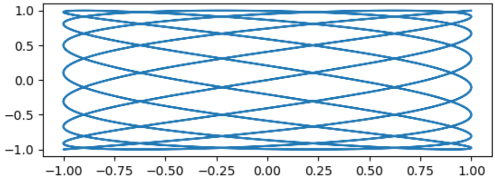
Using Python to support A-Level Mathematics

Lissajous figures

David de Oliveira   Hugh Shercliff

Cambridge University Engineering Department



Learning objectives

- To extend my knowledge of sinusoids and their properties,
- and to explore parametric equations,
- in order to find out what Lissajous figures are.

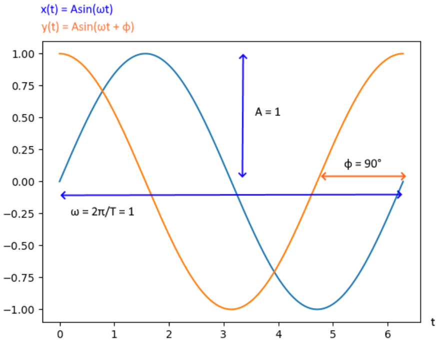
- To use Python to further explore Lissajous figures,
- and to see how they are used, especially in engineering contexts.

Sinusoids – a quick summary

- Sinusoids are simply sine waves, following the mathematical equation

$$f(t) = A\sin(\omega t + \phi) \tag{1}$$

- A is the amplitude,
- ω is the angular frequency (i.e. $\omega = \frac{2\pi}{T}$ where T is the time period)
- and ϕ is the phase



Activity 1 – Plotting sinusoids

This activity is just a quick warm-up in using Python to plot some graphs. Hints, tips and possible solutions for this and the following activities can be found in the LissajousHints document.

1. Using the `Numpy` and `Matplotlib` packages in Python, plot a graph of
$$y = \sin(x)$$
for $0 \leq x \leq 4\pi$, working in radians
2. How are the graphs of
$$y_1 = \sin(2x) \qquad \text{and}$$
$$y_2 = \sin(x + \pi)$$
different to the one you have just plotted? Try sketching these graphs out first, then plot them with Python

Activity 1 continued

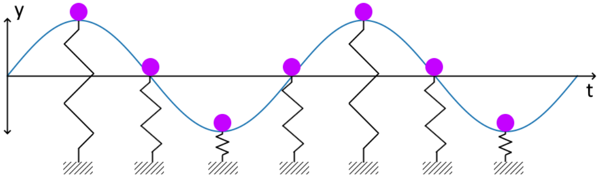
You may have noticed the graphs in question (2) are simple transformations of the original graph – a scaling and a translation respectively.

Predict and sketch what happens in the following transformations, and then plot them using Python to check your predictions:

1. $y = -3\sin(x)$
2. $y = \sin(-x) + 2$
3. $y = -\sin(-5x)$
4. $y = \frac{1}{2}\sin(\frac{x}{2} - \frac{3\pi}{2})$

Sinusoids to Lissajous figures

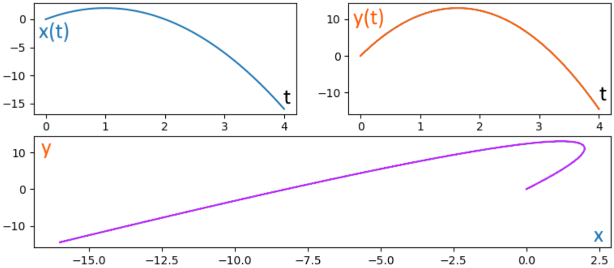
- Note how a sinusoid can describe simple harmonic motion – e.g. a swinging pendulum, or a mass-spring system:



- Some systems instead follow *complex* harmonic motion
- A Lissajous figure is one way to describe complex harmonic motion
- To understand these figures, a quick look at parametric equations will be taken on the next slide

Parametric equations

- Parametric equations are a system of equations which express quantities as functions of other independent variables
- *Example*: x and y are both functions of the parameter t
  - For each t, the point $((x(t), y(t))$ can be plotted on a set of Cartesian coordinates, as shown in the bottom graph:



Activity 2 – Parametric equations

In kinematics it is useful to look at motion in the x-, y- and z-directions separately, and to make each a function of time, t.

Let the following equations describe the motion of a particle in 2D space:

$$x = 2t \text{ and } y = 112.5 + 2t - 4.9t^2$$

1. Plot these functions, for $0 \leq t \leq 5$, on separate graphs
2. Predict and sketch what the overall motion of the particle looks like
3. Plot the overall motion, i.e. $\big(x(t),\ y(t)\big)$ and compare it to your prediction

Activity 2 continued

Parametric equations also allow for many interesting geometric shapes to be easily described.

Plot the following for $0 \leq t \leq 2\pi$:

1. $x = \cos(t),\ y = \sin(t)$
2. $x = \cos(4t)\cos(t),\ y = \cos(4t)\sin(t)$
3. $x = \sin(t)\left(e^{\cos(t)} - 2\cos(4t) - \sin^5\left(\frac{t}{12}\right)\right)$,
   $y = \cos(t)\left(e^{\cos(t)} - 2\cos(4t) - \sin^5\left(\frac{t}{12}\right)\right)$ for $0 \leq t \leq 12\pi$
4. Try your own!

Extension: Note that parametric equations can be converted to the form of $f(x,\ y) = 0$ and sometimes even to the form of $y = f(x)$. Try converting the kinematic system from the previous slide to the form of $y = f(x)$ and then plot it to verify you get the same graph.

Lissajous figures

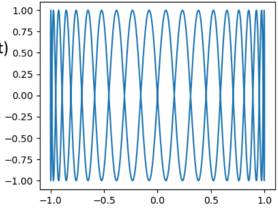
- A Lissajous figure is the set of parametric equations:

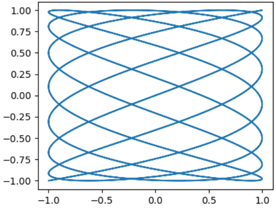
$$x(t) = A\sin(at) \tag{2}$$
$$y(t) = B\sin(bt + \phi) \tag{3}$$

where

- A and B are the amplitudes,
- a and b are the angular frequencies,
- and ϕ is the phase shift

x(t) = sin(t)
y(t) = sin(20t)





x(t) = sin(15t)
y(t) = sin(7t)

Activity 3 – Investigating Lissajous figures

A program, `lissajous.py`, has been provided which plots the resulting Lissajous figure for the angular frequencies and phase shift specified in the control panel, i.e.:

$$x(t) = \sin(\omega_x t)$$
$$y(t) = \sin(\omega_y t + \phi)$$

where ω_x and ω_y are the angular frequencies in the x and y directions respectively, and ϕ is the phase shift.

1. Run `lissajous.py`
2. Play around with the controls and see if you can determine any patterns/relationships between the control settings and the resulting Lissajous figures

Activity 3 continued

You may have noticed a relationship between the ratio $\frac{\omega_x}{\omega_y}$ and the number of loops in the resulting Lissajous figure, and also between ω_x, ω_y and whether the curve is closed or not.

For the following, first plot $x(t)$ against t and $y(t)$ against t $(0 \leq t \leq 2\pi)$ using Python, and then predict and sketch what the resulting Lissajous figure looks like. Finally verify your prediction by plotting the figure using Python.

1. $x = \sin(t),\ y = \sin(t)$
2. $x = \sin(t),\ y = \sin(t + \frac{\pi}{2})$
3. $x = \sin(5t),\ y = \sin(3t)$
4. $x = \sin(5t),\ y = \sin(3t + \frac{\pi}{2})$
5. $x = \sin(16t),\ y = \sin(12t)$
6. $x = \sin(1.6t),\ y = \sin(1.2t)$
7. $x = \sin(29t),\ y = \sin(13t)$
8. $x = \sin(2.9t),\ y = \sin(1.3t)$

You may notice the curves in (6) and (8) are incomplete – try to figure out why and then re-plot them to give complete figures

Lissajous figures – its uses

- Lissajous figures find many applications in science and engineering:
- Lissajous orbits are sometimes used by spacecraft when carrying out their mission
  - Examples include the *Gaia*, *Planck* and *WMAP* spacecraft

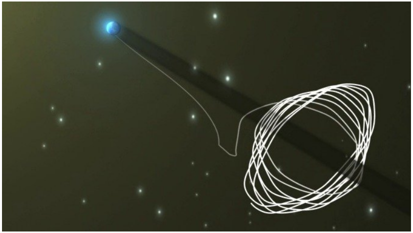


Figure: Gaia's orbit (Source: ESA)

- One reason for using this type of orbit is that it enables the spacecraft to maintain thermal stability

Lissajous figures – its uses

- Electrical engineers can use Lissajous figures to determine phase shifts in electrical systems
  - Using an oscilloscope in its *X-Y mode*, the phase shift can easily be found
  - *Challenge*: Build a simple ac circuit consisting of a capacitor and resistor in series and use this method to determine the phase shift of current with respect to voltage

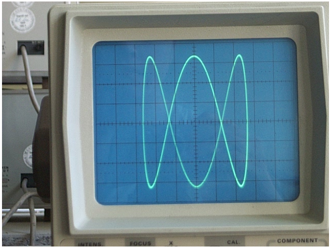


Figure: Oscilloscope showing an example of a Lissajous figure (Source: Oliver Kurmis)

An engineering problem – laser surface heat treatment

- One of the major tasks in engineering design is selecting an appropriate material, depending on its properties, for the required task
  - A civil engineer might choose steel for their bridge,
  - whereas an aerospace engineer might choose titanium for their jet turbine blade
- Materials can also be processed to alter their properties. For example:
  - some metals can be annealed in order to make them softer,
  - some metals can be tempered to increase their toughness

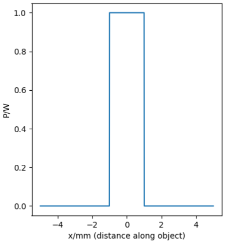
Figure: Top: The Severn Bridge uses steel for its strength; Bottom: Colours achieved when tempering steel

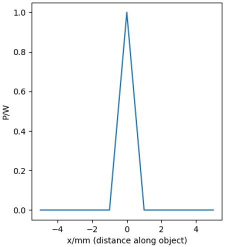
Laser surface heat treatment

- Metals can be surface hardened by heating up the surface of the metal, and then cooling it very quickly
- Lasers can be used to do this effectively, by scanning its light across the material
  - The parts hit by the laser will heat up, and then, once the laser light has gone by, quickly cool down
  - The scanning pattern can be chosen to control how the material is hardened
- In the final activity, you will explore how a Lissajous figure can be used as a scan pattern for the heat treatment of materials
  - We will start off by working in 1D, and then build up to 2D

Activity 4 – Lissajous laser scanning

Recall from science/physics lessons that power is the rate of energy transfer, e.g. $P = \frac{E}{t}$





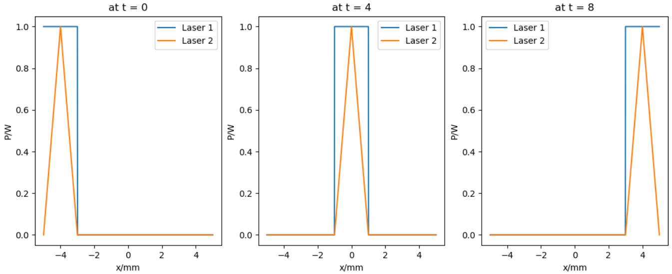
The graphs on the left show the power distributions for two lasers with finite spot size *(i.e. the size of the spot visible on an object when laser light is incident on it)*.
Each laser is shone on an object for $t = 5$ seconds. The centre of each laser spot is at $x = 0$ for the entire time. What is the total amount of energy transferred at $x = 0$ and at $x = 3$ when:

1. laser 1 (top) is used?
2. laser 2 (bottom) is used?

Activity 4 continued

Now each laser from the previous slide is moved along the x axis at a constant rate of 1 mm/s, from $x = -5$ to $x = 5$.
What is the total amount of energy transferred at $x = 0$ when:

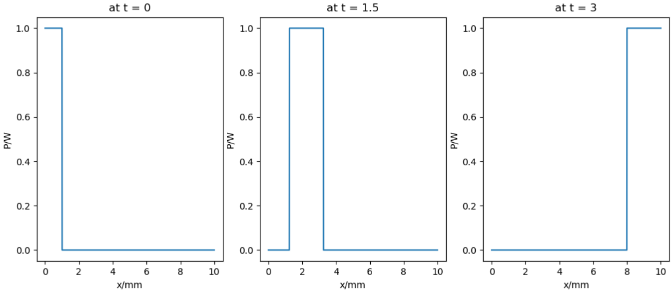
1. laser 1 is used?
2. laser 2 is used?



Activity 4 continued

Now consider a laser where its position, x, is a function of time, e.g. at time t, the laser's position is $x(t) = t^2$. This laser has the same power distribution as laser 1 in the previous questions.
The laser is shone at an object for $t = 3$ seconds.

1. What is the total amount of energy transferred at the point $x = 2$?



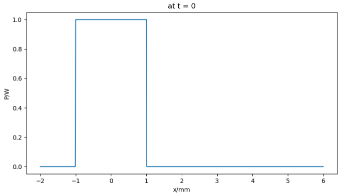
Activity 4 continued

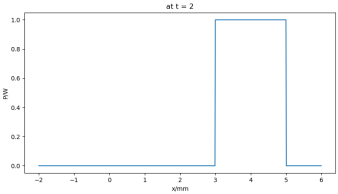
One way of solving the previous question is to think of power as a function of position and time, i.e. $p(x,t)$. For the power distribution of laser 1, this would be:

$$p(x,t) = \begin{cases} 1 & \text{for } -1 \leq x - t^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Here is one way of understanding the equation above:

- By holding t constant at a value, say 0, the graph of p against x will be as shown below, left
- If we now hold t constant at 2, the graph of p against x will be shifted to the right by $t^2 = 4$, as show below, right

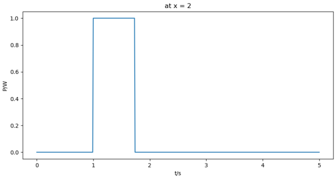


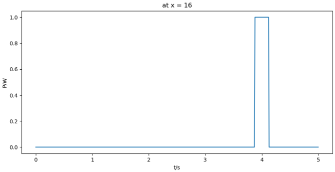


Activity 4 continued

$$p(x,t) = \begin{cases} 1 & \text{for } -1 \leq x - t^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

- By holding x constant at a value, say 2, the graph of p against t will be as shown below, left
- By holding x constant at 16, the graph of p against t will be as shown below, right





Activity 4 continued

$$p(x, t) = \begin{cases} 1 & \text{for } -1 \leq x - t^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

- Varying t effectively moves the position of the laser spot along the object
- Varying x effectively moves the position of interest along the object
- Thus, at a specific point x_0, to get the total energy absorbed, E_x, we integrate p with respect to t (from $P = \frac{E}{t}$):

$$E_x = \int p(x_0, t) \, \mathrm{d}t$$

This is easily extended to 2D. The last part of the activity will model a Lissajous laser scanner for heat treatment in 2D *(though many simplifying assumptions have been made in this model)*

Activity 4 continued

Open up the `gaussPow.py` script in an editor. This script defines the power distribution as:

$$p(x, y, t) = \exp\left\{-\frac{1}{2}\left(\frac{(x - tx)^2}{\sigma_x^2} + \frac{(y - ty)^2}{\sigma_x^2}\right)\right\}$$

where tx and ty are simply the x and y values given by the Lissajous parametric equations.

1. Plot this distribution using Python, for when $tx = ty = 0$
2. Run `gaussPow.py` and try to explain the output
3. Predict what the output will be for different Lissajous figures, then modify the script and run it to verify your prediction
4. What happens when the σ_x^2 and σ_y^2 values are changed?
5. An engineer must uniformally harden a flat piece of metal. Using this model of laser-induced heating, how can the engineer achieve this?