Solutions to selected exercises of Chapters 12–15

Bas Luttik

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This document contains solutions to the following exercises in the book [1]:

> 12.4(d), 13.1(b), 13.2(b), 14.2(a), 14.6, 14.8(b,c), 14.9(b), 14.10(b), 15.5, 15.8, 15.9.

We **strongly** advise you to first try all these exercises by yourself, before looking at all at the solutions below. There is not a lot of variation possible in the way solutions to exercises should be written down. So if your solution in one way or another deviates from a solution below, then consider discussing the differences with your instructor.

12.4 (d) The following derivation shows that the formula

$$(P \Rightarrow Q) \Rightarrow ((R \Rightarrow (P \Rightarrow Q)) \wedge ((P \wedge R) \Rightarrow Q))$$

is a tautology:

{ Assume; }

(1) $\boxed{P \Rightarrow Q}$

{ Assume: }

(2) $\boxed{R}$

{ Still valid: }

(3) $P \Rightarrow Q$

{ $\Rightarrow$-intro on (2) and (3): }

(4) $R \Rightarrow (P \Rightarrow Q)$

{ Assume: }

(5) $\boxed{P \wedge R}$

{ $\wedge$-elim on (5): }

(6) P

{ $\Rightarrow$-elim on (1) and (6): }

(7) Q

{ $\Rightarrow$-intro on (5) and (7): }

(8) $(P \wedge R) \Rightarrow Q$

{ $\wedge$-intro on (4) and (8): }

(9) $(R \Rightarrow (P \Rightarrow Q)) \wedge ((P \wedge R) \Rightarrow Q)$

{ $\Rightarrow$-intro on (1) and (9): }

(10) $(P \Rightarrow Q) \Rightarrow ((R \Rightarrow (P \Rightarrow Q)) \wedge ((P \wedge R) \Rightarrow Q))$

13.1 (b)

(1):	valid from (1) to (9)
(2):	valid from (2) to (3)
(3):	only valid on (3)
(4):	valid from (4) to (9)
(5):	valid from (5) to (7)
(6):	valid from (6) to (7)
(7):	only valid on (7)
(8):	valid from (8) to (9)
(9):	only valid on (9)
(10):	valid forever.

13.2 (b)

(1):	context consists of hypothesis on (1)
(2):	context consists of hypotheses on (1) and (2)
(3):	context consists of hypotheses on (1) and (2)
(4):	context consists of hypothesis on (1)
(5):	context consists of hypotheses on (1) and (5)
(6):	context consists of hypotheses on (1) and (5)
(7):	context consists of hypotheses on (1) and (5)
(8):	context consists of hypothesis on (1)
(9):	context consists of hypothesis on (1)
(10):	context is empty.

14.2 (a) The following derivation shows that the formula

$$(P \Rightarrow \neg Q) \Rightarrow ((P \Rightarrow Q) \Rightarrow \neg P)$$

is a tautology:

{ Assume: }

(1) $\boxed{P \Rightarrow \neg Q}$

{ Assume: }

(2) $\boxed{P \Rightarrow Q}$

{ Assume: }

(3) $\boxed{P}$

{ $\Rightarrow$-elim on (1) and (3): }

(4) $\neg Q$

{ $\Rightarrow$-elim on (2) and (3): }

(5) Q

{ $\neg$-elim on (4) and (5): }

(6) `False`

{ ¬-intro on (3) and (6): }

(7) $\neg P$

{ ⇒-intro on (2) and (7): }

(8) $(P \Rightarrow Q) \Rightarrow \neg P$

{ ⇒-intro on (1) and (8): }

(9) $(P \Rightarrow \neg Q) \Rightarrow ((P \Rightarrow Q) \Rightarrow \neg P)$

14.6 (b) The following derivation shows that the formula

$$(P \Rightarrow Q) \vee \neg Q$$

is a tautology:

{ Assume: }

(1) $\boxed{\neg(P \Rightarrow Q)}$

{ Assume: }

(2) $\boxed{Q}$

{ Assume: }

(3) $\boxed{P}$

{ Still valid: }

(4) Q

{ ⇒-intro on (3) and (4): }

(5) $P \Rightarrow Q$

{ ¬-elim on (1) and (5): }

(6) `False`

{ ¬-intro on (2) and (6): }

(7) $\neg Q$

{ ∨-intro on (1) and (7): }

(8) $(P \Rightarrow Q) \vee \neg Q$

14.8 (b) We prove that $(x \geq 2 \vee x = -1) \Rightarrow x^3 - 3x - 2 \geq 0$ with the help of case distinction:

{ Assume: }

(1) $\boxed{x \geq 2 \vee x = -1}$

{ Assume: }

(2) $\boxed{x \geq 2}$

{ Mathematics on (2): }

(3) $(x - 2) \geq 0$

{ Mathematics on (2): }

(4) $(x + 1) \geq 0$

{ Mathematics on (3) and (4): }

(5) $x^3 - 3x - 2 = (x - 2)(x + 1)(x + 1) \geq 0$

{ $\Rightarrow$-intro on (2) and (5): }

(6) $(x \geq 2) \Rightarrow (x^3 - 3x + 2 \geq 0)$

{ Assume: }

(7) $\boxed{x = -1}$

{ Substitute -1 for x: }

(8) $x^3 - 3x - 2 = (-1)^3 - 3 \cdot (-1) - 2 = 0 \geq 0$

{ $\Rightarrow$-intro on (7) and (8): }

(9) $(x = -1) \Rightarrow (x^3 - 3x - 2 \geq 0)$

{ Case distinction on (1), (6) and (9): }

(10) $x^3 - 3x - 2 \geq 0$

{ $\Rightarrow$-intro on (1) and (11): }

(11) $(x \geq 2 \vee x = -1) \Rightarrow (x^3 - 3x - 2 \geq 0)$

The case-distinction tautology is used with $P = (x \geq 2)$, $Q = (x = -1)$ and $R = (x^3 - 3x - 2 \geq 0)$. (NB: the exercise does not explicitly ask for a derivation, so the argument may be written otherwise, but it should precisely indicate how case distinction is used; see Remark 14.8.1 in the book [1] for an example of a more informal argument.)

(c) We need to prove that $x^2 = y^2 \Leftrightarrow (x = y \vee x = -y)$, using case distinction.

To prove the bi-implication, we establish both the implication from left to right (i.e., $x^2 = y^2 \Rightarrow (x = y \vee x = -y)$) and the implication from right to left (i.e., $(x = y \vee x = -y) \Rightarrow x^2 = y^2$) separately.

To prove the implication from left to right we do not need case distinction. Suppose that $x^2 = y^2$. Then $x = \sqrt{y^2} = y$ or $x = -\sqrt{y^2} = -y$. So $x = y \vee x = -y$. Thereby, we have established the implication $(x^2 = y^2) \Rightarrow (x = y \vee x = -y)$.

To prove the implication from right to left, suppose that $x = y \vee x = -y$. We now use case distinction to establish $x^2 = y^2$, taking $P = (x = y)$, $Q = (x = -y)$ and $R = (x^2 = y^2)$. Note that $P \vee Q$ holds by the supposition that $x = y \vee x = -y$. To see, on the one hand, that $P \Rightarrow R$ holds, suppose that $x = y$. Then it immediately follows that $x^2 = y^2$

holds too. On the other hand, to see that $Q \Rightarrow R$ holds, suppose that $x = -y$. Then $x^2 = (-y)^2 = y^2$. We conclude that $P \vee Q$, $P \Rightarrow R$ and $Q \Rightarrow R$, so, by case distinction, R, i.e., $x^2 = y^2$, holds.

14.9 (b) The following derivation shows that the formula

$$(P \Leftrightarrow Q) \Rightarrow (\neg P \Leftrightarrow \neg Q)$$

is a tautology:

{ Assume: }

(1) $\boxed{P \Leftrightarrow Q}$

{ Assume: }

(2) $\boxed{\neg P}$

{ Assume: }

(3) $\boxed{Q}$

{ $\Leftrightarrow$-elim on (1): }

(4) $Q \Rightarrow P$

{ $\Rightarrow$-elim on (4) and (3): }

(5) P

{ $\neg$-elim on (2) and (5): }

(6) `False`

{ $\neg$-intro on (3) and (6): }

(7) $\neg Q$

{ $\Rightarrow$-intro on (2) and (7): }

(8) $\neg P \Rightarrow \neg Q$

{ Assume: }

(9) $\boxed{\neg Q}$

{ Assume: }

(10) $\boxed{P}$

{ $\Leftrightarrow$-elim on (1): }

(11) $P \Rightarrow Q$

{ $\Rightarrow$-elim on (11) and (10): }

(12) Q

{ $\neg$-elim on (9) and (12): }

(13) `False`

{ $\neg$-intro on (10) and (13): }

(14) $\neg P$

{ $\Rightarrow$-intro on (9) and (14): }

(15) $\neg Q \Rightarrow \neg P$

{ $\Leftrightarrow$-intro on (8) and (15): }

(16) $\neg P \Leftrightarrow \neg Q$

{ $\Rightarrow$-intro on (1) and (16): }

(17) $(P \Leftrightarrow Q) \Rightarrow (\neg P \Leftrightarrow \neg Q)$

14.10 (b) To prove with a calculation that the formula $(P \Leftrightarrow Q) \Rightarrow (\neg P \Leftrightarrow \neg Q)$ is a tautology, it suffices to establish with a calculation that

$$P \Leftrightarrow Q \overset{val}{\models} \neg P \Leftrightarrow \neg Q \quad :$$

$$\begin{array}{ll} & (P \Leftrightarrow Q) \\ \overset{val}{=\!=} & \{ \text{ Bi-implication } \} \\ & (P \Rightarrow Q) \wedge (Q \Rightarrow P) \\ \overset{val}{=\!=} & \{ \text{ Contraposition } (2\times) \text{ } \} \\ & (\neg Q \Rightarrow \neg P) \wedge (\neg P \Rightarrow \neg Q) \\ \overset{val}{=\!=} & \{ \text{ Bi-implication } \} \\ & \neg P \Leftrightarrow \neg Q \end{array}$$

(NB: the calculation above actually establishes the stronger result that $(P \Leftrightarrow Q) \overset{val}{=\!=} (\neg P \Leftrightarrow \neg Q)$ from which it follows that $(P \Leftrightarrow Q) \Leftrightarrow (\neg P \Leftrightarrow \neg Q)$ is a tautology.)

15.5 (a) We rename the x and y bound by the universal quantifiers respectively to u and v, resulting in the formula: $\neg \exists_x \exists_y [P(x, y)] \Rightarrow \forall_u \forall_v [\neg P(u, v)]$, and then establish with a derivation that it is a tautology:

{ Assume: }

(1) $\boxed{\neg \exists_x \exists_y [P(x, y)]}$

{ Assume: }

(2) $\boxed{\textbf{var } u; \ \texttt{True}}$

{ Assume: }

(3) $\boxed{\textbf{var } v; \ \texttt{True}}$

{ Assume: }

(4) $\boxed{P(u, v)}$

{ Assume: }

(5) $\boxed{\forall_x [\neg \exists_y [P(x, y)]]}$

{ $\forall$-elim on (5) and (2): }

(6) $\neg\exists_y[P(u,y)]$

{ Assume: }

(7) $\boxed{\forall_y[\neg P(u,y)]}$

{ $\forall$-elim }

(8) $\neg P(u,v)$

{ $\neg$-elim on (8) and (4): }

(9) `False`

{ $\exists$-intro on (7) and (9): }

(10) $\exists_y[P(u,y)]$

{ $\neg$-elim on (6) and (10): }

(11) `False`

{ $\exists$-intro on (5) and (11): }

(12) $\exists_x\exists_y[P(x,y)]$

{ $\neg$-elim on (1) and (12): }

(13) `False`

{ $\neg$-intro on (4) and (13): }

(14) $\neg P(u,v)$

{ $\forall$-intro on (3) and (14): }

(15) $\forall_v[\neg P(u,v)]$

{ $\forall$-intro on (2) and (15): }

(16) $\forall_u\forall_v[\neg P(u,v)]$

{ $\Rightarrow$-intro on (1) and (16): }

(17) $\neg\exists_x\exists_y[P(x,y)] \Rightarrow \forall_u\forall_v[\neg P(u,v)]$

(b) We rename the occurrences of x bound by the existential quantfiers respectively to y and z, resulting in the formula:

$$\forall_x[P(x) : Q(x)] \Rightarrow (\exists_y[P(y)] \Rightarrow \exists_z[Q(z)]) \ ,$$

and then establish with a derivation that it is a tautology:

{ Assume: }

(1) $\boxed{\forall_x[P(x) : Q(x)]}$

{ Assume: }

(2) $\boxed{\exists_y[P(y)]}$

{ Assume: }

(3) $\boxed{\forall_z[\neg Q(z)]}$

{ Assume: }

(4) $\boxed{\textbf{var } y;\ \texttt{True}}$

{ Assume: }

(5) $\boxed{P(y)}$

{ $\forall$-elim on (1) and (5): }

(6) $Q(y)$

{ $\forall$-elim on (3) and (4): }

(7) $\neg Q(y)$

{ $\neg$-elim on (7) and (6): }

(8) `False`

{ $\neg$-intro on (5) and (8): }

(9) $\neg P(y)$

{ $\forall$-intro on (4) and (9): }

(10) $\forall_y[\neg P(y)]$

{ $\exists$-elim on (2) and (10): }

(11) `False`

{ $\exists$-intro on (3) and (11): }

(12) $\exists_z[Q(z)]$

{ $\Rightarrow$-intro on (2) and (12): }

(13) $\exists_y[P(y)] \Rightarrow \exists_z[Q(z)]$

{ $\Rightarrow$-intro on (1) and (13): }

(14) $\forall_x[P(x) : Q(x)] \Rightarrow (\exists_y[P(y)] \Rightarrow \exists_z[Q(z)])$

15.8 (a) We prove with a derivation that the formula $\exists_x\forall_y[P(x,y)]\Rightarrow\forall_v\exists_u[P(u,v)]$ is a tautology:

{ Assume: }

(1) $\boxed{\exists_x\forall_y[P(x,y)]}$

{ Assume: }

(2) $\boxed{\textbf{var } v;\ \texttt{True}}$

{ $\exists^*$-elim on (1): }

(3) Pick an x with (`True` and) $\forall_y[P(x,y)]$

{ $\forall$-elim on (3) and (2): }

(4) $P(x,v)$

{ $\exists^*$-intro on (4): }

(5) $\exists_u[P(u,v)]$

{ $\forall$-intro on (2) and (5): }

(6) $\forall_v[\exists_u[P(u,v)]]$

{ $\Rightarrow$-intro on (1) and (6): }

(7) $\exists_x\forall_y[P(x,y)] \Rightarrow \forall_v\exists_u[P(u,v)]$

(b) We prove with a derivation that the formula

$$\forall_y[Q(y) \Rightarrow (P(y) \Rightarrow \exists_x[P(x) \wedge Q(x)])]$$

is a tautology:

{ Assume: }

(1) $\boxed{\textbf{var } y;\ \texttt{True}}$

{ Assume: }

(2) $\boxed{Q(y)}$

{ Assume: }

(3) $\boxed{P(y)}$

{ $\wedge$-intro on (3) and (2): }

(4) $P(y) \wedge Q(y)$

{ $\exists^*$-intro on (4): }

(5) $\exists_x[P(x) \wedge Q(x)]$

{ $\Rightarrow$-intro on (3) and (5): }

(6) $P(y) \Rightarrow \exists_x[P(x) \wedge Q(x)]$

{ $\Rightarrow$-intro on (2) and (6): }

(7) $Q(y) \Rightarrow (P(y) \Rightarrow \exists_x[P(x) \wedge Q(x)])$

{ $\forall$-intro on (1) and (7): }

(8) $\forall_y[Q(y) \Rightarrow (P(y) \Rightarrow \exists_x[P(x) \wedge Q(x)])]$

15.9 We prove with a derivation that the formula

$$(\forall_x[x \in \mathbb{N} : P(x)] \Rightarrow \exists_y[y \in \mathbb{N} : Q(y)]) \Rightarrow \exists_z[z \in \mathbb{N} : P(z) \Rightarrow Q(z)]$$

is a tautology.

{ Assume: }

(1) $\boxed{\forall_x[x \in \mathbb{N} : P(x)] \Rightarrow \exists_y[y \in \mathbb{N} : Q(y)]}$

{ Assume: }

(2) $\boxed{\neg\exists_x[x \in \mathbb{N} : \neg P(x)]}$

{ Assume: }

(3) $\boxed{\mathbf{var}\ x; x \in \mathbb{N}}$

{ Assume: }

(4) $\boxed{\neg P(x)}$

{ $\exists^*$-intro on (3) and (4): }

(5) $\exists_x[x \in \mathbb{N} : \neg P(x)]$

{ $\neg$-elim on (2) and (5): }

(6) `False`

{ $\neg$-intro on (4) and (6), followed by $\neg\neg$-elim: }

(7) $P(x)$

{ $\forall$-intro on (3) and (7): }

(8) $\forall_x[x \in \mathbb{N} : P(x)]$

{ $\vee$-intro on (2) and (9): }

(9) $\forall_x[x \in \mathbb{N} : P(x)] \vee \exists_x[x \in \mathbb{N} : \neg P(x)]$

{ Assume: }

(10) $\boxed{\forall_x[x \in \mathbb{N} : P(x)]}$

{ $\Rightarrow$-elim on (1) and (10): }

(11) $\exists_y[y \in \mathbb{N} : Q(y)]$

{ $\exists^*$-elim on (11): }

(12) Pick a y with $y \in \mathbb{N}$ and $Q(y)$

{ Assume: }

(13) $\boxed{P(y)}$

{ Still valid: }

(14) $Q(y)$

{ $\Rightarrow$-intro on (13) and (14): }

(15) $P(y) \Rightarrow Q(y)$

{ $\exists^*$-intro on (12) and (15): }

(16) $\exists_z[z \in \mathbb{N} : P(z) \Rightarrow Q(z)]$

{ $\Rightarrow$-intro on (10) and (16): }

(17) $\forall_x[x \in \mathbb{N} : P(x)] \Rightarrow \exists_z[z \in \mathbb{N} : P(z) \Rightarrow Q(z)]$

{ Assume: }

(18) $\boxed{\exists_x[x \in \mathbb{N} : \neg P(x)]}$

{ $\exists^*$-elim on (18): }

(19) Pick an x with $x \in \mathbb{N}$ and $\neg P(x)$

{ Assume: }

(20) $\boxed{P(x)}$

{ $\neg$-elim on (19) and (20): }

(21) `False`

{ `False`-elim on (21): }

(22) $Q(x)$

{ $\Rightarrow$-intro on (20) and (22): }

(23) $P(x) \Rightarrow Q(x)$

{ $\exists^*$-intro on (19) and (23): }

(24) $\exists_z[z \in \mathbb{N} : P(z) \Rightarrow Q(z)]$

{ $\Rightarrow$-intro on (18) and (24): }

(25) $\exists_x[x \in \mathbb{N} : \neg P(x)] \Rightarrow \exists_z[z \in \mathbb{N} : P(z) \Rightarrow Q(z)]$

{ Case distinction on (9), (17) and (25): }

(26) $\exists_z[z \in \mathbb{N} : P(z) \Rightarrow Q(z)]$

{ $\Rightarrow$-intro on (1) and (27): }

(27) $(\forall_x[x \in \mathbb{N} : P(x)] \Rightarrow \exists_y[y \in \mathbb{N} : Q(y)]) \Rightarrow \exists_z[z \in \mathbb{N} : P(z) \Rightarrow Q(z)]$

References

[1] Rob Nederpelt and Fairouz Kamareddine. *Logical Reasoning: A First Course*, volume 3 of *Texts in Computing*. King's College Publications, second revised edition edition, 2011.