MAT 352 Assignment

Computer Science Department

Question:

Proof of the Inclusion-Exclusion Rule

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Proof of the Inclusion-Exclusion Rule

The Inclusion-Exclusion States that:

$$\begin{aligned}P(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n) = &\sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2}) + \sum_{i_1<i_2<i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) + \cdots + \\ &(-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) + \cdots + \\ &(-1)^{n} \sum_{i_1<i_2<\cdots<i_{n-1}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{n-1}}) + \\ &(-1)^{n+1} P(E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_n)\end{aligned} \tag{1}$$

where the summation $(-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r})$ is taken over all the $\binom{n}{r}$ possible combinations.

Proof by Principle of Mathematical Induction

Proof by PMI involves two steps,the **Basic Step** where the rule shown for the base value/unit and the **Inductive step** which also involve two steps:

1. Making an assumption that the rule/relation holds for an arbitrary value $n = k$
2. Using this assumption to prove that the rule/relation holds for the $(n = k + 1)^{th}$ value.

0.1 The Basic Step

For the Induction-Exclusion Rule, the most basic value (base case) is when the number of events is 2 (i.e $n = 2$), therefore, the rule is applied as:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \tag{2}$$

0.2 The Inductive Step

The assumption that it the rule holds for some $n = k$

It is assumed that the rule/relation holds for a $n = k$ number of events. Therefore the equation below is noted.

$$\begin{aligned}P(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k) = &\sum_{i=1}^{k} P(E_i) - \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2}) + \sum_{i_1<i_2<i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) + \cdots + \\ &(-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) + \cdots + \\ &(-1)^{k} \sum_{i_1<i_2<\cdots<i_{k-1}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{k-1}}) + \\ &(-1)^{k+1} P(E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_k)\end{aligned} \tag{3}$$

The proof that the rule holds for $n = k + 1$

Now, the rule is also applied for $n = k + 1$ number of events:

$$\begin{aligned}P(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k \cup E_{k+1}) = &\sum_{i=1}^{k+1} P(E_i) - \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2}) + \sum_{i_1<i_2<i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) \\ &+ \cdots + (-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) \\ &+ \cdots + (-1)^{k+1} \sum_{i_1<i_2<\cdots<i_k} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) \\ &+ (-1)^{k+2} P(E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_k \cap E_{k+1})\end{aligned} \tag{4}$$

Consider that $P(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k \cup E_{k+1})$ can be written as $P([E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k] \cup E_{k+1})$ wherer by "$E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k$" is taken as a single event.

$$P(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k \cup E_{k+1}) = P([E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k] \cup E_{k+1}) \tag{5}$$

By applying the Base Relation of the Inclusion-Exclusion Rule as in equation (2) to the RHS of the equation (5) above and comparing with equation (4) the rule/relation can be proven.

So now:

$$\begin{aligned} P([E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k] \cup E_{k+1}) =& P(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k) + P(E_{k+1}) \\ & - P([E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k] \cap E_{k+1}) \end{aligned} \tag{6}$$

The last term in equation (6) above can be expanded as:

$$P([E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k] \cap E_{k+1}) = P([E_1 \cap E_{k+1}] \cup [E_2 \cap E_{k+1}] \cup \cdots \cup [E_k \cap E_{k+1}]) \tag{7}$$

The RHS of equation (7) above can be further simplified by applying the Inclusion-Exclusion here again noting that $|P([E_1 \cap E_{k+1}] \cup [E_2 \cap E_{k+1}] \cup \cdots \cup [E_k \cap E_{k+1}])| = k$

$$\begin{aligned} P([E_1 \cap E_{k+1}] \cup [E_2 \cap E_{k+1}] \cup \cdots \cup [E_k \cap E_{k+1}]) = \sum_{i=1}^{k} P(E_i \cap E_{k+1}) \\ - \sum_{i_1<i_2} P([E_{i_1} \cap E_{k+1}] \cap [E_{i_2} \cap E_{k+1}]) \\ + \cdots + (-1)^r \sum_{i_1<i_2 \cdots i_{r-1}} P([E_{i_1} \cap E_{k+1}] \cap [E_{i_2} \cap E_{k+1}] \cap \cdots \cap [E_{i_{r-1}} \cap E_{k+1}]) \\ + \cdots + (-1)^k \sum_{i_1<i_2 \cdots i_{k-1}} P([E_{i_1} \cap E_{k+1}] \cap [E_{i_2} \cap E_{k+1}] \cap \cdots \cap [E_{i_{k-1}} \cap E_{k+1}]) \\ + (-1)^{k+1} P([E_1 \cap E_{k+1}] \cap [E_2 \cap E_{k+1}] \cap \cdots \cap [E_k \cap E_{k+1}]) \end{aligned} \tag{8}$$

From Set Theory:

$$\begin{aligned} (A \cap B) \cap C =& A \cap B \cap C \\ (A \cap B) \cap (A \cap C) =& A \cap B \cap C \end{aligned}$$

Some term in equation (8) can be rewritten as:

$$\begin{aligned} & P([E_{i_1} \cap E_{k+1}] \cap [E_{i_2} \cap E_{k+1}]) = P(E_{i_1} \cap E_{i_2} \cap E_{k+1}) \\ & P([E_{i_1} \cap E_{k+1}] \cap [E_{i_2} \cap E_{k+1}] \cap \cdots \cap [E_{i_{r-1}} \cap E_{k+1}]) = P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{r-1}} \cap E_{k+1}) \\ & P([E_{i_1} \cap E_{k+1}] \cap [E_{i_2} \cap E_{k+1}] \cap \cdots \cap [E_{i_{k-1}} \cap E_{k+1}]) = P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{k-1}} \cap E_{k+1}) \\ & P([E_1 \cap E_{k+1}] \cap [E_2 \cap E_{k+1}] \cap \cdots \cap [E_k \cap E_{k+1}]) = P(E_1 \cap E_2 \cap \cdots \cap E_k \cap E_{k+1}) \end{aligned}$$

Therefore equation (8) is simplified as:

$$\begin{aligned} P([E_1 \cap E_{k+1}] \cup [E_2 \cap E_{k+1}] \cup \cdots \cup [E_k \cap E_{k+1}]) = \sum_{i=1}^{k} P(E_i \cap E_{k+1}) \\ - \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2} \cap E_{k+1}) \\ + \cdots + (-1)^r \sum_{i_1<i_2 \cdots i_{r-1}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{r-1}} \cap E_{k+1}) \\ + \cdots + (-1)^k \sum_{i_1<i_2 \cdots i_{k-1}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{k-1}} \cap E_{k+1}) \\ + (-1)^{k+1} P(E_1 \cap E_2 \cap \cdots \cap E_k \cap E_{k+1}) \end{aligned} \tag{9}$$

By substituting equation (3) and above equation (9) and in equation (6), equation (6) is then evaluated as:

$$
\begin{aligned}
&P([E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k] \cup E_{k+1}) \\
&\qquad = P(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k) + P(E_{k+1}) - P([E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k] \cap E_{k+1})
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{k} P(E_i) - \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2}) + \sum_{i_1<i_2<i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) + \cdots + \\
&(-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) + \cdots + \\
&(-1)^{k} \sum_{i_1<i_2<\cdots<i_{k-1}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{k-1}}) + \\
&(-1)^{k+1} P(E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_k) + P(E_{k+1}) \\
&- \left[\sum_{i=1}^{k} P(E_i \cap E_{k+1}) - \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2} \cap E_{k+1}) \right. \\
&+ \cdots + (-1)^{r} \sum_{i_1<i_2\cdots i_{r-1}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{r-1}} \cap E_{k+1}) \\
&+ \cdots + (-1)^{k} \sum_{i_1<i_2\cdots i_{k-1}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{k-1}} \cap E_{k+1}) \\
&\left. + (-1)^{k+1} P(E_1 \cap E_2 \cap \cdots \cap E_k \cap E_{k+1}) \right]
\end{aligned} \tag{11}
$$

Note that the term "$(-1)^k \sum_{i_1<i_2<\cdots<i_{k-1}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{k-1}})$" can be contained in the terms "$(-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) + \cdots +$" and therefore may be ignored. The equation is then reduced to:

$$
\begin{aligned}
&P([E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k] \cup E_{k+1}) \\
&\quad = \sum_{i=1}^{k} P(E_i) - \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2}) + \sum_{i_1<i_2<i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) + \cdots + \\
&\quad (-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) + \cdots + \\
&\quad (-1)^{k+1} P(E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_k) + P(E_{k+1}) \\
&\quad - \left[\sum_{i=1}^{k} P(E_i \cap E_{k+1}) - \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2} \cap E_{k+1}) \right. \\
&\quad + \cdots + (-1)^{r} \sum_{i_1<i_2\cdots i_{r-1}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{r-1}} \cap E_{k+1}) \\
&\quad + \cdots + (-1)^{k} \sum_{i_1<i_2\cdots i_{k-1}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{k-1}} \cap E_{k+1}) \\
&\quad \left. + (-1)^{k+1} P(E_1 \cap E_2 \cap \cdots \cap E_k \cap E_{k+1}) \right]
\end{aligned} \tag{12}
$$

Expand the bracket in eqation (12) above and distribute the negative sign noting that $-1 \times (-1)^{a+1} = (-1)^{a+2}$

$$
\begin{aligned}
&P([E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k] \cup E_{k+1}) \\
&\quad = \sum_{i=1}^{k} P(E_i) - \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2}) + \sum_{i_1<i_2<i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) + \cdots + \\
&\quad (-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) + \cdots + \\
&\quad (-1)^{k+1} P(E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_k) + P(E_{k+1}) \\
&\quad - \sum_{i=1}^{k} P(E_i \cap E_{k+1}) + \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2} \cap E_{k+1}) \\
&\quad + \cdots + (-1)^{r+1} \sum_{i_1<i_2\cdots i_{r-1}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{r-1}} \cap E_{k+1}) \\
&\quad + \cdots + (-1)^{k+1} \sum_{i_1<i_2\cdots i_{k-1}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{k-1}} \cap E_{k+1}) \\
&\quad + (-1)^{k+2} P(E_1 \cap E_2 \cap \cdots \cap E_k \cap E_{k+1})
\end{aligned} \tag{13}
$$

Collecting like terms:

$$
\begin{aligned}
&P([E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k] \cup E_{k+1}) \\
&\quad = \sum_{i=1}^{k} P(E_i) + P(E_{k+1}) \\
&\quad - \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2}) - \sum_{i=1}^{k} P(E_i \cap E_{k+1}) \\
&\quad + \sum_{i_1<i_2<i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) + \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2} \cap E_{k+1}) + \cdots + \\
&\quad (-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) + \\
&\quad (-1)^{r+1} \sum_{i_1<i_2\cdots i_{r-1}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{r-1}} \cap E_{k+1}) + \cdots + \\
&\quad (-1)^{k+1} P(E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_k) + \\
&\quad (-1)^{k+1} \sum_{i_1<i_2\cdots i_{k-1}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{k-1}} \cap E_{k+1}) + \\
&\quad (-1)^{k+2} P(E_1 \cap E_2 \cap \cdots \cap E_k \cap E_{k+1})
\end{aligned} \tag{14}
$$

Considering the collected like terms in equation (14) above, the terms can be collapsed as follows (including summation bounds):

$$\sum_{i=1}^{k} P(E_i) + P(E_{k+1}) = \sum_{i=1}^{k+1} P(E_i) \tag{15}$$

$$\sum_{1\le i_1<i_2\le k} P(E_{i_1} \cap E_{i_2}) + \sum_{i=1}^{k} P(E_i \cap E_{k+1}) = \sum_{1\le i_1<i_2\le k+1} P(E_{i_1} \cap E_{i_2}) \tag{16}$$

$$\sum_{1\le i_1<i_2<i_3\le k} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) + \sum_{1\le i_1<i_2\le k} P(E_i \cap E_{k+1}) = \sum_{1\le i_1<i_2<i_3\le k+1} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) \tag{17}$$

$$
\begin{aligned}
&\sum_{1\le i_1<i_2<\cdots<i_{r\le k}\le k} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) + \sum_{1\le i_1<i_2\cdots i_{r-1\le k}\le k} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{r-1}} \cap E_{k+1}) \\
&\qquad = \sum_{1\le i_1<i_2\cdots i_{r\le k+1}\le k+1} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r})
\end{aligned} \tag{18}
$$

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) + \sum_{i_1 < i_2 \cdots i_{k-1} \leq k} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{k-1}} \cap E_{k+1})$$
$$= \sum_{i_1 < i_2 \cdots < i_k \leq k+1} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) \quad (19)$$

Substituting the collapsed equations (15) to (19) in equation (14):

$$\begin{aligned} P([E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_k] \cup E_{k+1}) = & \sum_{i=1}^{k+1} P(E_i) - \sum_{1 \leq i_1 < i_2 \leq k+1} P(E_{i_1} \cap E_{i_2}) \\ & + \sum_{1 \leq i_1 < i_2 < i_3 \leq k+1} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) \\ & + \cdots + (-1)^{r+1} \sum_{1 \leq i_1 < i_2 \cdots i_{r \leq k+1} \leq k+1} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) \\ & + \cdots + (-1)^{k+1} \sum_{i_1 < i_2 \cdots < i_k \leq k+1} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) \\ & + (-1)^{k+2} P(E_1 \cap E_2 \cap \cdots \cap E_k \cap E_{k+1}) \end{aligned} \quad (20)$$

Equation (20) above is equivalent to equation (4) (for the $n = k + 1$ number of events) and this concludes the proof for the Inclusion-Exclusion Rule.

Closing Statement

With the proof shown above, it can be said that for any n number of events, the Inclusion-Exclusion Rule:

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n) = & \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2}) + \sum_{i_1 < i_2 < i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) + \cdots + \\ & (-1)^{r+1} \sum_{i_1 < i_2 < \cdots < i_r} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) + \cdots + \\ & (-1)^{n} \sum_{i_1 < i_2 < \cdots < i_{n-1}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{n-1}}) + \\ & (-1)^{n+1} P(E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_n) \end{aligned}$$

is true!!!