Bounding VC Dimension in Non-abelian Groups

David Zeng and Tora Ozawa

YMC 2023

August 15, 2023

Vapnik–Chervonenkis Dimension

Definition (Shattered Set)

Given a ***Ground*** set X and a family of sets $\mathcal{S} \subset \mathcal{P}(X)$, we say a set $A \subset X$ is shattered by $\mathcal{S}$ if, for every subset $A' \subset A$, there exists $S \in \mathcal{S}$ with $S \cap A = A'$.

Example

Let $X = \mathbb{R}$, and let $\mathcal{S} = \{[a, b] : a, b \in \mathbb{R}\}$.

Vapnik–Chervonenkis Dimension

Definition (Vapnik–Chervonenkis Dimension)

We say the VC dimension $\mathrm{VC}_X(\mathcal{S})$ is the following:

$$\mathrm{VC}_X(\mathcal{S}) = \max\{n : A \subset X, |A| = n, \mathcal{S} \text{ shatters } A\}$$

If such maximum does not exist, then we say $\mathrm{VC}_X(\mathcal{S}) = \infty$.

Example

- $\mathrm{VC}_{\mathbb{R}}([a, b]) = 2$
- $\mathrm{VC}_{\mathbb{R}^2}(\text{convex polygons}) = \infty$

VC Dimension in Groups

Definition (VC Dimension of a set in a Group)

Given a group G and a subset $A \subset G$, our set family will be the (left) translates:

$$\mathrm{VC}^l_G(A) := \mathrm{VC}_G(\{gA : g \in G\})$$

One can similarly define $\mathrm{VC}^r_G(A)$.

Example

- $\mathrm{VC}^l_{\mathbb{R}}([a, b]) = 2$
- For a subgroup $H \lneq G$, $\mathrm{VC}^l_G(H) = 1$

Motivation

- Introduced by Vladimir Vapnik and Alexey Chervonenkis in 1970
- Used in Statistical Learning Theory
  - Binary Classifiers
  - PAC learnabillity
- Already lots of exploration in abelian settings ($\mathbb{R}^n$, $\mathbb{F}_p^q$)
- Non-abelian setting is not well understood
- Bounded VC-Dimension $\Rightarrow$ nice things
  - NIP structures
  - Schur-Erdős problem

Current Work: Metric Balls

If there is a distance d on G, then we may look the metric ball: $B_r(e) = \{g \in G : d(e, g) \leq r\}$. On discrete groups we have the word metric.

Definition (Word Metric)

If G is finitely generated and $G = \langle x_1, \cdots, x_k \rangle$, the word metric with respect to $x_1, \cdots, x_k$ is

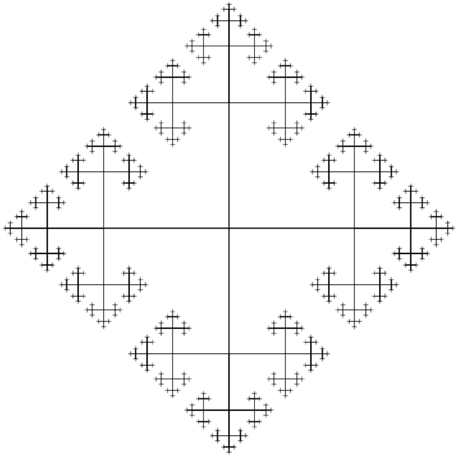
$$d(g, h) = \min\{n : g_n \cdots g_1 = gh^{-1}, g_i \in \{x_1^{\pm 1}, \cdots, x_k^{\pm 1}\}\}$$

Theorem

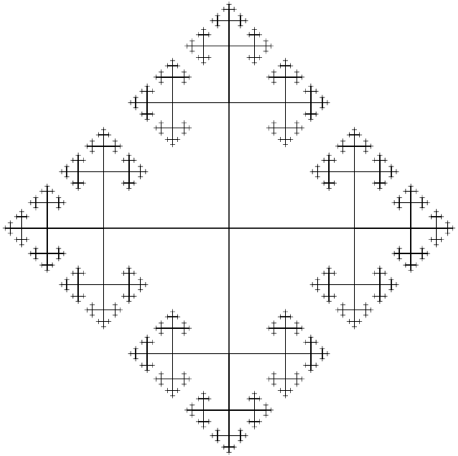
Given the standard generators of the free group $F_k = \langle x_1, \cdots, x_k \rangle$ and the word metric d with respect to these generators, for all $r \geq 1$,

$$\mathrm{VC}_{F_k}(B_r(e)) = 2$$

Proof Sketch

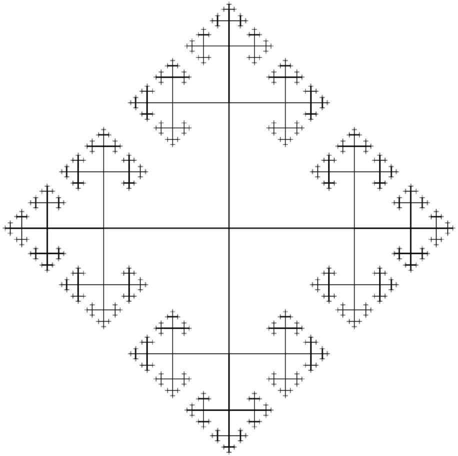


Proof Sketch

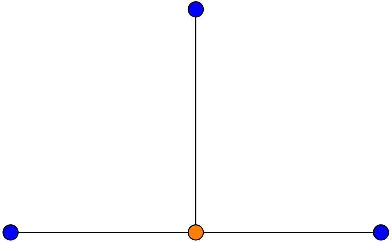


$\Rightarrow$

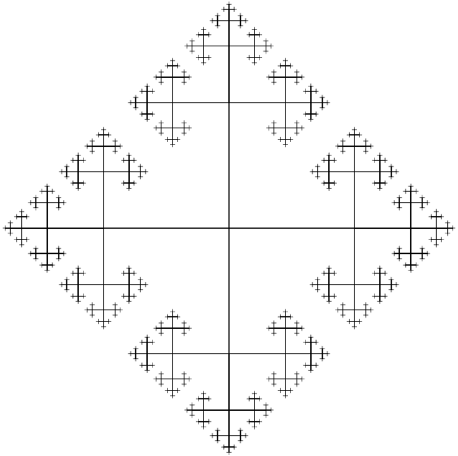
Proof Sketch



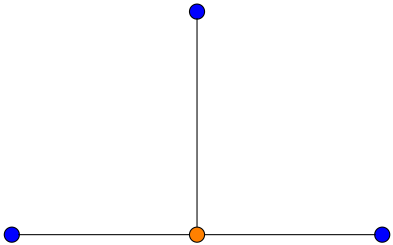
$\Rightarrow$



Proof Sketch



$\Rightarrow$



There will always be a unique (perhaps trivial) path to this tree.

Current Work: Non-abelian Progressions

Arithmetic progression in $\mathbb{Z}$
$$\{x_0 + ax_1 : -L \leq a \leq L\}$$

$$\Downarrow$$

Generalized arithmetic progression in abelian group G
$$\{x_0 + a_1x_1 + \cdots + a_kx_k : -L_i \leq a_i \leq L_i\}$$

$$\Downarrow$$

Definition (Non-abelian Progression)

Given marked elements $x_1, \cdots, x_k \in G$, a non-abelian progression is

$$P(\mathrm{L}) = P(L_1, \cdots, L_k) = \{g_n^{\pm 1} \cdots g_1^{\pm 1} : g_i \in \{x_1, \cdots, x_k\}, \text{ each } x_j \text{ appears at most } L_j \text{ times}\}$$

Theorem

Given the standard generators of the free group $F_k = \langle x_1, \cdots, x_k \rangle$, for all $\mathrm{L} \in \mathbb{N}^k$ and $k \geq 2$

$$k + 2 \leq \mathrm{VC}_{F_k}(P(\mathrm{L})) \leq 3k - 1$$

Questions and Partial Progress

- Narrowed down on possible configurations for a 5 point set in F_2.
- Is it possible to do better than the "almost trivial" lower bound?
- Does this set live in an NIP/stable language extension of the free group?

Definition (Discrete Heisenberg Group)

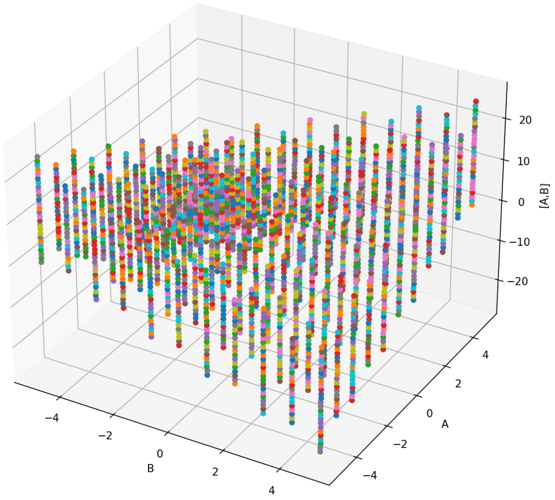
The discrete Heisenberg group H is $\left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$.

Properties

The group H can be generated by $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Further, $H = F(2,2)$, "the free 2-step nilpotent group of 2 generators."

Non-abelian Progression in Nilpotent Groups



Non-abelian Progression in Nilpotent Groups

Theorem

Given the Heisenberg group G and the generators A, B, we have, for all $\mathsf{L} \in \mathbb{N}^2$

$$\mathrm{VC}_H(P(\mathsf{L})) \leq C$$

for some constant C independent of L.

Questions

- Can this be generalized to all choices of generators $A', B' \in H$ with $H = \langle A', B' \rangle$?
- Can the current method be generalized at all to other nilpotent groups?

Future Directions

Questions

- Given a finitely generated group G and $\langle g_1, \cdots, g_k \rangle = G$ a finite generating set, define

$$\mathcal{Q} = \{gP(L) : g \in G, L \in \mathbb{N}^k\}$$

For $G = F_2$ or H, our current methods suggest that $\mathrm{VC}_G(\mathcal{Q}) < \infty$, implying $\mathrm{VC}_G(P(L)) < \infty$. Is this true in general?

- If further G is nilpotent of rank r and step s, is it true that

$$\mathrm{VC}_G(P(L)) \leq O_{r,s}(1)$$

uniformly for all such G?