ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

Linear inverse problems in imaging

Computational imaging 2024-25

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Linear inverse problems in imaging

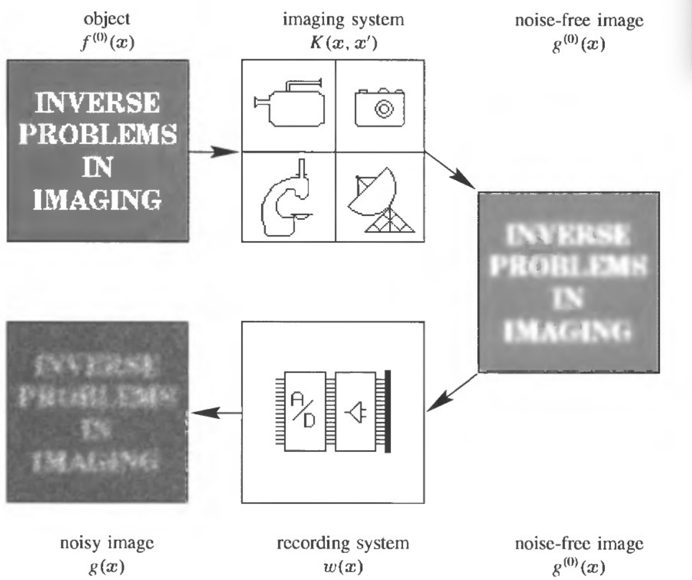


Figure 3.2. Schematic representation of the formation of the noisy image g.

Mathematical formulation of a linear inverse problem in imaging:

$$Ax = y + e$$

Where:

- A is a matrix, discretization of the continuos operator representing the imaging system
- y contains the data (supposed of dimension M)
- e is the noise
- x is the image of the images object to be recovered (supposed of dimension N).

Examples: image deblur

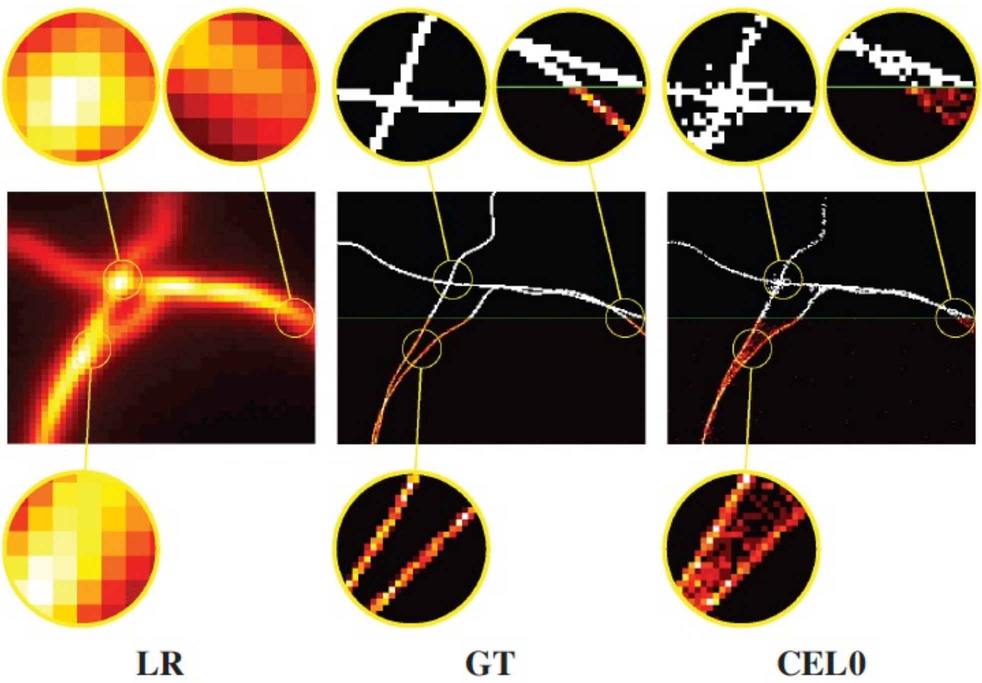
original | Blurred noisy | restored

Out of focus blur

Motion blur

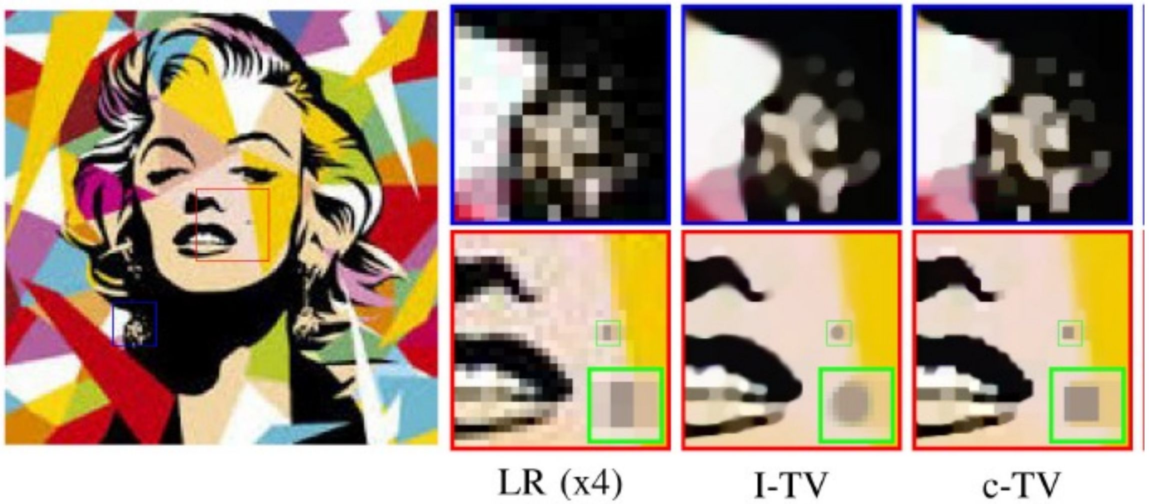
Examples: image superresolution

Fluorescence microscopy



Examples: image superresolution

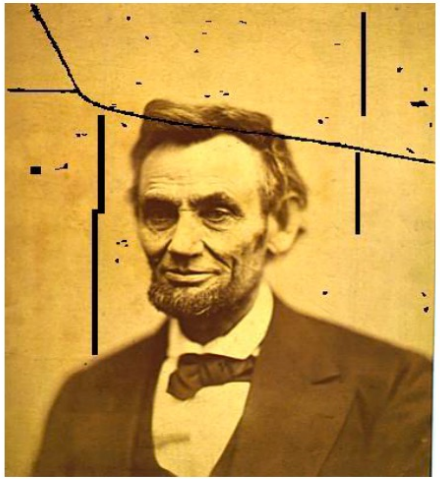
original data reconstructions



Examples: image inpainting

original

data

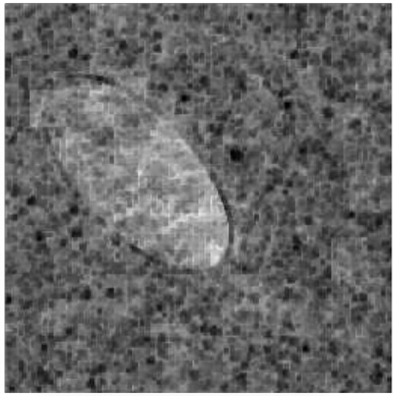


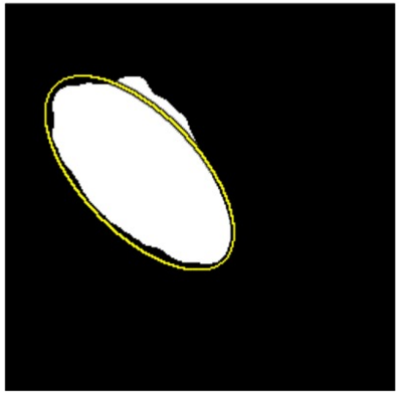
reconstruction

Examples: image segmentation

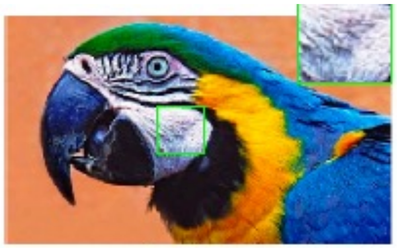
original

data





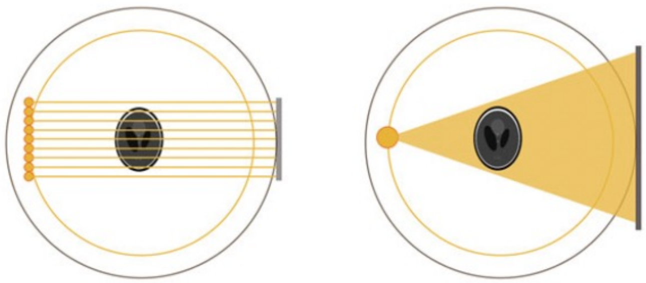
original



Segmented image

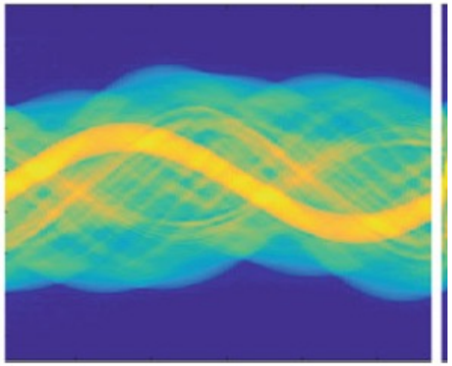
Examples: tomographic image reconstruction

Data acquisition for parallel and fan beam rays

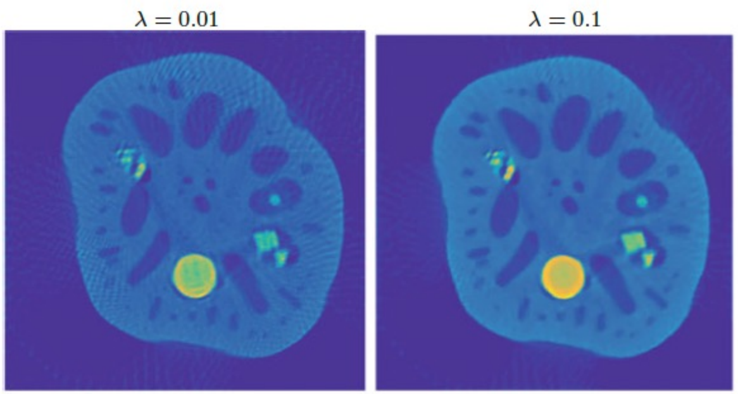


Object scanned

Noisy data (sinogram)



Two different reconstructions



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Linear inverse problems in imaging

Digital images always are affected by errors (noise):

- ✓ false light, defects in the recoding process, analog-to-digital conversion (salt-and-pepper)

- ✓ truncation errors (quantization): integer approximation of a continuous quantity (uniform noise)

- ✓ Noise generated during the conversion of the light to an electrical signal (Gaussian noise)

- ✓ Short noise: a type of electronic noise due to the discrete nature of electric charges (Poisson noise)

We have only statistical information about the noise

Ill-posed Problems

What is a well-posed problem?

A problem is defined well-posed if:
1. it exists a solution for arbitrary data
2. this solution is unique.
3. the solution continuously depends on the data.

This means that it small varies for small variations of the data.

The direct problem is well-posed (from physical considerations).

What is an ill-posed problem?

A problem when at least one of the three conditions is not satisfied.

Linear inverse imaging problems are ill-posed.

Ill-posed Problems

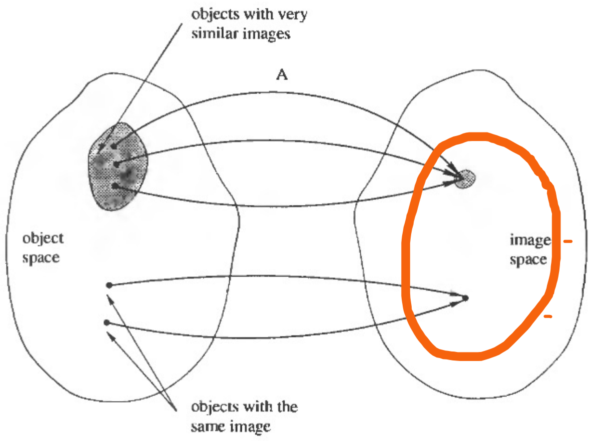


Figure 1.1. Schematic representation of the relationship between objects and images. The shaded subsets in $\mathcal{X}$ and $\mathcal{Y}$ illustrate the loss of information due to the imaging process.

- A is a linear operator mapping an image in the Image space X to a noise free image in the image space Y (orange subset).
- The set of noise free images is usually called range of A

1. It is possible that two or more objects have exactly the same image. It is related with objects whose Image is exactly zero (invisible objects). Given an object in X If we add an invisible object to it we obtain the same image.

2. It is possible that two distant objects have very similar images.

Ill-posed Problems

Solving the linear system

$$Ax = y + e$$

Is a quite simple numerical task.
Suppose A of size M x N.

1. If M=N and A non singular, the system has a unique solution, but usually there is no continuos dependence from the data.

2. If M>N the linear system has no solution and in place of it we solve the least square problem:

$$min\|Ax - (y+e)\|_2^2$$

3. If M<N the system has infinite possible solutions.

Ill-posed Problems

«The result is that the numerical inversion produces results that are physically unacceptable even Mathematically acceptable»

In inverse problems data are always affected by noise and the solution method amplifies that noise, Producing a large and wildly oscillating function that completely hides the physical solution Corresponding to noise-free data.

Even if the solution of the linear system exists and is unique (case 1.) it is completely corrupted due To the small noise on the data (ill-conditioned system).

Ill-posed Problems

Why the naive approach fails?

$$x_{naive} = A^{-1}y + A^{-1}e = x + A^{-1}e$$

Case of A square (M=N)

$$x_{naive} = min\|Ax - (y+e)\|_2^2$$

Case of A rectangular (M >N or M<N)

Hence the naive image cionsiste of the sum of two images:

- The first component x is the exact image
- The second component $A^{-1}e$ is the inverted noise.

The inverted noise contaminates the exct image x.

Ill-posed Problems

We use the Singular Value Decomposition (SVD) of A to analyse the contribution of the inverted noise.

Theorem

Any matrix A of size $m\times$ and rank k ¡ min(m,n) can be decomposed as:

$$A = U\Sigma V^T$$

where:

- U is an orthogonal matrix of size $m \times m$
- V is an orthogonal matrix of size $n \times n$
- Σ is a diagonal matrix with diagonal entries:

$$\sigma_1 \geq \sigma_2 \geq \ldots \sigma_k \geq \sigma_{k+1} = \ldots = \sigma_{min(m,n)} = 0$$

Ill-posed Problems: why the naive solution fails?

If we write in terms of SVD representation:

$$A^{-1} = V\Sigma^{-1}U^T = \sum_{i=1}^{k} \frac{1}{\sigma_i} v_i u_i^T$$

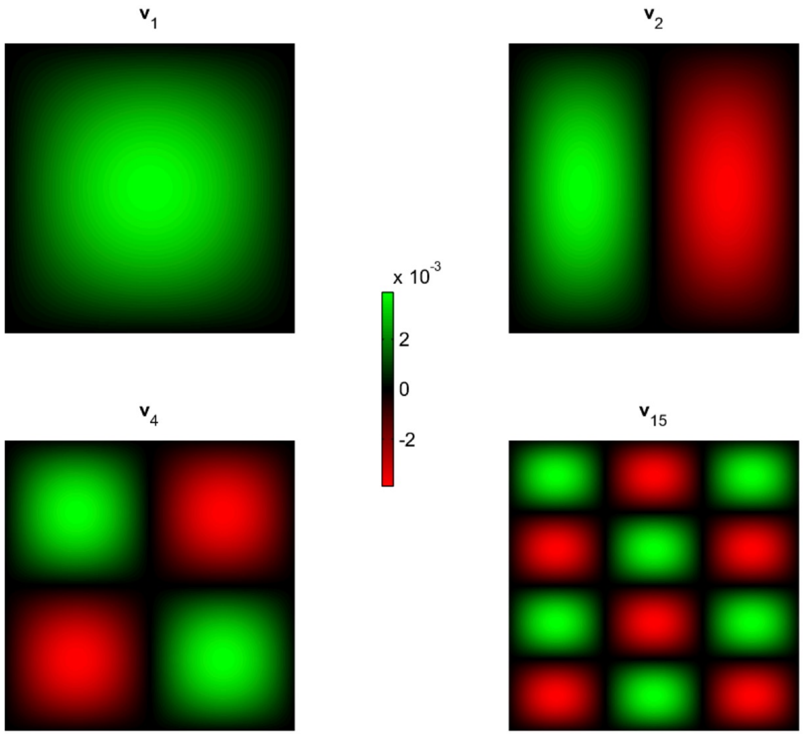
The naive solution can be written as:

$$x_{naive} = A^{-1}(y+e) = x + A^{-1}e = x + \sum_{i=1}^{k} \frac{1}{\sigma_i} v_i u_i^T e$$

- The error components $|u_i^T e|$ are small and typically of the same order of magnite for all i.
- The singular values decay to a value very close to zero.
- The singular vectors corresponding to the smaller singular values typically
tend to have more sign changes.

Ill-posed Problems

The singular vectors corresponding to the smaller singular values typically represent higher frequency information. That is, as i increases, the vectors u_i and. v_i tend to have more sign changes.



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Ill-posed Problems: why the naive solution fails?

- When we divide by a small singular value such as , σ_N we greatly magnify the corresponding error component. $u_N^T e$, which in turn contributes a large multiple of the high frequency information contained in v_N to the computed solution.

- We can examine the coefficents of the naive solution.

$$x_{naive} = A^{-1}y = V\Sigma^{-1}U^T y = \sum_{i=1}^{k} \frac{u_i^T y}{\sigma_i} v_i$$

- The quantities $\frac{u_i^T y}{\sigma_i}$ are expansion coefficients of the basis vectors v_i.

⇩

When these quantities are small in magnitude, the solution has very little contribution from vi , but when σ_i is very small, these quantities can be large due to the presence of the noise in the data term $y + e$

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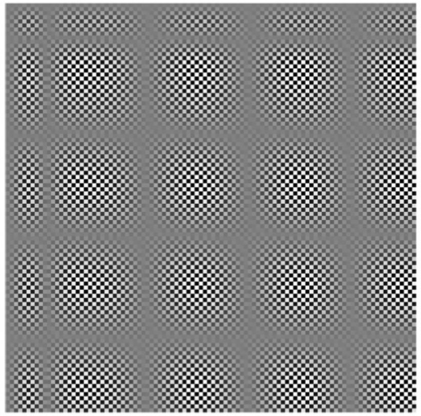
Ill-posed Problems: why the naive solution fails?

Hence in the presence of errors on the data the solution is dominated by noise.

x

y+e



Naive solution

Ill-posed Problems

How to cure ill-posedness?

1. By searching approximate solutions satisfying additional constraints deriving from the physics of the problem.

- The information added is also called *a priori or prior image information*.

- In mathematical models this a priori information is called *regularization.*

- It is added by means of one or more regularization terms, each multplied by a positive scalar calle *regularization parameter*.

Ill-posed Problems

The simplest regularization model with a single regularization function is expressed as:

$$min\|Ax-(y+e)\|_2^2+\lambda R(x)$$

Where:

- R(x) is the regularization function that incrporates a priori infromation on the solution

- λ is the regularization parameter representing the trade off between the fit-to-data and the regularization terms.

In the following, we will first consider the previuous model with different possible functions R(x).

Regularization approach

Where:

- R(x) is the regularization function that incrporates a priori infromation on the solution

- λ is the regularization parameter representing the trade off between the fit-to-data and the regularization terms.

Now we need some basics on optimization.