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Quantum optical correlations in the absence of intensity correlations

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Abstract

The simplest way to characterize optical states is based on intensity measurements leading also to the birth of intensity correlations when the system is investigated by using suitable interferometric setups. However, it is possible to have correlations beyond the intensity ones. In this thesis we study a particular class of states that, when sent through a beam splitter, do not give rise to intensity correlation but to quantum ones. We apply the modern tools of quantum information to characterize such states and discuss their possible application in quantum communication.

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Notation

The next list describes several symbols that will be used within the body of the thesis

Functions

$P(\alpha) := W(\alpha, 1)$ Glauber-Sudarshan P-representation

$Q(\alpha) := W(\alpha, -1)$ Husimi or Q-function

$W(\alpha) := W(\alpha, 0)$ Wigner function

$W(\alpha, p)$ Quasi-probability distribution

$W(\alpha, p)[\hat{\varrho}]$ Quasi-probability distribution of a density operator $\hat{\varrho}$

$\chi(\lambda) := \chi(\lambda, 0)$ symmetric ordered characteristic function

$\chi(\lambda, p)$ p-ordered characteristic function

$L_n(x)$ Laguerre polynomials of order n

Miscellanea

$\alpha^*; \alpha \in \mathbb{C}$ complex conjugate of α

$|ab\rangle = |a\rangle \otimes |b\rangle$ tensor product for a two mode state, a and b

$d^2z; z \in \mathbb{C}$ $d(\mathfrak{Re}\{z\})\, d(\mathfrak{Im}\{z\})$

Operators

$\delta^n(x)$ n-dimensional Dirac Delta

$\hat{a}_l$ annihilation operator on l-mode

$\hat{a}_l^\dagger$ creation operator on l-mode

$\hat{n} = \hat{a}^\dagger \hat{a}$ number operator

$\hat{x}_\theta = \hat{a}\, e^{-i\theta} + \hat{a}^\dagger\, e^{i\theta}$ quadrature operator

$\mathrm{Tr}[\hat{A}]$ Trace of an operator $\hat{A}$

Introduction

Optical correlations play a major role in quantum information and quantum technologies. An optical state can be investigated using an interferometric setup, that consists of a two-modes state mixing in a beam splitter, whose two outputs are revealed by photodetectors. These measurements, in general, give rise to a nonzero intensity correlation function and quantum correlations.

It is known that if we mix two equals thermal states in a balanced beam splitter, the resulting intensity correlation of the two outputs vanishes [1]. This effect is due to the particular dependence of the energy variance on the energy mean value. The thermal states class is classical and arises whenever we need to describe the optical properties of a radiation field at thermal equilibrium, e.g. black body radiation, but it can be generated also in quantum optics laboratories by suitably manipulating the optical fields, for instance, by means of rotating ground glass (Arecchi's disk) [2].

In this thesis we study the quantum properties of an optical state, expressed by the following diagonal density operator: $\hat{\varrho}_m = p_0 |0\rangle \langle 0| + p_1 |1\rangle \langle 1| + p_2 |2\rangle \langle 2|$ where $|n\rangle$ is a Fock state. We want this state to have the same first two moments i.e. mean and variance, of a thermal state, in order to have a vanishing intensity correlation, once we mix two $\hat{\varrho}_m$ in a balanced beam splitter.

With these assumptions, we can investigate the quantum or classical correlations that might arise after a beam splitter interaction. Our purpose is to study the nonclassicality and the nonlocality of this state quantitatively.

The nonclassicality of a quantum light state is related to its impossibility to be described by the classical Maxwell equations and can be quantified by the so-called nonclassical depth τ [3]. It turns out that this parameter is maximized, namely $\tau = 1$, by Fock states, and minimized,

$\tau = 0$, by coherent states, making the Fock states $|n\rangle$ the most nonclassical states among all the optical states. This parameter can thus be related to the number of thermal photons that have to be added to a quantum state in order to erase all its quantum features.
In order to compute this parameter, we introduced a class of functions, the quasi-probability function, and the Wigner function, that allow us to put some sufficient conditions about the nonclassicality of our state. In particular, we want to evaluate whenever the quasi-probability function is always positive-defined and can be considered as a classical probability distribution function [4]. If this measure of nonclassicality had to give positive results, our state would be certainly nonclassical, and we might expect it to show some nonlocal behaviors such as entanglement, as nonclassicality is a necessary condition for a state to be nonlocal [5, 6]. This quantum feature is related to the possibility of nonlocal interactions by two causal disconnected subsystems, in our case, after the interaction in the beam splitter.

In this context, the Bell theorems show that a local theory must satisfy certain inequalities, experimental verifiable, on the other hand, if the same inequalities are violated it means that the considered system is entangled. The Bell inequalities gave a quantitative criterion in order to measure the nonlocality of a physics system and to experimentally verify this feature. We chose to investigate the nonlocality of $\hat{\varrho}_m$ with the CHSH inequality: it states that, if it holds $2 < |\mathcal{B}| < 2\sqrt{2}$, with $\mathcal{B}$ the Bell parameter, then we are dealing with a nonlocal system. This is, therefore, a sufficient condition.
The considered cases are about a beam splitter mix between $\hat{\varrho}_m$ with two kinds of states: the vacuum state $|0\rangle\langle 0|$, and secondly with another copy of itself.

The thesis is structured as follows. In Chapter 1, we give an introduction to all those quantum optics concepts as Fock states and thermal states. We also define the intensity correlation function of a state after a beam splitter mix. In the 2$^{\text{nd}}$ Chapter we expose the Wigner formalism, aiming to provide all the required notions, such as quasi-probability function and Gaussian states, that will be useful in order to define quantitative parameters for nonclassicality and nonlocality. Thus we define the nonclassical depth τ and the Bell parameter $\mathcal{B}$ in Chapter 3, which will give, respectively, a measure of nonclassicality and nonlocality of our state. Finally, in Chapter 4 we discuss the properties of the state $\hat{\varrho}_m$, defining it and we will study in detail all his nonclassical features.

Chapter 1

Basics of quantum optics

The principal purpose of this chapter is to introduce the key concepts of quantum optics and to give a solid base to those concepts that will be used all along with the thesis, as Fock states and intensity correlation function.

We are starting from the quantization of the electromagnetic field, defining the Hamiltonian for a multimode electromagnetic field [7] with some of its properties [8], with the consequent introduction of all the quantum optical states with their characteristics [8, 9]. Then, we will give an introduction of the beam splitter, an optical device used to mix the modes of different optical states, finally, we will give the definition of intensity correlation function [1].

1.1 Quantization of electromagnetic field

In this Section we present a brief discussion of the quantization of the electromagnetic field. We start with the case of a single-mode field and later generalized to a multimode field. Afterwards we will analyze the properties of the quadrature and its relation with the quadrature operators and its consequences.

In classic theory, the electromagnetic field is described entirely by Maxwell equations:

$$\nabla \cdot \mathbf{E} = 0, \tag{1.1a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.1b}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.1c}$$

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}, \tag{1.1d}$$

that we reported in absence of sources, in the vacuum and in with $c = 1$, **E** and **B** are the electric and magnetic vector fields.
The classical field energy of a single-mode field in a one-dimensional cavity of length L and volume V, is given by:

$$\begin{aligned} H &= \frac{\varepsilon_0}{2} \int dV \left[\mathbf{E}^2(\mathbf{r}, t) + \mathbf{B}^2(\mathbf{r}, t) \right], \\ &= \frac{1}{2}(p^2 + \omega^2 q^2), \end{aligned} \tag{1.2}$$

where we integrate on the volume of the cavity. The last equality can be proven if we naively take the fields mutually orthogonal between them, characterizing a beam of light propagating in the z direction:

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= E_x(z, t) = E_0 \; q(t) \sin(kz), \\ \mathbf{B}(\mathbf{r}, t) &= B_y(z, t) = B_0 \; p(t) \cos(kz), \end{aligned} \tag{1.3}$$

where $E_0 = \omega \sqrt{\frac{2}{V \varepsilon_0}}$, $B_0 = \frac{E_0}{\omega} = \sqrt{\frac{2}{V \varepsilon_0}}$, with ε_0 the vacuum permittivity, in natural units it is $\varepsilon_0 = \mu_0^{-1}$. The oscillating frequency is $\omega = kc$ and, in natural units, equals the wave number. We also have to assume that the electric field vanish at the boundaries: $E_x(0, t) = E_x(L, t) = 0$. The quantities $q(t)$ and $p(t)$ have the dimension of a length and a momentum. We shall see [8] that $q(t)$ and $\dot{q}(t)$ will play the role of the canonical position and momentum: $p(t) = \dot{q}(t)$.
The single-mode field Hamiltonian resembles that of a harmonic oscillator of frequency ω and unit mass, where $q(t)$ and $p(t)$ play exactly the role of the canonical position and momentum. In the process of quantization, once we recognized the canonical variables q, p, we can just replace them with the corresponding operators $\hat{q}, \hat{p}$ that satisfy:

$$[\hat{q}, \hat{p}] = i, \tag{1.4}$$

also in this case we left $\hbar = 1$. In this way we can replace all the operators:

$$\begin{aligned} \hat{E}_x(z, t) &= E_0 \; \hat{q}(t) \sin(kz), \\ \hat{B}_y(z, t) &= B_0 \; \hat{p}(t) \cos(kz), \end{aligned} \tag{1.5}$$

and the Hamiltonian:

$$\hat{\mathcal{H}} = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2). \tag{1.6}$$

It is useful to introduce two non-hermitian operators: $\hat{a}$ and $\hat{a}^\dagger$, the so-called annihilation and creation operators

$$\begin{aligned}\hat{a} &= \sqrt{\frac{\omega}{2}}\hat{q} + i\sqrt{\frac{1}{2\omega}}\hat{p},\\ \hat{a}^\dagger &= \sqrt{\frac{\omega}{2}}\hat{q} - i\sqrt{\frac{1}{2\omega}}\hat{p},\end{aligned} \tag{1.7}$$

that satisfy:

$$[\hat{a}, \hat{a}^\dagger] = 1, \tag{1.8}$$

and the corresponding generalized coordinates operators:

$$\begin{aligned}\hat{q} &= \sqrt{\frac{1}{2\omega}}(\hat{a} + \hat{a}^\dagger),\\ \hat{p} &= -i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^\dagger).\end{aligned} \tag{1.9}$$

In this way we can rewrite the (1.5) as:

$$\begin{aligned}\hat{E}_x(z,t) &= \mathcal{E}_0(\hat{a} + \hat{a}^\dagger)\sin(kz),\\ \hat{B}_y(z,t) &= -i\mathcal{B}_0(\hat{a} - \hat{a}^\dagger)\cos(kz),\end{aligned} \tag{1.10}$$

where $\mathcal{E}_0 = \mathcal{B}_0 = \sqrt{\frac{\omega}{\varepsilon_0 V}}$.

Finally, the Hamiltonian takes the form:

$$\hat{\mathcal{H}} = \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{1.11}$$

We can now generalize it [7], to a multimode field, generalizing all the operators to multimode operators, each of them acting independently on its own mode. In this way we have $\hat{a}_l^\dagger$ and $\hat{a}_l$ as the creation and the annihilation operator for the l-th mode of oscillation, respectively, with the following properties:

$$[\hat{a}_l, \hat{a}_k^\dagger] = \delta_{l,k}, \qquad [\hat{a}_l, \hat{a}_k] = [\hat{a}_l^\dagger, \hat{a}_k^\dagger] = 0. \tag{1.12}$$

Afterwards the Hamiltonian (1.11) reads:

$$\hat{\mathcal{H}} = \sum_l \omega_l\left(\hat{a}_l^\dagger\hat{a}_l + \frac{1}{2}\right), \tag{1.13}$$

that is the Hamiltonian of a set of independent quantum harmonic oscillators with frequencies ω_l.

1.1.1 Quadrature operators

We can think to an electromagnetic wave equation in isotropic insulating medium:

$$\nabla^2 \mathbf{E}(\mathbf{r},t) - \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = 0, \tag{1.14}$$

whose solution can be written as:

$$\mathbf{E}(\mathbf{r},t) = [\alpha(\mathbf{r})e^{-i\omega t} + \alpha^*(\mathbf{r})e^{i\omega t}]\mathbf{p}(\mathbf{r}), \tag{1.15}$$

where $\mathbf{r}$ and t represents the position and the time respectively, $\alpha(\mathbf{r})$ is the complex amplitude, and ω still is the frequency of the oscillation, $\mathbf{p}(\mathbf{r})$ is the polarization vector.
The general expression of the amplitude reads: $\alpha = |\alpha|e^{i\phi(\mathbf{r})}$, where $|\alpha|$ is the magnitude of the field, and $\phi(\mathbf{r})$ determines the shape of the wave front. With this consideration, the (1.15) becomes:

$$\mathbf{E}(\mathbf{r},t) = 2|\alpha(\mathbf{r})|\cos(\phi(\mathbf{r}) - \omega t). \tag{1.16}$$

It is the moment to introduce the classical quadratures:

$$\begin{aligned} x_1(\mathbf{r}) &= |\alpha(\mathbf{r})| + |\alpha^*(\mathbf{r})|, \\ x_2(\mathbf{r}) &= -i\left(|\alpha(\mathbf{r})| - |\alpha^*(\mathbf{r})|\right), \end{aligned} \tag{1.17}$$

in order to write (1.15) as follows:

$$\mathbf{E}(\mathbf{r},t) = E_0[x_1(\mathbf{r})\cos(\omega t) + x_2(\mathbf{r})\sin(\omega t)]\mathbf{p}(\mathbf{r}), \tag{1.18}$$

where we can express [7] the complex amplitude as: $\alpha(t) = |\alpha| + \delta x_1(t) + i\delta x_2(t)$, with $|\alpha| \gg \delta x_1(t), \delta x_2(t)$. With this formalism we can think at x_1 as the amplitude and x_2 as the phase of an electromagnetic wave.
The same discussion can be done, without loss of generality, by promoting the classical quadratures to quadrature operator:

$$\begin{aligned} \hat{x}_1 &= \hat{a} + \hat{a}^\dagger, \\ \hat{x}_2 &= -i(\hat{a} - \hat{a}^\dagger). \end{aligned} \tag{1.19}$$

In order to write the electric field operator as:

$$\begin{aligned} \hat{E}_x(z,t) &= \mathcal{E}_0(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t})\sin(kz), \\ &= \mathcal{E}_0[\hat{x}_1\cos(\omega t) + \hat{x}_2\sin(\omega t)]\sin(kz), \end{aligned} \tag{1.20}$$

where we explicitly expressed the time dependence of the creation and annihilation operators. This fact brings us to define a generic quadrature operators:

$$\begin{aligned}\hat{x}_\theta &= \hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}, \\ &= \hat{x}_1\cos(\theta) + \hat{x}_2\sin(\theta).\end{aligned} \tag{1.21}$$

Is clear that $\hat{x}_1$ and $\hat{x}_2$, and hence $\hat{x}_\theta$, are related to the field amplitudes oscillating out of phase each other by 90°, that's because of the "quadrature". This fact has to be kept in mind when we will discuss the statistical properties of pure states.

1.2 Principal optical states

Afterwards the quantization of the electromagnetic field, it is possible to introduce a large variety of states. For our purpose it is necessary to see some properties of the most common states, as Fock, coherent, thermal and squeezed states. We saw that the Hamiltonian of a multimode field can be written as (1.13):

$$\begin{aligned}\hat{\mathcal{H}} &= \sum_l \omega_l \left(\hat{a}_l^\dagger \hat{a}_l + \frac{1}{2}\right), \\ &= \sum_l \omega_l \left(\hat{n}_l + \frac{1}{2}\right),\end{aligned} \tag{1.22}$$

where we introduced the number operator $\hat{n}_l = \hat{a}_l^\dagger \hat{a}_l$ that will play an important role in the definition of the following optical states.

Fock states

Once the electromagnetic field has been quantized, it is possible to introduce a space of a bosons system, namely the Fock space, whose basis states are the so-called the Fock states. Particularly, the eigenstates of the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$ are called Fock states or number states. The corresponding eigenvalues are integer numbers n. We can write this relation with the following notation:

$$\hat{n}|n\rangle = n|n\rangle, \qquad n \in \mathbb{N}. \tag{1.23}$$

In this space the annihilation and creation operators have the properties:

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{1.24}$$

In this way we can notice that a generic Fock state can be written as follows:

$$|n\rangle = \frac{\left(\hat{a}^{\dagger}\right)^{n}}{\sqrt{n!}} |0\rangle, \tag{1.25}$$

where the state $|0\rangle$ is called vacuum state and doesn't contain any quanta of radiation field. The number n, in a single mode of the radiation field, is the number of photons in the field, so, the energy increases with n.

Furthermore, the Fock states $|n\rangle$ are orthogonal and complete:

$$\langle n|m\rangle = \delta_{nm}, \tag{1.26a}$$

$$\sum_{n} |n\rangle\langle n| = \mathbb{I}. \tag{1.26b}$$

At the end we can calculate the mean and the variance of the number operator $\hat{n}$ and the expectation value for the quadrature operator given the Fock state $|n\rangle$:

$$\langle \hat{n}\rangle = \langle n|\hat{n}|n\rangle = n, \tag{1.27a}$$

$$\Delta^2(\hat{n}) = \langle n|\hat{n}^2|n\rangle - \langle n|\hat{n}|n\rangle^2 = 0, \tag{1.27b}$$

$$\langle \hat{x}_\theta\rangle = \langle n|\hat{x}_\theta|n\rangle = 0, \tag{1.27c}$$

$$\Delta^2(\hat{x}_\theta) = \langle \hat{x}_\theta^2\rangle = 2n+1. \tag{1.27d}$$

The property (1.27c) tells us that the Fock states could not represent fields with well-defined amplitudes and phase at a classical level, as its expectation value vanish. In fact, in classical theory, these properties are always well-defined and they could not be represented by a state that makes null its mean value. We also know that a field produced by a single mode laser is said to be coherent, i.e. has a well-defined amplitude and phase.

Coherent states

This observation pushes us to define another class of states that can describes a coherent field. Those states are named coherent states and were defined for the first time by Glauber [10, 11] as it follows:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \qquad \alpha \in \mathbb{C}. \tag{1.28}$$

We can immediately check the expectation value of the quadrature operator:

$$\langle \hat{x}_\theta\rangle = \langle \alpha|\hat{x}_\theta|\alpha\rangle = \langle \alpha|\hat{a}|\alpha\rangle e^{-i\theta} + \langle \alpha|\hat{a}^{\dagger}|\alpha\rangle e^{i\theta} = 2|\alpha|\mathfrak{Re}\{e^{-i\theta}\}, \tag{1.29}$$

that is the same expression as (1.16), that was what we were aiming: we can use coherent states to describe a collimated beam of light with well-defined amplitudes and phase.

We also compute the variance of the quadrature operator:

$$
\begin{aligned}
\langle \hat{x}_\theta^2 \rangle &= \langle \alpha | (\hat{a} e^{-i\theta} + \hat{a}^\dagger e^{i\theta})^2 | \alpha \rangle \, , \\
&= \langle \alpha | \hat{a}^2 e^{-2i\theta} + \hat{a}^{\dagger 2} e^{2i\theta} + 2\hat{a}^\dagger \hat{a} + 1 | \alpha \rangle \, , \\
&= \alpha^2 e^{-2i\theta} + \alpha^{*2} e^{-2i\theta} + 2\alpha\alpha^* + 1,
\end{aligned}
\tag{1.30}
$$

that, with (1.29), brings to:

$$
\mathrm{Var}(\hat{x}_\theta) = \langle \hat{x}_\theta^2 \rangle - \langle \hat{x}_\theta \rangle^2 = 1. \tag{1.31}
$$

The coherent states are nonorthogonal and overcomplete:

$$
\langle \alpha | \beta \rangle = \exp\left(\alpha^* \beta - \frac{|\alpha|^2 + \beta|^2}{2} \right) \neq 0, \tag{1.32a}
$$

$$
\frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \; |\alpha\rangle\langle\alpha| = \mathbb{I}. \tag{1.32b}
$$

A proof of (1.32b) can be found in Appendix B. The non orthogonality let a given coherent state to be written in terms of other coherent states:

$$
|\alpha\rangle = \frac{1}{\pi} \int d^2\beta \; \langle \alpha | \beta \rangle | \beta \rangle. \tag{1.33}
$$

We can always generate a coherent state by displacing the vacuum state:

$$
|\alpha\rangle = \hat{D}(\alpha) |0\rangle, \tag{1.34}
$$

$$
\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}. \tag{1.35}
$$

Where we defined $\hat{D}(\alpha)$ as the displacement operator whose main properties are written in the Appendix A. An easy proof of (1.34) is in the Appendix A.

At the end, we can solve the eigenvalue equation (1.28) in order to calculate the photon number distribution and the statistics of the system. We can expand a coherent state in terms of the Fock states, because they form a complete set. Using the completeness relation:

$$
\begin{aligned}
|\alpha\rangle &= \sum_{n=0}^{+\infty} |n\rangle\langle n|\alpha\rangle, \\
&= \sum_{n=0}^{+\infty} \langle n|\alpha\rangle |n\rangle, \\
&= \langle 0|\alpha\rangle \sum_{n=0}^{+\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,
\end{aligned}
\tag{1.36}
$$

with:

$$\langle n|\alpha\rangle = \langle 0|\frac{(\hat{a})^n}{\sqrt{n!}}|\alpha\rangle = \langle 0|\alpha\rangle\,\frac{\alpha^n}{\sqrt{n!}} = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}},$$

leading to the condition : $|\langle 0|\alpha\rangle| = e^{-|\alpha|^2/2}$ for the normalization.
Definitely we can write:

$$p(n) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}. \tag{1.37}$$

This is the probability of finding n photons in a coherent state, and this probability is thus given by the Poissonian distribution with mean and variance $|\alpha|^2$:

$$\langle\hat{n}\rangle = |\alpha|^2, \qquad \Delta^2(\hat{n}) = |\alpha|^2. \tag{1.38}$$

1.2.1 Mixed states

Now, it is possible to build a new variety of states, using superpositions of Fock states or coherent states. In order to describe efficiently these kind of states, we have to introduce a density operator $\hat{\varrho}$:

$$\hat{\varrho} = \sum_n p_n\,|\psi_n\rangle\,\langle\psi_n|, \tag{1.39}$$

with the constraints that:

- $\hat{\varrho}$ is an hermitian operator: $\hat{\varrho} = \hat{\varrho}^\dagger$;
- it is positive semi-definite: $\hat{\varrho} \geq 0$;
- $\mathrm{Tr}[\hat{\varrho}] = \sum_n p_n = 1$.

In particular, the states $|\psi_n\rangle$ don't have to be mutually orthogonal, but it is possible to choose states such that $\langle\psi_n|\psi_n\rangle = \delta_{nm}$ and (1.39) continues to hold.
In this context, the mean value of an operator $\hat{A}$ is given in [4], by:

$$\langle\hat{A}\rangle = \mathrm{Tr}[\hat{\varrho}\hat{A}], \tag{1.40}$$

where $\mathrm{Tr}[\hat{A}]$ is the trace of a matrix and its definition is in the Appendix D. We can see how (1.40) can be proved using the trace operation:

$$\begin{aligned}
\mathrm{Tr}[\hat{\varrho}\hat{A}] &= \sum_m \langle e_m| \left(\sum_n p_n |\psi_n\rangle \langle\psi_n| \hat{A} \right) |e_m\rangle, \\
&= \sum_n p_n \langle\psi_n| \hat{A} \sum_m |e_m\rangle \langle e_m|\psi_n\rangle, \\
&= \sum_n p_n \langle\psi_n| \hat{A} |\psi_n\rangle \,, \\
&= \langle\hat{A}\rangle \,,
\end{aligned} \tag{1.41}$$

where in the last step we used the definition of expectation value. From these passages it is evident how (1.41) is independent from the chosen basis $\{|e_m\rangle\}$ to compute $\langle\hat{A}\rangle$, so, any basis can be chosen. Furthermore, from (1.41) we can deduce a similar normalization condition for $\hat{\varrho}$, if we compute $\mathrm{Tr}[\hat{\varrho}]$:

$$\mathrm{Tr}[\hat{\varrho}] = \sum_n p_n = 1, \tag{1.42}$$

for definition.

Finally, if we put $\hat{\varrho}$ in $\hat{A}$, (1.41) becomes:

$$\begin{aligned}
\mathrm{Tr}[\hat{\varrho}^2] &= \sum_m p_m \langle\psi_m| \left(\sum_n p_n |\psi_n\rangle \langle\psi_n| \right) |\psi_m\rangle, \\
&= \sum_m \sum_n p_m p_n | \langle\psi_n|\psi_m\rangle |^2 \leq 1,
\end{aligned} \tag{1.43}$$

where in the last step we used the Cauchy-Schwarz inequality in Appendix D.

Therefore $\mathrm{Tr}[\hat{\varrho}^2] \leq 1$ and in particular $\mathrm{Tr}[\hat{\varrho}^2] = 1$ if and only if $\hat{\varrho} = |\psi_n\rangle \langle\psi_n|$. In this case $\hat{\varrho}^2 = \hat{\varrho}$ and the system is said to be in a pure state.

Thermal states

A single mode of a radiation field at thermal equilibrium, such as the black body radiation, corresponds to a mixed state of the radiation field. This mixed state is described in terms of the density matrix $\hat{\varrho}$:

$$\hat{\varrho}_{\mathrm{th}} = \frac{e^{-\beta\hbar\omega\hat{a}^\dagger\hat{a}}}{\mathrm{Tr}[e^{-\beta\hbar\omega\hat{a}^\dagger\hat{a}}]}, \tag{1.44}$$

that can be rewritten in terms of the number states $|\hat{n}\rangle$:

$$\hat{\varrho}_{\mathrm{th}} = \sum_n p_n |n\rangle\langle n| = \frac{1}{1+N_{\mathrm{th}}} \sum_{n=0}^{+\infty} \left(\frac{N_{\mathrm{th}}}{1+N_{\mathrm{th}}} \right)^n |n\rangle \langle n|, \tag{1.45}$$

where $p_n = \frac{N_{\text{th}}^n}{(1+N_{\text{th}})^{n+1}}$ with $\langle\hat{n}\rangle = N_{\text{th}} = (e^{\beta\hbar\omega} - 1)^{-1}$. The distribution p_n is the so-called Bose-Einstein distribution and depends on N_{th} i.e. the mean number of photons $\langle\hat{n}\rangle$, that is also the average thermal quanta at the equilibrium. The other parameters are: ω the radiation frequency, $\beta = \frac{1}{\kappa_B T}$ and T the temperature.

It is shown in Appendix B that the statistics for a thermal state $\hat{\varrho}_{\text{th}}(N_{\text{th}})$ obeys the following statements:

$$\langle\hat{n}\rangle = N_{\text{th}}, \tag{1.46a}$$

$$\Delta^2(\hat{n}) = N_{\text{th}}(N_{\text{th}} + 1), \tag{1.46b}$$

$$\langle\hat{x}_\theta\rangle = 0, \tag{1.46c}$$

$$\Delta^2(\hat{x}_\theta) = 2N_{\text{th}} + 1. \tag{1.46d}$$

This theoretic discussion can be supported by several experimental techniques that achieves to generate thermal states with any energy N_{th}. This aim can be achieved by suitably manipulating the optical fields, for instance, by means of rotating ground glass (Arecchi's disks) [2].

Squeezed states

Squeezed states arise in a quantum model, whose quadrature variance is less than the coherent state one (1.31): $\text{Var}(\hat{x}_\theta) = 1$, for some value of θ, in this case we can say that the state is "squeezed". The simplest single-mode squeezed states are generated by the action on the vacuum state $|0\rangle$ of the squeezing operator:

$$\hat{S}(\xi) = \exp\left(\frac{\xi}{2}(\hat{a}^\dagger)^2 - \frac{\xi^*}{2}\hat{a}^2\right). \tag{1.47}$$

Here $\xi = re^{i\psi}$ is any complex number. The form of the squeezing operator with $\hat{a}^2$ in place of $\hat{a}$ of the displacement operator, can referred to a two photons coherent state. Another useful way to write the squeezing operator is:

$$\hat{S}(\xi) = \exp\left(\frac{\nu}{2\mu}(\hat{a}^\dagger)^2\right)\mu^{-(\hat{a}^\dagger\hat{a}+\frac{1}{2})}\exp\left(-\frac{\nu^*}{2\mu}\hat{a}^2\right), \tag{1.48}$$

with $\mu = \cosh r$ and $\nu = \sinh re^{i\psi}$.

So the application of the squeezing operator to the vacuum state allows us to find:

$$|\xi\rangle = \hat{S}(\xi)|0\rangle = \frac{1}{\sqrt{\mu}}\sum_{n=0}^{+\infty}\left(\frac{\nu}{2\mu}\right)^n\frac{\sqrt{(2n)!}}{n!}|2n\rangle, \tag{1.49}$$

whose first two terms are:

$$|\xi_2\rangle = \frac{1}{\sqrt{\cosh r}} \left(|0\rangle + \frac{\tanh r}{\sqrt{2}} \, |2\rangle \right). \tag{1.50}$$

Furthermore the operator $\hat{S}(\xi)$ is unitary:

$$\hat{S}^{\dagger}(\xi) = \exp\left(-\frac{\xi}{2} (\hat{a}^{\dagger})^2 + \frac{\xi^*}{2} \hat{a}^2 \right) = \hat{S}(-\xi) = \hat{S}^{-1}(\xi),$$

with the following properties:

$$\hat{S}^{\dagger}(\xi) \; \hat{a} \; \hat{S}(\xi) = \mu \hat{a} + \nu \hat{a}^{\dagger},$$
$$\hat{S}^{\dagger}(\xi) \; \hat{a}^{\dagger} \; \hat{S}(\xi) = \mu \hat{a}^{\dagger} + \nu \hat{a}.$$

Also in this case we can compute the statistics in the case $\xi = r \in \mathbb{R}$:

$$\langle \hat{n} \rangle = \sinh^2 r, \tag{1.51a}$$

$$\Delta^2(\hat{n}) = 2 \sinh^2 r (\sinh^2 r + 1), \tag{1.51b}$$

$$\langle \hat{x}_{\theta} \rangle = 0, \tag{1.51c}$$

$$\Delta^2(\hat{x}_{\theta}) = e^{2r} \cos^2 \theta + e^{-2r} \sin^2 \theta. \tag{1.51d}$$

As we can notice, the quadrature variance (1.51d), ranges from a minimum of e^{-2r} to a maximum of e^{2r} and will be less than 1 if $\cos^2 \theta < (e^{2r} + 1)^{-1}$ [4]. We can also compare the quadrature variance with the vacuum variance, it turns out that this condition is respected whenever: $\cos^2 \theta < \frac{e^{4r}-3}{4(e^{4r}-1)}$.

These observations bring us to introduce a relation that will be useful in Chapter 3: using the properties of hyperbolic functions, we can always say:

$$\begin{aligned} e^{-2r} &= \cosh 2r - \sinh 2r, \\ &= 1 + 2 \sinh^2 r - 2 \sinh r \cosh r, \\ &= 1 + 2\langle \hat{n} \rangle - 2\sqrt{\langle \hat{n} \rangle (\langle \hat{n} \rangle + 1)}, \end{aligned}$$

that can be rewritten, in our case, in the following way:

$$e^{-2r} = 1 + 2 \sinh^2 r - 2\sqrt{\sinh^2 r (\sinh^2 r + 1)}. \tag{1.52}$$

Also in this case it's worth saying how this class of states can be experimentally generated [12]. The majority of squeezed states are generated by nonlinear crystals [13]. In addition, a part of squeezed state can be generated by the optical parametric oscillator (OPO) [14]. By the way in a squeezed state there is a reduction on the phase fluctuations of a laser beam, by contrast is reveled an increase of the amplitude fluctuations.

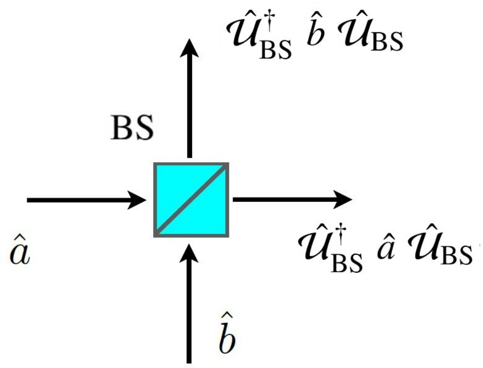


Figure 1.1: Beam splitter, . The two modes $\hat{a}$ and $\hat{b}$ are the input modes.

1.3 Beam splitter

The beam splitter is a very useful and almost necessary device in quantum optics experiments, it is also widely used in interferometers. It consists of a semi-reflective mirror that allows splitting a beam of light in two different beams, as sketched in Fig 1.1, it also allows to mix two optical states: that's because it generates a *two-mode mixing interaction*. In our case, it is interesting to see if a particular state mixed with other states will give birth to intensity and quantum correlations.

The Hamiltonian of a two-modes $\hat{a}$ and $\hat{b}$ state sent through a beam splitter can be written as:

$$\hat{\mathcal{H}}_{\mathrm{BS}} = g\hat{a}\hat{b}^\dagger + g^*\hat{a}^\dagger\hat{b}, \qquad g \in \mathbb{C}, \tag{1.53}$$

being bilinear in its two modes, all the Gaussian states are preserved in this kind of interaction [15]. Its evolution is directly derived:

$$\begin{aligned} \hat{\mathcal{U}}_{\mathrm{BS}}(\xi) &= e^{-\frac{i}{\hbar}\hat{\mathcal{H}}_{\mathrm{BS}}t}, \\ &= e^{\xi\hat{a}^\dagger\hat{b}-\xi^*\hat{a}\hat{b}^\dagger}, \end{aligned} \tag{1.54}$$

where the two modes commute: $[\hat{a}, \hat{b}] = 0$ and in general ξ is a complex number: $\xi = \phi e^{i\theta}$, it is proportional to the interaction length (time) and to the linear susceptibility of the medium. We can also define a new parameter, namely the transmittance:

$$\varepsilon = \cos^2\phi, \tag{1.55}$$

where, for a balanced beam splitter 50 : 50, i.e. that splits the beam of light into two beams of the same intensity, $\phi = \frac{\pi}{4}$, the transmittance equals $\varepsilon = \frac{1}{2}$. It is useful to compute the

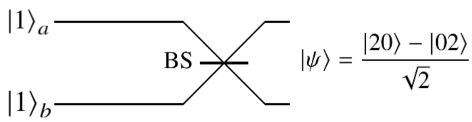


Figure 1.2: Schematization of the Hong-Ou-Mandel effect, as figured out, the output states can't be $|1\rangle_a \otimes |1\rangle_b$, but only a superposition of $|2\rangle_a \otimes |0\rangle_b = |20\rangle$ and $|0\rangle_a \otimes |2\rangle_b = |02\rangle$.

Heisenberg time evolution of two states through a beam splitter, see Appendix A:

$$\hat{c} = \hat{\mathcal{U}}^{\dagger}_{\mathrm{BS}}(\xi)\ \hat{a}\ \hat{\mathcal{U}}_{\mathrm{BS}}(\xi) = \hat{a}\cos\phi + \hat{b}e^{i\theta}\sin\phi, \tag{1.56a}$$

$$\hat{d} = \hat{\mathcal{U}}^{\dagger}_{\mathrm{BS}}(\xi)\ \hat{b}\ \hat{\mathcal{U}}_{\mathrm{BS}}(\xi) = \hat{b}\cos\phi - \hat{b}e^{-i\theta}\sin\phi, \tag{1.56b}$$

that, for a balanced beam splitter and without loss of generality, we can also take $\xi \in \mathbb{R}$ and the (1.56) become:

$$\begin{cases} \hat{a} \to \hat{c} = \frac{\hat{a}+\hat{b}}{\sqrt{2}}, \\ \hat{b} \to \hat{d} = \frac{\hat{b}-\hat{a}}{\sqrt{2}}, \end{cases} \tag{1.57}$$

letting us define two related number operators: $\hat{n}_c = \hat{c}^{\dagger}\hat{c}$ and $\hat{n}_d = \hat{d}^{\dagger}\hat{d}$.
One of the most spectacular effects that can be seen about a system going through a beam splitter is the Hong-Ou-Mandel effect described in the following section.

1.3.1 Hong-Ou-Mandel effect

This effect has been demonstrated in 1987 [16] and it is a two photons interference effect. We consider the two-mode output state of two photons interacting through a balanced beam splitter, i.e. a beam splitter with a transmittance $\varepsilon = \cos^2\phi = \frac{1}{2}$, as outlined in Fig. 1.2. We can say that the two-mode input state is:

$$|11\rangle = \hat{a}^{\dagger}\hat{b}^{\dagger}\,|00\rangle\,. \tag{1.58}$$

The output state can be found acting with $\hat{\mathcal{U}}_{\mathrm{BS}}$ on the starting state $|11\rangle$:

$$\begin{aligned}
\hat{\mathcal{U}}_{\mathrm{BS}}\,|11\rangle &= \hat{\mathcal{U}}_{\mathrm{BS}}\hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\mathcal{U}}^{\dagger}_{\mathrm{BS}}\hat{\mathcal{U}}_{\mathrm{BS}}\,|00\rangle\,, \\
&= \left(\frac{\hat{a}^{\dagger}-\hat{b}^{\dagger}}{\sqrt{2}}\right)\left(\frac{\hat{b}^{\dagger}+\hat{a}^{\dagger}}{\sqrt{2}}\right)|00\rangle\,, \\
&= \frac{1}{2}\left(a^{\dagger 2}-b^{\dagger 2}\right)|00\rangle\,, \\
&= \frac{|20\rangle-|02\rangle}{\sqrt{2}}.
\end{aligned} \tag{1.59}$$

So none of the output state corresponds to the $|11\rangle$ state. This effect is one of the quantum entanglement correlation effect we are interested in, it represents a significant example of nonclassical state, as far as it describes an entangled state [16], the destructive interference between the two $|11\rangle$ states, due to a phase shift corresponding to a factor -1, makes them delete.

This spectacular effect pushes us to make similar considerations about possible quantum effects on a state. To do this, we need first to introduce the intensity correlation function, as an index of how much two interacting beams of light display correlations in intensity.

1.4 Intensity correlation function

In typical quantum optical experiments, we observe variables like the number of photons of the two output modes of a beam splitter mixing, formalized by (1.56): $\hat{n}_c = \hat{c}^{\dagger}\hat{c}$ and $\hat{n}_d = \hat{d}^{\dagger}\hat{d}$. So, being interested in measuring intensity correlation between the two output radiation modes, according to [1], we can define a correlation intensity function as:

$$\Gamma = \frac{\langle\hat{n}_c\hat{n}_d\rangle - \langle\hat{n}_c\rangle\langle\hat{n}_d\rangle}{\sqrt{\mathrm{Var}(\hat{n}_c)\mathrm{Var}(\hat{n}_d)}}, \tag{1.60}$$

where the averages are made over the input state and $\mathrm{Var}(n) = \langle n^2\rangle - \langle n\rangle^2$ is the usual statistical variance. This expression allows us to quantify the amount of intensity correlation or anti-correlation if the statistics of the two modes are not independent. In general, the number operators of the output modes are defined as:

$$\hat{n}_c = \varepsilon\hat{n}_a + (1-\varepsilon)\hat{n}_b + \sqrt{\varepsilon(1-\varepsilon)}(\hat{q}_a\hat{q}_b + \hat{p}_a\hat{p}_b) = \frac{\hat{n}_a+\hat{n}_b}{2} + \frac{\hat{q}_a\hat{q}_b+\hat{p}_a\hat{p}_b}{2}, \tag{1.61a}$$

$$\hat{n}_d = \varepsilon\hat{n}_b + (1-\varepsilon)\hat{n}_a - \sqrt{\varepsilon(1-\varepsilon)}(\hat{q}_a\hat{q}_b + \hat{p}_a\hat{p}_b) = \frac{\hat{n}_a+\hat{n}_b}{2} - \frac{\hat{q}_a\hat{q}_b+\hat{p}_a\hat{p}_b}{2}, \tag{1.61b}$$

where $\hat{q}_k$ and $\hat{p}_k$ were defined in (1.9) and in the last equality we took $\varepsilon = \frac{1}{2}$.
It is possible to simplify this general expression if we consider a particular class of state and for our purpose, we will see primarily phase-insensitive states.

In particular, we are going to verify that a certain state presents a null intensity correlation function, this allows us to infer that our light state is a quantum one and we will be sure that all the correlations that will arise are of quantum origins. After the beam splitter, if the intensity correlation equals zero, it is possible to factorize the output states: $\hat{\varrho}_a \otimes \hat{\varrho}_b$, really as the beam splitter didn't exist. At this point, we are ready to study quantum correlations.

1.5 Phase-space symmetric states

Phase-space symmetric states is a class of states defined as a symmetric mixture of a state $\hat{\varrho}_0$ and the π shifted state $\hat{\varrho}_\pi = \hat{\Pi}\hat{\varrho}_0\hat{\Pi}$, where $\hat{\Pi} = e^{i\pi n_k} = (-1)^{\hat{k}^\dagger \hat{k}}$ is the parity operator that reverses the parity of a bosonic mode $\hat{k} = \hat{a}, \hat{b}$, namely:

$$\hat{\Pi}\hat{k}\hat{\Pi} = \sum_n \sqrt{n} e^{-i\pi\hat{k}^\dagger\hat{k}} \left|n\right\rangle \left\langle n+1\right| e^{i\pi\hat{k}^\dagger\hat{k}} = -k. \tag{1.62}$$

For this class of state, because of its symmetric nature, the average value of observables like $\langle \hat{x}_k \rangle = \langle \hat{k}\ \hat{n}_k \rangle = \langle \hat{n}_k\ \hat{k} \rangle = 0$. It can be proven [1] that:

$$\frac{\langle \hat{n}_c \hat{n}_d \rangle - \langle \hat{n}_c \rangle \langle \hat{n}_d \rangle}{\varepsilon(1-\varepsilon)} = \mathrm{Var}(\hat{n}_a) + \mathrm{Var}(\hat{n}_b) - \langle X_{a,b} \rangle, \tag{1.63a}$$

$$\frac{\mathrm{Var}(\hat{n}_c)}{\varepsilon(1-\varepsilon)} = \frac{\varepsilon}{1-\varepsilon}\mathrm{Var}(\hat{n}_a) + \frac{1-\varepsilon}{\varepsilon}\mathrm{Var}(\hat{n}_b) + \langle X_{a,b} \rangle, \tag{1.63b}$$

$$\frac{\mathrm{Var}(\hat{n}_d)}{\varepsilon(1-\varepsilon)} = \frac{\varepsilon}{1-\varepsilon}\mathrm{Var}(\hat{n}_b) + \frac{1-\varepsilon}{\varepsilon}\mathrm{Var}(\hat{n}_a) + \langle X_{a,b} \rangle, \tag{1.63c}$$

where we introduced a new term:

$$\langle X_{a,b} \rangle = 2\langle \hat{n}_a \rangle \langle \hat{n}_b \rangle + \langle \hat{n}_a \rangle + \langle \hat{n}_b \rangle + \langle \hat{a}^2 \rangle \langle \hat{b}^{\dagger 2} \rangle + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{b}^2 \rangle. \tag{1.64}$$

1.5.1 Phase-insensitive states

In addiction we can consider a further phase symmetric states class, that is the phase-insensitive states class: particular states whose representation in phase-space, through the Wigner function, depends only from the radius and not from the phase. States that belong to this class are diagonal on the Fock basis, i.e. $\hat{\varrho} = \sum_n p_n \left|n\right\rangle \left\langle n\right|$, and possible density probability distribution are:

- $p_n = \delta_{n,n_k}$;
- thermal state: $p_n = \frac{1}{1+N_{\text{th}}} \left(\frac{N_{\text{th}}}{1+N_{\text{th}}}\right)^n$;
- phase averaged coherent states (PHAV): $p_n = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}$.

In this case we can further simplify the expression of (1.63) in the sense that $\langle X_{a,b}\rangle$ reduces to $= 2\langle \hat{n}_a\rangle \langle \hat{n}_b\rangle + \langle \hat{n}_a\rangle + \langle \hat{n}_b\rangle$, and introducing the Q-Mandel parameter:

$$Q_k = \frac{\text{Var}(\hat{n}_k)}{\langle \hat{n}_k\rangle} - 1, \qquad k = a, b; \tag{1.65}$$

the (1.63) become:

$$\frac{\langle \hat{n}_c \hat{n}_d\rangle - \langle \hat{n}_c\rangle \langle \hat{n}_d\rangle}{\varepsilon(1-\varepsilon)} = \langle \hat{n}_a\rangle Q_a + \langle \hat{n}_b\rangle Q_b - 2\langle \hat{n}_a\rangle \langle \hat{n}_b\rangle, \tag{1.66a}$$

$$\frac{\text{Var}(\hat{n}_c)}{\varepsilon(1-\varepsilon)} = \langle \hat{n}_a\rangle \left(\frac{Q_a\varepsilon + 1}{1-\varepsilon}\right) + \langle \hat{n}_b\rangle \left(\frac{Q_b(1-\varepsilon)+1}{\varepsilon}\right) + 2\langle \hat{n}_a\rangle \langle \hat{n}_b\rangle, \tag{1.66b}$$

$$\frac{\text{Var}(n_d)}{\varepsilon(1-\varepsilon)} = \langle \hat{n}_a\rangle \left(\frac{Q_a(1-\varepsilon)+1}{\varepsilon}\right) + \langle \hat{n}_b\rangle \left(\frac{Q_b\varepsilon + 1}{1-\varepsilon}\right) + 2\langle \hat{n}_a\rangle \langle \hat{n}_b\rangle. \tag{1.66c}$$

1.5.2 Thermal state

Now, we consider the intensity correlation function for a thermal state $\hat{\varrho}_{\text{th}}$, since it is a phase-insensitive state, we can consider (1.66) with $\varepsilon = 1/2$, in order to compute the intensity correlation function:

$$\Gamma_{\text{th}} = \frac{\frac{\langle \hat{n}_c \hat{n}_d\rangle - \langle \hat{n}_c\rangle\langle \hat{n}_d\rangle}{\varepsilon(1-\varepsilon)}}{\sqrt{\frac{\text{Var}(\hat{n}_c)}{\varepsilon(1-\varepsilon)} \frac{\text{Var}(\hat{n}_d)}{\varepsilon(1-\varepsilon)}}},$$

$$= \frac{\text{Var}(\hat{n}_a) + \text{Var}(\hat{n}_b) - \langle X_{a,b}\rangle}{\sqrt{\left(\frac{\varepsilon}{1-\varepsilon}\text{Var}(\hat{n}_a) + \frac{1-\varepsilon}{\varepsilon}\text{Var}(\hat{n}_b) + \langle X_{a,b}\rangle\right)\left(\frac{\varepsilon}{1-\varepsilon}\text{Var}(\hat{n}_b) + \frac{1-\varepsilon}{\varepsilon}\text{Var}(\hat{n}_a) + \langle X_{a,b}\rangle\right)}},$$

$$= \frac{\text{Var}(\hat{n}_a) + \text{Var}(\hat{n}_b) - \langle X_{a,b}\rangle}{\text{Var}(\hat{n}_a) + \text{Var}(\hat{n}_b) + \langle X_{a,b}\rangle},$$

$$= \frac{\text{Var}(\hat{n}_a) + \text{Var}(\hat{n}_b) - 2\langle \hat{n}_a\rangle\langle \hat{n}_b\rangle - \langle \hat{n}_a\rangle - \langle \hat{n}_b\rangle}{\text{Var}(\hat{n}_a) + \text{Var}(\hat{n}_b) + 2\langle \hat{n}_a\rangle\langle \hat{n}_b\rangle + \langle \hat{n}_a\rangle + \langle \hat{n}_b\rangle},$$

we recall that the mean and variance for a thermal state are given by (1.46a) and (1.46b):

$$\langle \hat{n}_a\rangle = \langle \hat{n}_b\rangle = N_{\text{th}},$$

$$\text{Var}(\hat{n}_a) = \text{Var}(\hat{n}_b) = N_{\text{th}}(N_{\text{th}} + 1),$$

so the expression simplifies a lot:

$$\Gamma_{\text{th}} = \frac{2N_{\text{th}}(N_{\text{th}}+1) - 2N_{\text{th}}^2 - 2N_{\text{th}}}{2N_{\text{th}}(N_{\text{th}}+1) + 2N_{\text{th}}^2 + 2N_{\text{th}}} = 0, \tag{1.67}$$

that means that in a thermal state mixed in a beam splitter there is no intensity correlation:

$$\Gamma_{\text{th}} = 0. \tag{1.68}$$

This fact brings us to say that the outputs of a beam splitter are uncorrelated and can be expressed as a factorization between two states, so all the eventual correlations will have quantum nature. In particular, it's interesting to study if actually, quantum correlations exist in this scenario.
At this point is the moment to introduce the theoretic apparatus to study the quantum features of the state, such that nonclassicality and nonlocality. In particular, in the next Chapter, we will introduce the required functions that will establish sufficient conditions in order to say whether a state displays quantum features and will set a hierarchy among the quantum states.

Chapter 2

Wigner formalism

In this Chapter we want to give a general introduction to the Wigner formalism, that will be used within the thesis. We start with the definition of characteristic and quasi-probability functions with their properties, and then with some keys results [8]. We are interested in this formalism in order to have a more complete statistical description of our physics and find some useful properties aiming to characterising it satisfactorily [4].
Furthermore the Wigner formalism is used to study the nonclassicality and nonlocality features of optical states, that will be enunciated in Chapter 3.

2.1 Characteristic function

Given a state $\hat{\varrho}$ we can always define the p-order characteristic function, that is always a defined complex-valued function, as:

$$\begin{aligned} \chi(\lambda, p) &= \mathrm{Tr}[\hat{\varrho}\ \hat{D}(\lambda)]e^{p|\lambda|^2/2}, \\ &= \mathrm{Tr}[\hat{\varrho}\ e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}]e^{p|\lambda|^2/2}, \\ &= \mathrm{Tr}[\hat{\varrho}\ e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}}]e^{(p-1)|\lambda|^2/2}. \end{aligned} \tag{2.1}$$

Here, the last expression has been obtained using the BCH Theorem (see Appendix A, and $\hat{D}(\lambda)$ is the usual displacement operator, the parameter p refers to a specific order of the creation and annihilation operators in the expression, in particular we have:

$$\chi(\lambda,1) = \mathrm{Tr}[\hat{\varrho}e^{\lambda\hat{a}^\dagger}e^{-\lambda^*\hat{a}}], \qquad \text{normal ordered;} \tag{2.2a}$$

$$\chi(\lambda,0) = \mathrm{Tr}[\hat{\varrho}e^{\lambda\hat{a}^\dagger-\lambda^*\hat{a}}], \qquad \text{symmetric ordered;} \tag{2.2b}$$

$$\chi(\lambda,-1) = \mathrm{Tr}[\hat{\varrho}e^{-\lambda^*\hat{a}}e^{\lambda\hat{a}^\dagger}], \qquad \text{antinormal ordered;} \tag{2.2c}$$

Through (2.2b) we can see that the characteristic function is the definition of the expectation value of the displacement operator:

$$\chi(\lambda,0) = \langle\hat{D}(\lambda)\rangle = \mathrm{Tr}[\hat{\varrho}\hat{D}(\lambda)]. \tag{2.3}$$

In the next calculation we will often use the symmetric ordered characteristic function : $\chi(\lambda) = \chi(\lambda,0)$. This formalism allows us to write any expectation value of a function of the product of any m,n degree between $\hat{a}^\dagger$ and $\hat{a}$:

$$\left(\frac{\partial}{\partial\lambda}\right)^m\left(-\frac{\partial}{\partial\lambda^*}\right)^n\chi(\lambda,p)\Big|_{\lambda=0} = \langle(\hat{a}^\dagger)^m\,\hat{a}^n\rangle_p. \tag{2.4}$$

And the more useful for our purpose :

$$\begin{aligned}\left(\frac{\partial}{\partial\lambda}\right)^m\left(-\frac{\partial}{\partial\lambda^*}\right)^n\chi(\lambda)\Big|_{\lambda=0} &= \langle(\hat{a}^\dagger)^m\,\hat{a}^n\rangle_0\,,\\ &= \langle\big[(\hat{a}^\dagger)^m\,\hat{a}^n\big]_S\rangle\,,\\ &= \frac{1}{2}\,\langle(\hat{a}^\dagger)^m\,\hat{a}^n + \hat{a}^n\,(\hat{a}^\dagger)^m\rangle\,,\\ &= \langle(\hat{a}^\dagger)^m\,\hat{a}^n\rangle + \frac{1}{2}.\end{aligned} \tag{2.5}$$

Another important property that allow us to express the density operator $\hat{\varrho}$ in terms of its characteristic function is the Glauber formula:

$$\hat{\varrho} = \frac{1}{\pi}\int_{\mathbb{C}} d^2\lambda\;\chi(\lambda)\hat{D}^\dagger(\lambda) = \frac{1}{\pi}\int_{\mathbb{C}} d^2\lambda\;\mathrm{Tr}[\hat{\varrho}\hat{D}(\lambda)]\hat{D}^\dagger(\lambda). \tag{2.6}$$

Let's briefly see how we can write the characteristic function for our states (Fock and thermal states). It is useful to start from the third expression of (2.1).

The p-ordered characteristic function for a Fock state $|n\rangle$ can be written as (see Appendix C):

$$\chi_n(\lambda,p) = \langle n|\,e^{\lambda\hat{a}^\dagger}e^{-\lambda^*\hat{a}}\,|n\rangle\,e^{(p-1)|\lambda|^2/2} = L_n(|\lambda|^2)e^{(p-1)|\lambda|^2/2}. \tag{2.7}$$

Thanks to this expression, we can always obtain the characteristic function for any state, whose density matrix is diagonal in the number state basis in the form of a weighted sum:

$$\chi(\lambda, p) = \sum_{n=0}^{+\infty} p_n \; \chi_n(\lambda, p). \tag{2.8}$$

The last important property of the characteristic function is the trace rule:

$$\mathrm{Tr}[\hat{A}\,\hat{B}] = \frac{1}{\pi}\int_{\mathbb{C}} d^2\lambda \; \chi_{\hat{A}}(\lambda)\chi_{\hat{B}}(-\lambda). \tag{2.9}$$

Gaussian states

We can compute the characteristic function for the vacuum state and the thermal state $\hat{\varrho}_{\mathrm{th}}(N_{\mathrm{th}})$:

$$\chi_{\mathrm{vac}}(\lambda, p) = \chi_0(\lambda, p) = e^{(p-1)|\lambda|^2/2}, \tag{2.10}$$

$$\chi_{\mathrm{th}}(\lambda, p) = e^{(1+2N_{\mathrm{th}}-p)|\lambda|^2/2}, \tag{2.11}$$

a complete demonstration can be found in Appendix C. How we can see, these states have Gaussian characteristic function, that's because they are called Gaussian.
Gaussian states cover an important role in quantum optics and quantum information, processing continuous variables, because the vacuum state of quantum electrodynamics is itself a Gaussian state. In addiction the quantum evolutions achievable with current technology are described by Hamiltonian operators at most bilinear in the quantum fields [17]. Furthermore, Gaussian states exhibit extremality properties: among the continuous variable states, they tend to be extremal if we impose some constraints on the covariance matrix [18].
For our purpose we need to know that it is possible to conserve the covariance matrix and the first-moment vector that characterize a Gaussian state during a linear or bilinear transformation in the two modes, in our case, during a beam splitter evolution [17, 18, 19].
Mathematically a general covariance matrix Σ of n-mode interacting Gaussian state $\hat{\varrho}_\nu$ is defined [17] by an $2n \times 2n$ matrix:

$$\Sigma_n = \begin{pmatrix} \Delta^2(\hat{x}_1)\;\mathbb{I}_2 & \varsigma_{12} & \cdots & \varsigma_{1n} \\ \varsigma_{21} & \Delta^2(\hat{x}_2)\;\mathbb{I}_2 & \cdots & \varsigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varsigma_{n1} & \varsigma_{n2} & \cdots & \Delta^2(\hat{x}_n)\;\mathbb{I}_2 \end{pmatrix}, \tag{2.12}$$

where $\Delta^2(\hat{x}_i)$ is the usual variance: $\Delta^2(\hat{x}_i) = \mathrm{Var}(\hat{x}_i) = \langle \hat{x}_i^2 \rangle - \langle \hat{x}_i \rangle^2$ and ς_{ij} is a 2×2 matrix in which compare the covariance σ_{ij} between $\hat{x}_i$ and $\hat{x}_j$, and it is related to the classical or quantum correlations between the mode i and j [18]. Its general expression is:

$$\sigma_{ij} = \frac{\langle \hat{x}_i \hat{x}_j + \hat{x}_j \hat{x}_i \rangle - 2 \langle \hat{x}_i \rangle \langle \hat{x}_j \rangle}{2}, \tag{2.13}$$

if the two operators commute, it simplifies as the most known covariance:

$$\sigma_{ij} = \sigma_{ji} = \langle \hat{x}_i \hat{x}_j \rangle - \langle \hat{x}_i \rangle \langle \hat{x}_j \rangle \,. \tag{2.14}$$

If $\varsigma_{ij} = 0$, then $\hat{\varrho}_{ij} = \hat{\varrho}_i \otimes \hat{\varrho}_j$, that means that the two modes are uncorrelated and Σ is diagonal. A one-mode thermal state is given by a diagonal 2×2 matrix:

$$\Sigma_1^{\mathrm{th}} = \begin{pmatrix} 1 + 2N_{\mathrm{th}} & 0 \\ 0 & 1 + 2N_{\mathrm{th}} \end{pmatrix}, \tag{2.15}$$

For this states we can pick $\Delta^2(x_\theta) = 1 + 2N_{\mathrm{th}}$ from (1.46d). For a squeezed state,using (1.51d) we have:

$$\Sigma_1^{\xi} = \begin{pmatrix} e^{2r} & 0 \\ 0 & e^{-2r} \end{pmatrix}, \tag{2.16}$$

It turns out that a Gaussian state is preserved with a linear or bilinear Hamiltonian in the modes [18]. For the case of a bilinear Hamiltonian, as the beam splitter one (1.53), the covariance matrix Σ_2 is a 4×4 matrix and evolves following:

$$\Sigma \rightarrow S \, \Sigma \, S^T \,. \tag{2.17}$$

where S is a symplectic matrix. In this case S takes the form [18]:

$$S = \begin{pmatrix} \cos\phi \; \mathbb{I}_2 & \sin\phi \; \mathcal{R}_\theta \\ -\sin\phi \; \mathcal{R}_\theta^T & \cos\phi \; \mathbb{I}_2 \end{pmatrix}, \tag{2.18}$$

where $\mathcal{R}_\theta$ is the usual rotation matrix, given by:

$$\mathcal{R}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2.19}$$

This kind of transformations leave the kinematics of a system invariant: if we have a two-mode Gaussian state before the beam splitter, its mean value and variance remains unchanged after a beam splitter interaction.

2.2 Quasi-probability distributions

Alternatively to the characteristic function, we can define, in general, the quasi-probability distribution, as the Fourier transform of the corresponding p-ordered characteristic function, it is real valued but it is not always positive, that's because it can't be always interpreted as a probability distribution [7, 4]:

$$W(\alpha, p) = \frac{1}{\pi^2} \int_{\mathbb{C}} d^2\lambda \chi(\lambda, p) e^{\alpha\lambda^* - \alpha^*\lambda}. \tag{2.20}$$

Or, in another way, without going through the expression of the characteristic function:

$$W(\alpha, p) = \mathrm{Tr}\left[\hat{\varrho}\hat{D}(\alpha)\hat{T}(p)\hat{D}^\dagger(\alpha)\right]; \qquad p < 1. \tag{2.21}$$

Where $\hat{D}$ is the usual displacement operator and $\hat{T}$ is the following operator :

$$\hat{T}(p) = \frac{2}{\pi(1-p)} e^{-\frac{2}{1-p}\hat{a}^\dagger\hat{a}}. \tag{2.22}$$

In this way we can rewrite the quasi-probability distribution as follows:

$$W(\alpha, p) = \frac{2}{\pi(1-p)} \sum_{n=0}^{+\infty} \left(-\frac{1+p}{1-p}\right)^n \langle n|D^\dagger \hat{\varrho} D|n\rangle. \tag{2.23}$$

Properties

- **Normalization**

We can demonstrate the normalization property starting from:

$$\frac{1}{\pi^2} \int_{\mathbb{C}} d^2\lambda e^{\alpha\lambda^* - \alpha^*\lambda} = \delta^{(2)}(\alpha), \tag{2.24}$$

in order to show that $W(\alpha, p)$ is normalized:

$$\int_{\mathbb{C}} d^2\alpha\; W(\alpha, p) = \frac{1}{\pi^2} \int_{\mathbb{C}} d^2\lambda\; \chi(\lambda, p) \int_{\mathbb{C}} d^2\alpha\; e^{\alpha\lambda^* - \alpha^*\lambda} = \chi(0, p) = \mathrm{Tr}[\hat{\varrho}] = 1.$$

- **Moment generator function**

As we expect from a probability distribution (integrating by parts, max$[m, n]$ times), we can find the moments of any desired p-ordered product of the creation and annihilation operators:

$$\begin{aligned}
\int_{\mathbb{C}} d^2\alpha\; W(\alpha, p)(\alpha^*)^m\; \alpha^n &= \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\alpha\; \int_{\mathbb{C}} d^2\lambda \chi(\lambda, p) e^{\alpha\lambda^* - \alpha^*\lambda}(\alpha^*)^m\; \alpha^n, \\
&= \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\lambda\; \chi(\lambda, p)\left(\frac{\partial}{\partial\lambda}\right)^m \left(-\frac{\partial}{\partial\lambda^*}\right)^n \int_{\mathbb{C}} d^2\alpha\; e^{\alpha\lambda^* - \alpha^*\lambda}, \\
&= \int_{\mathbb{C}} d^2\lambda\; \chi(\lambda, p)\left(\frac{\partial}{\partial\lambda}\right)^m \left(-\frac{\partial}{\partial\lambda^*}\right)^n \delta^{(2)}(\lambda), \qquad (2.25)\\
&= \left(\frac{\partial}{\partial\lambda}\right)^m \left(-\frac{\partial}{\partial\lambda^*}\right)^n \chi(\lambda, p)\big|_{\lambda=0}, \\
&= \langle(\hat{a}^\dagger)^m\; \hat{a}^n\rangle_p .
\end{aligned}$$

- **Convolution**

Following [7], we recall that, for $q < p$:

$$\chi(\lambda, q) = \chi(\lambda, p) e^{-(p-q)|\lambda|^2/2}, \tag{2.26}$$

and, in order to write a relation between $W(\alpha, q)$ and $W(\alpha, p)$, namely:

$$W(\alpha, q) = \frac{2}{\pi(p-q)}\int_{\mathbb{C}} d^2\beta\, W(\beta, p)\; \exp\left[-\frac{2|\alpha-\beta|^2}{p-q}\right]; \qquad q < p. \tag{2.27}$$

In the case of $p = 1, q = 0$ the expression reads:

$$W(\alpha) = \frac{2}{\pi}\int_{\mathbb{C}} d^2\beta\; P(\beta)\; \exp\left[-2|\alpha-\beta|^2\right], \tag{2.28}$$

that is a convolution between the P-Glauber function : $W(\alpha, 1) = P(\alpha)$ and a Gaussian with variance $\tau = 1/2$.

We recall that a convolution between two integrable function $f(t)$ and $g(t)$, can be written as:

$$(f * g)(t) := \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{+\infty} f(t-\tau)g(\tau)d\tau. \tag{2.29}$$

Generally for $p = 1, 0 < q < 1$, (2.27) is a convolution between the P-Glauber function and a Gaussian distribution with variance $\tau = \frac{p-q}{2} = \frac{1-q}{2}$.

- **Ordering**

It is often convenient to choose an appropriate value of p, and usually it is $p = 1, 0, -1$, as for the characteristic function, giving birth to normal, symmetric or antinormal ordered moments. A special notation has been adopted for these three case:

- $W(\alpha, 0) = W(\alpha)$ the Wigner function;
- $W(\alpha, -1) = Q(\alpha)$ the Husimi or Q-function;
- $W(\alpha, 1) = P(\alpha)$ the Galuber-Sudarshan P-representation.

2.2.1 Wigner function

Now let's consider the Wigner function defined as $W(\alpha) = W(\alpha, 0)$, so (2.20) reads:

$$W(\alpha, p) = \frac{1}{\pi^2} \int_{\mathbb{C}} d^2\lambda \, \mathrm{Tr}[\hat{\varrho}\hat{D}(\lambda)] e^{\alpha\lambda^* - \alpha^*\lambda}, \tag{2.30}$$

furthermore we can write it as:

$$W[\hat{\varrho}](\alpha) = \mathrm{Tr}\left[\hat{\varrho}\hat{D}\hat{\Pi}\hat{D}^\dagger\right], \qquad \hat{\Pi} = (-1)^{a^\dagger a}, \tag{2.31}$$

and making all the terms explicit for a Fock basis density operator $\hat{\varrho} = \sum_n p_n |n\rangle\langle n|$:

$$W(\alpha) = \frac{2}{\pi} \sum_{n=0}^{+\infty} (-1)^n \langle nD^\dagger \hat{\varrho}\hat{D}|n\rangle = \frac{2}{\pi} e^{-2|\alpha|^2} \sum_{n=0}^{+\infty} (-1)^n p_n L_n(4|\alpha|^2). \tag{2.32}$$

Here, $L_n(x)$ are the Laguerre polynomials, whose definition and properties can be found in Appendix C.

One can see that the Wigner function (2.32) has only radial dependence and does not depend on the phase, in this sense, diagonal states in the Fock basis are called phase-insensitive states.

Trace rule

A useful property that will be used in the next section is the trace rule for the Wigner functions: given two operators $\hat{A}$ and $\hat{B}$, and the corresponding Wigner function, $W[\hat{A}](\alpha)$ and $W[\hat{B}](\alpha)$, we have that

$$\mathrm{Tr}[\hat{A}\hat{B}] = \pi \int_{\mathbb{C}} d^2\alpha \, W[\hat{A}](\alpha) \, W[\hat{B}](\alpha). \tag{2.33}$$

Composition

We want now a process in order to compute the Wigner function of an evolved state, namely a state that mix in a beam splitter with another state.

The evolution of a state $\hat{\varrho}$ in general is given by: $\hat{\mathcal{U}}\hat{\varrho}\hat{\mathcal{U}}^\dagger$ where $\hat{\mathcal{U}}$ is the unitary evolution operator, that in the case of a beam splitter can be taken as (1.53), and the evolution of its Wigner function (2.30) reads:

$$W[\hat{\mathcal{U}}\hat{\varrho}\hat{\mathcal{U}}^\dagger](\alpha,\beta) = \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\lambda \int_{\mathbb{C}} d^2\xi \; \mathrm{Tr}[\hat{\mathcal{U}}\hat{\varrho}\hat{\mathcal{U}}^\dagger \hat{D}_a(\lambda)\hat{D}_b(\lambda)] e^{\alpha\lambda^* - \alpha^*\lambda} e^{\alpha\xi^* - \alpha^*\xi}, \quad (2.34)$$

with the cyclic property of the trace (see Appendix D):

$$\mathrm{Tr}[\hat{\mathcal{U}}\hat{\varrho}\, \hat{\mathcal{U}}^\dagger \hat{D}_a(\lambda)\hat{D}_b(\lambda)] = \mathrm{Tr}[\hat{\varrho}\hat{\mathcal{U}}^\dagger \hat{D}_a(\lambda)\, \hat{\mathcal{U}}\hat{\mathcal{U}}^\dagger \hat{D}_b(\lambda)\hat{\mathcal{U}}]. \quad (2.35)$$

The evolution of the two displacement operator have been computed in Appendix A and we report them here:

$$\hat{\mathcal{U}}^\dagger \hat{D}_a(\lambda)\, \hat{\mathcal{U}} = \hat{D}_a(\lambda\cos\phi)\hat{D}_b(-\lambda\sin\phi), \quad (2.36a)$$

$$\hat{\mathcal{U}}^\dagger \hat{D}_b(\xi)\, \hat{\mathcal{U}} = \hat{D}_a(\xi\sin\phi)\hat{D}_b(\xi\cos\phi) \quad (2.36b)$$

thus, (2.34) becomes:

$$\frac{1}{\pi^2}\int_{\mathbb{C}} d^2\lambda \int_{\mathbb{C}} d^2\xi \; \mathrm{Tr}[\hat{\varrho}\hat{D}_a(\lambda\cos\phi + \xi\sin\phi)\hat{D}_b(\xi\cos\phi) - \lambda\sin\phi)] e^{\alpha\lambda^* - \alpha^*\lambda} e^{\alpha\xi^* - \alpha^*\xi}, \quad (2.37)$$

that for $\phi = \frac{\pi}{4}$, that's the case of a balanced beam splitter:

$$\begin{aligned} W[\hat{\mathcal{U}}\hat{\varrho}\hat{\mathcal{U}}^\dagger](\alpha,\beta) &= \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\lambda \int_{\mathbb{C}} d^2\xi \; \mathrm{Tr}\left[\hat{\varrho}\hat{D}_a\left(\frac{\xi+\lambda}{\sqrt{2}}\right)\hat{D}_b\left(\frac{\xi-\lambda}{\sqrt{2}}\right)\right] e^{\alpha\lambda^* - \alpha^*\lambda} e^{\alpha\xi^* - \alpha^*\xi}, \\ &= W[\hat{\varrho}']\left(\frac{\alpha+\beta}{\sqrt{2}}, \frac{\beta-\alpha}{\sqrt{2}}\right). \end{aligned} \quad (2.38)$$

Whenever we have to mix two states $\hat{\varrho}_1$ and $\hat{\varrho}_2$ in a balanced beam splitter, we need to compose the two Wigner functions as follows:

$$W^\pm[\hat{\varrho}_1 \otimes \hat{\varrho}_2](\alpha,\beta) = W[\hat{\varrho}_1]\left(\frac{\beta \pm \alpha}{\sqrt{2}}\right) \cdot W[\hat{\varrho}_2]\left(\frac{\beta \mp \alpha}{\sqrt{2}}\right), \quad (2.39)$$

this relation will be very useful in the next section, where we will mix two different states: we need the Wigner function to study its properties.

2.2.2 P-representation

With this quasi-probability function it is possible to express directly the density probability operator $\hat{\varrho}$ in terms of $P(\alpha)$, namely:

$$\hat{\varrho} = \int_{\mathbb{C}} d^2\alpha P(\alpha)\, |\alpha\rangle\, \langle\alpha|, \quad (2.40)$$

where $|\alpha\rangle$ are coherent states.

In this case we can compute the expectation value of the normal ordered product:

$$\begin{aligned}
\int_{\mathbb{C}} d^2\alpha \; P(\alpha)(\alpha^*)^m \; \alpha^n &= \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha \; \langle\alpha|\, \hat{a}^{\dagger m} \hat{\varrho} a^n \,|\alpha\rangle, \\
&= \mathrm{Tr}[a^n \hat{\varrho} \hat{a}^{\dagger m}], \\
&= \mathrm{Tr}[\hat{\varrho} \hat{a}^{\dagger m} a^n], \\
&= \langle \hat{a}^{\dagger m} \; \hat{a}^n \rangle \, .
\end{aligned} \tag{2.41}$$

2.2.3 Q-function

It is possible to obtain a simple expression for $Q(\alpha)$ in terms of the density matrix, putting $p = -1$ in (2.23), only the first term survives:

$$Q(\alpha) = W(\alpha, -1) = \frac{1}{\pi} \langle 0|\, \hat{D}^\dagger(\alpha) \hat{\varrho} \hat{D}(\alpha) \,|0\rangle = \frac{1}{\pi} \langle\alpha|\, \hat{\varrho} \,|\alpha\rangle \, . \tag{2.42}$$

One of the main consequence of this expression is that $Q(\alpha)$ is positive semi-definite, in fact it is an expectation value of an hermitian operator $\hat{\varrho}$ with positive semi-definite eigenvalues, meaning that $Q(\alpha)$ can be considered as a probability distribution associated with the probability for the results of joint measurements of the two in-quadrature components of the field, $\hat{x}_\theta$ and $\hat{x}_{\theta+\frac{\pi}{2}}$.

In addiction, it's easy to show the antinormal ordered version of (2.25):

$$\begin{aligned}
\int_{\mathbb{C}} d^2\alpha \; Q(\alpha)(\alpha^*)^m \; \alpha^n &= \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha \; \langle\alpha|\, \hat{a}^{\dagger^m} \hat{\varrho} a^n \,|\alpha\rangle, \\
&= \mathrm{Tr}[\hat{a}^{\dagger^m} \hat{\varrho} a^n], \\
&= \langle \hat{a}^n \; \hat{a}^{\dagger^m} \rangle \, .
\end{aligned} \tag{2.43}$$

2.3 Some properties

Let's now compute the quasi-probability functions for a Fock state $|n\rangle$. With the definition (2.20) and using (2.7) we can writes:

$$\begin{aligned}
W_n(\alpha, p) &= \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\lambda \chi_n(\lambda, p) e^{\alpha\lambda^* - \alpha^*\lambda}, \\
&= \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\lambda \; L_n(|\lambda|^2) e^{(p-1)|\lambda|^2/2} e^{\alpha\lambda^* - \alpha^*\lambda},
\end{aligned} \tag{2.44}$$

putting $\lambda = xe^{i\theta}$ and $\alpha = |\alpha|e^{i\phi}$, and going to polar coordinates, the integration over the angular dimension θ becomes:

$$\int_0^{2\pi} d\theta\ e^{2i|\alpha|x\sin(\theta-\phi)} = 2\pi J_0(2|\alpha|x). \tag{2.45}$$

Where J_0 is the Bessel function of order zero (see Appendix C).

Then, writing this result back into (2.44):

$$\begin{aligned} W_n(\alpha,p) &= \frac{2}{\pi}\int_0^{+\infty} dx\ xL_n(x^2)e^{(p-1)x^2/2}J_0(2|\alpha|x), \\ &= \frac{2}{\pi(1-p)}(-1)^n\left(\frac{1+p}{1-p}\right)^n \exp\left(-\frac{2|\alpha|^2}{1-p}\right)L_n\left(\frac{4|\alpha|^2}{1-p^2}\right), \end{aligned} \tag{2.46}$$

for $p \neq -1$.

In fact for $p = -1$, we can take the limit of $p \to -1$ of (2.46), that gives (2.47c), as shown in Appendix (C.15).

$$W_n(\alpha,1) = P_n(\alpha) = \sum_{m=0}^{n}\binom{n}{m}\frac{1}{m!}\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)^m\delta^{(2)}(\alpha), \tag{2.47a}$$

$$W_n(\alpha,0) = W_n(\alpha) = \frac{2}{\pi}(-1)^n e^{-2|\alpha|^2}L_n\left(4|\alpha|^2\right), \tag{2.47b}$$

$$W_n(\alpha,-1) = Q_n(\alpha) = \frac{1}{\pi}e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \tag{2.47c}$$

The calculation for $p = 0$ is a trivial consequence of (2.46) and their graphs are plotted in Fig 2.1 for $n = 0$, $n = 1$ and $n = 2$. How we can see, except the $|0\rangle\langle 0|$ state Wigner function, the others assume negative values, too. This shows and explains because they can't be used as probability distribution function, they are not positive-definite. We can also notice a trend for increasing n: they become more and more singular in the origin, with increasing different-sign region. We recall that all the function are normalized to 1.

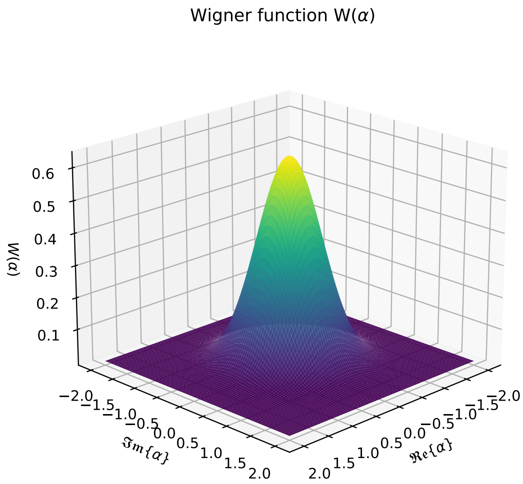
For the $p = 1$ case, we have to start from (2.20) with (2.7) and $p = 1$:

$$\begin{aligned} P_n(\alpha) = W_n(\alpha,1) &= \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\lambda L_n(|\lambda|^2)e^{\alpha\lambda^*-\alpha^*\lambda}, \\ &= \frac{1}{\pi^2}\sum_{m=0}^{n}\frac{(-1)^m}{m!}\binom{n}{m}\int_{\mathbb{C}} d^2\lambda\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)^m e^{\alpha\lambda^*-\alpha^*\lambda}, \\ &= \sum_{m=0}^{n}\binom{n}{m}\frac{1}{m!}\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)^m\delta^{(2)}(\alpha), \end{aligned} \tag{2.48}$$

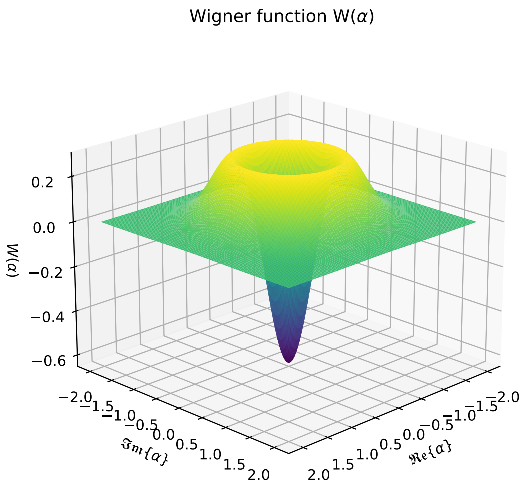
where we used the series expansion of the Laguerre polynomials (see Appendix C and (2.24). Finally, if we have a general diagonal mixture of Fock states $\hat{\varrho} = \sum_n p_n |n\rangle\langle n|$, following (2.8), we have to consider the linear combination:

$$W(\alpha, p) = \sum_n p_n W_n(\alpha, p). \tag{2.49}$$

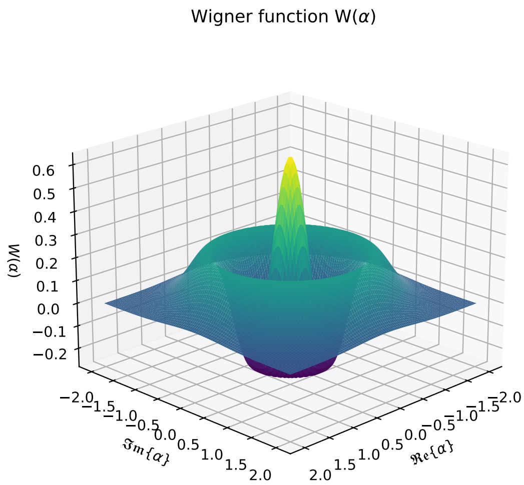
Given the main definitions of Wigner function and quasi-probability function, it is now possible to move on the characterization of an optical state. We want to study its quantum features as nonclassicality and nonlocality by quantitative parameters as nonclassical depth and Bell parameter that use the quasi-probability and Wigner function properties. In particular we will see that the nonclassical depth can be related to making the quasi probability function a negative function, and thus couldn't be used as a classic probability function. The Bell parameter, instead, is defined through the Wigner functions.



(a) *State* $|0\rangle\langle 0|$



(b) *State* $|1\rangle\langle 1|$



(c) *State* $|2\rangle\langle 2|$

Figure 2.1: *Wigner functions for $n = 0, n = 1$ and $n = 2$ Fock states. Except for the $n = 0$ case, the Wigner functions assume negative values in a finite region, that explains because the Wigner function can't be used as probability distribution function. We can also notice that for increasing n, the functions become more and more singular at the origin. These facts introduce us to the quasi-probability distribution of a quantum state and to set a hierarchy among them: the more negative or singular is a Wigner function, the more quantum is the relative state, namely, the wider convolution with a plane wave it needs to become positive, the more quantum the state is.*

Chapter 3

Nonclassicality and nonlocality

The main goal of this chapter, is to build a solid hierarchy among the nonclassical state and in particular to set a way to measure the nonclassicality [3]. Therefore, understanding how and how much the nonclassicality is related with the nonlocality. Not all the nonclassical state are also nonlocal.

In general we define the nonclassicality of a radiation field with the impossibility of dealing with it through Maxwell equation [19].

Nonlocality, instead, is related to the quantum entanglement that we observe in a beam splitter mixing two different states [20]. More specifically if a theory describes two separated systems that don't have any long-range interactions, i.e. they can't influence each other instantaneously if they are far enough apart, it is said to be local. Even in this case we need a strong formalism to describe quantitatively the nonlocality and try to set up a hierarchy, understanding under which conditions a nonclassical state is nonlocal, too. The Bell inequalities can help us with this task, providing a useful tool in this way, setting sufficient conditions.

3.1 Nonclassical depth

A method to quantify the nonclassicality of a state, is the nonclassical depth parameter τ [17, 19, 3]. One way to introduce this parameter is through a slightly modified version of (2.27):

$$R(\alpha,\tau) = \frac{1}{\tau\pi}\int_{\mathbb{C}} d^2\beta \; P(\beta)\; \exp\left(-\frac{|\alpha-\beta|^2}{\tau}\right); \tag{3.1}$$

where we relabelled $W(\alpha, q) \rightarrow R(\alpha, \tau)$ and set:

$$\tau = \frac{p-q}{2} = \frac{1-q}{2}, \tag{3.2}$$

with $p = 1$. It's worth noting that for the special cases of $\tau = 0, \frac{1}{2}, 1$, the R function corresponds to P, W, Q functions introduced in the previous chapter.
In general this expression is true and allows us to employ it for our purposes. Here q is the minimum in the set of values $\{q\}$ for which the quasi-probability function R associated with the state can no more be considered a probability distribution i.e. positive semi-definite and at most singular as a Dirac Delta. This parameter can be interpreted as the minimum number of thermal photons that has to be added to the quantum state in order to remove all its quantum characteristics [19, 3]. How we said in (2.47a), the P-Glauber function of quantum states is more singular than a Dirac Delta and not positive definite, this is one of the cause of nonclassical effect. How we noticed in Section (2.2), the quasi-probability distribution function (3.1) is a convolution between a P-Glauber function and a plane wave of variance τ, $\exp\left(-\frac{|\alpha-\beta|^2}{\tau}\right)$. The consequences of the convolution of a negative function or a singular one, with plane waves, i.e. a Gaussian, is to make them more positive and smoother, respectively [3]. The exact necessary quantity of nonclassical depth τ that makes R a classical distribution function is the quantitative measure that we were looking for!
We can now determine the nonclassical depths for the family of states that we encountered in the first chapter. First of all we can say by definition that a coherent state $|\alpha\rangle$ has $\tau = 0$, because its P function is:

$$P(\alpha) = \pi\delta^{(2)}(\alpha), \tag{3.3}$$

where $\delta^{(2)}(\alpha)$ is the two dimensional Dirac Delta. Furthermore for $\tau = 1$ we recall that we have $R(\alpha, 1) = Q(\alpha)$ that is always a valid classical probability distribution.
For a Fock number state, the discussion is a bit more difficult. We have to expand $R(\alpha, \tau)$ as in [3] and re-write (3.1) in the following way:

$$R_n(\alpha, \tau) = \frac{1}{\tau}\left(\frac{\tau-1}{\tau}\right)^n \exp\left(-\frac{|\alpha|^2}{\tau}\right) L_n\left(\frac{|\alpha|^2}{\tau(1-\tau)}\right), \tag{3.4}$$

where L_n are the usual Laguerre polynomials. It is informative to list the first three terms:

$$R_0(\alpha,\tau) = \frac{1}{\tau} e^{-\frac{|\alpha|^2}{\tau}}, \tag{3.5a}$$

$$R_1(\alpha,\tau) = \frac{1}{\tau} e^{-\frac{|\alpha|^2}{\tau}} \left(\frac{1}{\tau^2}|\alpha|^2 - \frac{1-\tau}{\tau} \right), \tag{3.5b}$$

$$R_2(\alpha,\tau) = \frac{1}{\tau} e^{-\frac{|\alpha|^2}{\tau}} \left[\frac{1}{2\tau^4}|\alpha|^4 - \frac{2(1-\tau)}{\tau^3}|\alpha|^2 + \left(\frac{1-\tau}{\tau} \right)^2 \right], \tag{3.5c}$$

in order to compare them with the obtained quasi-probability distribution in Section 4.2.2. These functions are positive definite for $\tau \geq 1$ so must be $\tau = 1$ for the Fock states. This means that this state is the most nonclassical between all the quantum states, τ is at his maximum: a thermal state with at least one photon destroys its nonclassical characteristics.
The same argument can be carried on [17] for the squeezed states, leading to:

$$\tau = \frac{1 - e^{-2r}}{2}, \tag{3.6}$$

where $r = |\xi|$ is the modulus of the in the squeezing parameter ξ in the squeezing operator $\hat{S}(\xi)$ (1.47).
In general, [19] provides a relation between the eigenvalues of covariance matrix Σ (2.12) and the nonclassical depth τ for Gaussian states.

3.1.1 Sufficient conditions for nonclassicality

It's necessary to list some of the sufficient condition for a state to bee nonclassical, in relation with its nonclassical depth and characteristics of its Wigner and quasi-probability function.

- **Wigner function**

If we take (3.1) with $q = 0$, we turn back to the definition of the Wigner function:

$$R\left(\alpha, \frac{1}{2}\right) = W(\alpha) = \frac{2}{\pi} \int_{\mathbb{C}} d^2\beta \, P(\beta) \, e^{-2|\alpha-\beta|^2}; \tag{3.7}$$

that has $\tau = \frac{1}{2}$. We saw in Fig 2.1 that these function can also be negative. In general they could describe a nonclassical state whether the weighted sum is somewhere negative. If this doesn't happen, we don't have to worry, because this is only a sufficient condition and we have to see other cases.

- **Quasi-probability function**

The more general condition than the previous is with the negativity of the quasi-probability distribution (3.1) and generic $0 < q < 1$. Following what we said in Section 3.1, we have to vary q and the first value at which $R(\alpha, \tau)$ becomes negative, we found our state to be nonclassical and R can no more be considered a probability distribution as assumes negative values. We found the nonclassical depth τ, that is the parameter we were looking for.
Other sufficient condition for a state to be nonclassical exists [9], also considering other parameters, nevertheless our focus will be only on this way.

Given a nonclassical state, we can ask if it has some quantum behavior as entanglement, too. In particular we can investigate its quantum correlations as nonlocality. As a matter of fact the nonclassicality is a necessary condition for a state to be nonlocal [5, 6]. Nonetheless it is not a sufficient condition, thus can exist nonclassical states that don't display any nonlocal behaviours.

3.2 Nonlocality and Bell inequalities

The formalism behind the nonlocality is given by the Bell inequalities [21, 22]. They are a family of theorems that gives conditions under which the nonlocality of theory can be evaluated quantitatively, making use of the correlation function $\mathcal{E}(\alpha, \beta)$. It takes in account that two systems interact, in this case in a beam splitter, and this property might be revealed by the Bell inequalities.
In particular, we want to study what happens when a single state of light is made interact with another state, even the vacuum state, in a $50 : 50$ beam splitter, and then revealed by displaced on/off photodetection.
The measuring apparatus, proposed for the first time in [23], can bee seen in Fig 3.1.
Among these Bell inequalities, in this thesis are considered the CHSH inequality [22], named after Clauser, Horne, Shimony and Holt, starting from an on/off photodetection measure and through the Wigner functions formalism, in order to test the nonlocality of an entangled state.

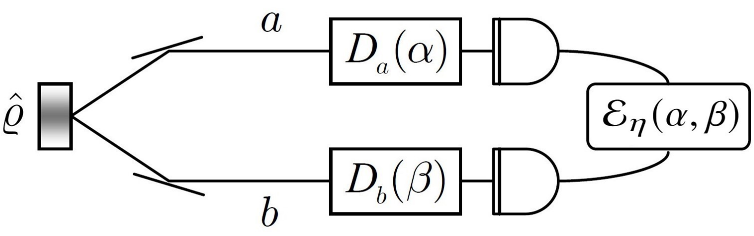


Figure 3.1: Apparatus of a nonlocality test based on displaced on/off photodetection, adapted from [20]. The two a and b modes of a mixed state $\hat{\varrho}$ are locally displaced by α and β, respectively, and then revealed through an on/off photodetection.

In this kind of Bell inequality, we consider four observables:

$$\hat{A}(\alpha) = \hat{A}_1, \qquad \hat{A}(\alpha') = \hat{A}_2,$$
$$\hat{B}(\beta) = \hat{B}_1, \qquad \hat{B}(\beta') = \hat{B}_2,$$

that corresponds to self-adjoint operators with eigenvalues ± 1, and with the following properties:

$$[\hat{A}_i \hat{B}_j] = 0, \qquad i, j = 1, 2;$$
$$\hat{A}_i^2 = \mathbb{I},$$
$$\hat{B}_i^2 = \mathbb{I},$$

the last two equalities means that $\hat{A}$ and $\hat{B}$ are projectors. In this case the correlation function $\mathcal{E}$ indicates the expectation value of all the possible outcome on the observables $\hat{A}$ and $\hat{B}$, namely:

$$\mathcal{E}(\alpha, \beta) = \langle \hat{A}_1 \hat{B}_1 \rangle\,, \tag{3.8}$$

and the same holds for the other parameters.

The Bell inequalities, through the Bell parameter $\mathcal{B}$, provide a numerical test to evaluate its nonlocality after the mixing in the beam splitter. Two photon counters can do the counting measurements distinguishing the number of absorbed photons, giving the different values of $+1$ or -1 whether the photons are in an even or odd number.

The measures are affected by errors that can be parameterized introducing a parameter η, called quantum efficiency, it takes into account the effect of dark counts. The ideal case is when $\eta = 1$ and there are no dark counts.

The mathematics behind the on/off detection starts from the definition of two projectors:

$$\begin{aligned}\Pi_{0,\eta} &= \sum_{n=0}^{+\infty} (1-\eta)^n \left|n\right\rangle \left\langle n\right|, \\ \Pi_{1,\eta} &= \mathbb{I} - \Pi_{0,\eta},\end{aligned} \tag{3.9}$$

where the measurement on $\hat{a}$ and $\hat{b}$ modes are described from:

$$\Pi^{\eta}_{h,k}(\alpha,\beta) = \Pi^{\eta}_{h}(\alpha) \otimes \Pi^{\eta}_{k}(\beta), \tag{3.10}$$

where $h, k = \{0, 1\}$, and we have:

$$\Pi^{\eta}_{j}(\alpha) = \hat{D}(\alpha)\Pi_{j,\eta}\hat{D}^{\dagger}(\alpha), \tag{3.11}$$

where $\hat{D}(\alpha)$ is the displacement operator and we put $j = \{h, k\}$. With $j = 0$ indicates a measurement of -1, if $j = 1$, the measurement gave +1. Now we have to compute the Wigner function of (3.9):

$$\begin{aligned}W_0(\alpha) &= W[\Pi_{0,\eta}](\alpha) = \frac{2}{\pi}\frac{1}{2-\eta} e^{\frac{2\eta}{2-\eta}|\alpha|^2}, \\ W_1(\alpha) &= W[\Pi_{1,\eta}](\alpha) = W[\mathbb{I}](\alpha) - W[\Pi_{0,\eta}](\alpha),\end{aligned} \tag{3.12}$$

where $W[\mathbb{I}](\alpha) = \frac{1}{\pi}$. We can also use the property that

$$W[\hat{D}(\alpha)\hat{\varrho}\hat{D}^{\dagger}(\alpha)](z) = W[\hat{\varrho}](z-\alpha), \tag{3.13}$$

in order to write $W[\hat{D}(\alpha)\Pi_{j,\eta}\hat{D}^{\dagger}(\alpha)]$ in a more convenient way:

$$W[\Pi_{0,\eta}](z-\alpha) = \frac{2}{\pi}\frac{1}{2-\eta} e^{-\frac{2\eta}{2-\eta}[|z|^2+|\alpha|^2-(z^*\alpha+\alpha^* z)]}, \tag{3.14a}$$

$$W[\Pi_{1,\eta}](z-\alpha) = W[\mathbb{I}](z-\alpha) - W[\Pi_{0,\eta}](z-\alpha) \tag{3.14b}$$

Now, in order to follow the way to the Bell parameter $\mathcal{B}$, it is possible to evaluate the correlation function, defined as follows:

$$\begin{aligned}\mathcal{E}_{\eta}(\alpha,\beta) &= \sum_{m,n} mn\, P_{m,n}(\alpha,\beta), \\ &= P_{++}(\alpha,\beta) + P_{--}(\alpha,\beta) - P_{+-}(\alpha,\beta) - P_{-+}(\alpha,\beta),\end{aligned} \tag{3.15}$$

where m, n indicate the $+1, -1$ measure of the counter and $P_{m,n}(\alpha,\beta) = \mathrm{Tr}[\hat{\varrho}\Pi^{\eta}_{h}(\alpha)\otimes\Pi^{\eta}_{k}(\beta)] = \langle\Pi^{\eta}_{h}(\alpha) \otimes \Pi^{\eta}_{k}(\beta)\rangle$ is the correlated probability of simultaneous measurements. This correlation function takes the results of the two measurements of the photodetectors.

In the case of (3.10), we can write $P_{m,n}(\alpha, \beta) = \langle \Pi^{\eta}_{h,k}(\alpha, \beta) \rangle$ and the correlation function is expressed as follows:

$$\begin{aligned}
\mathcal{E}_{\eta}(\alpha, \beta) &= \sum_{h,k} hk \; P_{h,k}(\alpha, \beta), \\
&= \sum_{h,k=0}^{1} (-1)^{h+k} \, \langle \Pi^{\eta}_{h,k}(\alpha, \beta) \rangle, \\
&= 1 + 4 \, \langle \Pi^{\eta}_{0,0}(\alpha, \beta) \rangle - 2 \left[\langle \Pi^{\eta}_{0}(\alpha) \otimes \mathbb{I} \rangle + \langle \mathbb{I} \otimes \Pi^{\eta}_{0}(\beta) \rangle \right], \\
&\equiv 1 + 4\mathcal{I}(\alpha, \beta) - 2 \left[\mathcal{G}(\alpha) + \mathcal{Y}(\beta) \right],
\end{aligned} \tag{3.16}$$

where we defined the following three quantities:

$$\mathcal{I}(\alpha, \beta) = \langle \Pi^{\eta}_{0,0}(\alpha, \beta) \rangle = \mathrm{Tr}[\hat{\varrho} \Pi^{\eta}_{0}(\alpha) \otimes \Pi^{\eta}_{0}(\beta)], \tag{3.17a}$$

$$\mathcal{G}(\alpha) = \langle \Pi^{\eta}_{0}(\alpha) \otimes \mathbb{I} \rangle = \mathrm{Tr}[\hat{\varrho} \Pi^{\eta}_{0}(\alpha) \otimes \mathbb{I}_b], \tag{3.17b}$$

$$\mathcal{Y}(\beta) = \langle \mathbb{I} \otimes \Pi^{\eta}_{0}(\beta) \rangle = \mathrm{Tr}[\hat{\varrho} \mathbb{I}_a \otimes \Pi^{\eta}_{0}(\beta)], \tag{3.17c}$$

that, we can further expand using the trace rule for the Wigner functions (2.33)as:

$$\mathcal{I}(\alpha, \beta) = \pi^2 \int_{\mathbb{C}^2} d^2z \; d^2w \; W[\hat{\varrho}](z, w) W[\Pi_{0,\eta}](z - \alpha) W[\Pi_{0,\eta}](w - \beta), \tag{3.18a}$$

$$\mathcal{G}(\alpha) = \pi^2 \int_{\mathbb{C}^2} d^2z \; d^2w \; W[\hat{\varrho}](z, w) W[\Pi_{0,\eta}](z - \alpha) W[\mathbb{I}](w - \beta), \tag{3.18b}$$

$$\mathcal{Y}(\beta) = \pi^2 \int_{\mathbb{C}^2} d^2z \; d^2w \; W[\hat{\varrho}](z, w) W[\mathbb{I}](z - \alpha) W[\Pi_{0,\eta}](w - \beta), \tag{3.18c}$$

Here we used the (3.11) and (3.13) to simplify the expressions.

Finally we are able to write the final expression of the Bell parameter $\mathcal{B}$:

$$\mathcal{B} = \mathcal{E}_{\eta}(\alpha, \beta) + \mathcal{E}_{\eta}(\alpha, \beta') + \mathcal{E}_{\eta}(\alpha', \beta) - \mathcal{E}_{\eta}(\alpha', \beta'). \tag{3.19}$$

The CHSH Bell inequality states that a physical system, in order to be described by local theories, must satisfy:

$$|\mathcal{B}| < 2. \tag{3.20}$$

We recall that this is only a sufficient condition, if a measurement of an experiment gives $|\mathcal{B}| < 2$, it do not allow us to say anything about the system's nonlocality. In quantum mechanics is also possible to give an upper bound to the maximal violation of the Bell inequalities in the CHSH form. This limit it's given by the Cirel'son inequality [24] and it states that:

$$|\mathcal{B}| < 2\sqrt{2}. \tag{3.21}$$

So, in general, any quantum experiment whose the outcome gives $2 < |\mathcal{B}| < 2\sqrt{2}$ confirm the nonlocality of the considered physical system.

At this point we have all the mathematical tools in order to analyze a particular state, that is a superposition of three Fock states. In particular is interesting to evaluate its quantum features as it arise a null intensity correlation function once it is mixed in a beam splitter.

Chapter 4

Quantum state without intensity correlations: a case study

In this chapter we consider a particular state $\hat{\varrho}_m$, defined as the following mixture:

$$\hat{\varrho}_m = \sum_{n=0}^{2} p_n |n\rangle\langle n| = p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1| + p_2|2\rangle\langle 2|. \tag{4.1}$$

We want to impose that our state has the first two moments, i.e. mean and variance, of a thermal state.

$$\langle \hat{n} \rangle = N_{\text{th}},$$
$$\Delta^2(\hat{n}) = N_{\text{th}}(N_{\text{th}} + 1).$$

We make this choice because a thermal state have a null intensity correlation function, as shown in Section (1.68), so once two states are mixed in a beam splitter, we can think at the beam splitter as a source of two uncorrelated mixed thermal states and study the quantum correlations only.

Furthermore the moments of a probability distribution remains unchanged in a beam splitter interaction, as seen in Section 2.1, so we are sure that the state continue to have the first two moments of a thermal state. In addiction, thermal states are phase-insensitive states, because its Wigner function depends only on the modulus $|\alpha|$ as seen in (2.32).

Finally it is a very reproducible state in a laboratory and in general, is capable of maximizing the transported information in a canal.

4.1 Characterizing the state

In order to find the three coefficient p_n let's consider the following conditions:

- **Normalization:** $\mathrm{Tr}[\hat{\varrho}_m] = \sum_{n=0}^{2} p_n = p_0 + p_1 + p_2 = 1$;
- **Mean:** $\langle n \rangle := \mathrm{Tr}[\hat{\varrho}_m \hat{n}] = p_1 + 2p_2 \equiv N_{\text{th}}$;
- **Variance:** $\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 := \mathrm{Tr}[\hat{\varrho}_m \hat{n}^2] - \mathrm{Tr}[\hat{\varrho}_m \hat{n}]^2 = p_1 + 4p_2 - (p_1 + 2p_2)^2 \equiv N_{\text{th}}(N_{\text{th}} + 1)$.

With this constraints we find

$$\begin{cases} p_0 = N_{\text{th}}^2 - N_{\text{th}} + 1, \\ p_1 = N_{\text{th}} - 2N_{\text{th}}^2, \\ p_2 = N_{\text{th}}^2, \end{cases} \tag{4.2}$$

with the further condition: $p_n > 0 \leftrightarrow 0 < N_{\text{th}} < \frac{1}{2}$.
So we have :

$$\hat{\varrho}(N_{\text{th}}) = (N_{\text{th}}^2 - N_{\text{th}} + 1)|0\rangle\langle 0| + (N_{\text{th}} - 2N_{\text{th}}^2)|1\rangle\langle 1| + N_{\text{th}}^2|2\rangle\langle 2|, \tag{4.3}$$

with boundary values:

- $\hat{\varrho}_m(0) = |0\rangle\langle 0|$, vacuum state;
- $\hat{\varrho}_m(1/2) = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|2\rangle\langle 2|$;

4.2 Nonclassicality

In order to evaluate the nonclassicality of $\hat{\varrho}_m$, we make use of the sufficient conditions in Section 3.1.1 on the Wigner function and the quasi-probability function.

4.2.1 Wigner function

Now, we have to compute the Wigner function of the state. This will be useful when we will mix the state in a beam splitter, we will need the Wigner function $W[\hat{\varrho}_m](\alpha)$ and compose them following (2.39). In order to fulfill this request, we have to use directly (2.49) with (2.47b). In

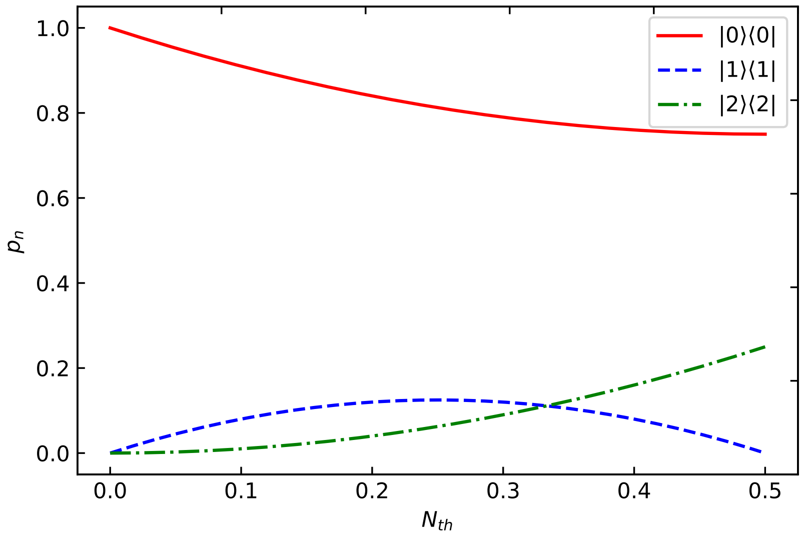


Figure 4.1: Probability distribution p_n in function of the energy N_{th}.

our case:

$$\begin{aligned} W(\alpha) &= \sum_{n=0}^{2} p_n W_n(\alpha), \\ &= \frac{2}{\pi} e^{-2|\alpha|^2} \sum_{n=0}^{2} p_n (-1)^n L_n(4|\alpha|^2), \\ &= \frac{2}{\pi} e^{-2|\alpha|^2} \left[p_0 - p_1 \left(-4|\alpha|^2 + 1 \right) + p_2 \left(8|\alpha|^4 - 8|\alpha|^2 + 1 \right) \right]. \end{aligned} \tag{4.4}$$

In particular, plotting these functions, we can notice in Fig 4.2 that all of them are always positive. This feature doesn't say anything about our nonclassicality. In fact, having the Wigner function negative is just a sufficient condition for our state to be nonclassical.

In order to go deeper on this way, we have to compute the quasi probability function.

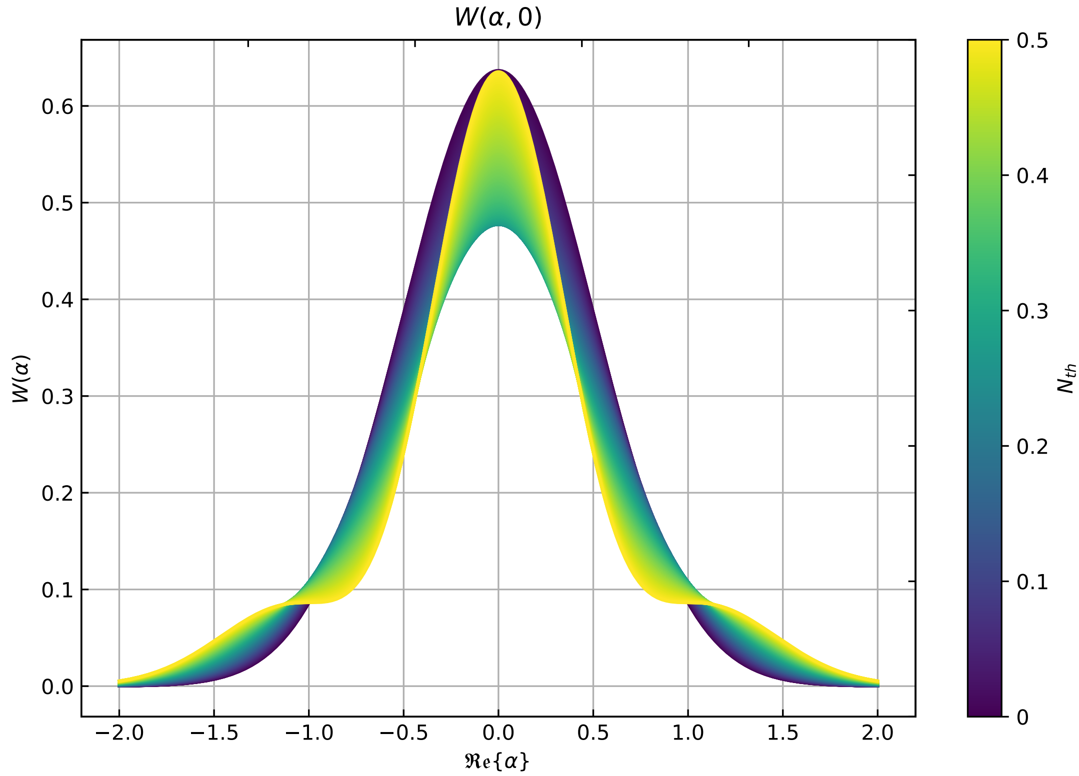


Figure 4.2: Transverse section of the Wigner function $W(\alpha, 0)$ in function of the real part of α, $\Re e\{\alpha\}$ with fixed N_{th} in the range $[0, 0.5]$

4.2.2 Quasi-probability distribution

We saw in Section 3.1.1 that another sufficient condition ensuring that our state is nonclassical is given by the negativity of the quasi-probability function.

Our purpose is to see whenever this Wigner function becomes negative. Namely the first negative function. We can satisfy this assignment using (2.27).

$$W(\alpha, q) = \frac{2}{\pi(p-q)} \int_{\mathbb{C}} d^2\beta \, W(\beta, p) \, \exp\left[-\frac{2|\alpha-\beta|^2}{p-q}\right]; \qquad q < p$$

As we already shown in Fig 4.2, the Wigner function with $q = 0$ is always non-negative, now let's find a generalization of the previous formula, by continuing to use (2.27), for a generic

q, p, but then set $p = 1$.

$$
\begin{aligned}
W(\alpha, q) &= \frac{2}{\pi(p-q)} \int_{\mathbb{C}} d^2\beta \, W(\beta, 1) \, \exp\left(-2\frac{|\alpha-\beta|^2}{p-q}\right), \\
&= \frac{2}{\pi(p-q)} \int_{\mathbb{C}} d^2\beta \, \sum_{n=0}^{2} P_n(\beta) \, p_n \, \exp\left(-2\frac{|\alpha-\beta|^2}{p-q}\right), \\
&= \frac{2}{\pi(p-q)} \int_{\mathbb{C}} d^2\beta \, \exp\left(-2\frac{|\alpha-\beta|^2}{p-q}\right) \sum_{n=0}^{2} p_n \sum_{m=0}^{n} \binom{n}{m} \frac{1}{m!} \left(\frac{\partial^2}{\partial\beta\partial\beta^*}\right)^m \delta^{(2)}(\beta), \\
&= \frac{2}{\pi(p-q)} \int_{\mathbb{C}} d^2\beta \, \exp\left(-2\frac{|\alpha-\beta|^2}{p-q}\right) \left(p_0 \delta^{(2)}(\beta) + p_1 \left[\delta^{(2)}(\beta) + \left(\frac{\partial^2}{\partial\beta\partial\beta^*}\right) \delta^{(2)}(\beta)\right]\right. \\
&\qquad \left. + p_2 \left[\delta^{(2)}(\beta) + 2\left(\frac{\partial^2}{\partial\beta\partial\beta^*}\right)\delta^{(2)}(\beta) + \frac{1}{2}\left(\frac{\partial^2}{\partial\beta\partial\beta^*}\right)^2 \delta^{(2)}(\beta)\right]\right),
\end{aligned}
\tag{4.5}
$$

making the calculation, we can find:

$$
\begin{aligned}
W(\alpha, q) = \frac{2}{\pi(p-q)} \exp\left(-2\frac{|\alpha|^2}{p-q}\right) \Bigg[p_0 - p_1 \left(\frac{-4|\alpha|^2}{(p-q)^2} + \frac{2}{p-q} - 1\right) + \\
+ p_2 \left(\frac{8|\alpha|^4}{(p-q)^4} - \frac{16|\alpha|^2}{(p-q)^3} + \frac{4+8|\alpha|^2}{(p-q)^2} - \frac{4}{p-q} + 1\right)\Bigg].
\end{aligned}
\tag{4.6}
$$

We can check that these calculation are correct comparing them with the linear combination of (3.5), in fact:

$$
W(\alpha, q) = \sum_{n=0}^{2} p_n \, R_n(\alpha, \tau), \tag{4.7}
$$

that is respected.

All these functions depend on α, q and N_{th}, that we remember is inside the expression of p_n. We also remember that the normalization condition for the functions is respected. If we plot these functions, we observe that they become negative for a value of q, this fact allows us to say that the state $\hat{\varrho}_m$ is nonclassical. In Fig. 4.3 we reported the first negative quasi-probability function. We can also set a relation between q and the energy N_{th}, this give birth to the nonclassical depth.

4.2.3 Nonclassical depth

For our purposes, the functions (4.6) are taken from real value of α, p is set $p = 1$, and we are interested in let varying $q \in [0, 1)$. Among all these functions, we are looking for the first becoming negative, fixed q, and varying N_{th}. In this way we can try to build a hierarchy among

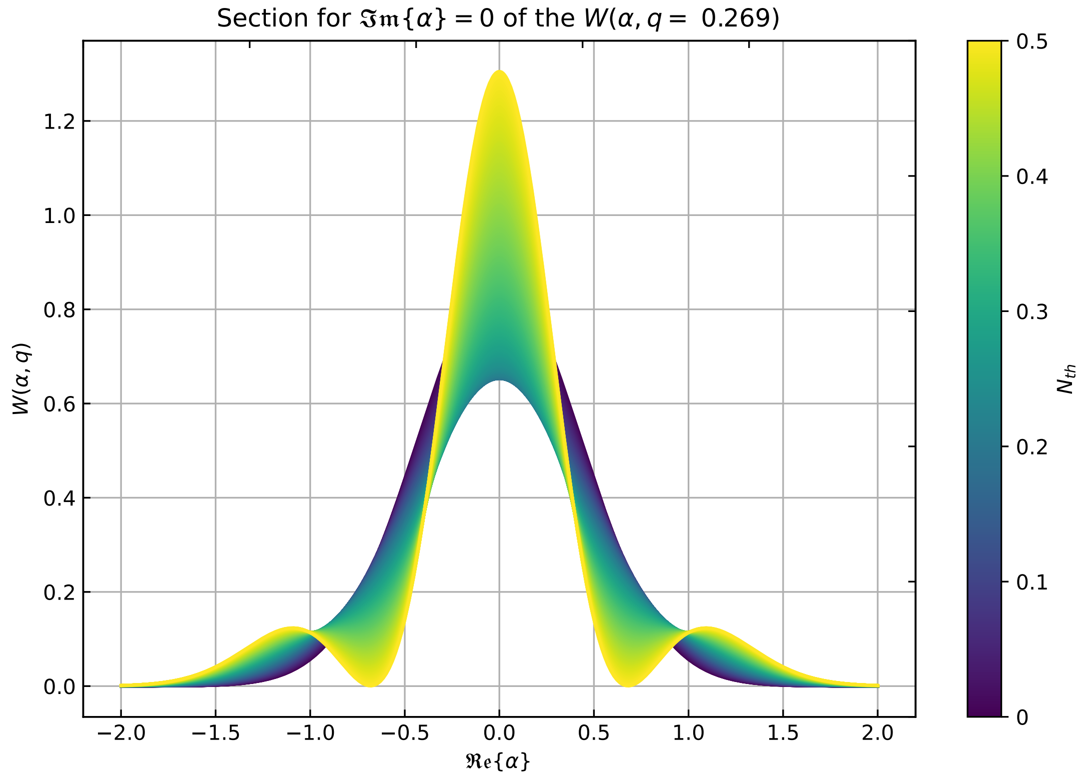


Figure 4.3: Transverse section ($\mathfrak{Im}\{\alpha\} = 0$) of the quasi-probability function $W(\alpha, q)$, with $q = 0.269$, in function of the real part of α, $\mathfrak{Re}\{\alpha\}$, with fixed N_{th} in the range $[0, 0.5]$

all the quasi-probability functions, in order to understand the nonclassicality of our state. In Fig 4.3 we see the first quasi-probability function becoming negative. It turns out that the first function becoming negative has: $q \simeq 0.269$ and $N_{\text{th}} = 0.5$. Increasing q, the functions becomes negative for smaller values of N_{th}. It is possible to get a graph of the relation between q and N_{th} for the first function becoming negative, how we can see in Fig 4.4.

As the expansion (1.50) of the first two terms of a squeezed state $|\xi\rangle$ is very similar to our state:

$$|\xi_2\rangle = \frac{1}{\sqrt{\cosh r}} \left(|0\rangle + \frac{\tanh r}{\sqrt{2}} |2\rangle \right), \tag{4.8}$$

we decided to compare the nonclassical depth of our state $\hat{\varrho}_m$ with a squeezed state's one

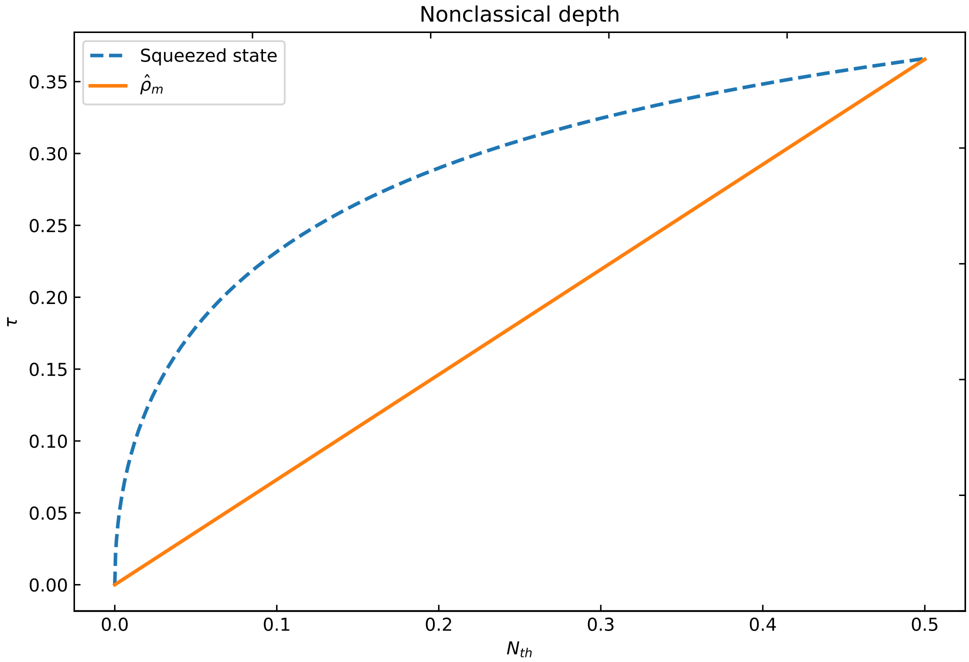


Figure 4.4: Nonclassical depth τ for our state $\hat{\varrho}_m$ and for a squeezed state in function of N_{th}. We notice that the nonclassical depth of our state is always smaller than the squeezed state's one, showing that our state is always less nonclassical, nonetheless, it provides a quantitative trend of nonclassicality as it is always $\tau > 0$.

using (3.6) and (1.52), in order to obtain:

$$\begin{aligned}\tau &= \frac{1-e^{-2r}}{2} = \frac{1-\left(1+2\langle\hat{n}\rangle - 2\sqrt{\langle\hat{n}\rangle(\langle\hat{n}\rangle+1)}\right)}{2}, \\ &= N_{\text{th}} + 2\sqrt{N_{\text{th}}(N_{\text{th}}+1)}.\end{aligned} \tag{4.9}$$

How we can see from Fig 4.4 our state $\hat{\varrho}_m$ is less nonclassical than a squeezed state with the same energy for $N_{\text{th}} \in [0, 0.5]$, but however shows a discrete degree of nonclassicality as $\tau > 0$.

In this section we evaluated the nonclassical degree of our state. This fact pushes us to study deeper quantum features of our state. In particular since our state is non classical, it fulfils the necessary conditions in order to be nonlocal. We are interested in to see if a particular sufficient measure of nonlocality is satisfied by our state.

4.3 Bell inequalities

In this section we investigate the nonlocality of our state. In fact we gave our state a measure of nonclassicality in the previous one. Now, our purpose is to see whether and how much our state is nonlocal. In order to do that we have to mix our state $\hat{\varrho}_m$ in some different configurations in the beam splitter, in this way it acts as a mode mixer.
We will see whether the state violate the CHSH Bell inequality (3.20) as in [20, 22]:

$$|\mathcal{B}| < 2. \tag{4.10}$$

Firstly we mix $\hat{\varrho}_m$ in a beam splitter with the vacuum state, namely $\hat{\varrho}_{vac} = |0\rangle\langle 0|$. Then we mix the $\hat{\varrho}_m$ state with itself. However the main point is to compute the Wigner function of the mixed final state with (2.39) in order to text the nonlocality of the possible entangled state:

$$W^{\pm}[\hat{\varrho}_1 \otimes \hat{\varrho}_2](\alpha, \beta) = W[\hat{\varrho}_1]\left(\frac{\beta \pm \alpha}{\sqrt{2}}\right) \cdot W[\hat{\varrho}_2]\left(\frac{\beta \mp \alpha}{\sqrt{2}}\right), \tag{4.11}$$

thus, we have to compute the Bell parameter $\mathcal{B}$, as in Section 3.2, in order to see if our state is nonlocal.

4.3.1 Mixing with the vacuum

Firstly we mixed the $\hat{\varrho}_m$ state with the vacuum in a beam splitter, namely we mixed our state with nothing in order to mix the modes. We also set the quantum efficiency $\eta = 1$ (ideal case).
The Wigner function is given by:

$$\begin{aligned} W^{\pm}_{\mathrm{tot}}[\hat{\varrho} \otimes |0\rangle\langle 0|](\alpha, \beta) &= \frac{4}{\pi^2} e^{-2(|\alpha|^2+|\beta|^2)} \left[p_0 - p_1\left(-2|\alpha+\beta|^2+1\right) + p_2\left(2|\alpha+\beta|^4 - 4|\alpha+\beta|^2+1\right)\right], \\ &= \frac{4}{\pi^2} e^{-2(|\alpha|^2+|\beta|^2)} \left[(p_0 - p_1 + p_2) + (2p_1 - 4p_2)|\alpha \pm \beta|^2 + 2p_2|\alpha \pm \beta|^4\right], \\ &= \frac{4}{\pi^2} e^{-2(|\alpha|^2+|\beta|^2)} \left(A + B|\alpha \pm \beta|^2 + C|\alpha \pm \beta|^4\right), \end{aligned} \tag{4.12}$$

where we set:

$$\begin{cases} A = p_0 - p_1 + p_2, \\ B = 2p_1 - 4p_2, \\ C = 2p_2, \end{cases} \tag{4.13}$$

We are now able to compute the (3.16):

$$
\begin{aligned}
\mathcal{I}(\alpha,\beta) &= \mathrm{Tr}[\hat{\varrho}'\ \Pi_0^{\eta}(\alpha)\otimes\Pi_0^{\eta}(\alpha)],\\
&= \pi^2\int_{\mathbb{C}^2} d^2w\ d^2z\ W^{\pm}_{\mathrm{tot}}[\hat{\varrho}\otimes|0\rangle\langle 0|](z,w)\ W[\Pi_{0,\eta}](z-\alpha)W[\Pi_{0,\eta}](w-\beta),\\
&= e^{-(|\alpha|^2+|\beta|^2)}\left(A+\frac{B}{4}\left(2+|\alpha\pm\beta|^2\right)+\frac{C}{16}\left(8+8|\alpha\pm\beta|^2+|\alpha\pm\beta|^4\right)\right).
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
\mathcal{G}(\alpha) &= \mathrm{Tr}[\hat{\varrho}'\ \Pi_0^{\eta}(\alpha)\otimes\mathbb{I}],\\
&= \pi^2\int_{\mathbb{C}^2} d^2w\ d^2z\ W^{\pm}_{\mathrm{tot}}[\hat{\varrho}\otimes|0\rangle\langle 0|](z,w)\ W[\Pi_{0,\eta}](z-\alpha)W[\mathbb{I}](w),\\
&= e^{-|\alpha|^2}\left(A+\frac{B}{4}\left(3+|\alpha|^2\right)+\frac{C}{16}\left(18+12|\alpha|^2+|\alpha|^4\right)\right).
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
\mathcal{Y}(\beta) &= \mathrm{Tr}[\hat{\varrho}'\ \mathbb{I}\otimes\Pi_0^{\eta}(\beta)],\\
&= \pi^2\int_{\mathbb{C}^2} d^2w\ d^2z\ W^{\pm}_{\mathrm{tot}}[\hat{\varrho}\otimes|0\rangle\langle 0|](z,w)\ W[\mathbb{I}](z)W[\Pi_{0,\eta}](w-\beta),\\
&= e^{-|\beta|^2}\left(A+\frac{B}{4}\left(3+|\beta|^2\right)+\frac{C}{16}\left(18+12|\beta|^2+|\beta|^4\right)\right),
\end{aligned}
\tag{4.16}
$$

where we noticed $\hat{\varrho}' = \hat{\varrho}\otimes|0\rangle\langle 0|$ and used:

$$
\begin{aligned}
W[\mathbb{I}](z-\alpha) &= \frac{1}{\pi},\\
W[\Pi_{0,\eta}](z-\alpha) &= \frac{2}{\pi}\frac{1}{2-\eta}e^{-\frac{2\eta}{2-\eta}(|z|^2+|\alpha|^2-(z^*\alpha+\alpha^* z))},
\end{aligned}
\tag{4.17}
$$

from (3.14a), and the important relations in order to differentiate under the integral:

$$
\begin{aligned}
\int_{\mathbb{C}} d^2z\ e^{a|z|^2+b_1z+b_2z^*} &= \frac{\pi}{a}e^{\frac{b_1b_2}{a}},\\
|z|^2e^{a|z|^2+b_1z+b_2z^*} &= (\partial_{b_1}\partial_{b_2})e^{a|z|^2+b_1z+b_2z^*},
\end{aligned}
\tag{4.18}
$$

with $a = 4$ and $b_1 = 2\alpha^*$, $b_2 = 2\alpha$. At this point we are able to consider the correlation function $\mathcal{E}(\alpha,\beta)$ and the $\mathcal{B}$ parameter as defined in Eqs (3.16), (3.19).

The results are shown in Fig 4.5. The Bell parameter has been chosen to be plotted with the following parameterization: $J = |\alpha| = -|\beta| = k|\alpha'| = -k|\beta'|$ as these are phase-insensitive states and does not depend on their phases.

However the Bell inequalities haven't been violated in any case. This means that we can say nothing about nonlocality behaviours when our state is mixed with the vacuum in a beam splitter.

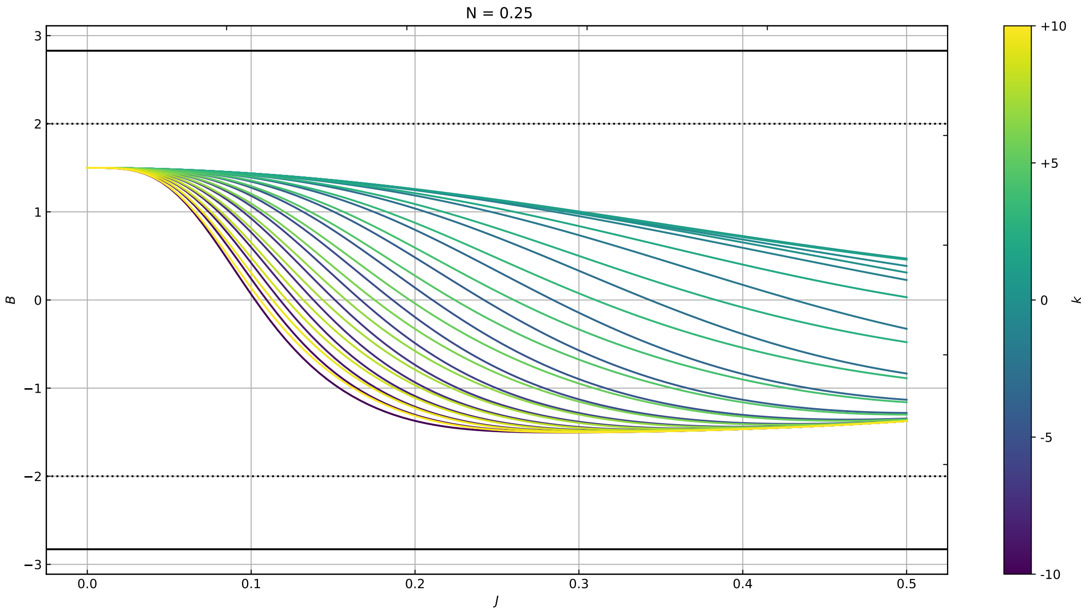


Figure 4.5: Bell parameter for a $\hat{\varrho}_{\text{th}} \otimes |0\rangle\langle 0|$ mixture, in function of $J = |\alpha| = -|\beta| = k|\alpha'| = -k|\beta'|$ for fixed $N_{\text{th}} = 0.25$ with k varying in $[-10, +10]$. As shown, the Bell parameter $\mathcal{B}$ is always in the range $(-2, +2)$, meaning a non violation of the CHSH Bell inequality.

4.3.2 Two state mixing

In this case we attempted to see if mixing the state $\hat{\varrho}_m$ with a copy of itself, in a beam splitter, would give some quantum correlations.
The total Wigner function, as in the previous case is given by:

$$\begin{aligned} W^{\pm}_{\text{tot}}[\hat{\varrho} \otimes \hat{\varrho}](\alpha, \beta) &= \frac{4}{\pi^2} e^{-2(|\alpha|^2+|\beta|^2)} \left(A + B|\alpha \pm \beta|^2 + C|\alpha \pm \beta|^4\right) \left(A + B|\alpha \mp \beta|^2 + C|\alpha \mp \beta|^4\right), \\ &= \frac{4}{\pi^2} e^{-2(|\alpha|^2+|\beta|^2)} \Big[A^2 + AB\left(|\alpha \pm \beta|^2 + |\alpha \mp \beta|^2\right) + \\ &\quad + AC\left(|\alpha \pm \beta|^4 + |\alpha \mp \beta|^4\right) + B^2|\alpha \pm \beta|^2|\alpha \mp \beta|^2 + \\ &\quad + BC\left(|\alpha \pm \beta|^2|\alpha \mp \beta|^4 + |\alpha \mp \beta|^2|\alpha \pm \beta|^4\right) + C^2|\alpha \pm \beta|^4|\alpha \mp \beta|^4\Big]. \end{aligned} \tag{4.19}$$

Even in this case we set (4.13) and the results for $\mathcal{I}, \mathcal{G}, \mathcal{Y}$ follow as before:

$$
\begin{aligned}
\mathcal{I}(\alpha,\beta) &= \mathrm{Tr}[\hat{\varrho}'\ \Pi_0^{\eta}(\alpha)\otimes\Pi_0^{\eta}(\alpha)],\\
&= \pi^2\int_{\mathbb{C}^2} d^2w\ d^2z\ W_{\mathrm{tot}}^{\pm}[\hat{\varrho}\otimes\hat{\varrho}](z,w)\ W[\Pi_{0,\eta}](z-\alpha)W[\Pi_{0,\eta}](w-\beta),\\
&= e^{-(|\alpha|^2+|\beta|^2)}\Big\{A^2+\frac{AB}{4}\left[4+|\alpha\pm\beta|^2+|\alpha\mp\beta|^2\right]+\\
&\quad+\frac{AC}{16}\left[16+8(|\alpha\pm\beta|^2+|\alpha\mp\beta|^2)+|\alpha\pm\beta|^4+|\alpha\mp\beta|^4\right)+\\
&\quad+\frac{B^2}{16}\left[4+4(|\alpha|^2+|\beta|^2)+|\alpha\pm\beta|^2|\alpha\mp\beta|^2\right]+\\
&\quad+\frac{BC}{64}\Big[32+8(|\alpha\pm\beta|^2+|\alpha\mp\beta|^2)+32(|\alpha|^2+|\beta|^2)+2(|\alpha\pm\beta|^4+|\alpha\mp\beta|^4)+\\
&\qquad+16(|\alpha\pm\beta|^2|\alpha\mp\beta|^2)+|\alpha\pm\beta|^4|\alpha\mp\beta|^2+|\alpha\pm\beta|^2|\alpha\mp\beta|^4\Big]+\\
&\quad+\frac{C^2}{256}\Big[64+128(|\alpha|^2+|\beta|^2)+16\left(3(|\alpha\pm\beta|^2|\alpha\mp\beta|^2)+2(|\alpha|^2+|\beta|^2)^2\right)+\\
&\qquad+16\left(|\alpha\pm\beta|^2|\alpha\mp\beta|^2(|\alpha|^2+|\beta|^2)\right)+|\alpha\pm\beta|^4|\alpha\mp\beta|^4\Big]\Big\}.
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
\mathcal{G}(\alpha) &= \mathrm{Tr}[\hat{\varrho}'\ \Pi_0^{\eta}(\alpha)\otimes\mathbb{I}],\\
&= \pi^2\int_{\mathbb{C}^2} d^2w\ d^2z\ W_{\mathrm{tot}}^{\pm}[\hat{\varrho}\otimes\hat{\varrho}](z,w)\ W[\Pi_{0,\eta}](z-\alpha)W[\mathbb{I}](w),\\
&= e^{-|\alpha|^2}\Big[A^2+\frac{AB}{4}\left(6+2|\alpha|^2\right)+\frac{AC}{16}\left(36+24|\alpha|^2+2|\alpha|^4\right)+\frac{B^2}{16}\left(10+4|\alpha|^2+|\alpha|^4\right)+\\
&\quad+\frac{BC}{64}\left(132+68|\alpha|^2+22|\alpha|^4+2|\alpha|^6\right)+\frac{C^2}{256}\left(472+224|\alpha|^2+104|\alpha|^4+16|\alpha|^6+|\alpha|^8\right)\Big].
\end{aligned}
\tag{4.21}
$$

$$
\begin{aligned}
\mathcal{Y}(\beta) &= \mathrm{Tr}[\hat{\varrho}'\ \mathbb{I}\otimes\Pi_0^{\eta}(\beta)],\\
&= \pi^2\int_{\mathbb{C}^2} d^2w\ d^2z\ W_{\mathrm{tot}}^{\pm}[\hat{\varrho}\otimes\hat{\varrho}](z,w)\ W[\mathbb{I}](z)W[\Pi_{0,\eta}](w-\beta),\\
&= e^{-|\beta|^2}\Big[A^2+\frac{AB}{4}\left(6+2|\beta|^2\right)+\frac{AC}{16}\left(36+24|\beta|^2+2|\beta|^4\right)+\frac{B^2}{16}\left(10+4|\beta|^2+|\beta|^4\right)+\\
&\quad+\frac{BC}{64}\left(132+68|\beta|^2+22|\beta|^4+2|\beta|^6\right)+\frac{C^2}{256}\left(472+224|\beta|^2+104|\beta|^4+16|\beta|^6+|\beta|^8\right)\Big].
\end{aligned}
\tag{4.22}
$$

The results can be viewed in Fig. 4.6. Also the mixture $\hat{\varrho}\otimes\hat{\varrho}$ doesn't violate the Bell inequality.

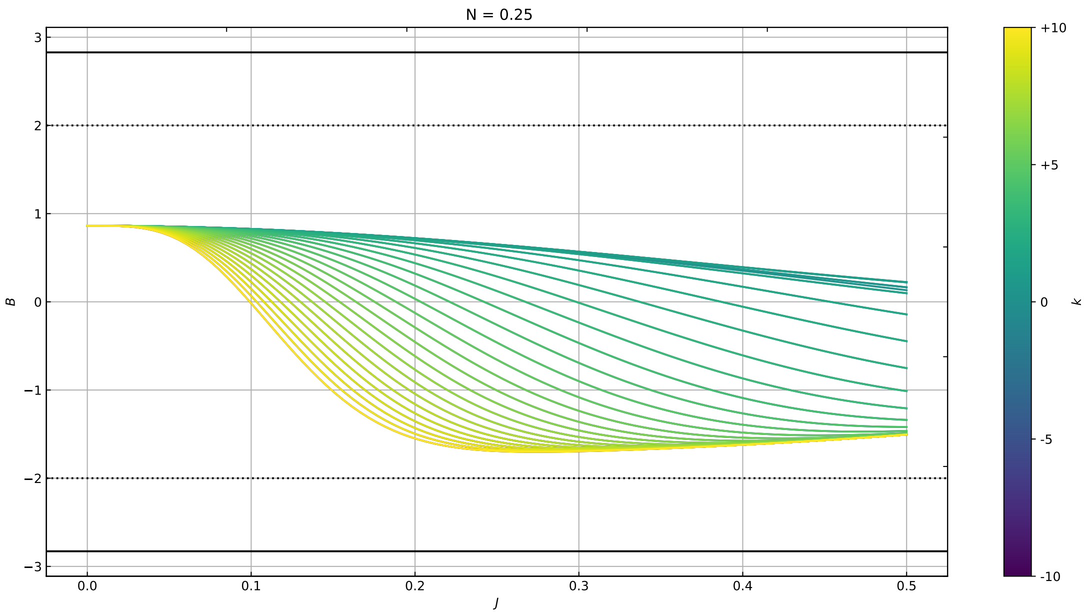


Figure 4.6: Bell parameter for a $\hat{\varrho}_{\text{th}} \otimes \hat{\varrho}_{\text{th}}$ mixture, in function of $J = |\alpha| = -|\beta| = k|\alpha'| = -k|\beta'|$ for fixed $N_{\text{th}} = 0.25$ with k varying in $[-10, +10]$. As shown, the Bell parameter $\mathcal{B}$ is always in the range $(-2, +2)$, meaning a non violation of the CHSH Bell inequality.

Appendix A

Operator ordering theorems

A.1 Baker-Campbell-Hausdorff theorems

Given two operators, $\hat{A}$ and $\hat{B}$, it can be proven that:

Theorem 1 *[1st BCH formula]*

If $[\hat{A}, \hat{B}] \in \mathbb{C}$*:*

$$\begin{aligned} e^{\hat{A}+\hat{B}} &= e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]}, \\ &= e^{\hat{B}} e^{\hat{A}} e^{\frac{1}{2}[\hat{A},\hat{B}]}. \end{aligned} \tag{A.1}$$

Theorem 2 *[2nd BCH formula]*

$$e^{-\hat{A}} \hat{B} e^{\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!} \left[\hat{A}, [\hat{A}, \hat{B}]\right] + \frac{1}{3!} \left[\hat{A}, \left[\hat{A}, [\hat{A}, \hat{B}]\right]\right] + \dots \, . \tag{A.2}$$

A proof of this formulas can be found in [8] and in [4].

A.2 Displacement operator

$\hat{D}(\alpha)$ is called displacement operator and is defined by:

$$\hat{D}(\lambda) = e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}. \tag{A.3}$$

With the following properties:

$$\hat{D}(\lambda) = e^{-\frac{1}{2}|\lambda|^2} e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}} = e^{\frac{1}{2}|\lambda|^2} e^{-\lambda^* \hat{a}} e^{\lambda \hat{a}^\dagger}, \qquad \text{using (A.1),} \tag{A.4a}$$

$$\hat{D}^{-1}(\lambda) = \hat{D}^\dagger(\lambda) = D(-\lambda), \tag{A.4b}$$

$$\hat{D}^\dagger(\lambda)\ \hat{a}\ \hat{D}^(\lambda) = \hat{a} + \lambda, \qquad \text{with (A.2),} \tag{A.4c}$$

$$\hat{D}(\lambda)\ \Pi\ \hat{D}^\dagger(\lambda) = \hat{D}(2\lambda)\Pi = \Pi\hat{D}^\dagger(2\lambda), \qquad \text{with } \Pi = (-1)^{\hat{a}^\dagger \hat{a}} \tag{A.4d}$$

- **Displacement of the vacuum state**

$$\begin{aligned} |\alpha\rangle &= e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} e^{\alpha^* \hat{a}} |0\rangle, \\ &= e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} |0\rangle, \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{+\infty} \frac{\alpha^n}{n!} \left(\hat{a}^\dagger\right)^n |0\rangle, \end{aligned} \tag{A.5}$$

where we used the fact that $\hat{a}^n|0\rangle = 0$.

- **Heisenberg evolution**

The Heisenberg evolution of the displacement operator in general can be viewed as:

$$\begin{aligned} \hat{\mathcal{U}}^\dagger \hat{D}(\lambda) \hat{\mathcal{U}} &= \hat{\mathcal{U}}^\dagger e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}} \hat{\mathcal{U}}, \\ &= e^{\lambda \hat{\mathcal{U}}^\dagger \hat{a}^\dagger \hat{\mathcal{U}} - \lambda^* \hat{\mathcal{U}}^\dagger \hat{a} \hat{\mathcal{U}}}, \\ &= e^{\lambda(\hat{a}^\dagger \cos\phi - \hat{b}^\dagger \sin\phi) - \lambda^*(\hat{a}\cos\phi - \hat{b}\sin\phi)}, \\ &= \hat{D}_a(\lambda\cos\phi)\ \hat{D}_b(-\lambda\sin\phi). \end{aligned} \tag{A.6}$$

In order to proof this property we have to use the precedent BCH theorem (A.2) , (1.56) and the relations between an analytic function and the operators $\hat{A}$ and $\hat{B}$:

$$e^{\hat{A}} f(\hat{B}) e^{-\hat{A}} = f(e^{\hat{A}} \hat{B} e^{-\hat{A}}). \tag{A.7}$$

Appendix B

Expectation values

B.1 Thermal states

We recall that a thermal state is defined as

$$\hat{\varrho}_{\text{th}}(N_{\text{th}}) = \frac{1}{1+N_{\text{th}}}\sum_{n=0}^{+\infty}\left(\frac{N_{\text{th}}}{1+N_{\text{th}}}\right)^n |n\rangle\langle n|. \tag{B.1}$$

We want to carry out statistical characteristics such as mean and variance for the number and the quadrature operator. In every calculation we define $q \doteq \frac{N_{\text{th}}}{1+N_{\text{th}}}$.

- **Number operator**
  - *Mean*

$$\begin{aligned}
\langle\hat{n}\rangle &= \text{Tr}[\hat{\varrho}_{\text{th}}\hat{n}],\\
&= \frac{1}{1+N_{\text{th}}}\text{Tr}\left[\sum_{n=0}^{+\infty}\left(\frac{N_{\text{th}}}{1+N_{\text{th}}}\right)^n \hat{n}\,|n\rangle\langle n|\right],\\
&= \frac{1}{1+N_{\text{th}}}\left[\sum_{n=0}^{+\infty} q^n n\right],\\
&= \frac{1}{1+N_{\text{th}}} q\frac{\partial}{\partial q}\left[\sum_{n=0}^{+\infty} q^n\right],\\
&= \frac{1}{1+N_{\text{th}}} q\frac{\partial}{\partial q}\frac{1}{1-q},\\
&= \frac{1}{1+N_{\text{th}}}\frac{N_{\text{th}}}{1+N_{\text{th}}}(1+N_{\text{th}})^2,\\
&= N_{\text{th}}.
\end{aligned} \tag{B.2}$$

– *Variance*

The variance follows similar considerations:

$$
\begin{aligned}
\Delta^2(\hat{n}) &= \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 , \\
&= \mathrm{Tr}[\hat{\varrho}_{\mathrm{th}} \hat{n}^2] - N_{\mathrm{th}}^2, \\
&= \frac{1}{1+N_{\mathrm{th}}} \left[\sum_{n=0}^{+\infty} \left(\frac{N_{\mathrm{th}}}{1+N_{\mathrm{th}}} \right)^n n^2 \right] - N_{\mathrm{th}}^2, \\
&= \frac{1}{1+N_{\mathrm{th}}} \left[q^2 \frac{\partial^2}{\partial q^2} + q \frac{\partial}{\partial q} \right] \left(\sum_{n=0}^{+\infty} q^n \right) - N_{\mathrm{th}}^2, \\
&= \frac{1}{1+N_{\mathrm{th}}} \left[q^2 \frac{2}{(1-q)^3} + q \frac{1}{(1-q)^2} \right] - N_{\mathrm{th}}^2, \\
&= \frac{1}{1+N_{\mathrm{th}}} \frac{q^2+q}{(1-q)^3} - N_{\mathrm{th}}^2, \\
&= \frac{1}{1+N_{\mathrm{th}}} \frac{N_{\mathrm{th}}}{1+N_{\mathrm{th}}} \left(\frac{N_{\mathrm{th}}}{1+N_{\mathrm{th}}} + 1 \right) (1+N_{\mathrm{th}})^3 - N_{\mathrm{th}}^2, \\
&= N_{\mathrm{th}}(N_{\mathrm{th}}+1).
\end{aligned}
\tag{B.3}
$$

- **Quadrature operator**

Using the definition of the quadrature operator:

$$
\hat{x}_\theta = \hat{a} e^{-i\theta} + \hat{a}^\dagger e^{i\theta}. \tag{B.4}
$$

The same quantities for a thermal state can be computed:

– *Mean*

$$
\begin{aligned}
\langle \hat{x}_\theta \rangle &= \mathrm{Tr}[\hat{\varrho}_{\mathrm{th}} \hat{x}_\theta], \\
&= \frac{1}{1+N_{\mathrm{th}}} \mathrm{Tr} \left[\sum_{n=0}^{+\infty} \left(\frac{N_{\mathrm{th}}}{1+N_{\mathrm{th}}} \right)^n \left(|n\rangle \langle n| \, \hat{a} e^{-i\theta} + |n\rangle \langle n| \, \hat{a}^\dagger e^{i\theta} \right) \right], \\
&= \frac{1}{1+N_{\mathrm{th}}} \mathrm{Tr} \left[\sum_{n=0}^{+\infty} \left(\frac{N_{\mathrm{th}}}{1+N_{\mathrm{th}}} \right)^n \left(|n\rangle \langle n+1| \sqrt{n+1} e^{-i\theta} + |n\rangle \langle n-1| \sqrt{n} e^{i\theta} \right) \right], \\
&= 0.
\end{aligned}
\tag{B.5}
$$

– *Variance*

$$
\begin{aligned}
\Delta(\hat{x}_\theta) &= \langle \hat{x}_\theta^2 \rangle - \langle \hat{x}_\theta \rangle^2 = \mathrm{Tr}[\hat{\varrho}_{\mathrm{th}} \hat{x}_\theta^2],\\
&= \frac{1}{1+N_{\mathrm{th}}} \mathrm{Tr}\left[\sum_{n=0}^{+\infty} \left(\frac{N_{\mathrm{th}}}{1+N_{\mathrm{th}}} \right)^n \left(|n\rangle \langle n| \, \hat{a} e^{-i\theta} + |n\rangle \langle n| \, \hat{a}^\dagger e^{i\theta} \right) \left(\hat{a} e^{-i\theta} + \hat{a}^\dagger e^{i\theta} \right) \right],\\
&= \frac{1}{1+N_{\mathrm{th}}} \mathrm{Tr}\left[\sum_{n=0}^{+\infty} \left(\frac{N_{\mathrm{th}}}{1+N_{\mathrm{th}}} \right)^n \left(|n\rangle \langle n+1| \sqrt{n+1} e^{-i\theta} + |n\rangle \langle n-1| \sqrt{n} e^{i\theta} \right) \left(\hat{a} e^{-i\theta} + \hat{a}^\dagger e^{i\theta} \right) \right],\\
&= \frac{1}{1+N_{\mathrm{th}}} \mathrm{Tr}\left[\sum_{n=0}^{+\infty} \left(\frac{N_{\mathrm{th}}}{1+N_{\mathrm{th}}} \right)^n \left(|n\rangle \langle n| \, (n+1) + |n\rangle \langle n| \, n \right) \right],\\
&= \frac{1}{1+N_{\mathrm{th}}} \left[\sum_{n=0}^{+\infty} \left(\frac{N_{\mathrm{th}}}{1+N_{\mathrm{th}}} \right)^n (2n+1) \right],\\
&= \frac{1}{1+N_{\mathrm{th}}} \left(2q \frac{\partial}{\partial q} + 1 \right) \left[\sum_{n=0}^{+\infty} q^n \right],\\
&= \frac{1}{1+N_{\mathrm{th}}} \left(\frac{2q}{(1-q)^2} + \frac{1}{1-q} \right),\\
&= \frac{1}{1+N_{\mathrm{th}}} \frac{1+q}{(1-q)^2},\\
&= 2N_{\mathrm{th}} + 1.
\end{aligned}
\tag{B.6}
$$

B.2 Coherent states overcompleteness

In order to demonstrate (1.32b) we have to use the completeness relation of the Fock states (1.26b):

$$
\begin{aligned}
\frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \; |\alpha\rangle \langle \alpha| &= \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \; e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^n (\alpha^*)^m}{\sqrt{n!m!}} \, |n\rangle \langle m|,\\
&= \frac{1}{\pi} \sum_{n,m} \frac{|n\rangle \langle m|}{\sqrt{n!m!}} \int_0^{+\infty} r \; dr \; e^{-r^2} r^{n+m} \int_0^{2\pi} d\theta \; e^{i(n-m)\theta},\\
&= \frac{1}{\pi} \sum_{n,m} \frac{|n\rangle \langle m|}{\sqrt{n!m!}} \, 2\pi \delta_{n,m} \int_0^{+\infty} r \; dr \; e^{-r^2} r^{n+m},\\
&= \sum_{n} \frac{|n\rangle \langle n|}{n!} \int_0^{+\infty} 2r dr \; e^{-r^2} r^{2n},\\
&= \sum_{n} \frac{|n\rangle \langle n|}{n!} \int_0^{+\infty} dt \; e^{-t} t^n = \mathbb{I}
\end{aligned}
\tag{B.7}
$$

where $\int_0^{2\pi} d\theta \; e^{i(n-m)\theta} = 2\pi\delta_{n,m}$, we also made the substitution $t = r^2$, $dt = 2rdr$. The last integral is the gamma function Γ: $\Gamma(n+1) \doteq \int_0^{+\infty} dt \; e^{-t} t^n = n!$. Furthermore we used that $\sum_n |n\rangle \langle n|$ is the resolution of the identity.

Appendix C

Special functions

In this Appendix we summarize the properties and the definitions of the special functions required in the thesis, such as Laguerre polynomials, and the calculations that make use of these functions.

C.1 Laguerre polynomials

The first special function are the Laguerre polynomials. There are multiple way to define them. The Laguerre polynomials form a orthogonal system, satisfying the orthogonality relation:

$$\int_0^\infty e^{-x} L_n(x) L_m(x) \; dx = \delta_{nm}. \tag{C.1}$$

They also satisfy the differential equation:

$$x \frac{d^2 L_n(x)}{dx^2} + (1-x)\frac{dL_n(x)}{dx} + nL_n(x) = 0, \tag{C.2}$$

where $n \in \mathbb{N}$ and $x \geq 0$.

The series expansion of the Laguerre polynomials is:

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{m} \frac{(-x)^k}{k!}, \tag{C.3}$$

so become easy to compute the first terms of $L_n(x)$:

$$L_0(x) = 1; \qquad L_1(x) = -x + 1; \qquad L_2(x) = \frac{1}{2}x^2 - 2x + 1. \tag{C.4}$$

The generating function reads:

$$\frac{1}{1-z} e^{\frac{z x}{z-1}} = \sum_{n=0}^{+\infty} z^n \; L_n(x), \tag{C.5}$$

with $|z| < 1$.

Another way to define the Laguerre polynomials is with the relation:

$$L_n(x) = \frac{e^{-x}}{n!}\frac{d^n}{dx^n}(x^n e^{-x}). \tag{C.6}$$

C.2 Bessel Function

A similar discussion can be made for the Bessel functions $J_n(x)$. They satisfy the following differential equation:

$$x^2\frac{d^2J_n(x)}{dx^2} + x\frac{dJ_n(x)}{dx} + (x^2 - n^2)J_n(x) = 0, \tag{C.7}$$

and can also be expressed in power series:

$$J_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}. \tag{C.8}$$

The generating function is:

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{+\infty} z^n\, J_n(x), \tag{C.9}$$

where $J_{-n}(x) = (-1)^n J_n(x)$ The integral form of $J_n(x)$ is:

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)d\theta, \tag{C.10}$$

and in particular we have:

$$\begin{aligned} J_0(x) &= \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)d\theta, \\ &= \frac{1}{2\pi}\int_0^{2\pi} \cos(x\sin\theta)d\theta, \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta}d\theta, \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin(\theta-\phi)}d\theta, \end{aligned} \tag{C.11}$$

where we noticed that $\int_0^{2\pi}\sin(x\sin\theta)d\theta = 0$ and $\sin\theta$ is periodic 2π, so it can be shifted by an arbitrary angle ϕ. We can also express some relations between $L_n(x)$ and $J_n(x)$:

$$\int_0^{\infty} u^m e^{x-u} J_0(2\sqrt{ux})du = m!L_m(x), \tag{C.12}$$

that can be used to evaluate the integral in (2.46) in Section 2.3:

$$\begin{aligned}&\frac{2}{\pi}\int_0^{\infty} xe^{-(1-p)\frac{x^2}{2}} L_n(x^2) J_0(2|\alpha|x)dx,\\&= \frac{2}{\pi(1-p)}(-1)^n\left(\frac{1+p}{1-p}\right)^n \exp\left(-\frac{2|\alpha|^2}{1-p}\right) L_n\left(\frac{4|\alpha|^2}{1-p^2}\right),\end{aligned} \tag{C.13}$$

a detailed demonstration of this calculation can be found at pag.232-233 of [4]. Since this expression is valid for $p < 1$, we want to take the limit for $p \to 1$:

$$\lim_{p\to -1}(1+p)^n L_n\left(\frac{4|\alpha^2|}{1-p^2}\right) = \lim_{p\to -1}(1+p)^n \frac{1}{n!}\left(-\frac{4|\alpha^2|}{1-p^2}\right)^n = \frac{(-2|\alpha|^2)^n}{n!}, \tag{C.14}$$

so for $p = -1$ case we have:

$$\begin{aligned}&\lim_{p\to -1}\frac{2}{\pi(1-p)}(-1)^n\left(\frac{1+p}{1-p}\right)^n \exp\left(-\frac{2|\alpha|^2}{1-p}\right) L_n\left(\frac{4|\alpha|^2}{1-p^2}\right),\\&= \frac{2}{2\pi}(-1)^n \frac{e^{-|\alpha|^2}}{2^n}\frac{(-2|\alpha|^2)^n}{n!},\\&= \frac{1}{\pi}e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!},\end{aligned} \tag{C.15}$$

as in (2.47c).

C.3 Characteristic function

We want to demonstrate the statement (2.11) in Section 2.1 that express the characteristic function for a thermal state:

$$\chi_{\text{th}}(\lambda, p) = e^{(1+2N_{\text{th}}-p)|\lambda|^2/2}. \tag{C.16}$$

We have to start from the general expression for a characteristic function (2.8) as a series of weighted terms:

$$\begin{aligned}\chi(\lambda, p) &= \sum_{n=0}^{+\infty} p_n\, \chi_n(\lambda, p),\\&= \frac{1}{1+N_{\text{th}}}\sum_{n=0}^{+\infty}\left(\frac{N_{\text{th}}}{1+N_{\text{th}}}\right)^n L_n(|\lambda|^2)e^{(p-1)|\lambda|^2/2},\\&= \frac{1}{1+N_{\text{th}}}e^{(p-1)|\lambda|^2/2}\frac{1}{1-x}e^{\frac{|\lambda|^2 x}{x-1}},\\&= \frac{1}{1+N_{\text{th}}}(N_{\text{th}}+1)e^{(p-1)|\lambda|^2/2}e^{-N_{\text{th}}|\lambda|^2},\\&= e^{(1+2N_{\text{th}}-p)|\lambda|^2/2},\end{aligned} \tag{C.17}$$

where we used the expansion in terms of the Laguerre polynomials (2.7), the generating function of Laguerre polynomials (C.5) and $x = \frac{N_{\text{th}}}{1+N_{\text{th}}}$. Furthermore we can write an expression for $\chi_n(\lambda, p)$ for the Fock state $|n\rangle$ required in (2.7):

$$\begin{aligned}
\chi_n(\lambda, p) &= \langle n|\, e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}}\, |n\rangle\, e^{(p-1)|\lambda|^2/2}, \\
&= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{\lambda^l (-\lambda^*)^m}{l! m!} \langle n|\hat{a}^{\dagger l} \hat{a}^m |n\rangle\, e^{(p-1)|\lambda|^2/2}, \\
&= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{\lambda^l (-\lambda^*)^m}{l! m!} \langle n-l| \left(\frac{n!}{(n-l)!}\right)^{\frac{1}{2}} \left(\frac{n!}{(n-m)!}\right)^{\frac{1}{2}} |n-m\rangle e^{(p-1)|\lambda|^2/2}, \\
&= \sum_{m=0}^{\infty} \frac{(-|\lambda|^2)^m}{(m!)^2} \frac{n!}{(n-m)!} e^{(p-1)|\lambda|^2/2}, \\
&= L_n(|\lambda|^2) e^{(p-1)|\lambda|^2/2}.
\end{aligned} \tag{C.18}$$

Appendix D

Trace

D.1 Cauchy-Schwarz inequality

The Cauchy-Schwarz inequality:

$$|\langle\phi_1|\phi_2\rangle|^2 \leq \langle\phi_1|\phi_1\rangle\langle\phi_2|\phi_2\rangle\,, \tag{D.1}$$

which is a direct consequence of $\langle\psi|\psi\rangle > 0$ with:

$$|\psi\rangle = |\phi_2\rangle - \frac{\langle\phi_1|\phi_2\rangle}{\langle\phi_1|\phi_1\rangle}|\phi_1\rangle\,. \tag{D.2}$$

These statements imply that:

$$|\langle\psi_n|\psi_m\rangle|^2 \leq \langle\psi_n|\psi_n\rangle\langle\psi_m|\psi_m\rangle = 1. \tag{D.3}$$

D.2 Trace

Given a matrix $A = a_{ij}$ we can define the trace, that is the sum over all the diagonal elements, namely:

$$\mathrm{Tr}[A] = \sum_n a_{nn}. \tag{D.4}$$

We can extend this definition on all the operators acting on a infinite-dimensional space. The Trace operation is independent on the basis on which we choose to write the operator $\hat{A}$. In fact, any basis having a complete orthonormal set of states is good:

$$\mathrm{Tr}[\hat{A}] = \sum_m \langle e_m|\hat{A}|e_m\rangle. \tag{D.5}$$

We can also say some of the properties of the trace operation:

$$
\begin{aligned}
\text{Linearity :} \quad & \mathrm{Tr}[\hat{A} + \hat{B}] = \mathrm{Tr}[\hat{A}] + \mathrm{Tr}[\hat{B}], && \text{(D.6a)} \\
& \mathrm{Tr}[z\hat{A}] = z\mathrm{Tr}[\hat{A}], && \text{(D.6b)} \\
\text{Cyclic :} \quad & \mathrm{Tr}[\hat{\varrho}\hat{A}\hat{B}] = \mathrm{Tr}[\hat{B}\hat{\varrho}\hat{A}] = \mathrm{Tr}[\hat{A}\hat{B}\hat{\varrho}], && \text{(D.6c)} \\
\text{Invariance under unitary transformation:} \quad & \mathrm{Tr}[\mathcal{U}\hat{A}\mathcal{U}^{\dagger}] = \mathrm{Tr}[\hat{A}] && \text{(D.6d)}
\end{aligned}
$$

Conclusions

In this thesis we analyzed a particular state, namely: $\hat{\varrho}_m = p_0 |0\rangle \langle 0| + p_1 |1\rangle \langle 1| + p_2 |2\rangle \langle 2|$, that once sent in a beam splitter give birth to a null intensity correlation function.
This fact is used to evaluate its nonclassical properties, making use of quantitative parameters, such as the nonclassical depth τ and Bell parameter $\mathcal{B}$.

We saw that this state turned out to be nonclassical, as its quasi-probability function became negative at a certain point. This characteristic has been related to its nonclassical depth τ and we observed that the nonclassical depth of our state is a linear positive function of the energy.
However, our state has been compared with another similar state, the squeezed state, showing a smaller τ. This allowed us to set a hierarchy among the nonclassical states.
This feature pushed us to consider other nonclassical characteristics, as entanglement between two states mixed in a beam splitter. For this purpose, we considered the CHSH Bell inequality. We computed that in every case it is $|\mathcal{B}| < 2$. The output state didn't show any violation of the considered Bell inequality, which is just a sufficient but not necessary condition to have nonlocality, and, thus, other approaches are needed to investigate the possible nonlocal feature of the state.

This work puts the basis for further studies, as we deal with a nonclassical state. In particular, could be interesting to analyze Bell inequalities, different from the CHSH one, or make consideration at entropy or information level. This state might display interesting quantum features that could be used in quantum cryptography or quantum communication.

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