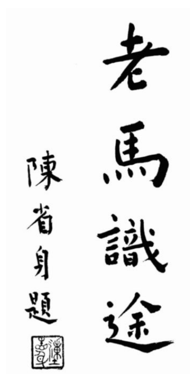
# Crypto 6

**Note: this material is not intended to replace the live lecture for students.**

**Contents**





## 6.1 Cyclic groups and Discrete Logarithm

**What is a Group ?**

Roughly speaking a group is a set with two operations $*$ and inversion.

Formally: By a group $(G, *)$ we mean a set $G$ and a operation $*$ between elements of $G$ such that:

i) any two elements $a, b$ of $G$ can be used to get another element $a * b \in G$;

ii) associativity : $(a * b) * c = a * (b * c)$;

iii) there is a neutral element $e$ i.e. for all $a \in G$

$$a * e = e * a = a$$

iv) any element $a$ has an inverse i.e. there is $b \in G$ such that $a * b = e$.

The cardinal $|G|$ is called **order of the group** $G$.

**Exercise 6.1.1**

Show:
i) The neutral element $e$ is unique.

ii) Given $a$ the inverse is unique.

**NOTE 6.1.2**

Commutativity is not compulsory. The symmetric group $S_n$ is not commutative if $n > 2$.

**Exercise 6.1.3**

Show that $S_3$ is not commutative.

### 6.1.4 Cyclic Group

A group $(G, *)$ is cyclic if there is $a \in G$ such that any other element $b$ of $G$ is of the form

$$b = \underbrace{a * a * a * a * a * \cdots * a}_{n-times}$$

In this case the element $a$ is called **generator** of $G$. It is usual to write $b = a^n$.

### 6.1.5 Cyclic Subgroup generated by $a$

For $a$ in $G$ the cyclic subgroup or cyclic group generated by $a$ and denoted by $\langle a \rangle$ is the set

$$\langle a \rangle = \{\cdots, a^{-3}, a^{-1}, 1, a, a^2, a^3, \cdots\}$$

The **order** of $a$ is $|\langle a \rangle|$.

**NOTE 6.1.6**

The group $\mathbb{Z}_p^* = \mathrm{GF}(p) \setminus \{0\}$ with $p$ a prime number is cyclic. A generator $g$ is also called **primitive element**.

### 6.1.1 Computing powers $a^n$

**Exercise 6.1.7**

Compute
$$8^e \pmod p$$
where $p = 2^{20} - 2^3 - 1$ and $e = \frac{p+1}{4}$.

**Exercise 6.1.8**

Compute
$$8^e \pmod p$$
where $p = 2^{256} - 2^{32} - 977$ and $e = \frac{p+1}{4}$.

http://mathworld.wolfram.com/LandauSymbols.html
Let $n = (d_t d_{t-1} d_{t-2} \cdots d_0)_2$, $d_t = 1$, the binary representation (base-2).

**6.1.9 Efficient computation of powers: square-and-multiply algorithm**

```
T=e
For i=t downto i=0
   T = T * T
   if d_i = 1
      T = T * a
return(T)
```

Here is the **python** to compute $a^n \pmod q$:

**6.1.10 Square-and-multiply in $\mathbb{Z}_q$**

```
def squaremultiply(a,n,q):
    bina='{0:b}'.format(n)
    T=1
    amq=a%q
    for d in bina:
        T=(T*T)%q
        if d=='1': T=(T*amq)%q
    return(T)
```

### 6.1.2 Order of a group and of an element

Here we take $g \in \mathbb{Z}_n^*$ and consider the cyclic group $\langle g \rangle$ generated by $g$ :

$$\langle g \rangle = \{g, g^2, g^3, \cdots, 1\}$$

Observe that there is a power $w$ such that $g^w = 1$. The first such $w$ is called the order (or period) of $g$. Notice that $w = |\langle g \rangle|$ i.e. the order of the element is the order of the cyclic subgroup generated by itself.

**Lagrange's theorem**

the order $w$ is a divisor of $\phi(n)$.
The order of a subgroup divides the order of the group.

For example, let $g = 3$ in $\mathbb{Z}_{44}$. Then

$$\langle 3 \rangle = \{3, 9, 27, 37, 23, 25, 31, 5, 15, 1\}$$

So 3 has order 10 in $\mathbb{Z}_{44}$ and $10|\phi(44) = \phi(4) \times \phi(11) = 3 \times 10$.

**Exercise 6.1.11**

Let $a \in \mathbb{Z}_n^*$. Show that $a^{\phi(n)} = 1 \pmod n$.

**Exercise 6.1.12**

Find $x \in \mathbb{Z}_{44}$ of order 3.

**Exercise 6.1.13**

By using the above table about $\mathbb{Z}_{24} \approx \mathbb{Z}_3 \times \mathbb{Z}_8$ compute a similar table for $\mathbb{Z}_{24}^* \approx \mathbb{Z}_3^* \times \mathbb{Z}_8^*$. Is $\mathbb{Z}_{24}^*$ a cyclic group ?

**Exercise 6.1.14**

Show that $\mathbb{Z}_{22}^*$ is a cyclic group.

## 6.2 Solving the first degree equation $a \cdot x = 1$ (mod n)

$29 \cdot x = 1 \pmod{45}$

| | |
|---|---|
| 29 | 1 |
| 45 | 0 |
| 29 | 1 |
| 16 | -1 |
| 13 | 2 |
| 16 | -1 |
| 13 | 2 |
| 3 | -3 |
| 1 | 14 |
| 3 | -3 |
| 1 | 14 |
| 0 | -45 |

So $x = 14$.

You can also use the following **python**:

### 6.2.1 Inverse in Python

```
''' inverse(r,N) = x ,
          such that r x = 1 (mod N).

'''
def inverse(r,N):
    x=1
    while (r*x)%N != 1 and  x<N:
              x+=1
    if  (r*x)%N == 1:
        return x
    else: return 'non invertible'
```

**Exercise 6.2.2**

Set $N = 16^{30} - 1$. Compute $x$ such that $2x = 1 \pmod N$.

**Exercise 6.2.3**

Is the above **python** script 6.2.1 efficient ? That is to say, how many iterations are (in average) necessary to compute `inverse(r,N)` ?

**Exercise 6.2.4**

Let $F_n$ be the sequence of Fibonacci $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. Compute $x$ such that

$$F_{100} \cdot x \equiv 1 \pmod{F_{101}}$$

### 6.2.1 Greatest common divisor (gcd)

**6.2.5 gcd in Python**

```
''' gcd(a,b) = [d,x,y] ,
        where $d$ is the g.c.d(a,b).
        The integers $x,y$ are such that $d=ax+by$.

'''
def gcd(a,b):
    import numpy as np
    M = np.array([[a,1,0],[b,0,1]])
    while M[0,0]*M[1,0] != 0:
            if M[0,0]<M[1,0]:
                M[1]=M[1] - (M[1,0] // M[0,0])*M[0]
            else:
                M[0]=M[0] - (M[0,0] // M[1,0])*M[1]
    if M[0,0] == 0:
            return M[1]
    else:
            return M[0]
```

**Exercise 6.2.6**

Is the above **python** script 6.2.5 efficient ? That is to say, how many iterations are (in average) necessary to compute `gcd(a,b)` ?

## 6.3 Quadratic residues: solving $x^2 = r$ (mod n)

**Teorema 6.3.1 ▶ Euler's Criterion**

Let $p$ be a prime number and $r$ an integer non divisible by $p$. Then $r$ is a quadratic residue modulo $p$ if and only if

$$r^{\frac{p-1}{2}} \equiv 1 \pmod{p}$$

**Group of quadratic residues**

1) The set $G$ of quadratic residues is a subgroup of $\mathbb{Z}_n^*$.

2) In case $n$ is a prime number $n = 2p + 1$ where $p$ is also a prime number $G$ has order $p$.

3) If $r$ is a quadratic residue modulo a prime $p \equiv 3 \pmod{4}$ then a square root of $r$ is $r^{\frac{p+1}{4}}$.

4) If $r$ is a quadratic residue modulo a prime $p \equiv 1 \pmod{4}$ then there is an efficient algorithm to find a square root e.g. Algorithm of Tonelli-Shanks.

**Exercise 6.3.2**

Find $x$ such that $x^2 = 2082 \pmod{6121}$.

### 6.3.3 Cipolla's algorithm

Let $r$ be a quadratic residue modulo $p$. To find a square root $u \in \mathbb{Z}_p$ find $s \in \mathbb{Z}_p$ such that $s^2 - r$ is not a quadratic residue. Set $\mathrm{A} = \begin{bmatrix} s & s^2 - r \\ 1 & s \end{bmatrix}$. Compute $u$ as

$$\mathrm{A}^{\frac{p+1}{2}} = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$$

**Exercise 6.3.4**

Find $u$ such that $u^2 = 4799 \pmod{1002257}$. Hint: **python**.

**NOTE 6.3.5**

To explain why Cipolla's method works it is necessary to develop Galois Theory but that is beyond this course.

## 6.4 The Chinese Remainder Theorem (CRT)

The CRT allows us to compute in $\mathbb{Z}_{nm}$ regarding its elements as pairs. Here is an example:

**$\mathbb{Z}_{24} \approx \mathbb{Z}_3 \times \mathbb{Z}_8$**

| | |
|---|---|
| 0 | (0, 0) |
| 1 | (1, 1) |
| 2 | (2, 2) |
| 3 | (0, 3) |
| 4 | (1, 4) |
| 5 | (2, 5) |
| 6 | (0, 6) |
| 7 | (1, 7) |
| 8 | (2, 0) |
| 9 | (0, 1) |
| 10 | (1, 2) |
| 11 | (2, 3) |
| 12 | (0, 4) |
| 13 | (1, 5) |
| 14 | (2, 6) |
| 15 | (0, 7) |
| 16 | (1, 0) |
| 17 | (2, 1) |
| 18 | (0, 2) |
| 19 | (1, 3) |
| 20 | (2, 4) |
| 21 | (0, 5) |
| 22 | (1, 6) |
| 23 | (2, 7) |

So to compute $13 \times 18 \pmod{24}$ we compute $(1, 5) \times (0, 2) = (0, 10) = (0, 2)$ hence

$$13 \times 18 = 18 \pmod{24}$$

**Exercise 6.4.1**

Find $x < 3 \times 5 \times 7$ such that

$$\begin{cases} x = 2 \pmod 3 \\ x = 3 \pmod 5 \\ x = 2 \pmod 7 \end{cases}$$

**Exercise 6.4.2**

Compute $x$ such that $17 \times x = 1 \pmod{24}$.

**Exercise 6.4.3**

Compute ALL $x$ such that $13 \times x = 18 \pmod{24}$.

**Exercise 6.4.4**

Compute ALL $x$ such that $x^2 = 1 \pmod{24}$.

**Exercise 6.4.5**

Find $x < 809933$ such that

$$\begin{cases} x^2 = 62953 \pmod{809933} \\ x^2 = 504539 \pmod{854429} \end{cases}$$

### 6.4.1 CRT and Euler's $\phi$ function

Euler's $\phi(N)$ (also called totient) function counts the positive integers up to a given integer $N$ that are relatively prime to $N$. That is to say, $\phi(N)$ is the cardinal of $\mathbb{Z}_N^*$ the invertible elements w.r.t. the product.
Let $N = p^r \times q^s \times \cdots$ the factorization of $N$ into powers of different prime numbers $p, q, \ldots$. By computing with pairs we get

$$\phi(N) = \phi(p^r) \times \phi(q^s) \times$$

$|\mathbb{Z}_{p^r}^*|$

$$\boxed{\phi(p^r) = (p-1) \times p^{r-1}}$$

For example $\phi(7^2) = 6 \times 7 = 42$ because:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| 29 | 30 | 31 | 32 | 33 | 34 | 35 |
| 36 | 37 | 38 | 39 | 40 | 41 | 42 |
| 43 | 44 | 45 | 46 | 47 | 48 | 49 |

namely, $\phi(7^2) = 7^2 - 7 = 7 \times (7-1) = 7 \times 6 = 42$.

**NOTE 6.4.6**

Observe that to compute $\phi(n)$ using the above formulae it is necessary to factorize the number $n$.

## 6.5 Prime and pseudoprime numbers

**Teorema 6.5.1 ▶ Fermat's Little Theorem**

Let $p$ be a prime number and $r$ any integer. Then

$$r^p \equiv r \pmod p$$

### 6.5.1 Tests, Criteria and Certificates

**Rabin-Miller: starting clue**

Let $n$ an odd number. Set $n - 1 = 2^s \cdot d$ where $d$ odd. If $n$ is a prime number and $a \not\equiv 0 \pmod n$ then either

$$a^d \equiv 1 \pmod n$$

or

$$a^{2^r \cdot d} \equiv -1 \pmod n$$

for some $0 \leq r \leq s - 1$.

**6.5.2 Rabin-Miller probabilistic test**

```
Input #1: n > 3, an odd integer to be tested for primality
Input #2: k, the number of rounds of testing to perform
Output: "composite" if n is found to be composite, "probably prime" otherwise

write n as 2^r·d + 1 with d odd (by factoring out powers of 2 from n - 1)
WitnessLoop: repeat k times:
   pick a random integer a in the range [2, n - 2]
   x ← a^d mod n
   if x = 1 or x = n - 1 then
      continue WitnessLoop
   repeat r - 1 times:
      x ← x^2 mod n
      if x = n - 1 then
         continue WitnessLoop
   return "composite"
return "probably prime"
```

**NOTE 6.5.3**

The running time is $O(k \cdot \log^3(n))$. The mistake probability declaring prime a composite number is at most $4^{-k}$ [Rabin77, Theorem 2, page 134].

https://inventwithpython.com/cracking/chapter22.html

### 6.5.2 Generation of large prime numbers

Problem: How to construct a random n-bit prime number $p$ ?

Idea of the solution: Pick a n-bit random number $X$ from

$$[2^{n-1}, 2^n - 1]$$

and check if it is a prime number.

Iteration until the test gives a positive answer.

## Running time ?

**Teorema 6.5.4 ▶ Prime Number Theorem**

The total number of prime $< N$ is (roughly)

$$\frac{N}{\log(N)}$$

So the number of primes in the interval $[2^{n-1}, 2^n - 1]$ is roughly

$$\frac{2^{n-1}}{n}$$

hence the probability that a random number in that range is a prime is $\frac{1}{n}$.

The failing probability after $t$ trials is $(1 - \frac{1}{n})^t$. By putting $t = \alpha \cdot n$ we can estimate ($n >> 0$):

$$(1 - \frac{1}{n})^t = (1 - \frac{1}{n})^{\alpha \cdot n} = \left((1 - \frac{1}{n})^n\right)^\alpha \approx (e^{-1})^\alpha = e^{-\alpha}$$

So for $n = 1000$ and $t = 3000$ we have $e^{-\alpha} \approx 0.05$.

https://www.cs.purdue.edu/homes/hmaji/teaching/Fall%202018/lectures/11.pdf

## 6.6 Bibliography

Books I used to prepare this note:

[Paar10] Paar, Christof, Pelzl, Jan, *Understanding Cryptography, A Textbook for Students and Practitioners*, Springer-Verlag, 2010.

Here a list of papers:

[Rabin77] Michael O Rabin,; *Probabilistic algorithm for testing primality*, Journal of Number Theory, Volume 12, Issue 1, 1980, Pages 128-138. https://www.sciencedirect.com/science/article/pii/0022314X80900840#!

and some interesting links:

http://gauss.math.luc.edu/greicius/Math201/Fall2012/Exercises/ChineseRemainderThm.pdf

https://www.whitman.edu/mathematics/higher_math_online/section03.07.html

https://pynative.com/python-range-function/

https://inventwithpython.com/cracking/chapter22.html

https://cseweb.ucsd.edu/~mihir/papers/gb.pdf

https://en.wikipedia.org/wiki/Euler%E2%80%93Jacobi_pseudoprime

https://en.wikipedia.org/wiki/Baillie%E2%80%93PSW_primality_test

https://en.wikipedia.org/wiki/Strong_pseudoprime

https://en.wikipedia.org/wiki/Solovay%E2%80%93Strassen_primality_test

http://www.dtc.umn.edu/~odlyzko/doc/discrete.logs.hff.pdf

https://en.wikipedia.org/wiki/Fermat_primality_test