Markov Decision Processes

Riccardo Brioschi

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Foundations of Reinforcement Learning

Introduction

Many RL papers contain a background section like the following one:

The Option-Critic Architecture

Pierre-Luc Bacon, Jean Harb, Doina Precup
Reasoning and Learning Lab, School of Computer Science
McGill University
{pbacon, jharb, dprecup}@cs.mcgill.ca

Preliminaries and Notation

A Markov Decision Process consists of a set of states $\mathcal{S}$, a set of actions $\mathcal{A}$, a transition function $\mathrm{P} : \mathcal{S} \times \mathcal{A} \to (\mathcal{S} \to [0,1])$ and a reward function $r : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$. For convenience, we develop our ideas assuming discrete state and action sets. However, our results extend to continuous spaces using usual measure-theoretic assumptions (some of our empirical results are in continuous tasks). A (Markovian stationary) *policy* is a probability distribution over actions conditioned on states, $\pi : \mathcal{S} \times \mathcal{A} \to [0,1]$. In discounted problems, the value function of a policy π is defined as the expected return: $V_\pi(s) = \mathbb{E}_\pi\left[\sum_{t=0}^{\infty} \gamma^t r_{t+1} \mid s_0 = s\right]$ and its action-value function as $Q_\pi(s,a) = \mathbb{E}_\pi\left[\sum_{t=0}^{\infty} \gamma^t r_{t+1} \mid s_0 = s, a_0 = a\right]$, where $\gamma \in [0,1)$ is the *discount factor*. A policy π is *greedy* with respect to a given action-value function Q if

In this lecture you will learn

1. What a Markov Decision Process is.
2. How MDPs can be solved with dynamic programming.
3. How future discounted MDPs can be solved with value iteration or policy iteration.

Recommended reading:
Sutton & Barto, Chapters 3 & 4

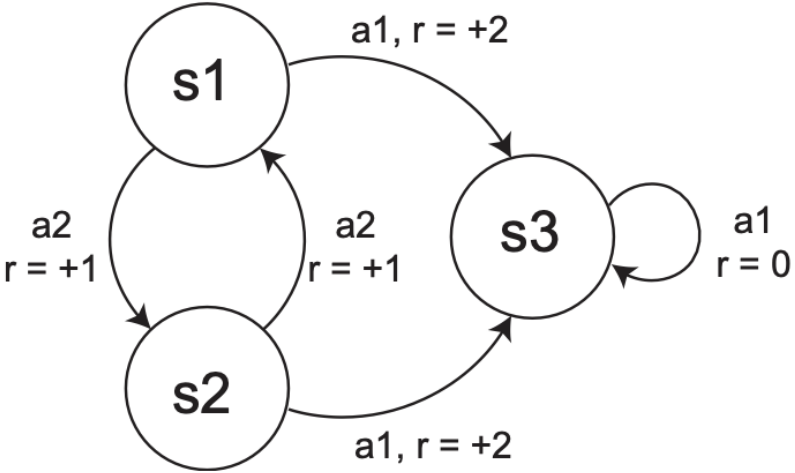
Markov Decision Processes

We define as a **Markov Decision Process (MDP)** a stochastic process characterized by:

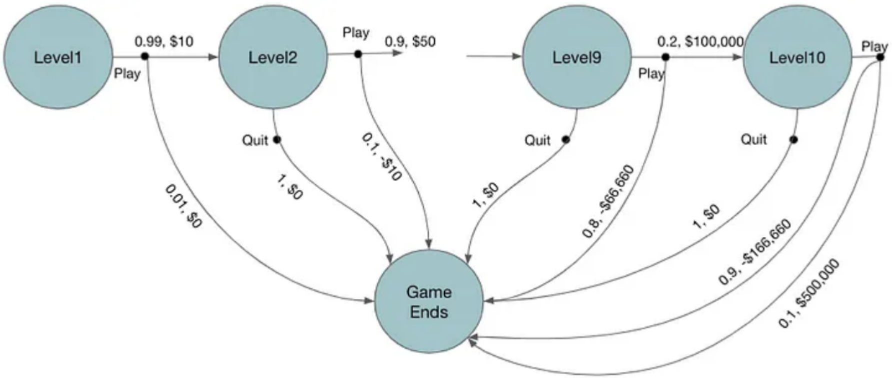
- finite **state space** $\mathcal{S}$ with $|\mathcal{S}| < \infty$,
- finite **action spaces** $\{\mathcal{A}_s | s \in \mathcal{S}\}$ with $|\mathcal{A}_s| < \infty$,
- **transition probabilities** $p^a_{s_i \to s_j} \in [0, 1]$ (alternatively, $P(s_j | s_i, a)$),
- **immediate rewards** $R^a_{s_i \to s_j}$ and $r^a_{s_i} = \sum_{s_j} p^a_{s_i \to s_j} \mathbb{E}[R^a_{s_i \to s_j}] \in \mathbb{R}$,
- **discount factor** $\gamma \in [0, 1]$,
- and **initial state probabilities** $p^{(0)}_{s_i}$.

For a sequence (or trajectory) of state-action-reward tuples, we will use the notation $\tau = (S_0, A_0, R_1, S_1, A_1, \ldots, R_T)$ where $R_i = R^{A_{i-1}}_{S_{i-1} \to S_i}$.

MDPs: An Example



MDPs: An Example



Policies

A policy π selects an action based on the previous history $\tau = (S_{0:T}, A_{0:T})$

- In this course, we only consider **Markov Policies**, i.e. dependent only on the current state
- It is a mapping $\pi : \mathcal{S} \to A$ or $\pi : \mathcal{S} \to \Delta(A)$, $\Delta(\mathcal{A})$ denotes the simplex over A

Families of policies

Deterministic Policy

- Stationary policy $\pi : \mathcal{S} \to \mathcal{A}$, $a_t = \pi(s_t)$
- Markov policy $\pi_t : \mathcal{S} \to \mathcal{A}$, $a_t = \pi_t(s_t)$

Randomized Policy:

- Stationary policy $\pi : \mathcal{S} \to \Delta(\mathcal{A})$, $a_t \sim \pi(\cdot|s_t)$
- Markov policy $\pi_t : \mathcal{S} \to \Delta(\mathcal{A})$, $a_t \sim \pi_t(\cdot|s_t)$

Returns

Acting on a MDP results in immediate rewards $R^a_{s_i \to s_j}$. Accumulating these rewards, we obtain the return.

- **Finite time horizon T**: $\mathbb{E}_\pi \left[\sum_{t=1}^{T} \gamma^{(t-1)} R_t \right]$
- **Discounted Reward**: $J(\pi) = \mathbb{E}_\pi \left[\sum_{t=1}^{\infty} \gamma^{(t-1)} R_t \right]$

These quantities are obtained averaging the returns over the trajectories obtained moving in the MDP, according to the policy and the transition probabilities of the environment.

Q-Values and V-Values

- It is useful to introduce **Q-values** and **V-values**

$$Q_\gamma^{(T)}(\pi, s, a) = \mathbb{E}\left[\sum_{t=1}^{T} \gamma^{(t-1)} R_t \middle| S_0 = s, A_0 = a\right]$$

$$V_\gamma^{(T)}(\pi, s) = \mathbb{E}\left[\sum_{t=1}^{T} \gamma^{(t-1)} R_t \middle| S_0 = s\right]$$

- These two quantities are strongly related, indeed:

$$V_\gamma^{(T)}(\pi, s) = \sum_{a \in \mathcal{A}_s} \pi(a|s) Q_\gamma^{(T)}(\pi, s, a)$$

- These quantities can also be generalized to the infinite horizon setting

Value Functions and Objectives

The goal is to find a **policy** $\pi^{(t)}(a|s) \in [0,1]$ that maximizes some objective. We define the horizon-T **value function (V-value)**

$$
\begin{aligned}
V_\gamma^{(T)}(\pi, s) &= \mathrm{E}_\pi\left[\sum_{t=1}^{T} \gamma^{(t-1)} R_t \middle| S_0 = s\right] \\
&= \sum_{A_0, S_1, A_1, \ldots, A_{T-1}} \Pi(\tau)\left(r_s^{A_0} + \gamma r_{S_1}^{A_1} + \cdots + \gamma^{T-1} r_{S_{T-1}}^{A_{T-1}}\right) \\
&= \sum_a \pi(a|s)\left\{r_s^a + \gamma \sum_{s'} p_{s \to s'}^a V_\gamma^{(T-1)}(\pi, s')\right\}
\end{aligned}
$$

where $\Pi(\tau) = \Pi_{t=0}^{T-1} \pi^{(t)}(A_t|s_t) p_{s_t \to S_t}^{A_t}$ is the probability of trajectory τ

- **Observation**: The last expression is obtained by recursion, noticing that part of the sum can be expressed as the value function in the state s'.

The Optimal Fixed Horizon Policy

The policy π^* can be found with **Dynamic Programming**.

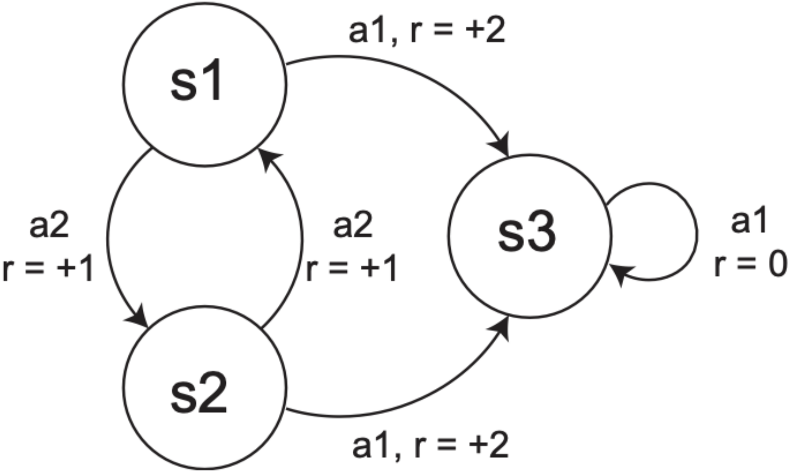
1. The optimal horizon-1 values are $V_\gamma^{(1)}(\pi^*, s) = \max_{a \in \mathcal{A}_s} r_s^a$.
2. The optimal horizon-$(t+1)$ values are

$$V_\gamma^{(t+1)}(\pi^*, s) = \max_{a \in \mathcal{A}_s} Q_\gamma^{(t+1)}(\pi^*, s, a) = \max_{a \in \mathcal{A}_s} \left(r_s^a + \gamma \sum_{s' \in \mathcal{S}} p_{s \to s'}^a V_\gamma^{(t)}(\pi^*, s') \right)$$

This result is based on **Bellman's Principle of Optimality**.

The horizon-T policy is not stationary, in general, i.e. $\pi^{(t)}(a|s) \neq \pi^{(t')}(a|s)$ for $t \neq t'$, but it can be chosen to be deterministic. Any idea why?

Optimal Fixed Horizon Policy: An Example



The Optimal Infinite Horizon Policy

We are now looking for a policy π which maximizes

$$V_\gamma^{(\infty)}(\pi, s) = \mathbb{E}_\pi\left[\sum_{t=1}^{\infty} \gamma^{(t-1)} R_t \middle| S_0 = s\right], \forall s \in S$$

- The optimal policy is now stationary
- As done before, we look for a deterministic policy, for which

$$V_\gamma^{(\infty)}(\pi, s) = \max_{a \in \mathcal{A}_s}\left(r_s^a + \gamma \sum_{s' \in \mathcal{S}} p_{s \to s'}^a V_\gamma^{(\infty)}(\pi, s')\right) \tag{1}$$

Fixed-Point Iterations and Banach's Fixed Point Theorem

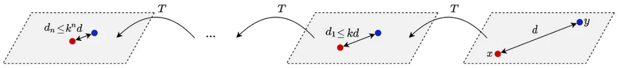
Some equations of the form $x = T(x)$ can be solved with a fixed point iteration:

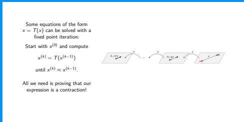
Start with $x^{(0)}$ and compute

$$x^{(k)} = T(x^{(k-1)})$$

until $x^{(k)} \approx x^{(k-1)}$.

All we need is proving that our expression is a contraction!





Notes

Let (X, d) be a complete metric space. Then a map $T : X \to X$ is called a **contraction mapping** on X if there exists $q \in [0, 1)$ such that $d(T(x), T(y)) \leq qd(x, y)$ for all $x, y \in X$.

Banach Fixed Point Theorem. Let (X, d) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed-point x^* in X (i.e. $T(x^*) = x^*$). Furthermore, x^* can be found as follows: start with an arbitrary element $x_0 \in X$ and define a sequence $(x_n)_{n\in\mathbb{N}}$ by $x_n = T(x_{n-1})$ for $n \geq 1$. Then $\lim_{n\to\infty} x_n = x^*$.

Maximizing Future Discounted Values with Dynamic Programming

Let us define the mapping (sometimes called **Bellman operator**)

$$T_\gamma : \mathbb{R}^{|\mathcal{S}|} \to \mathbb{R}^{|\mathcal{S}|}, T_\gamma(X)_s = \max_{a \in \mathcal{A}_s} \left(r_s^a + \gamma \sum_{s' \in \mathcal{S}} p_{s \to s'}^a X_{s'} \right) . \tag{2}$$

- One can show that the mapping T_γ is a contraction mapping and Banach's fixed point theorem can be applied. Hence, there is a unique fixed point $X^* = T_\gamma(X^*)$.
- The optimal policy is to choose actions in $\arg\max_{a \in \mathcal{A}_s} Q_\gamma^\infty(\pi^*, s, a)$.
- This policy is **stationary** and it can be chosen to be **deterministic**.

Value Iteration

Iteratively compute horizon-t values until $\max_{s \in \mathcal{S}} |V_\gamma^{(t+1)}(\pi^*, s) - V_\gamma^{(t)}(\pi^*, s)| < \theta$, where $\theta > 0$ is some convergence criterion. The optimal stationary policy picks actions in $\arg\max_{a \in \mathcal{A}_s} Q_\gamma^{t^*}(\pi^*, s, a)$, where t^* is the stopping iteration.

Value Iteration, for estimating $\pi \approx \pi_*$

Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
| $\Delta \leftarrow 0$
| Loop for each $s \in \mathcal{S}$:
| $v \leftarrow V(s)$
| $V(s) \leftarrow \max_a \sum_{s',r} p(s', r | s, a)\big[r + \gamma V(s')\big]$
| $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Output a deterministic policy, $\pi \approx \pi_*$, such that
  $\pi(s) = \operatorname{argmax}_a \sum_{s',r} p(s', r | s, a)\big[r + \gamma V(s')\big]$

What is the relationship to Reinforcement Learning?

“Solving” an MDP amounts to a solving an optimal control problem, i.e. finding the optimal policy, where the dynamics is known, i.e. $p^a_{s_i \rightarrow s_j}$ and r^a_s are assumed to be known. On the contrary, as we will see, in Reinforcement Learning, one assumes that the dynamics and rewards are unknown.

References

- https://towardsdatascience.com/real-world-applications-of-markov-decision-process-mdp-a39685546026
- Reinforcement Learning: An Introduction, Sutton and Barto
- Reinforcement Learning course (CS-456), EPFL