# Warm up: A simple language for arithmetic expressions

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5 October 2020

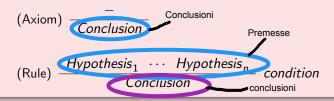
# Logic and software engineering

- Logic is the mathematical basis for software engineering
- We can make the following statement:

```
logic : sw engineering = calculus : mechanical/civil engineering
```

- Induction will be a foundational concept
- For instance, inductively defined sets and relations or inductive proofs are the basis of software verification

### Anatomy of an inference system:



# A Language for Arithmetic Expression: Syntax

$$E ::= n \mid E + E \mid E \times E \mid \cdots$$

#### where

- n ranges over the domain of numerals Num: 0, 1, · · ·
- E ranges over the domain of arithmetic expressions Exp
- numerals 0, 1, · · · are part of the syntax of our language
- they are piece of our syntax and they should not be confused with numbers  $(0,1,2,\ldots\in\mathbb{N})$  which are mathematical objects
- instead of 0, 1, ··· we could have used in our language the terminals zero, uno, ···; it would have been exactly the same.
- $\bullet$  +,  $\times$ ,  $\cdots$  are symbols for operations.

We will always work with abstract syntax. We will assume that we already did the parsing of our programs. So, the grammars we will use to define our languages define syntactic trees: parentheses are only used for disambiguation - they are not part of the grammar.

# Operational semantics for the language Exp

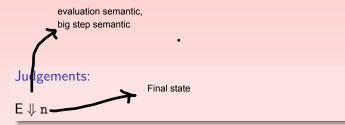
An operational semantics for Exp has the goal to evaluate an arithmetic expression of the language to get its associated numeral.

This can be done in two different manners:

- via a small-step (or structural) semantics that provides a method to evaluate an expression, step by step
- via a big-step (or natural) semantics that ignores the intermediate steps and directly provides the final result

In the following, we assume that there is an obvious correspondence between the numeral n and the number n. This is just to make things simple: In another language the numeral 3 might be associated to the number 42!

# Big-Step semantics for Exp



### Meaning:

The evaluation of expression E results in the numeral n.

# Big-Step Semantics for Exp

### Axioms and Rules for Exp

(B-Num) 
$$\frac{-}{\mathbf{n} \Downarrow \mathbf{n}}$$
 (B-Add)  $\frac{\mathsf{E}_1 \Downarrow \mathsf{n}_1 \quad \mathsf{E}_2 \Downarrow \mathsf{n}_2}{\mathsf{E}_1 + \mathsf{E}_2 \Downarrow \mathsf{n}_3} \quad \mathsf{n}_3 = \mathsf{add}(\mathsf{n}_1, \mathsf{n}_2)$ 

Similar rules for  $\times$ , -,  $\cdots$ 

IMPORTANT: add(-, -) is a semantic operator on numbers NOT numerals.

### How to read axioms

The axioms

says that:

The evaluation of numeral n is n itself.

In the axiom (B-Num) the symbol n is a kind of *variable* which can be replaced with any numeral  $0, 1, 2, \cdots$ . Those kinds of variables are called *meta-variables*.

### How to read rules

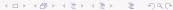
The rule

(B-Add) 
$$\frac{\mathsf{E}_1 \Downarrow \mathsf{n}_1 \quad \mathsf{E}_2 \Downarrow \mathsf{n}_2}{\mathsf{E}_1 + \mathsf{E}_2 \Downarrow \mathsf{n}_3} \quad \mathsf{n}_3 = \mathsf{add}(\mathsf{n}_1, \mathsf{n}_2)$$

should be read in the following manner:

- ullet given two expressions  $E_1$  and  $E_2$
- if it is the case that  $E_1 \Downarrow n_1$
- ullet and it is the case that  $E_2 \Downarrow n_2$
- then it follows that  $E_1 + E_2 \Downarrow n_3$
- where  $n_3$  is the numeral associated to the number  $n_3$ , such that  $n_3 = \operatorname{add}(n_1, n_2)$
- recall that add(-, -) is an operation on numbers NOT numerals.

In the rule (B-Add),  $E_1$ ,  $E_2$ ,  $n_1$ ,  $n_2$ ,  $n_3$  are meta-variables.



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### How to use axioms and rules

We can apply axioms and rules to *derive* judgements. Such derivations take the form of trees:

$$(\text{B-Add}) \quad \underbrace{ \begin{array}{cccc} (\text{B-Num}) & \frac{-}{3 \Downarrow 3} \\ \end{array} & (\text{B-Add}) & \underbrace{ \begin{array}{ccccc} (\text{B-Num}) & \frac{-}{2 \Downarrow 2} \\ \end{array} & (\text{B-Num}) & \frac{-}{1 \Downarrow 1} \\ \end{array}}_{3 + (2 + 1) \Downarrow 6}$$

For example, the derivation above allows us to derive the judgement:

$$3 + (2 + 1) \Downarrow 6$$

by applying three times the axiom (B-Num) and two times rule (B-Add).

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# Small-step Semantics for Exp

Judgements:  $E_1 \twoheadrightarrow E_2$ 

### Meaning:

After performing one-step of  $E_1$  the expression  $E_2$  remain to be evaluated.

## A Left-to-right Small-step Semantics for Exp

#### Inference rules

$$(S-Left) \ \frac{\mathsf{E}_1 \to \mathsf{E}_1'}{\mathsf{E}_1 + \mathsf{E}_2 \to \mathsf{E}_1' + \mathsf{E}_2}$$

(S-N.Right) 
$$\frac{\mathsf{E}_2 \to \mathsf{E}_2'}{\mathsf{n}_1 + \mathsf{E}_2 \to \mathsf{n}_1 + \mathsf{E}_2'}$$

We fix the evaluation order, from left to right. Something similar is not possible in a big-step semantics where expressions are evaluated in a single "big" step.

40 + 40 + 45 + 45 + 5 490

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# A Choice Small-step Semantics for Exp

A different small-step semantics for Exp is the following:

#### Inference rules:

$$\text{(S-Left)} \ \frac{\mathsf{E}_1 \to_{\mathsf{ch}} \mathsf{E}_1'}{\mathsf{E}_1 + \mathsf{E}_2 \to_{\mathsf{ch}} \mathsf{E}_1' + \mathsf{E}_2}$$

$$\begin{array}{c} \text{(S-Right)} \ \ \underline{ \begin{array}{c} E_2 \rightarrow_{ch} E_2' \\ \hline E_1 + E_2 \rightarrow_{ch} E_1 + E_2' \end{array} } \end{array} \label{eq:controller}$$

(S-Add) 
$$\frac{-}{n_1 + n_2 \to_{ch} n_3} n_3 = add(n_1, n_2)$$

Here, no precedence is established during the evaluation. Similar rules apply to the other operators  $\times$ , -,  $\cdots$ .

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# Executing small-step semantics

The relation  $\rightarrow^k$ , for  $k \in \mathbb{N}$  (k may be 0)

We write  $E \rightarrow^k E_k$  whenever:

$$E = E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \ldots \rightarrow E_k$$

The relation →\* Stella di Cliny, faccio n passi con n >= 0

We write  $E \rightarrow^{k} F$  if  $E \rightarrow^{k} F$  for some  $k \in \mathbb{N}$ .

The relation  $\rightarrow$ \* is called *reflexive and transitive closure* of  $\rightarrow$ . Reflexive because k may be 0.

### The final answer

We say that n is the final answer of E if  $E \rightarrow^* n$ .

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## Questions

### Internal consistency of semantics

• Is it possible to derive

$$E \Downarrow 3$$
 and  $E \Downarrow 7$ 

for some expression E?

• Is there some expression E which has no resulting value:

 $E \rightarrow^* n$  for no numeral n

# Questions

### Consistency between the different semantics

What is the relationship between the different judgements:

- $\bullet$  E  $\Downarrow$  n
- E →\* n
- E →<sup>\*</sup><sub>ch</sub> n

### Usefulness

Can these techniques be applied to realistic programming languages?