An introduction to the no-arbitrage pricing theorem

Introduction

Arbitrage is a concept used in all methods of pricing derivatives: it means taking different positions so that one can achieve a riskless rate of return higher than the risk free rate. It can be distinguished in two forms:

I). Arbitrage of the first kind: taking a free position today that yield a positive profit in the future;

II). Arbitrage of the second kind: taking a positive with negative commitment today, and non negative payoff in the future.

From these one can derive the fair price idea: a security is correctly (aka fairly) prices if there are no arbitrage opportunities.

The ingredients: asset prices, states of the world, returns and payoffs, portfolio

In order to develop the idea we introduce the notation for asset prices, states of the world, returns and payoffs, portfolio.

The securities are represented in a vector denoted by S_t, taking the form

$$S_t = \begin{bmatrix} S_1(t) \\ \vdots \\ S_N(t) \end{bmatrix}$$

From 1 to N we have all the possible securities: from the risk-free bond to derivatives.

The next concept we need is a bit more astract, the states of the world. Think about securities and what their prices might be at a defined time in the future: each possible price (and so scenario) represents one states of the world. They are mutually exclusive and at least one of them is required to happen.

$$W = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_N \end{bmatrix}$$

Another concept is payoffs, represented in a matrix since each security will have different payoff according to the state of the world.

$$D = \begin{bmatrix} d_{11} & \cdots & d_{1K} \\ \vdots & \vdots & \vdots \\ d_{N1} & \cdots & d_{NK} \end{bmatrix}$$

In the case of a risk-free bond, a stock and a call with 2 states of the world the payoff matrix will be

$$D_t = \begin{bmatrix} (1+r\Delta)B(t) & (1+r\Delta)B(t) \\ S_1(t+\Delta) & S_2(t+\Delta) \\ C_1(t+\Delta) & C_2(t+\Delta) \end{bmatrix}$$

Dividing a row of this matrix by S_i will give the rate of return for each state of world for the i-th security.

The last concept is the one portfolio, which takes into consideration the amount we invest in each security. θ_i represents the position in that security: positive ➔ buying, negative ➔ selling.

The no-arbitrage theorem

Theorem 1. *Given the S_t, D_t defined in (2.6) and (2.7), and given that the two states have positive probabilities of occurrence,*

1. *if positive constants ψ_1, ψ_2 can be found such that asset prices satisfy*

$$\begin{bmatrix} 1 \\ S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} (1+r\Delta) & (1+r\Delta) \\ S_1(t+\Delta) & S_2(t+\Delta) \\ C_1(t+\Delta) & C_2(t+\Delta) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{2.10}$$

then there are no-arbitrage possibilities[6]*; and*

2. *if there are no-arbitrage opportunities, then positive constants ψ_1, ψ_2 satisfying (2.10) can be found.*

The constants ψ_i can be interpreted in terms of insurance. ψ_1 is the price of $S_{(t)}$ if it pays 1 in ω_1 and 0 in ω_2. Such a security, paying 1 in one state of the world and 0 in all other states, is also called state-price security or Arrow-Debreu security. The same reasoning goes for ψ_2. Spending $\psi_1 + \psi_2$ one guarantees 1 in the future (as can be seen from 2.10, since we have the price of the risk free). ψ_i are called *state prices*.

Risk-adjusted probabilities

From

$$\begin{bmatrix} 1 \\ S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} (1+r\Delta) & (1+r\Delta) \\ S_1(t+\Delta) & S_2(t+\Delta) \\ C_1(t+\Delta) & C_2(t+\Delta) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

$\boxed{\Delta = 1 \text{ hypothesis}}$

we get

$$1 = (1+r)\psi_1 + (1+r)\psi_2$$

We define

$$Q_1 = (1+r)\psi_1$$
$$Q_2 = (1+r)\psi_2$$

$$\Rightarrow \quad 0 \leq Q_1 \leq 1$$
$$Q_1 + Q_2 = 1$$

Q_i can be interpreted as a probability, although it is not the "real" probability of the states of the world. They are called *risk-adjusted synthetic probabilities*, and they exist only if there aren't arbitrage opportunities. If there is no arbitrage we can guarantee to find positive ψ_i and multiplying those by $1 + r$ we can find Q_i.

They provide a very important framework: expectations calculated with them, once discounted by the risk free rate r, equal the current value of the assets.

We will now present some algebra related to risk neutral probabilities and arbitrage.

From the no-arbitrage theorem

$$\left\{\begin{array}{l} 1 = (1+r)\psi_1 + (1+r)\psi_2 \\ S(t) = \psi_1 S_1(t+1) + \psi_2 S_2(t+1) \\ C(t) = \psi_1 C_1(t+1) + \psi_2 C_2(t+1) \end{array}\right.$$

multiplying the right-hand side by $\frac{1+r}{1+r}$

$$\Rightarrow S(t) = (1+r)^{-1}\left[\overbrace{\boxed{(1+r)\psi_1}}^{\mathbb{Q}_1} S_1(t+1) + \overbrace{\boxed{(1+r)\psi_2}}^{\mathbb{Q}_2} S_2(t+1)\right]$$

$$\Rightarrow C(t) = (1+r)^{-1}\left[\overbrace{\boxed{(1+r)\psi_1}}^{\mathbb{Q}_1} C_1(t+1) + \overbrace{\boxed{(1+r)\psi_2}}^{\mathbb{Q}_2} C_2(t+1)\right]$$

$$\Rightarrow S(t) = (1+r)^{-1}\left[\mathbb{Q}_1 S_1(t+1) + \mathbb{Q}_2 S_2(t+1)\right]$$

$$C(t) = (1+r)^{-1}\underbrace{\left[\mathbb{Q}_1 C_1(t+1) + \mathbb{Q}_2 S_2(t+1)\right]}$$

$\Downarrow$

can be interpreted as expected value under $\mathbb{Q}$.

The current value of all assets becomes equal to their discounted (at the risk-free rate) expected (under the risk-neutral probability) expected payoffs. Note that the risk-free is used as a discounting factor even though the assets are risky: the risk-premium is incorporated into the risk-neutral probabilities.

We will consider now what happen when we use the “true” probabilities.

We will consider what happens using P.

$$\mathbb{E}^{true}[S(t+1)] = [P_1 S_1(t+1) + P_2 S_2(t+1)]$$

$$\mathbb{E}^{true}[C(t+1)] = [P_1 C_1(t+1) + C_2 S_2(t+1)]$$

Since they are risky assets

$$S(t) = \frac{1}{1+r} E^{Q}[S(t+1)] < \frac{1}{1+r} E^{P}[S(t+1)]$$

and

$$C(t) = \frac{1}{1+r} E^{Q}[C(t+1)] < \frac{1}{1+r} E^{P}[C(t+1)]$$

$S(t) = (1+r)^{-1} E^{Q}[S(t+1)]$ is not plausible since this would imply

$$\boxed{(1+r) = \frac{E^{Q}[S(t+1)]}{S(t)}}$$

↳ not possible since investors command a positive risk premium for risky assets.

We can then write:

$$(1+r+\text{risk premium for } S(t)) = \frac{E^{P}[S(t+1)]}{S(t)}$$

The same reasoning can be applied to $C(t)$.

We can write the equations

$$1+r = \frac{E^{Q}[S(t+1)]}{S(t)}$$

$$1+r = \frac{E^{Q}[C(t+1)]}{C(t)}$$

These equations internalize risk premiums and one does not need to calculate risk premiums if one uses risk adjusted expectations.

Martingales and sub-martingales

Martingales and sub-martingales are very important concept for pricing financial assets. Suppose being at time t and having information summarized by I_t. A random variable X_t satisfying for all $s > 0$ $E^P[X_{t+s}|X_t] = X_t$ is called a **martingale** with respect to the probability P. Instead, if we have $E^Q[X_{t+s}|X_t] \geq X_t$ we have a **sub-martingale** with respect to the probability Q. Considering what has been said before we have that asset prices discounted by the risk-free rate are sub-martingales under the true probabilities but become martingales under the risk-adjusted probabilities. Therefore, the fair asset value can be obtained by using the martingale equality $E^Q[X_{t+s}|I_t] = X_t$ where $X_{t+s} = \frac{S_{t+s}}{(1+r)^s}$. Finally there are two remarks about martingales:

1). They are always defined with respect to a probability;

2). In the case of this application to finance, we have that the process to obey the martingale is not S_t but rather the normalization of S_t by $(1+r)^s$

Source: *An introduction to the Mathematics of Financial Derivatives, Neftci Salih N.*