1 Fernandez-Villaverde Style Model

1.1 Labor Packer

Households have a variety of labor types $n_t(i)$ paid at a wage $w_t(i)$. The labor packer converts varying labor into homogeneous labor based on their production function:

$$n_t = \left(\int_0^1 n_t(i)^{\frac{\vartheta-1}{\vartheta}} di \right)^{\frac{\vartheta}{\vartheta-1}}$$

Where ϑ is the elasticity of substitution between labor types. The labor packer attempts to minimize their budget:

$$\min_{n_t(i)} \int_0^1 n_t(i) w_t(i) di$$

subject to their production function. Which yields the following first order condition:

$$n_t(i) = \left(\frac{w_t(i)}{w_t} \right)^{-\vartheta} n_t$$

This is the demand function, which relates the firms aggregate demand for labor to their individual demand over a given labor type. In solving this, we are able to derive the wage index by the zero-profit condition:

$$w_t = \left(\int_0^1 w_t(i)^{1-\vartheta} \right)^{\frac{1}{1-\vartheta}}$$

1.2 Households

The households have differentiated labor, that is packaged and sold to their firms by the labor packer. Households gain utility from consumption, c, and disutility from labor n, based on traditional preferences:

$$\max_{\{c_{t+j}, w_{t+j}(i), b_{t+j}\}_{j=0}^{\infty}} E_t \sum_{j=0}^{\infty} \beta^j \left[\frac{c_{t+j}^{1-\gamma}}{1-\gamma} - \chi \int_0^1 \frac{n_{t+j}(i)^{1+\eta}}{1+\eta} di \right]$$

subject to:

$$c_t + b_t + \int_0^1 \frac{\phi_w}{2} \left(\frac{w_t(i)\pi_t}{w_{t-1}(i)\bar{\pi}} - 1 \right)^2 y_t di = \int_0^1 w_t(i) n_t(i) di + \frac{r_{t-1} s_{t-1}}{\pi_t} b_{t-1} + d_t$$

$$w_t(i) \geq \frac{\omega w_{t-1}(i)}{\pi_t}$$

Where ω is the strictness of the downward nominal wage rigidity, $\pi \equiv \frac{p_t}{p_{t-1}}$ is the rate of inflation, and s_t is an AR(1) risk premium process. The consumer purchases consumption, bonds, and pays quadratic wage adjustment costs, while they are paid from labor, previous bonds returns, and dividends from firms.

The household has the following first order conditions for consumption, bonds, and the wage rate:

$$c_t^{-\gamma} = \lambda_t \tag{HH FOC C}$$

$$1 = r_t s_t E_t\left[\frac{x_{t+1}}{\pi_{t+1}}\right] \tag{HH FOC B}$$

$$\phi_w y_t\left(\frac{w_t\pi_t}{w_{t-1}\bar{\pi}} - 1\right)\left(\frac{w_t\pi_t}{w_{t-1}\bar{\pi}}\right) + \beta\omega\frac{w_t}{\lambda_t}E_t\left[\frac{\mu_{t+1}}{\pi_{t+1}}\right] =$$

$$\vartheta\frac{\chi}{\lambda_t}n_t^{\eta+1} - (\vartheta - 1)n_t w_t + E_t\left[\phi_w x_{t+1}y_{t+1}\left(\frac{w_{t+1}\pi_{t+1}}{w_t\bar{\pi}} - 1\right)\left(\frac{w_{t+1}\pi_{t+1}}{w_t\bar{\pi}}\right)\right] + \frac{w_t}{\lambda_t}\mu_t \tag{WPC}$$

$$\mu_t \geq 0 \perp w_t - \frac{\omega w_{t-1}}{\pi_t} \geq 0 \perp \mu_t(w_t - \frac{\omega w_{t-1}}{\pi_t}) \tag{KKT Condition}$$

Where λ_t is the Lagrange Multiplier for consumption, μ_t is the (KKT) Lagrange Multiplier of the DNWR, and $x_t \equiv \beta\frac{\lambda_t}{\lambda_{t-1}}$ is the stochastic discount factor.

1.3 Final Goods Firm

There is a competitive final goods firm that aggregates the production of the intermediate goods firms using a CES production function:

$$y_t = \left(\int_0^1 y_t(f)^{\frac{\theta-1}{\theta}}df\right)^{\frac{\theta}{\theta-1}}$$

Where θ is the elasticity of substitution between intermediate goods. Taking prices as given, the firm attempts to maximize their profit function:

$$p_t y_t - \int_0^1 p_t(f)y_t(f)df$$

subject to their production function. This yields two relationships for the final goods firm, the demand function:

$$y_t(f) = y_t\left(\frac{p_t(f)}{p_t}\right)^{-\theta}$$

and the price index:

$$p_t = \left[\int_0^1 p_t(f)^{1-\theta}df\right]^{\frac{1}{1-\theta}}$$

1.4 Intermediate Goods Firms

The intermediate goods are produced by a monopolistically competitive firm with the following production function:

$$y_t(f) = z_t n_t(f)^\alpha \implies y_t = z_t n_t^\alpha \tag{PF}$$

where α is output elasticity of labor and z is total factor productivity determined exogenously. Cost minimization implies that intermediate goods producers all have the same marginal cost:

$$\begin{aligned} TC_t &= w_t n_t \\ TC_t &= w_t \big(\frac{y_t}{z_t}\big)^{\frac{1}{\alpha}} \\ mc_t = \frac{\partial TC_t}{\partial y_t} &= \frac{w_t z_t^{-\frac{1}{\alpha}}}{\alpha} y_t^{\frac{1}{\alpha}-1} \\ \implies mc_t &= \frac{w_t}{\alpha z_t} n_t^{1-\alpha} \end{aligned} \tag{MCF}$$

The intermediate goods firm is tasked with maximizing profit:

$$\max_{p_t(f)} E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} \left[\frac{p_{t+j}(f)}{p_{t+j}} y_{t+j}(f) - mc_{t+j} y_{t+j}(f) - \frac{\phi_p}{2} \left(\frac{p_{t+j}(f)}{p_{t-1+j}(f)} - \bar{\pi} \right)^2 y_{t+j}(f) \right]$$

subject to:

$$y_t(f) = \left(\frac{p_t(f)}{p_t} \right)^{-\theta} y_t$$

Optimizing this problem yields the traditional Price-Phillips curve:

$$0 = 1 - \theta + \theta mc_t - \phi_p \pi_t (\pi_t - \bar{\pi}) + \frac{\theta \phi_p}{2} (\pi_t - \bar{\pi})^2 + \phi_p E_t \left[x_{t+1} \pi_{t+1} (\pi_{t+1} - \bar{\pi}) \frac{y_{t+1}}{y_t} \right] \tag{PPC}$$

1.5 Central Bank

To maintain a healthy level of inflation and output, the central bank sets the notational interest rate:

$$r_{n,t} = \rho_{r_n} r_{n,t-1} + (1 - \rho_{r_n})(\bar{r}_n + \varphi_\pi (\pi_t - \bar{\pi}) + \varphi_y \left(\frac{y_t}{\bar{y}} - 1 \right)) \tag{Taylor's}$$

This interest rate is subjected to the Zero-Lower-Bound:

$$r_t = 1 + \frac{1}{\psi} \ln(1 + e^{\psi(r_{n,t} - 1)}) \tag{ZLB}$$

Where ψ is the convergence parameter for the LogExpSum function into the max function.

1.6 Aggregation

We can aggregate the budget constraint to create the aggregate resource constraint:

$$c_t = \left[1 - \frac{\phi_w}{2}\left(\frac{w_t \pi_t}{w_{t-1}\bar{\pi}} - 1\right)^2 - \frac{\phi_p}{2}\left(\pi_t - \bar{\pi}\right)^2\right] y_t \tag{ARC}$$

1.7 Non-Linear System

This allows us to make a non-linear system to analyze:

$$c_t^{-\gamma} = \lambda_t \tag{1}$$

$$x_t = \beta \frac{\lambda_t}{\lambda_{t-1}} \tag{2}$$

$$1 = r_t s_t E_t\left[\frac{x_{t+1}}{\pi_{t+1}}\right] \tag{3}$$

$$\phi_w y_t \left(\frac{w_t \pi_t}{w_{t-1}\bar{\pi}} - 1\right)\left(\frac{w_t \pi_t}{w_{t-1}\bar{\pi}}\right) + \beta\omega \frac{w_t}{\lambda_t} E_t\left[\frac{\mu_{t+1}}{\pi_{t+1}}\right] = \tag{4}$$

$$\vartheta \frac{\chi}{\lambda_t} n_t^{\eta+1} - (\vartheta - 1) n_t w_t + E_t\left[\phi_w x_{t+1} y_{t+1}\left(\frac{w_{t+1}\pi_{t+1}}{w_t \bar{\pi}} - 1\right)\left(\frac{w_{t+1}\pi_{t+1}}{w_t \bar{\pi}}\right)\right] + \frac{w_t}{\lambda_t}\mu_t$$

$$\sqrt{\mu_t^2 + (w_t - \frac{\omega w_{t-1}}{\pi_t})^2} - (\mu_t + (w_t - \frac{\omega w_{t-1}}{\pi_t})) = 0 \tag{5}$$

$$y_t = z_t n_t^{\alpha} \tag{6}$$

$$mc_t = \frac{w_t}{\alpha z_t} n_t^{1-\alpha} \tag{7}$$

$$0 = 1 - \theta + \theta mc_t - \phi_p \pi_t(\pi_t - \bar{\pi}) + \frac{\theta \phi_p}{2}(\pi_t - \bar{\pi})^2 + \phi_p E_t\left[x_{t+1}\pi_{t+1}(\pi_{t+1} - \bar{\pi})\frac{y_{t+1}}{y_t}\right] \tag{8}$$

$$r_{n,t} = \rho_{r_n} r_{n,t-1} + (1 - \rho_{r_n})(\bar{r}_n + \varphi_\pi(\pi_t - \bar{\pi}) + \varphi_y\left(\frac{y_t}{\bar{y}} - 1\right)) \tag{9}$$

$$r_t = 1 + \frac{1}{\psi}\ln(1 + e^{\psi(r_{n,t} - 1)}) \tag{10}$$

$$c_t = \left[1 - \frac{\phi_w}{2}\left(\frac{w_t \pi_t}{w_{t-1}\bar{\pi}} - 1\right)^2 - \frac{\phi_p}{2}\left(\pi_t - \bar{\pi}\right)^2\right] y_t \tag{11}$$

$$\ln(z_t) = \rho_z \ln(z_{t-1}) + \sigma_z \epsilon_{z,t} \tag{12}$$

$$\ln(s_t) = \rho_s \ln(s_{t-1}) + \sigma_z \epsilon_{s,t} \tag{13}$$

Equations (13): MUC, SDF, FOC Bonds, FOC Wage, KKT DNWR, PF, MC, Phillip's Curve, Taylor's Rule, LogExpSum ZLB, ARC, TFP AR(1), RP AR(1)

Variables (13): $c, \lambda, \mu, x, r, \pi, n, w, y, mc, r_n, z, s$

Parameters (18): $\gamma, \beta, \chi.\eta, \omega, \phi_w, \vartheta, \alpha, \theta, \phi_p, \varphi_\pi, \varphi_y, \rho_z, \sigma_z, \rho_s, \sigma_s, \rho_{r_n}, \psi$

1.8 Proofs

We want to show that the KKT conditions derived in the households problem is logically equivalent to equation 5. We will show that:

$$\sqrt{\mu_t^2 + (w_t - \frac{\omega w_{t-1}}{\pi_t})^2} - (\mu_t + (w_t - \frac{\omega w_{t-1}}{\pi_t})) = 0 \iff \mu_t \geq 0 \perp w_t - \frac{\omega w_{t-1}}{\pi_t} \geq 0 \perp \mu_t(w_t - \frac{\omega w_{t-1}}{\pi_t}) = 0$$

To do so, first we will define $G_t = w_t - \frac{\omega w_{t-1}}{\pi_t}$ for simplicity then:

" $\implies$ "

$\sqrt{\mu_t^2 + G_t^2} - (\mu_t + G_t) = 0 \implies \sqrt{\mu_t^2 + G_t^2} = \mu_t + G_t \implies \mu_t^2 + G_t^2 = \mu_t^2 + 2\mu_t G_t + G_t^2 \implies \mu_t G_t = 0$

Which is our third condition. Then we can use that condition to get the first two:

$\sqrt{\mu_t^2 + G_t^2} \geq 0 \implies \mu_t + G_t \geq 0$. Since $\mu_t G_t = 0$ either $\mu_t = 0$, $G_t = 0$ or both equal 0. If both equal 0, $0 \geq 0 \implies G_t \geq 0$ and $\mu_t \geq 0$. If only one equals 0, it implies the other is greater than or equal to 0, WLOG, assume $G_t = 0$. Thus: $\mu_t + G_t \geq 0 \implies \mu_t \geq 0$. The same logic applies the other way.

" $\impliedby$ "

If $\mu_t G_t = 0$, $\mu_t \geq 0$, and $G_t \geq 0$, $\implies$ either $\mu_t = 0$ or $G_t = 0$. WLOG assume $G_t = 0$, then $\sqrt{\mu_t^2 + G_t^2} - (\mu_t + G_t) = |\mu_t| - \mu_t$ and since $\mu_t \geq 0$, $\sqrt{\mu_t^2 + G_t^2} - (\mu_t + G_t) = 0$. The logic similarly applies to when $\mu_t = 0$.