + \cdots + \theta_M f_M(x)) \tag{1}$$

where $f_i : \mathcal{X} \rightarrow \mathbb{R}$ is a *feature* of the input x, $\theta_1, \ldots, \theta_M$ are coefficients, and the function sign is defined by

$$\mathrm{sign}(z) = \begin{cases} 1, & z > 0 \\ 0, & z = 0 \\ -1, & z < 0 \end{cases}$$

The parameters θ are chosen by solving a least-squares problem

$$\text{minimize} \quad ||A\theta - y||_2 \tag{2}$$

where matrix $A \in \mathbb{R}^{N\times M}$ has $A_{ij} = f_j(x_i)$.

Clearly, the effectiveness of this approach will depend on the choice of features $f_1, \ldots, f_M$. As part of the project, you will explore how the choice of features impacts the classification accuracy.

The accuracy of the classifier is evaluated using test data. The test data consists of a collection of samples $\{(\hat{x}_1, \hat{y}_1), \ldots, (\hat{x}_R, \hat{y}_R)\}$ with $\hat{x}_i \in \mathcal{X}$, $\hat{y}_i \in \mathcal{C}$, and $\hat{y}_i = f(\hat{x}_i)$. We say that the learned classifier $\tilde{f}(x)$ has a classification error on sample $\hat{x}_i$ if $\tilde{f}(\hat{x}_i) \neq \hat{y}_i$. The *error rate* is the number of test samples with classification errors divided by the total number of test samples. The *false positive rate* is the number of test samples with $\tilde{f}(\hat{x}_i) = 1$ and $\hat{y}_i = -1$ divided by the number of test samples with $\hat{y}_i = -1$. The false negative rate is the number of samples with $\tilde{f}(\hat{x}_i) = -1$ and $\hat{y}_i = 1$ divided by the number of test samples with $\hat{y}_i = 1$.

3 Least Squares for Image Classification

We consider the problem of classifying images. An image can be represented as a vector $x \in \mathbb{R}^n$, where n is the number of pixels in the image and x_i represents the color intensity in the i-th pixel of the image. The class is a description of what is in the image.

In this project, you will work with the MNIST dataset [2]. MNIST is a collection of images, each of which contains a handwritten digit, i.e., '0', '1', ..., '9'. We will focus on determining whether each image is a 0 (corresponding to class 1) or one of the digits 1-9 (corresponding to class -1). The dataset contains four files:

- "train-images-idx3-ubyte": This file contains the images in the *training* dataset.
- "train-labels-idx1-ubyte": This file contains the labels in the *training* dataset.
- "t10k-images-idx3-ubyte": This file contains the images in the *test* dataset.
- "t10k-labels-idx1-ubyte": This file contains the images in the *test* dataset.

You are provided a Matlab file "readMNIST" that can read the contents of these files to Matlab arrays. The command

```
[imgs, labels] = readMNIST('train-images-idx3-ubyte', 'train-labels-idx1-ubyte',...
 num_imgs,0);
```

reads the first `num_imgs` entries in the files 'train-images-idx3-ubyte" and "train-labels-idx1-ubyte" into the arrays `imgs` and `labels`, respectively. `images` is a three-dimensional array, with `images(i,j,k)` equal to the intensity of the pixel in the i-th row and j-th column of image k. `labels` is a one-dimensional array with `labels(k)` equal to the type of digit (from 0-9) in the k-th image. A similar command can be used to extract arrays of images and labels from the test dataset files.

The features in the image classification problem are equal to the pixel intensities that are nonzero in at least 600 images.

4 Project Assignment

Please complete the following steps:

1. Download the training and test datasets. Use the provided Matlab function to load the first 5000 images and labels of the training dataset into Matlab.

2. Identify the row and column indices of the pixels that have nonzero intensities in at least 600 of the training images. These will be the features used for classification.

3. Construct the matrix A and vector y as in Eq. (2). Solve the least-square problem to obtain the parameter vector θ. Create a plot showing the values of the entries of θ at different pixel locations.

4. Load the first 5000 images of the test dataset. Using these images, compute the error rate, false positive rate, and false negative rate of your classifier.

5. Repeat steps 1-4, except use only the first 100 images and labels of the training dataset to train the classifier. Use the same features that you identified in Step 2. How did the error rate and false positive/negative rates change?

6. In what follows, we will experiment with changing the feature set. Let M_0 denote the number of features that you identified in Step 2, and let $M = 5000$. Choose a matrix $R \in \mathbb{R}^{M \times M_0}$ whose entries are in $\{1, -1\}$ with equal probability (This can be done using the Matlab command `R = sign(randn(M,M0))`). Letting $x_i \in \mathbb{R}^{M_0}$ denote the vector of features from Step 2 for the i-th image, the new feature vector is given by $\max\{Rx_i, 0\}$. Train a least-squares classifier using this new set of features.

7. Compute the error rate, false positive rate, and false negative rate for the new classifier that you constructed in the previous step. Repeat the process with $M = 20, 50, 1000, 5000, 10000$.

References

[1] S. Boyd and L. Vandenberghe, *Introduction to applied linear algebra: vectors, matrices, and least squares.* Cambridge university press, 2018.

[2] Y. LeCun, "The MNIST dataset of handwritten digits."