BINOMIAL TREE: A TOOL FOR OPTION PRICING

There are several tools for option pricing, of which binomial trees are one of the most basic. In this article we develop the idea behind it and some algebra.

1.1 A counter-intuitive beginning on option value

The binomial tree is based on the assumption that we can model the price of a stock in two different states: up and down, with a certain probability. Let's assume that from an initial price of 100 the stock can go to 101 or 99, with p=0.7 and p'=0.3 respectively. We introduce a call option on the stock, asking ourselves the value of the option. We would be inclined to say 0.7 taking the expectation (0.7*(101-100)), but we would be wrong.. In order to get the option value we use the concept of no arbitrage: there is not risk-free money to be made in our market. Now consider two different portfolios:

A). Long one option and short half (we'll later where this derives from) stock;

B). Risk-free security

With portfolio A, in the up scenario, the payoff would be (101-100)-0.5*100=-99/2; in the down scenario we would have 0-99/2=-99/2.

We have therefore constructed a risk free portfolio and assuming the interest rate are 0, the value (payoff) tomorrow must be the same as the ones from today.

Therefore, at time t=0, we have: option value – 100/2 = -99/2, finding the option value is ½.

What tricked us in the first place was the probability of the stock having an up movement: the value of an option does not depend on the probability of raising and falling. This is since we have hedged the option with a stock and therefore don't care whether the stocks rises or falls. What we care about is the stock's **volatility.**

½ is chosen as to hedge the portfolio, in general $\Delta = \frac{Range\ of\ option\ payoffs}{Range\ of\ stock\ prices}$

1.2 And the role of risk in pricing stocks..

The expected stock value tomorrow is 0.7*101+0.3*99 = 100.4. Why is it not the price of the stock today? Because the stock investment is risky and therefore a positive expected return is expected.

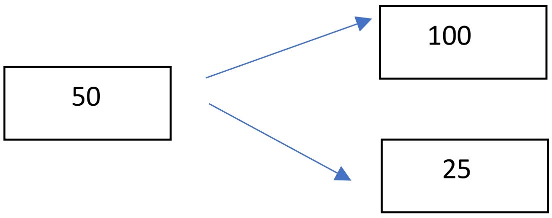
1.3 The real and risk-neutral worlds

The case we have presented is based in the **real-world** where we are sensitive to risk, expecting greater returns for taking risk. Another setting is the **risk-neutral world**, where risk is not taken into consideration and we price using expectations.

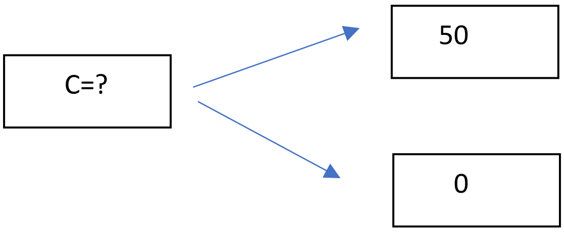
In this setting p(RN) and p'(RN) would be 0.5: the stock is currently priced at 100 today and the possible values are 101 and 99 in the future. p(RN)*101+p'(RN)*99 =100. These probabilities assumes that expectation are used for pricing and no allowance has been made for risk.

The model (Cox-Ross-Rubenstein)

Single-period replication: consider the following stock with its stock price evolution and the risk free rate is R = e^{rt}



Now consider a call with strike K=25.



Considering a long position of Δ in the stock and B dollars in the bond, we have two different payoffs:

UP: $\Delta uS + RB = 100\Delta + 1.25B$

DOWN: $\Delta dS + RB = 25\Delta + 1.25B$

Defining the payoff of the call in the UP as C_u and in the DOWN case as C_d we have that:

$C_u = \Delta uS + RB; C_d = \Delta dS + RB$

(Same as before where we considered Call – stock, but here there is interest rate); solving we have

$\Delta = \frac{C_u - C_d}{S(u-d)}; B = \frac{uC_d - dC_u}{(u-d)R}$

The formula for Delta is the same in paragraph 1.1

The value of the call is therefore:

$$C = \Delta S + B$$

Which substituting gives us

$$C = \frac{1}{R}\left(\frac{R-d}{u-d}C_u + \frac{u-R}{u-d}C_d\right)$$

Defining $p = \frac{R-d}{u-d}$ and $1 - p = \frac{u-R}{u-d}$ we have $C = \frac{1}{R}(pC_u + (1-p)C_d)$

P is the probability of the stock going up in a risk-neutral world.

References: Paul Wilmott on Quantitative Finance, cap. 3