Assignment 4

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I use MATLAB to calculate the result. Here is the result and code:

1. Use the formula in textbook P.264:

$$\frac{d}{dx}P_n(s) = \frac{1}{h}\left[\Delta f_i + \sum_{j=2}^{n}\left\{\sum_{k=0}^{j-1}\prod_{\substack{l=0\\l\neq k}}^{j-1}(s-l)\right\}\frac{\Delta^j f_i}{j!}\right], \qquad s = \frac{x-x_i}{h}$$

```
function result = P(x, start_i, order, table, h)
    s = (x - table(start_i, 1))/h;
    result = table(start_i, 3);
    for j = 2:order
        sum = 0;
        for k = 0:j-1
            product = 1;
            for l = 0:j-1
                if(l ~= k)
                    product = product * (s - l);
                end
            end
            sum = sum + product;
        end
        sum = sum * table(start_i, j+2)/factorial(j);
        result = result + sum;
    end
    result = result / h;
end
```

```
table = [0.30 0.3985 0.2613 -0.0064 -0.0022 0.0003;
         0.50 0.6598 0.2549 -0.0086 -0.0018 0.0004;
         0.70 0.9147 0.2464 -0.0104 -0.0014 0.0005;
         0.90 1.1611 0.2360 -0.0118 -0.0010      0;
         1.10 1.3971 0.2241 -0.0128       0      0;
         1.30 1.6212 0.2113       0       0      0;
         1.50 1.8325      0       0       0      0];
h = 0.2;

% a
x1 = 0.72;
i1 = 2; % 0.5, let x1 in the middle of points (0.5, 0.7, 0.9)
ans1 = P(x1, i1, 3, table, h);
fprintf('a: %.8f\n', ans1);

% b
x2 = 1.33;
i2 = 5; % 1.10, let x2 in the middle of point (1.1, 1.3)
ans2 = P(x2, i2, 2, table, h);
fprintf('b: %.8f\n', ans2);

%c
x3 = 0.50;
i3 = 2; % x3 equal to 0.5, let s be 0
ans3 = P(x3, i3, 4, table, h);
fprintf('c: %.8f\n', ans3);
```

a. To make x = 0.72 close to the middle, we choose i = 1, so we will use x = 0.5, 0.7, 0.9 in the table to calculate the $f'(0.72) = P'_3(\frac{0.72-0.5}{h})$.

```
a: 1.25015500
```

b. To make x = 1.33 close to the middle, we choose i = 5. However, the $\Delta^2 f_5$ that we need is not given. Therefore, we choose i = 4 to calculate $f'(1.33) = P'_2(\frac{1.33-1.10}{h})$.

```
b: 1.07890000
```

c. Since x = 0.5 is in the table, so we choose i = 1 to calculate $f'(0.5) = P'_2(0)$.

```
c: 1.29250000
```

2. Solve the $f'(x_0)$ first:

$$x_n = x_0 + nh \Rightarrow f(x_n) = f_n \,,\quad P(nh) = f_n \,,\quad f^{(k)}(x_0) = C_{-2}f_{-2} + C_{-1}f_{-1} + C_0 f_0 + C_1 f_1 + C_2 f_2$$

Case 1: $P(u) = 1$

$$f_{-2} = f_{-1} = f_0 = f_1 = f_2 = P(u) = 1$$

$$① \; f^{(k)}(x_0) = C_{-2} + C_{-1} + C_0 + C_1 + C_2 = P^{(k)}(u)$$

Case 2: $P(u) = u$

$$f_{-2} = P(-2h) = -2h,\; f_{-1} = P(-h) = -h,\; f_0 = P(0) = 0,\; f_1 = P(h) = h,\; f_2 = P(2h) = 2h$$

$$② \; f^{(k)}(x_0) = -2hC_{-2} - hC_{-1} + 0\cdot C_0 + hC_1 + 2hC_2 = P^{(k)}(u)$$

Case 3: $P(u) = u^2$

$$f_{-2} = P(-2h) = 4h^2,\; f_{-1} = P(-h) = h^2,\; f_0 = P(0) = 0,\; f_1 = P(h) = h^2,\; f_2 = P(2h) = 4h^2$$

$$③ \; f^{(k)}(x_0) = 4h^2 C_{-2} + h^2 C_{-1} + 0\cdot C_0 + h^2 C_1 + 4h^2 C_2 = P^{(k)}(u)$$

Case 4: $P(u) = u^3$

$$f_{-2} = P(-2h) = -8h^3,\; f_{-1} = P(-h) = -h^3,\; f_0 = P(0) = 0,\; f_1 = P(h) = h^3,\; f_2 = P(2h) = 8h^3$$

$$④ \; f^{(k)}(x_0) = -8h^3 C_{-2} - h^3 C_{-1} + 0\cdot C_0 + h^3 C_1 + 8h^3 C_2 = P^{(k)}(u)$$

Case 5: $P(u) = u^4$

$$f_{-2} = P(-2h) = 16h^4,\; f_{-1} = P(-h) = h^4,\; f_0 = P(0) = 0,\; f_1 = P(h) = h^4,\; f_2 = P(2h) = 16h^4$$

$$⑤ \; f^{(k)}(x_0) = 16h^3 C_{-2} + h^3 C_{-1} + 0\cdot C_0 + h^3 C_1 + 16h^3 C_2 = P^{(k)}(u)$$

$k=2$
① $P''(u) = 0 \Rightarrow f''(x_0) = 0$
② $P''(u) = 0 \Rightarrow f''(x_0) = 0$
③ $P''(u) = 2 \Rightarrow f''(x_0) = 2$
④ $P''(u) = 6u \overset{u=0}{=} 0 \Rightarrow f''(x_0) = 0$
⑤ $P''(u) = 12u^2 \overset{u=0}{=} 0 \Rightarrow f''(x_0) = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow f''(x_0) = \frac{1}{h^2}\left(-\frac{1}{12}f_{-2} + \frac{4}{3}f_{-1} - \frac{5}{2}f_0 + \frac{4}{3}f_1 - \frac{1}{12}f_2\right)$$

$k=3$
① $P^{(3)}(u) = 0 \Rightarrow f^{(3)}(x_0) = 0$
② $P^{(3)}(u) = 0 \Rightarrow f^{(3)}(x_0) = 0$
③ $P^{(3)}(u) = 0 \Rightarrow f^{(3)}(x_0) = 0$
④ $P^{(3)}(u) = 6 \Rightarrow f^{(3)}(x_0) = 6$
⑤ $P^{(3)}(u) = 24u \overset{u=0}{=} 0 \Rightarrow f^{(3)}(x_0) = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 0 \end{bmatrix}$$

$$\Rightarrow f^{(3)}(x_0) = \frac{1}{h^3}\left(-\frac{1}{2}f_{-2} + f_{-1} - f_1 + \frac{1}{2}f_2\right)$$

$k=4$
① $P(u) = 0 \Rightarrow f^{(4)}(x_0) = 0$
② $P(u) = 0 \Rightarrow f^{(4)}(x_0) = 0$
③ $P(u) = 0 \Rightarrow f^{(4)}(x_0) = 0$
④ $P(u) = 0 \Rightarrow f^{(4)}(x_0) = 0$
⑤ $P(u) = 24 \Rightarrow f^{(4)}(x_0) = 24$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 24 \end{bmatrix}$$

$$\Rightarrow f^{(4)}(x_0) = \frac{1}{h^4}\left(f_{-2} - 4f_{-1} + 6f_0 - 4f_1 + f_2\right)$$

Then, use Taylor expansion to find error term:

Find error term by Taylor expansion.

$$f(x+ih) = f(x) + (ih)f'(x) + \frac{(ih)^2}{2!}f''(x) + \frac{(ih)^3}{3!}f^{(3)}(x) + \frac{(ih)^4}{4!}f^{(4)}(x) + \frac{(ih)^5}{5!}f^{(5)}(x)\ldots$$

For $f''(x_0) = \frac{1}{h^2}\left(-\frac{1}{12}f_{-2} + \frac{4}{3}f_{-1} - \frac{5}{2}f_0 + \frac{4}{3}f_1 - \frac{1}{12}f_2\right)$

$$= \frac{1}{h^2}\left(\begin{array}{l} \frac{-1}{12}f(x) + \frac{8}{3}hf'(x) - \frac{1}{6}h^2f''(x) + \frac{1}{9}h^3f^{(3)}(x) - \frac{1}{18}h^4f^{(4)}(x) + \frac{1}{45}h^5f^{(5)}(x) - \frac{1}{135}h^6f^{(6)}(x)\ldots \\ + \frac{4}{3}f(x) - \frac{4}{3}hf'(x) + \frac{2}{3}h^2f''(x) - \frac{2}{9}h^3f^{(3)}(x) + \frac{1}{18}h^4f^{(4)}(x) - \frac{16}{45}h^5f^{(5)}(x) + \frac{1}{540}h^6f^{(6)}(x)\ldots \\ - \frac{5}{2}f(x) \\ + \frac{4}{3}f(x) + \frac{4}{3}hf'(x) + \frac{2}{3}h^2f''(x) + \frac{2}{9}h^3f^{(3)}(x) + \frac{1}{18}h^4f^{(4)}(x) + \frac{16}{45}h^5f^{(5)}(x) + \frac{1}{540}h^6f^{(6)}(x)\ldots \\ + \frac{-1}{12}f(x) - \frac{8}{3}hf'(x) - \frac{1}{6}h^2f''(x) - \frac{1}{9}h^3f^{(3)}(x) - \frac{1}{18}h^4f^{(4)}(x) - \frac{1}{45}h^5f^{(5)}(x) - \frac{1}{135}h^6f^{(6)}(x)\ldots \end{array}\right)$$

$$= \frac{1}{h^2}\left(0 + 0 + h^2f''(x) + 0 + 0 + 0 - \frac{1}{90}h^6f^{(6)}(x)\ldots\right.$$

$$= f''(x) - \frac{1}{90}h^4f^{(6)}(x) + \ldots$$

$\Rightarrow$ Error term $O(h^4)$

For $f^{(3)}(x) = \frac{1}{h^3}\left(-\frac{1}{2}f_{-2} + f_{-1} - f_1 + \frac{1}{2}f_2\right)$

$$= \frac{1}{h^3}\left(\begin{array}{l} \frac{-1}{2}f(x) + hf'(x) - h^2f''(x) + \frac{2}{3}h^3f^{(3)}(x) - \frac{1}{3}h^4f^{(4)}(x) + \frac{2}{15}h^5f^{(5)}(x) \\ f(x) - hf'(x) + \frac{1}{2}h^2f''(x) - \frac{1}{6}h^3f^{(3)}(x) + \frac{1}{24}h^4f^{(4)}(x) - \frac{1}{120}h^5f^{(5)}(x) \\ -f(x) - hf'(x) - \frac{1}{2}h^2f''(x) - \frac{1}{6}h^3f^{(3)}(x) - \frac{1}{24}h^4f^{(4)}(x) - \frac{1}{120}h^5f^{(5)}(x) \\ \frac{1}{2}f(x) + hf'(x) + h^2f''(x) + \frac{2}{3}h^3f^{(3)}(x) + \frac{1}{3}h^4f^{(4)}(x) + \frac{2}{15}h^5f^{(5)}(x) \end{array}\right)$$

$$= \frac{1}{h^3}\left(0 + 0 + 0 + h^3f^{(3)}(x) + 0 + \frac{1}{4}h^5f^{(5)}(x)\right.$$

$$= f^{(3)}(x) + \frac{1}{4}h^2f^{(5)}(x)$$

$\Rightarrow$ Error term $O(h^2)$

For $f^{(4)}(x) = \frac{1}{h^4}\left(f_{-2} - 4f_{-1} + 6f_0 - 4f_1 + f_2\right)$

$$= \frac{1}{h^4}\left(\begin{array}{l} f(x) - 2hf'(x) + 2h^2f''(x) - \frac{4}{3}h^3f^{(3)}(x) + \frac{2}{3}h^4f^{(4)}(x) - \frac{4}{15}h^5f^{(5)}(x) + \frac{4}{45}h^6f^{(6)}(x)\ldots \\ -4f(x) + 4hf'(x) - 2h^2f''(x) + \frac{2}{3}h^3f^{(3)}(x) - \frac{1}{6}h^4f^{(4)}(x) + \frac{1}{30}h^5f^{(5)}(x) - \frac{1}{180}h^6f^{(6)}(x)\ldots \\ 6f(x) \\ -4f(x) - 4hf'(x) - 2h^2f''(x) - \frac{2}{3}h^3f^{(3)}(x) - \frac{1}{6}h^4f^{(4)}(x) - \frac{1}{30}h^5f^{(5)}(x) - \frac{1}{180}h^6f^{(6)}(x)\ldots \\ f(x) + 2hf'(x) + 2h^2f''(x) + \frac{4}{3}h^3f^{(3)}(x) + \frac{2}{3}h^4f^{(4)}(x) + \frac{4}{15}h^5f^{(5)}(x) + \frac{4}{45}h^6f^{(6)}(x)\ldots \end{array}\right)$$

$$= \frac{1}{h^4}\left(0 + 0 + 0 + 0 + h^4f^{(4)}(x) + 0 + \frac{1}{6}h^6f^{(6)}(x)\right.\ldots$$

$$= f^{(4)}(x) + \frac{1}{6}h^2f^{(6)}(x) + \ldots$$

$\Rightarrow$ Error term $O(h^2)$

3. Proof:

Simpson's $\frac{1}{3}$ rule $\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2}\left(f_i + s\Delta f_i + \frac{s(s-1)}{2}\Delta^2 f_i\right)dx$

$$\text{Error} = \int_{x_0}^{x_2}\left(\frac{s(s-1)(s-2)}{3!}\Delta^3 f_i + \frac{s(s-1)(s-2)(s-3)}{4!}h^4f^{(4)}(\xi)\right)dx \qquad s = \frac{x-x_0}{h} \quad ds = \frac{1}{h}dx$$

$$= h\left(\frac{\Delta^3 f_i}{3!}\int_0^2 s(s-1)(s-2)\,ds + \frac{h^4}{4!}f^{(4)}(\xi)\int_0^2 s(s-1)(s-2)(s-3)\,ds\right)$$

$$= h\frac{\Delta^3 f_i}{6}\left[\frac{1}{4}s^4 - s^3 + s^2\right]_0^2 + \frac{h^5}{24}f^{(4)}(\xi)\left[\frac{1}{5}s^5 - \frac{3}{2}s^4 + \frac{11}{3}s^3 - 3s^2\right]_0^2$$

$$= 0 - f^{(4)}(\xi)\frac{h^5}{90}$$

If $f(x)$ is cubic that implys $f^{(4)}(x) = 0$, Error will be 0.

Therefore, the area under any cubic between $x=a$ and $x=b$ is identical to the area of a parabola that matches the cubic at $x=a$, $x=b$, and $x=\frac{a+b}{2}$.

4. Use Simpson's $\frac{1}{3}$ rule:

$$\int_0^1 \frac{\sin(x)}{x}dx \approx \sum \frac{h}{3}(f_i + 4f_{i+1} + f_{i+2})$$

```
% Calculate the analytical solution of function
g = @(x) sin(x) ./ x;
analytical_ans = integral(g, 0, 1);

a = 0;
b = 1;

h1 = 0.5;
h2 = 0.25;

% Calculate integral using Simpson's 1/3 rule
x1 = a:h1:b;
x2 = a:h2:b;

% 1 4 2 4 2 4 2...2 4 1
ans1= (h1/3) * (f(a) + 4*sum(f(x1(2:2:end-1))) + 2*sum(f(x1(3:2:end-2))) + f(b));
ans2= (h2/3) * (f(a) + 4*sum(f(x2(2:2:end-1))) + 2*sum(f(x2(3:2:end-2))) + f(b));

fprintf('Use h: %.4f, integral: %.8f\n', h1, ans1);
fprintf('Use h: %.4f, integral: %.8f\n', h2, ans2);
fprintf('Analytical solution: %.8f\n', analytical_ans);
```

```
Use h: 0.5000, integral: 0.94614588
Use h: 0.2500, integral: 0.94608693
Analytical solution: 0.94608307
```

Compared to the analytical solution, h=0.25 is more precise. According to the question 3, we know that error term of Simpson's $\frac{1}{3}$ rule for multiple segements is $O(h^4)$.

Using extrapolation $Better = more\ accurate + \frac{1}{(2^n-1)}(more - less)$, we can improve the error term to $O(h^5)$.

```
% Error term O(n^4)
n = 4;
ex_ans = ans2 + (ans2 - ans1)/(2^n-1);
fprintf('After extrapolation: %.8f\n', ex_ans);
```

```
After extrapolation: 0.94608300
```

5.

a. Using the trapezoidal rule:

$$For\ each\ y = y_i, \qquad Ix_i = \int_{xmin}^{xmax} f(x, y_i)dx \approx \sum \frac{h}{2}(f_i + 2f_{i+1} + f_{i+2})$$

$$\iint_R f(x,y)dxdy \approx \sum \frac{h}{2}(Ix_i + 2Ix_{i+1} + Ix_{i+2})$$

```
f = @(x, y) exp(x).*sin(2.*y);
xmin = -0.2; xmax = 1.4;
ymin =  0.4; ymax = 2.6;
h = 0.1;

% a
x = xmin:h:xmax;
y = ymin:h:ymax;

Ix = x;
for i = 1:length(x)
    Ix(i) = h/2 * (f(x(i), ymin) + 2*sum(f(x(i), y(2:end-1))) + f(x(i), ymax));
end
ans_a = h/2 * (Ix(1) + 2*sum(Ix(2:end-1)) + Ix(end));
fprintf('Trapezoidal rule: %.8f\n', ans_a);
```

```
Trapezoidal rule: 0.36833996
```

b. Using Simpson's $\frac{1}{3}$ rule:

$$For\ each\ y = y_i, \qquad Ix_i = \int_{xmin}^{xmax} f(x, y_i)dx \approx \sum \frac{h}{3}(f_i + 4f_{i+1} + f_{i+2})$$

$$\iint_R f(x,y)dxdy \approx \sum \frac{h}{3}(Ix_i + 4Ix_{i+1} + Ix_{i+2})$$

```
% b
Ix = x;
for i = 1:length(x)
    Ix(i) = h/3 * (f(x(i), ymin) + 4*sum(f(x(i), y(2:2:end-1))) ...
            + 2*sum(f(x(i), y(3:2:end-2))) + f(x(i), ymax));
end
ans_b = h/3 * (Ix(1) + 4*sum(Ix(2:2:end-1)) + 2*sum(Ix(3:2:end-2))+ Ix(end));
fprintf('Simpson 1/3 rule : %.8f\n', ans_b);
```

```
Simpson 1/3 rule : 0.36926852
```

c. Using Gaussian quadrature, three-term formulas:

Change to variable u and t for limits [-1, 1]

$$\iint_R f(x,y)dxdy = 0.8 \times 1.1 \times \iint_R f(t,u)dtdu$$

$$\approx 0.8 \times 1.1 \times \sum_{i=1}^{3}\sum_{j=1}^{3} w_i w_j f(0.8t_i + 0.6, 1.1u_j + 1.5)$$

And the weight and t are from textbook P.309:

Table 5.13 Values for Gaussian quadrature

Number of terms	Values of t	Weighting factor
2	−0.57735027	1.0
	0.57735027	1.0
3	−0.77459667	0.55555555
	0.0	0.88888889
	0.77459667	0.55555555
4	−0.86113631	0.34785485
	−0.33998104	0.65214515
	0.33998104	0.65214515
	0.86113631	0.34785485
5	−0.90617975	0.23692689
	−0.53846931	0.47862867
	0.0	0.56888889
	0.53846931	0.47862867
	0.90617975	0.23692689

```
% c
% Change to variable u and t for limits [-1, 1]:
% 0.8t + 0.6, dx = 0.8dt
% 1.1u + 1.5, dy = 1.1du
term = 3;
w = [5/9, 8/9, 5/9];
t = [-0.7745966692414834, 0, 0.7745966692414834];

I = 0;
for j = 1:term
    for i = 1:term
        I = I + w(j) * w(i) * f(0.8*t(j) + 0.6, 1.1*t(i) + 1.5);
    end
end
ans_c = I * 0.8 * 1.1;
fprintf('Gaussian quadrature, 3-term: %.8f\n', ans_c);
```

```
Gaussian quadrature, 3-term: 0.37237772
```

Compared these results to the analytical solution.

```
integral_value = integral2(f, xmin, xmax, ymin, ymax);
fprintf('Analytical solution: %.8f\n', integral_value);
```

```
Trapezoidal rule: 0.36833996
Simpson 1/3 rule : 0.36926852
Gaussian quadrature, 3-term: 0.37237772
Analytical solution: 0.36926502
```

We can see that Simpson's $\frac{1}{3}$ rule provides best estimation, the Trapezoidal rule is second, and Gaussian quadrature with 3-term performs poorly.

6. Use Monte Carlo Integration to estimate:

$$\int_{ymin}^{ymax}\int_{xmin}^{xmax} f(x,y)dxdy \approx (xmax - xmin)(ymax - ymin)\frac{1}{N}\sum_{i=1}^{N} f(x_i, y_i)$$

```
% Number of random points
N = 1e7;

% R
xmin = -2; xmax = 3;
ymin = -1; ymax = 2;

% Generate random points in R
rng(0);
x = xmin + (xmax - xmin) * rand(N, 1);
y = ymin + (ymax - ymin) * rand(N, 1);

% f(x, y) at the random points
f_values = ((x - 1).^2 + y.^2) / 16;

% The area of R
area_R = (xmax - xmin) * (ymax - ymin);
```

```
% A(R)/N * sigma(f(xi, yi))
integral_approx = mean(f_values) * area_R;

fprintf('Use Monte Carlo integration: %.8f\n', integral_approx);

% Define the function
f = @(x, y) ((x - 1).^2 + y.^2) / 16;

% Analytical solution
integral_value = integral2(f, xmin, xmax, ymin, ymax);

fprintf('Analytical solution: %.8f\n', integral_value);
```

ndow

```
Use Monte Carlo integration: 3.12478682
Analytical solution: 3.12500000
```