MSAS – Assignment #2: Modeling

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1 Questions

Question 1

1) List the stages of dynamic investigation and their meaning. 2) When going from the *real system* to the *physical model* a number of assumptions are made; report the most important ones along with their mathematical implications. 3) For each of the assumptions below, shortly state what sort of simplification may result: i) The gravity torque on a pendulum is taken proportional to the pendulum angle θ; ii) Only wind forces and gravity are assumed in studying the motion of an aircraft; iii) A temperature sensor is assumed to report the temperature exactly; iv) The pressure in a hydraulic actuator is assumed uniform throughout the chamber. 4) List the *effort* and *flow* variables for the domains treated and discuss their similarity.

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1) Dynamic investigation
The process of dynamic investigation is a model-based design which aims to understand the dynamic behaviour of a generic system reducing costs with respect to trial and error methods. It is made up of four stages: the first two belong to the *modeling* phase, the second one to the *analysis* and the last one to the *synthesis*. (a) At first it has to be specified the system to be studied and it has to be imagined a simple **physical model** whose behavior will match sufficiently closely the behavior of the real system. (b) Then a **mathematical model** which describes the physical model has to be derived, for example writing the differential and algebraic equations. (c) **System response** has to be obtained by solving the mathematical model, thus the dynamic behaviour is obtained. (d) To conclude, **design decisions** have to be made, as choosing physical parameters of the system so that it will behave as desired.

2) Assumptions
The step real to physical models requires making abstraction through approximations and engineering judgment. But most importantly implications have to be studied. (a) **Neglecting small effects** is the first approximation process. It leads to mathematical simplifications by reducing the number of variables, equations and the overall complexity of the equations of motion. Just remember that small effects have to be neglected on a relative basis. (b) **Independent environment** approximation means that the environment surrounding a physical system is unaffected by the behavior of the system, thus the mathematical equations are much more simpler since the input is not affected by the response. (c) **Lumped elements** instead of distributed physical characteristics lead from partial differential equations, which are difficult to be solved, to a system of ordinary differential equations, which are easier to be computed. (d) **Linear** differential equations can sometimes approximate nonlinear systems. When having a linear ordinary differential equation, the solution is general and holds for all magnitudes of motion, making it easier to understand the real behaviour of the system. Furthermore, superposition principle applies. (e) **Constant parameters** instead of time-variant parameters simplify the equation of motions. The approximation is valid only if parameters vary at a slow rate with respect to the characteristic time. (f) **Neglecting uncertainty and noise** means to have a deterministic approach to avoid the usage of statistics in the evaluation of the model.

3) Simplification results

i. By applying Newton's second law for rotational systems, the equation of motion for the pendulum may be obtained and is:

$$\ddot{\theta} + \frac{g}{l} sin(\theta) = 0$$

If the amplitude of angular displacement is small enough, so the small angle approximation $sin(\theta) \approx \theta$ holds true, then the equation of motion reduces to the equation of simple harmonic motion:

$$\ddot{\theta} + \frac{g}{l} \theta = 0$$

which has a simple harmonic solution. With the assumption of small angles, the frequency and the period of the pendulum are independent from the initial angular displacement amplitude.

ii. When aerodynamics and gravity forces are the only acting on an aircraft the assumption is to consider as negligible many other minor forces as the solar radiation pressure acting on the wings, the torques produced by Earth's magnetic field and by the gravity gradient. As already said in the first part, neglecting small effects lead to a reduction in the number of variable and equations.

iii. Assuming the reported temperature as exact means to use a deterministic approach and so to neglect uncertainty and noise to avoid statistical effects related to the measurement.

iv. Assuming uniform the pressure in a hydraulic actuator throughout the chamber means assuming the presence of constant parameters as the viscosity. Furthermore minor effect are neglected as mass flow rate leakages, resulting in a reduction of the number of variables. One last approximation is that the flow is considered fully developed.

4) Effort and Flow variables

In table 1 are reported all the effort and flow variables for each domain.

System	Effort	Flow
Mechanical	force/torque	velocity/angular velocity or position/rotation
Fluid	pressure	volume flow rate
Electrical	voltage	current
Thermal	temperature	heat flow

Table 1: Effort and Flow variables

The different domains can be represented considering the similarities with electrical elements: resistances, inductances and capacitors. In fact: a capacitor can model a spring (mechanical domain), fluid compressibility (fluid domain) and heat capacity of the material (thermal domain); an Inductance can model the mass inertia or the fluid inertia properties; a resistance can model frictions (mechanical domain), general losses (fluid domain) and dissipation of thermal energy (thermal domain). Generally effort and flow are linked as follow:

$$effort = R \cdot flow$$

$$effort = L \cdot \frac{d}{dt} flow$$

$$flow = C \cdot \frac{d}{dt} effort$$

Question 2

1) Derive from scratch the mathematical model for RC and RL circuits and express the system response in closed form. 2) Consider a real DC motor and a) sketch its physical model (list the assumptions made); b) derive its mathematical model; c) show how the motor constant depends on the physical parameters.

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1) RC and RL mathematical models

The RC physical model is shown in figure 1.

By using Ohm first law and the capacitor characteristic equation

$$\begin{cases} i = \frac{V_0 - V_C}{R} \\ i = C\dot{V}_C \end{cases} \quad (1)$$

then the mathematical model results in:

$$\dot{V}_c + \frac{V_c}{RC} - \frac{V_0}{RC} = 0 \quad (2)$$

By integrating this first-order ODE we get the system response in close form:

$$\begin{cases} V_C(t) = V_0 + (V_c(0) - V_0)e^{-\frac{t}{RC}} \\ i(t) = -\frac{V_C(0) - V_0}{R}e^{-\frac{t}{RC}} \end{cases} \quad (3)$$

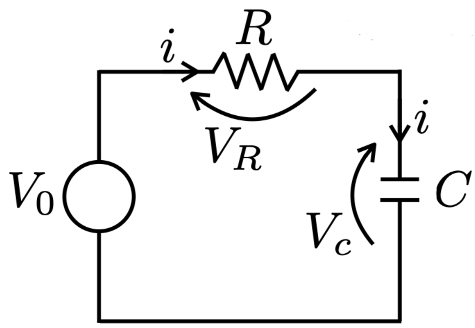

Figure 1: RC circuit

The RL physical model is shown in figure 2.

By using Ohm first law and the inductance characteristic equation

$$\begin{cases} i = \frac{V_0 - V_L}{R} \\ L\frac{di}{dt} = V_L \end{cases} \quad (4)$$

then the mathematical model results in:

$$\frac{di}{dt} + \frac{R}{L}i - \frac{V_0}{L} = 0 \quad (5)$$

By integrating this first-order ODE we get the system response in close form:

$$\begin{cases} i(t) = \frac{V_0}{R} + (i(0) - \frac{V_0}{R})e^{-\frac{tR}{L}} \\ V_L(t) = (V_0 - Ri(0))e^{-\frac{tR}{L}} \end{cases} \quad (6)$$

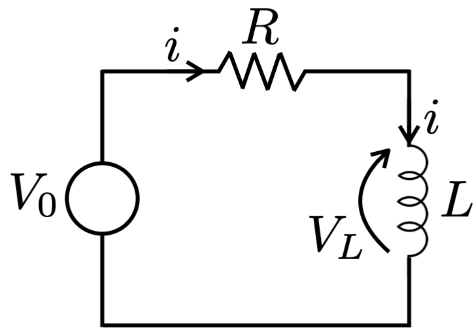

Figure 2: RL circuit

2) Real DC motor

A DC motor is a rotary electrical motor that converts direct current electrical energy into mechanical energy. The main goal is to study the dynamic response of a DC motor when a specified time-varying voltage is applied to its rotor windings, and some specified variable load is applied to the output drive.

a) With reference to figure 3 and 4 the physical model consists of a resistor, an inductance and an additional voltage drop proportional to the angular velocity representing the armature coil, through which the current flows. The DC motor drives a shaft connected to a disk which is disturbed by a load.

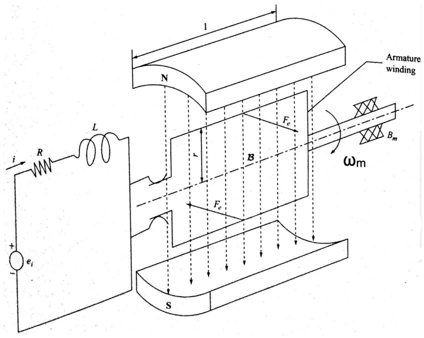

Figure 3: Physical model of DC motor - Electrical part

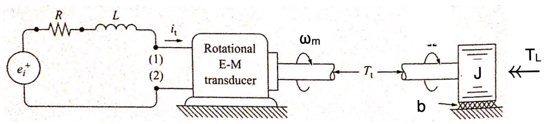

Figure 4: Physical model of DC motor

The assumption that has been made are: low intensity of the current to avoid complications due to the Joule effect; in the ideal case only an infinite number of coils would make the torque constant however we assume a constant torque even with a finite number of coils; shaft flexibility is considered negligible.

b) The mathematical model is derived combining the electrical and mechanical elements, as described by proper physical relationship, Kirchhoff's laws and force equilibrium:

$$\begin{cases} e_1 - e_2 = Ri \\ L\frac{di}{dt} = e_2 - e_m \\ T_m = Ki \\ e_m = K\omega_m \end{cases} \tag{7}$$

The final equations of motion are:

$$\begin{cases} L\frac{di}{dt} + Ri + K\omega_m = e_1 \\ J\dot{\omega}_m + b\omega_m - Ki = -T_L \end{cases} \tag{8}$$

c) To show the motor constant K, let's derive it. For the nomenclature refers to figure 3. Considering N turns of the winding the total torque is:

$$\begin{cases} T_m = 2 \cdot N \cdot F_e \cdot r \\ F_e = i \cdot l \cdot B \end{cases} \tag{9}$$

$$T_m = 2 \cdot N \cdot l \cdot B \cdot r \cdot i = K \cdot i \tag{10}$$

The motor constant can also be retrieved using the electromotive force definition and the velocity of a charge, obtaining -obviously- the same dependencies.

$$e_m = u \cdot B \cdot l = 2 \cdot N \cdot \omega_m \cdot r \cdot B \cdot l = K \cdot \omega_m \tag{11}$$

Explicitly:

$$K = 2 \cdot N \cdot r \cdot B \cdot l$$

Question 3

1) Write down the Fourier law and show how it is specialized in the case of conduction through a thin plate; discuss the concept of thermal resistance. 2) Report the equation for thermal radiation in case of a) black body and b) real body and discuss them.

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1) Fourier law

The law of heat conduction states that the rate of heat transfer through a material is proportional to the negative gradient in the temperature and to the area through which the heat flows. Mathematically it is expressed as follows:

$$\mathbf{q} = -k\nabla T \tag{12}$$

$\mathbf{q}$ is the local heat flux density ($[\frac{W}{m^2}]$), k(T) is the thermal conductivity ($[\frac{W}{mK}]$) and ∇T is the gradient of temperature ($[\frac{K}{m}]$).

Let's now consider a particular case of Fourier law in a 1D space. Assuming a uniform heat flux and a constant thermal conductivity the equation becomes:

$$\frac{Q}{A} = -k\frac{dT}{dx} \tag{13}$$

Integrating the equation for a thick plate having boundary temperatures equal to T_1 and T_2 the result is:

$$Q = \frac{kA}{L}\Delta T \tag{14}$$

$\frac{L}{kA}$ ($[\frac{K}{W}]$) is the thermal resistance which can be used to apply similarities with the electrical domain. As already explained in question number 1, it links the thermal effort with the thermal flow in that way: $T = RQ$. The thermal resistance can be derived even for different kind of heat transfer and geometries.

2) Thermal radiation

a) Thermal radiation in case of black body, which is a perfect emitter and receiver, the equation is:

$$\frac{Q}{A} = \sigma T^4 \tag{15}$$

where Q is the heat flux ([W]), A is the its surface area ($[m^2]$), T is the temperature ([K]) and $\sigma = 5.67 \cdot 10^{-8}$ is the Stefan-Boltzmann constant ($[\frac{W}{m^2K^4}]$).

b) Instead, a real body, has less emissivity than blackbody, in fact an emissivity factor has to be introduced. Another aspect to be considered is that the real body is not isolated in space, therefore it will exchange radiation with other objects. Since not all the radiation leaving a body impinges another body the viewing factor is added. The equation considering all this aspects becomes:

$$\frac{Q}{A} = F_e F_v \sigma(T_h^4 - T_l^4) \tag{16}$$

where F_e is the emissivity factor, F_v is the viewing factor, T_h and T_l are the hotter and the cooler temperatures ([K]).

2 Exercises

Exercise 1

A miniaturized reaction wheel can be modeled as a couple of massive disks, connected with a flexible shaft (Figure 5). The first disk is driven by a rigid shaft, linked to an electric motor. The motor provides a given torque, while the rigid shaft is subjected to viscous friction, due to motor internal mechanisms. At $t_0 = 0$, the motor provides the torque $T(t_0) = T_0$.
1) Write down the mathematical model from first principles. 2) Using the data given in the figure caption, and guessing a value for the flexible shaft stiffness k and the viscous friction coefficient b, compute the system response from t_0 to $t_f = 10$ s. 3) Two accelerometers placed on the two disks recorded samples at 100 Hz, which were saved in the file `samples.txt`; the samples are affected by measurement noise. Determine the values of k and b that allow retracing the experimental data, so avoiding parametric errors.

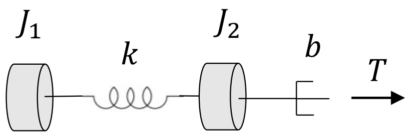


Figure 5: Physical model ($J_1 = 0.2\,\mathrm{kg{\cdot}m}$; $J_2 = 0.1\,\mathrm{kg{\cdot}m}$; $T_0 = 0.1\,\mathrm{Nm}$).

Mathematical model
In order to write down the mathematical model it is important to correctly select the model of the rotational damper: at first it was chosen a linear one but then the problem has been reformulated using a nonlinear model since the linear one could not retrace the experimental data as was requested in the last part of the exercise. Therefore, only the nonlinear model of the damper will be considered from now on.

The equations has been derived balancing:

1. the forces coming from the inertial disks: $J_i\ddot{\theta}_i$
2. the external torque (constant between the initial and the final time): T
3. the torque generated by the rotational damper: $sign(\dot{\theta}_2)b\dot{\theta}_2^2$
4. the torque generated by the rotational spring: $k(\theta_2 - \theta_1)$

Therefore, the equations representing the dynamics of the system are:

$$\begin{cases} J_1\ddot{\theta}_1 = k(\theta_2 - \theta_1) \\ J_2\ddot{\theta}_2 = -k(\theta_2 - \theta_1) - sign(\dot{\theta}_2)b\dot{\theta}_2^2 \end{cases} \tag{17}$$

This system of second order nonlinear differential equations, in order to be integrated in Matlab, has to be transformed into a system of first order differential equations. It can be used the state space representation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{u}(\mathbf{t}) \tag{18}$$

where, in our case, the state vector is: $\mathbf{x} = [\theta_1, \omega_1, \theta_2, \omega_2]^T$.

Explicitly, the system results in the following equations:

$$\begin{cases} \dot{\theta}_1 = \omega_1 \\ \dot{\omega}_1 = \frac{k}{J_1}(\theta_2 - \theta_1) \\ \dot{\theta}_2 = \omega_2 \\ \dot{\omega}_2 = -\frac{k}{J_2}(\theta_2 - \theta_1) - \frac{1}{J_2} sign(\omega_2) b\omega_2^2 + T \end{cases} \tag{19}$$

System response

System 19 has been linearized around an equilibrium point in order to compute the eigenvalues of the Jacobian matrix and to be able to properly select the Matlab integrator. The linearized state space representation results in:

$$\dot{\mathbf{x}} = J\mathbf{x} + \mathbf{b} \tag{20}$$

The resulting Jacobian is:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_1} & 0 & \frac{k}{J_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_2} & 0 & -\frac{k}{J_2} & -2\frac{b}{J_2}\omega_2^{eq} \end{pmatrix} \tag{21}$$

For $\omega_2 = 0$ the system eigenvalues are:

$$\lambda = \begin{pmatrix} 3.8730i \\ -3.8730i \\ 0 \\ 0 \end{pmatrix} \tag{22}$$

As we expect from this single domain system, we are dealing with a non-stiff problem. Since stringent tolerances are not imposed, *ode45* Matlab function has been selected as integrator. The guessed value to compute the system response are k = 1 N and b = 1 $kg\frac{m}{rad}$. In figure 6 are reported the results of the integration and the corresponding accelerations given the initial state $\mathbf{x}_0 = [0, 0, 0, 0]^T$.

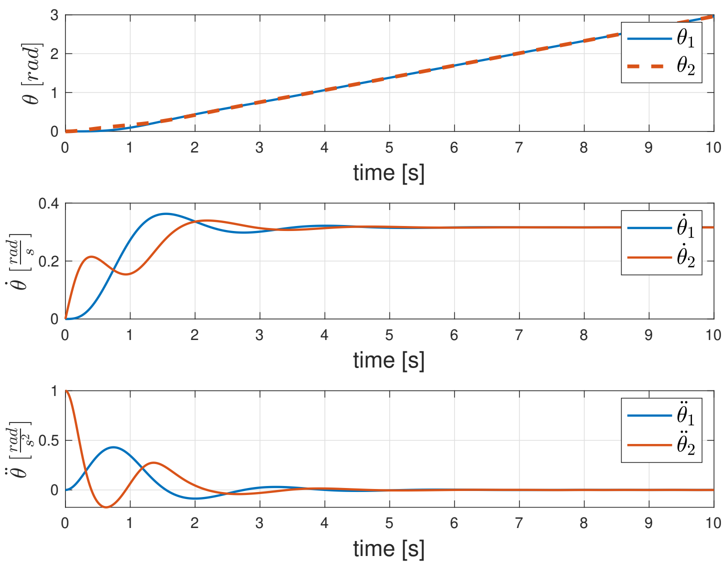


Figure 6: System Response and angular accelerations

Experimental data retracing

The selected values of k and b do not fit correctly the experimental data. In order to find the fitting parameters it has been written an objective function to be minimized with the Matlab *fminunc* function. The objective function has been stated as the sum of the absolute errors between the acceleration coming from the integrated results and the given samples:

$$\sum |\ddot{\theta} - \ddot{\theta}_{samples}|$$

In order to increase the accuracy of fitting parameters it has been considered the relative error, but in the end it has been excluded since it gave rise to numerical errors due to the asymptotic convergence to zero of the sample accelerations.

As shown in figure 7 the absolute error minimization leads to a good fitting. The resulting parameters are k = 3.14097 N and b = 2.71672 $kg\frac{m}{rad}$.

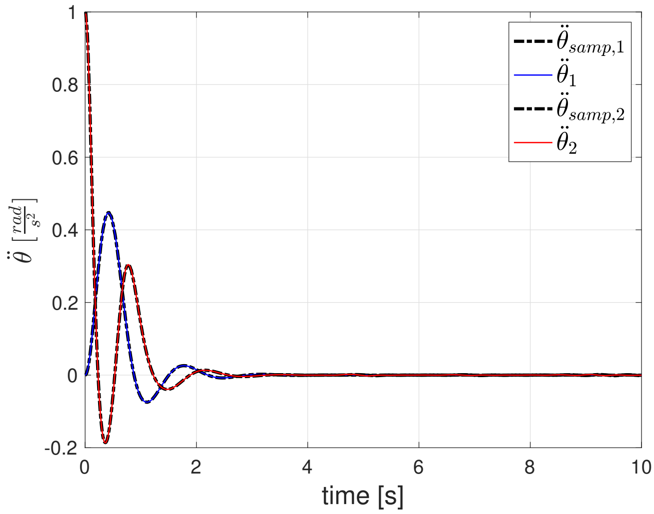


Figure 7: Acceleration Comparison

In figure 8 is represented the error of the experimental data retracing with the parameters derived from the minimization problem.

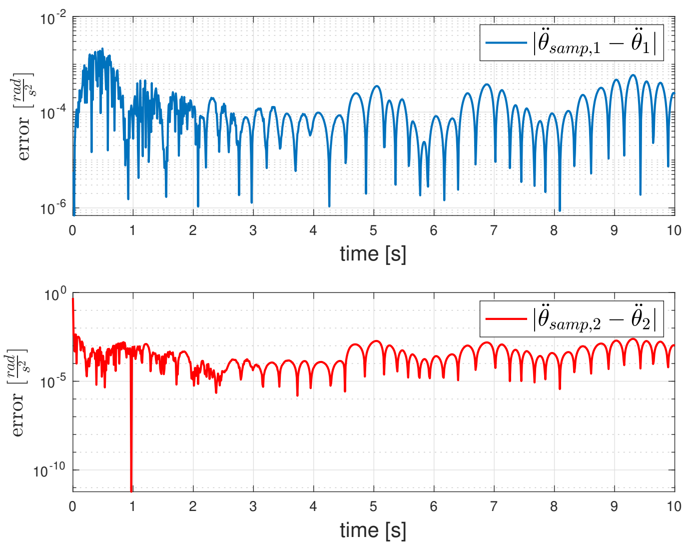


Figure 8: Absolute error between experimental data and solution

Exercise 2

Consider the ideal physical model network shown in Figure 9. The resistance R_2 varies its value with the value of the current, i.e., $R_2 = R_{2k}i$.The switch has been open for a long time. The capacitor is charged and has a voltage drop between its ends equal to 1 V. Then, at $t = 0$, the switch is closed. 1) Plot the subsequent time history of the voltage V_C across the capacitor. 2) Assume a voltage source characterized by $v(t) = \sin(2\pi ft)\arctan(t)$ having the positive terminal downward inserted in place of the switch. What is in this case the voltage history across the capacitor?

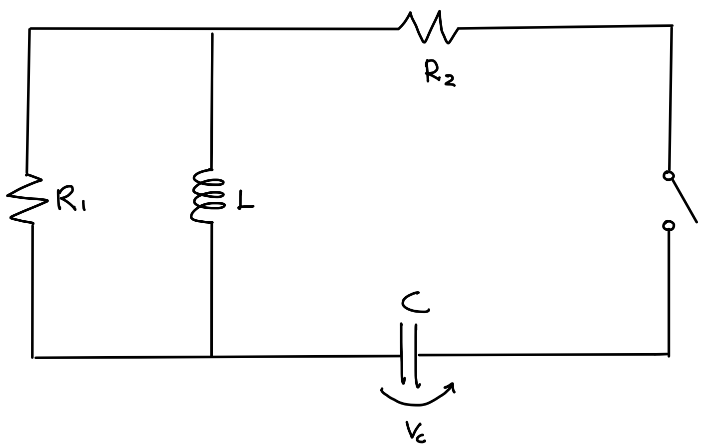


Figure 9: Circuit physical model ($R_1 = 100\,\Omega$; $R_{2k} = 10\,\Omega/\text{A}$; $L = 10\,\text{H}$; $C = 1\,\text{mF}$; $f = 5\,\text{Hz}$.)

Mathematical model

With reference to figure 9, i_1, i_2 and i_3 are the currents flowing respectively through R_1, R_2 and L, from bottom to top, top to bottom and left to right. The voltages are respectively V_{R_1}, V_L, V_{R_2}. Kirchhoff's laws can be written as follows:

$$\begin{cases} i_1 = i_2 + i_3 \\ V_{R_1} + V_L = 0 \\ V_L - V_{R_2} + v(t) - V_C = 0 \end{cases} \tag{23}$$

By combining equations 23 and the following relations:

$$\begin{cases} i_2 = C\dot{V}_C \\ V_L = L\dot{i}_3 \\ R_2 = R_{2k}i_2 \end{cases} \tag{24}$$

we obtain the final equation which results in a second order nonlinear differential equation.

$$\ddot{V}_c = \frac{\dot{V}_c + \frac{R_1 V_C}{L} + \frac{R_1 R_{2k} C^2 \dot{V}_C^2}{L} - \dot{v}(t)}{-R_1 C - 2R_{2k}C^2\dot{V}_C} \tag{25}$$

where

$$\dot{v}(t) = 2cos(2\pi ft)\pi ftan^{-1}(t) + \frac{1}{1+t^2}sin(2\pi ft)$$

is a contribute to be considered only in the second part of the exercise.

This second order nonlinear differential equation, in order to be integrated in Matlab, has to be transformed into a system of first order differential equations. The state variables are $\mathbf{x} = [V_C, Y]^T$ and the final system to be integrated is:

$$\begin{cases} \dot{V}_C = Y \\ \dot{Y} = \frac{Y + \frac{R_1 V_C}{L} + \frac{R_1 R_{2k} C^2 Y^2}{L} - \dot{v}(t)}{-R_1 C - 2R_{2k} C^2 Y} \end{cases} \tag{26}$$

System response

System 26 has been linearized around the equilibrium point $\mathbf{x} = [0, 0]^T$ in order to compute the eigenvalues of the Jacobian matrix and to be able to properly select the Matlab integrator. The resulting Jacobian evaluated at the equilibrium point is:

$$J = \begin{pmatrix} 0 & 1 \\ -\frac{1}{L} & -R_1 C \end{pmatrix} \tag{27}$$

Therefore the system eigenvalues are:

$$\lambda = \begin{pmatrix} -0.0500 + 0.3122i \\ -0.0500 - 0.3122i \end{pmatrix} \tag{28}$$

It was expected that the eigenvalues were complex and conjugate since the state vector is composed by the voltage and its derivative. The system is non-stiff and stringent tolerances are not imposed, therefore the chosen integrator is *ode45*. The initial condition are $\mathbf{x_0} = [1, 0]^T$. In figure 10 and 11 is reported the system response in terms of voltage drop across the capacitor and its derivative.

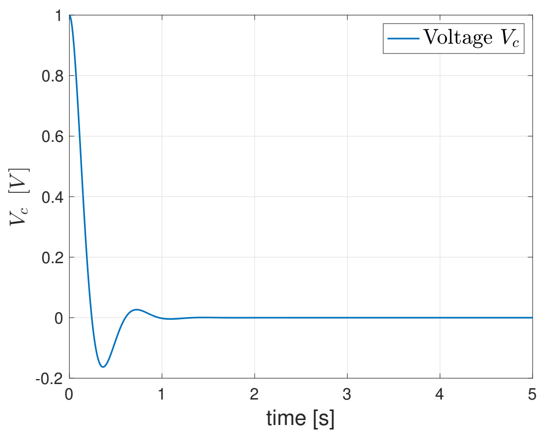


Figure 10: V_C WITHOUT voltage source

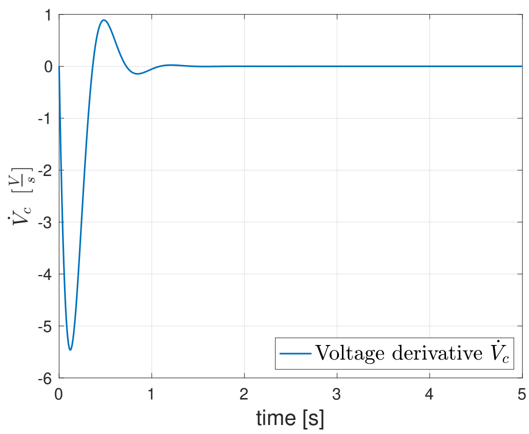


Figure 11: $\dot{V}_C$ WITHOUT voltage source

It can be noticed that starting from the initial time the voltage across the capacitor drops as it discharges. After one second the voltage drop can be considered null. In particular it can be observed the presence of an undershoot phenomenon which is due to the presence of an inductance and a capacitance and so we have a resonant circuit. Adding a damping resistor between source and load can dramatically reduce undershoot if restriction on the transition time are applied.

Let's now add the voltage source $v(t) = sin(2\pi ft)arctan(t)$. The system response is shown in figure 12. The capacitor voltage initially drops and then, differently from the other case, starts oscillating according to the voltage provided by the generator. The amplitude increases over time until it reaches a constant amplitude of 0.52 V.

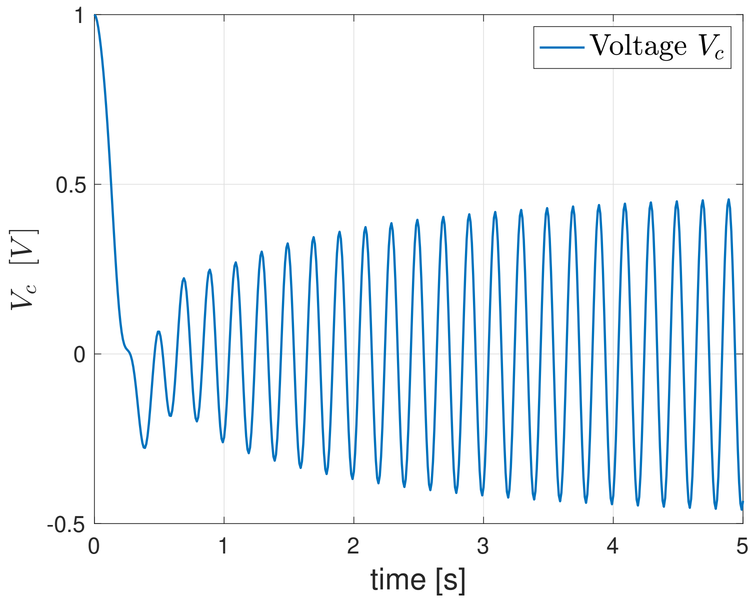


Figure 12: V_C WITH voltage source

One last remark has to be done regarding the period of oscillation after the transient: it can be noticed from 13 that the period is equal to the one induced by the voltage source and it is equal to $T = \frac{1}{f} = 0.2s$.

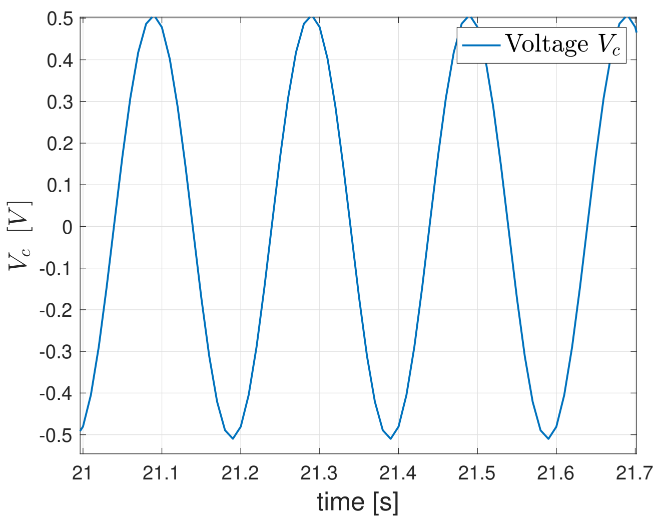


Figure 13: Zoom on V_C WITH voltage source

Exercise 3

The rocket engine in Figure 28a is fired in laboratory conditions. With reference to Figure 28a, the nozzle is made up of an inner lining (k_1), an inner layer having specific heat c_2 and high conductivity k_2, an insulating layer having specific heat c_4 and low conductivity k_4, and an outer coating (k_5). The interface between the conductor and the insulator layers has thermal conductivity k_3. 1) Select the materials of which the nozzle is made of[1], and therefore determine the values of k_i ($i = 1, \dots, 5$), c_2, and c_4. Assign also the values of ℓ_i (i=1,. . . ,5), L, and A in Figure 28a. 2) Derive a physical model and the associated mathematical model using one node per each of the five layers and considering that only the conductor and insulator layers have thermal capacitance. The inner wall temperature, T_i, as well as the outer wall temperature, T_o, are assigned. 3) Using the mathematical model at point 2), carry out a dynamic simulation to show the temperature profiles across the different sections. At initial time, $T_i(t_0) = T_o(t) = 20$ C°. When the rocket is fired, $T_i(t) = 1000$ C°, $t \in [t_1, t_f]$, following a ramp profile in $[t_0, t_1]$. Integrate the system using $t_1 = 1$ s and $t_f = 60$ s. 4) Repeat the simulation in point 3) using a mathematical model implementing two nodes for the conductor and insulator layers.

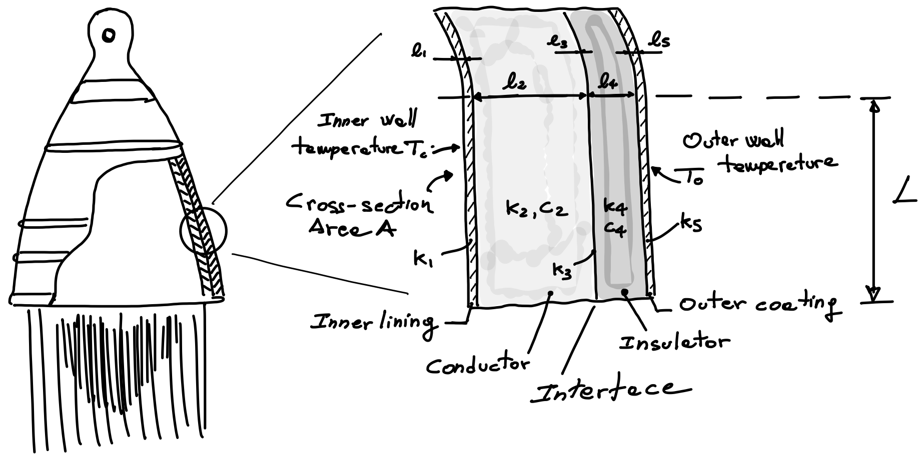


Figure 14: Real thermal system.

Material selection

Nozzle materials have been selected considering several aspects. The inner lining material is stainless steel A286 since it has a discrete thermal conductivity, a high melting temperature and an high strength. The conductor is C651 Silicon Bronze alloy which has high thermal conductivity, good corrosion resistance and moderate strength. Interface and insulator are Zirconium based material: a ceramic material pretty popular for its incredible low thermal conductivity. Finally, the outer coating is Inconel Alloy 718 which is a high-strength, corrosion-resistant nickel chromium material. In the table 2 are reported the main properties of the selected materials.

Preliminary model hypotheses

The problem has been solved considering a flat plate model. For that reason the overall thickness has to be much smaller than the planar dimensions. However the cross-section area has to be small enough so that the nozzle curvature can be considered negligible. In accordance with these assumptions the overall thickness is l = 44 mm, A = 0.1 m^2, L = 300 mm.

[1] The interface layer is not made of a physically existing material, though it produces a thermal resistance. For this layer, the value of the thermal resistance R_3 can be direcly assumed, so avoiding to choose k_3 and ℓ_3.

	l [mm]	k [$\frac{W}{Km}$]	c [$\frac{J}{kgK}$]	ρ [$\frac{kg}{m^3}$]
A286 Alloy	3	13		
651 Silicon Bronze alloy	30	57	380	8900
Zirconia	0.2	2.7		
Zirconia oxide	8	2	460	5700
Inconel Alloy 718	3	11.4		

Table 2: Nozzle materials properties

One-node per layer model

The physical model has been done considering the electrical analogy, so that the thermal conductivity can be expressed as a thermal resistance and the heat capacity as a capacitor.

In figure 15 is represented the physical model with one node per layer, thus resulting in five nodes placed at the center of each material. The two capacitors are all concentrated in the center of the layers.

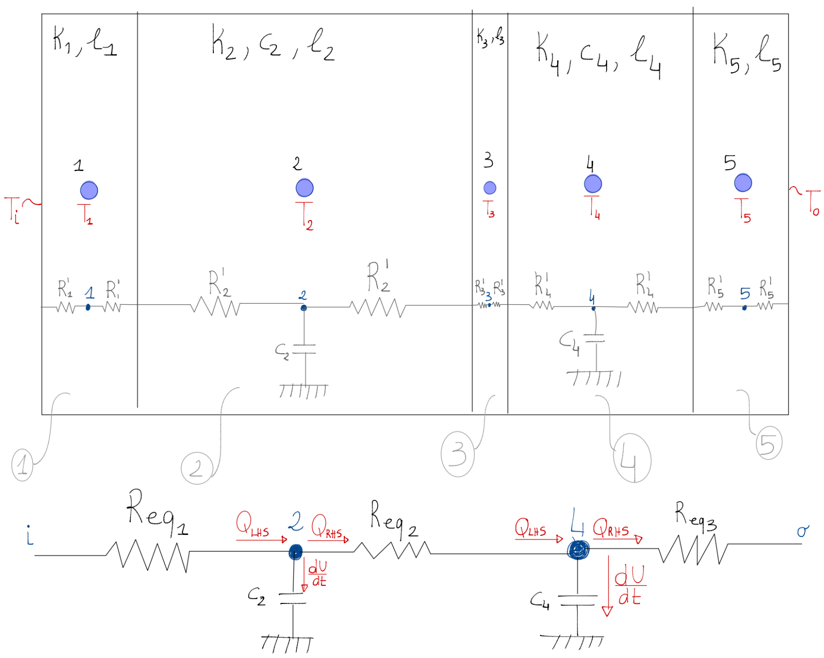


Figure 15: Physical Model: single node case

Referring to the figure, the resistances are:

$$R_i' = \frac{l_i}{2k_iA}$$

$$R_{eq_1} = \frac{l_1}{k_1A} + \frac{l_2}{2k_2A}$$

$$R_{eq_2} = \frac{l_2}{2k_2A} + \frac{l_3}{k_3A} + \frac{l_4}{2k_4A}$$

$$R_{eq3} = \frac{l_2}{2k_4A} + \frac{l_5}{k_5A}$$

The general heat exchange balance leads to a system of first order differential equations:

$$Q_{LHS} - Q_{RHS} = \frac{l_i}{2}\rho_i A c_i \dot{T}_i \quad (29)$$

In order to derive the temperature profiles across the sections the state vector $\mathbf{x} = [T_2, T_4]^T$ has been computed integrating the differential equations coming from the heat exchange balance. *Ode45* has been used since we are dealing with a single thermal domain, therefore we expect a non stiff problem.

$$\begin{cases} \dot{T}_2 = \frac{R_{eq_2}(T_i - T_2) - R_{eq_1}(T_2 - T_4)}{R_{eq_1} R_{eq_2} \rho_2 A l_2 c_2} \\ \dot{T}_4 = \frac{R_{eq_3}(T_2 - T_4) - R_{eq_2}(T_4 - T_o)}{R_{eq_2} R_{eq_3} \rho_4 A l_4 c_4} \end{cases} \tag{30}$$

The initial state conditions are $\mathbf{x}_0 = [293.15, 293.15]^T K$ since thermal equilibrium has been assumed along the materials at $t_{initial} = 0\ s$.
Once the state of the system has been retrieved, the temperature profiles at each node have been compute as follows:

$$\begin{cases} T_1 = T_i - \frac{R_1'(T_i - T_2)}{R_1 + R_2'} \\ T_3 = T_4 + \frac{(R_4' + R_3')(T_2 - T_4)}{R_2' + R_3 + R_4'} \\ T_5 = T_o + \frac{R_5'(T_4 - T_o)}{R_4' + R_5} \end{cases} \tag{31}$$

In figure 19 are shown the temperature profiles over time for the single-node model.

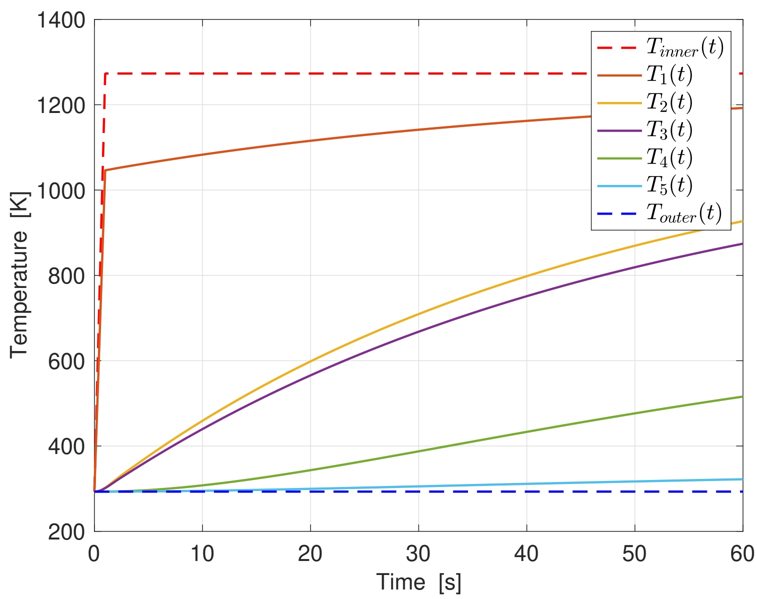


Figure 16: Temperature profiles: single node case

It can be noticed the initial ramp temperature, as imposed by the problem. Furthermore none of the temperatures reaches materials melting points. Temperatures are relatively high in the first conductive layer then decrease because of the contact resistance and of the the insulator.

To conclude the discussion of the results in figure 17 are reported the temperature profiles along the sections at nine different time instant. At a given time instant the temperature profile along a section decreases linearly depending on the material conductive resistance . It is just important to remember that when passing through a capacitor node there is a loss of heat and so a change in profile slope.

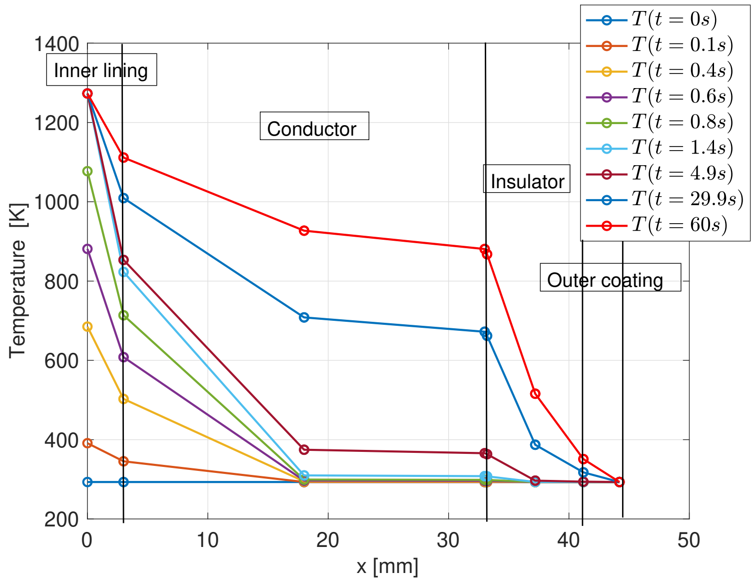


Figure 17: Temperature profiles along the sections : single node case

Two-nodes model

Let's now take into account the case in which the conductor and the insulator layers have two nodes placed at one-third and two-thirds of the length of the layer. The resulting physical model is represented in figure 18.

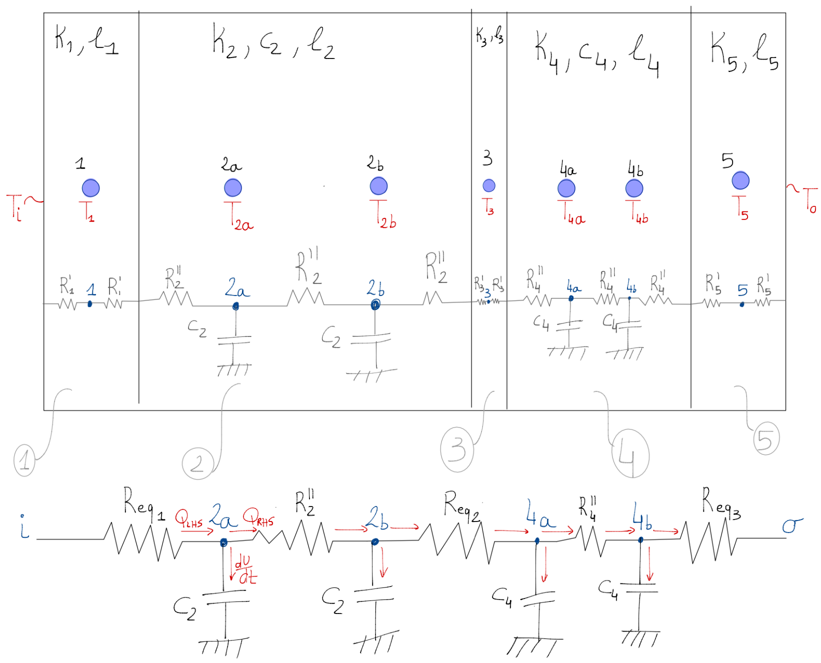


Figure 18: Physical Model: two-nodes model

The problem-solving approach is the same as the single-node model and so are reported only the main equations describing the mathematical model. As in the previous case thermal equilibrium has been assumed along the materials at $t_{initial} = 0\ s$. Considering $\mathbf{x} = [T_{2a}, T_{2b}, T_{4a}, T_{4b}]^T$ the state vector, the first ordinary differential equations are:

$$
\begin{cases}
\dot{T}_{2a} = \frac{R_2''(T_i - T_{2a}) - R_{eq_1}(T_{2a} - T_{2b})}{R_{eq_1} R_2'' \rho_2 A \frac{l_2}{2} c_2} \\
\dot{T}_{2b} = \frac{R_{eq_2}(T_{2a} - T_{2b}) - R_{2b}(T_{2b} - T_{4a})}{R_{eq_2} R_2'' \rho_2 A \frac{l_2}{2} c_2} \\
\dot{T}_{4a} = \frac{R_4''(T_{2b} - T_{4a}) - R_{eq_2}(T_{4a} - T_{4b})}{R_{eq_2} R_4'' \rho_4 A \frac{l_4}{2} c_4} \\
\dot{T}_{4b} = \frac{R_{eq_3}(T_{4a} - T_{4b}) - R_{4b}(T_{4b} - T_{out})}{R_{eq_3} R_4'' \rho_4 A \frac{l_4}{2} c_4}
\end{cases}
\tag{32}
$$

Once the state of the system has been retrieved, the temperature profiles at each node have been compute as follows:

$$
\begin{cases}
T_1 = T_i - \frac{R_1'(T_i - T_{2a})}{R_1 + R_2''} \\
T_3 = T_{4a} + \frac{(R_4'' + R_3')(T_{2b} - T_{4a})}{R_2'' + R_3 + R_4''} \\
T_5 = T_o + \frac{R_5'(T_{4b} - T_o)}{R_4'' + R_5}
\end{cases}
\tag{33}
$$

In figure 19 are shown the temperature profiles over time for the double-node model.

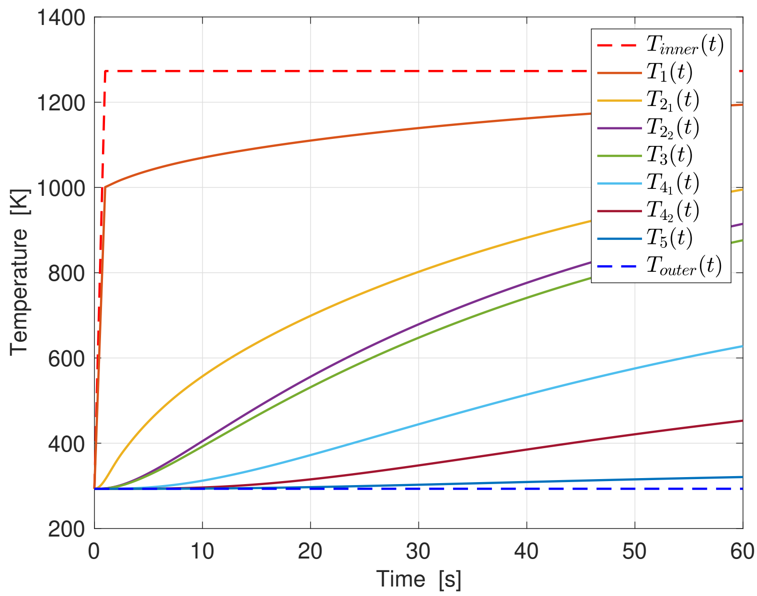


Figure 19: Temperature profiles: double node case

A couple of observations can be added to those already made before, comparing the two models. Having taken single nodes at the center of the material, their temperature will be contained among the other two resulting from the two nodes approach, which are instead at one third and two third of the material; the first node temperature has shifted downward since the capacitor node is closer.
Finally, to conclude the discussion of the results, in figure 20 are reported the temperature profiles along the sections at nine differet time instant.

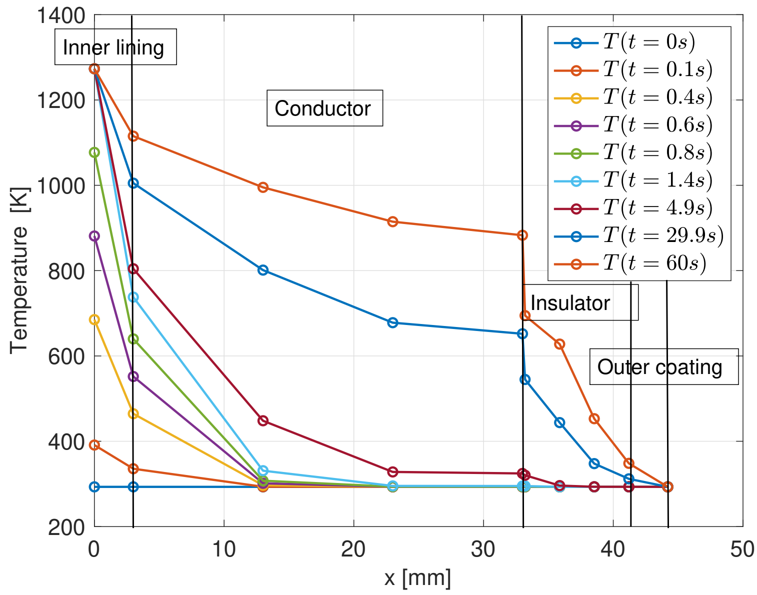


Figure 20: Temperature profiles along the sections : double node case

Exercise 4

The electro-mechanical system in Figure 21 is constituted by an electric circuit that drives a mechanical part. A voltage source $v(t) = v_0 \cos(\omega t)\ e^{-\beta t}$ is activated at $t = 0$. Assume a coefficient K_m such that $T(t) = K_m\ i(t)$ and $e(t) = K_m\ \Omega_1$, where T is the torque acting on the mechanical part, i the current in the circuit, and Ω_1 the angular velocity of the first disk. Considering $K_m = 20$; $R = 200$; $v_0 = 2$ V; $\omega = 5$ Hz; $\beta = 0.2$ Hz; $L = 2$ mH; $J_1 = 0.5$ kg m^2; $J_2 = 0.3$ kg m^2; $b = 0.1$ kg m^2 s^{-1}; $k = 0.5$ Nm, it is asked to:

1) Derive the mathematical model of the system;
2) Determine the system eigenvalues;
3) Select and motivate the most appropriate integration scheme;
4) Show the system response until $t = 30$ s;
5) Setup, discuss, and run a procedure to find the values of K_m and R that allow matching the second disk angular velocity profile sampled at 10 Hz given in the file `Profile.txt`

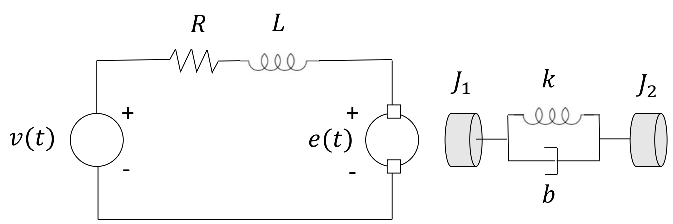


Figure 21: Electro-mechanical system.

Mathematical model

The mathematical model of the system is formulated considering the coupling of two different domains that have already been studied, therefore the main mathematical relations are the same used in exercise 1 and 2.

The state vector representing the dynamic of the system is:

$$\mathbf{x} = [i, \theta_1, \omega_1, \theta_2, \omega_2]$$

In order to properly select the damper model it has been done a dimensional analysis: in particular it can be noticed that $[\mathrm{b}] = [\frac{kgm^2}{s}]$ thus the damper will be modeled as linear.

The resulting mathematical model is:

$$\begin{cases} L\dot{I} = v(t) - Ri - K_m\dot{\theta}_1 \\ J_1\ddot{\theta}_1 = K_m i + k(\theta_2 - \theta_1) + b(\dot{\theta}_2 - \dot{\theta}_1) \\ J_2\ddot{\theta}_2 = -k(\theta_2 - \theta_1) - b(\dot{\theta}_2 - \dot{\theta}_1) \end{cases} \tag{34}$$

System 34 is composed by one first-order linear differential equation and two second-order linear differential equations that in order to be integrated in Matlab, has to be transformed into a set of first order differential equations. Using the state-space representation the linear system can be expressed as:

$$\mathbf{x} = A\mathbf{x} + \mathbf{b}$$

Specifically:

$$\begin{pmatrix} \dot{I} \\ \dot{\theta}_1 \\ \dot{\omega}_1 \\ \dot{\theta}_2 \\ \dot{\omega}_2 \end{pmatrix} = \begin{pmatrix} -\frac{R}{L} & 0 & -\frac{K_m}{L} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{K_m}{J_1} & -\frac{k}{J_1} & -\frac{b}{J_1} & \frac{k}{J_1} & \frac{b}{J_1} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{k}{J_2} & \frac{b}{J_2} & -\frac{k}{J_2} & -\frac{b}{J_2} \end{pmatrix} \begin{pmatrix} I \\ \theta_1 \\ \omega_1 \\ \theta_2 \\ \omega_2 \end{pmatrix} + \begin{pmatrix} \frac{v(t)}{L} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{35}$$

System eigenvalues and Integrator selection

In figure 22 are reported the system eigenvalues that differ by order of magnitude. This result was predictable, being an electro-thermal coupled system. Since the system is stiff, the selected integrator is *ode*15*s* which is a very efficient multi-step solver for stiff problems.

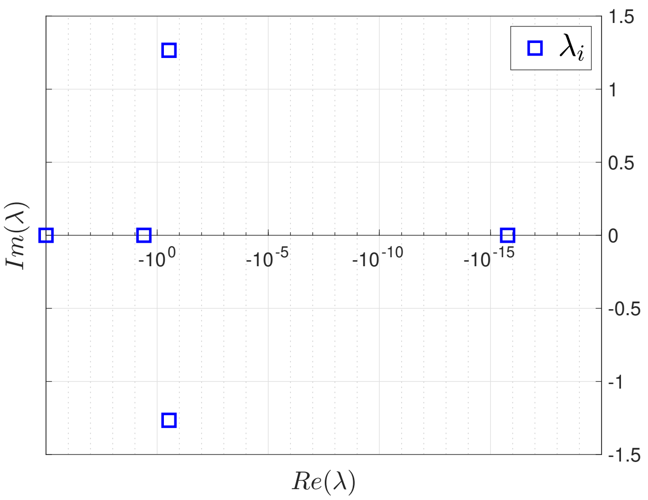


Figure 22: System eigenvalues

System response

In figure 23, 24 and 25 are reported the integration results with an initial state condition (at t = 0) $\mathbf{x}_0 = [0, 0, 0, 0, 0]^T$

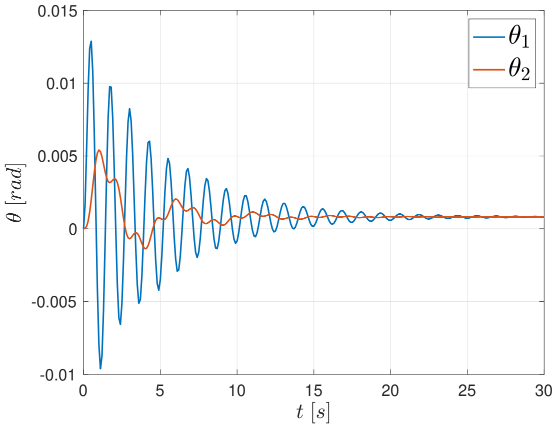


Figure 23: System response: angular rotation

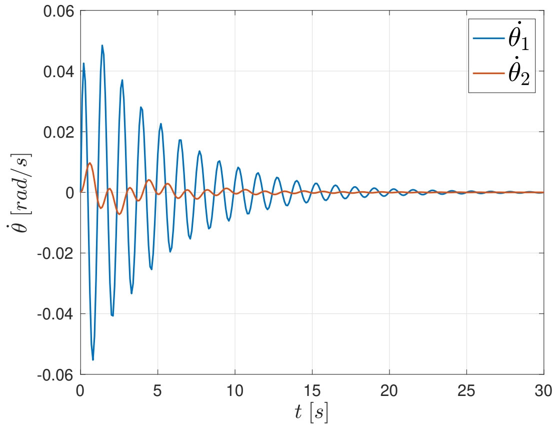


Figure 24: System response: angular velocity

With reference to the system response images, the evident damped oscillatory behavior is due to the forcing voltage source which is a decreasing sinusoidal exponential. Instead, the mechanical damper, imposes a condition on the relative speed of the two disks. In addition can be noticed the opposite phase between θ_1 and θ_2, due to the mechanical connection between the disks. After 25 seconds, the amplitude of the sine waves can be considered close to zero.

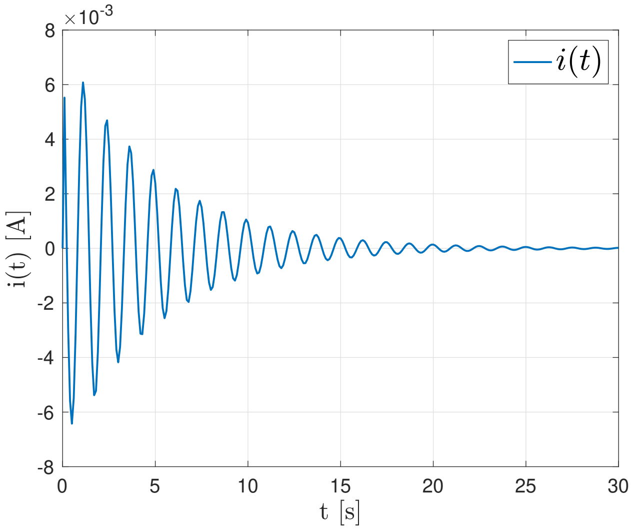


Figure 25: System response: current

Matching parameters

The procedure to find the parameters that allow the second disk angular velocity to be matched with sampled velocities is similar to that of exercise 1.

In order to find the fitting parameters it has been written an objective function to be minimized with the Matlab *fminunc* function. The objective function has been stated as the sum of the absolute errors between the velocities coming from the integrated results and the given samples:

$$\sum |\dot\theta_2 - \dot\theta_{profile}|$$

In order to increase the accuracy of fitting parameters it has been considered the relative error, but in the end it has been excluded since it gave rise to numerical errors due to the asymptotic convergence to zero of the sample velocities.

As shown in figure 26 the absolute error minimization leads to a good fitting. The resulting parameters are $K_m = 16.70678$ $[\frac{Nmrad}{A}]$ and R = 227.92970 [Ω].

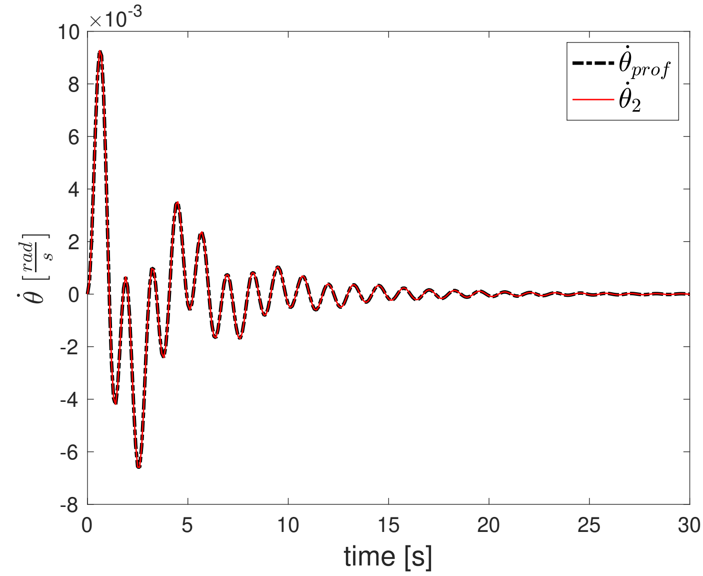


Figure 26: Velocities Comparison

In figure 27 is represented the absolute error made by using the found parameters.

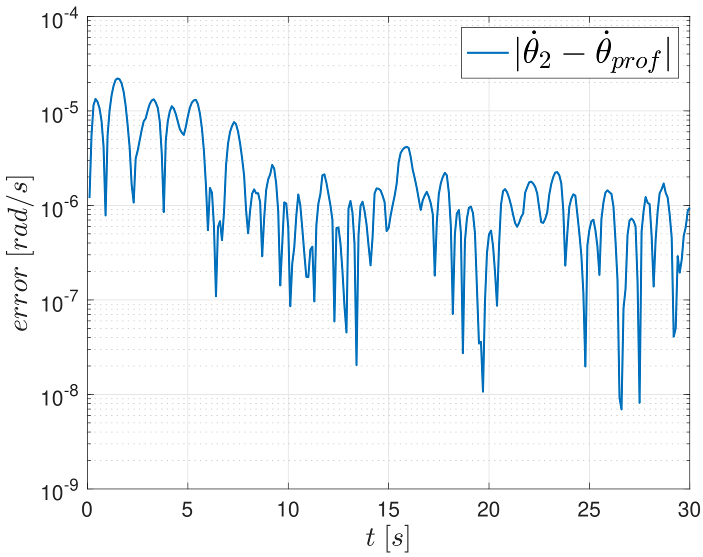


Figure 27: Absolute error between experimental data and solution

Exercise 5

The hydraulic system in Figure 28b is made of a tank, a pump, a check valve, a distribution valve, a filter and an heat exchanger, plus the lines. The heat exchanger is used to cool down an external system, having a temperature profile $T(t) = T_0 + k_T \cos(\omega t)$, and its wall is made up of three layers with different thermal properties, as depicted in Figure 28a. The pressure drop inside the heat exchanger can be modelled as a simplified Rayleigh flow, such that $P_{out} = e^{\dot{Q}/\kappa} P_{in}$. Assuming:

- Fluid: Incompressible fluid, $\rho = 1000$ kg/m^3, specific heat $c_w = 4186$ J/(kg $\cdot$ K)
- Lines: Coefficient of pressure drop across the check valve $k_{cv} = 2$, diameter of the lines $D = 20$ mm;
  - Branch T–1: Length $L_{T1} = 0.5$ m, friction factor $f_{T1} = 0.032$;
  - Branch 3–4: Length $L_{34} = 1.5$ m, friction factor $f_{34} = 0.032$;
  - Branch 5–6: Length $L_{56} = 2.7$ m, friction factor $f_{56} = 0.040$;
  - Branch 7–8: Length $L_{78} = 2.5$ m, friction factor $f_{78} = 0.028$;
  - Branch 9–T: Length $L_{9T} = 1$ m, friction factor $f_{9T} = 0.032$.
- Tank: Adiabatic tank with constant pressure $P_T = 0.1$ MPa;
- Pump:

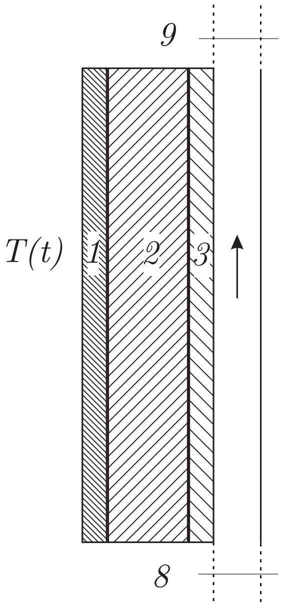
  Pistons:

  - Number: $N = 9$;
  - Diameter: $D_p = 0.7$ cm
  - Distance between the shaft of the shaft: $d_p = 1.5$ cm
  - Nominal pressure: 5 atm

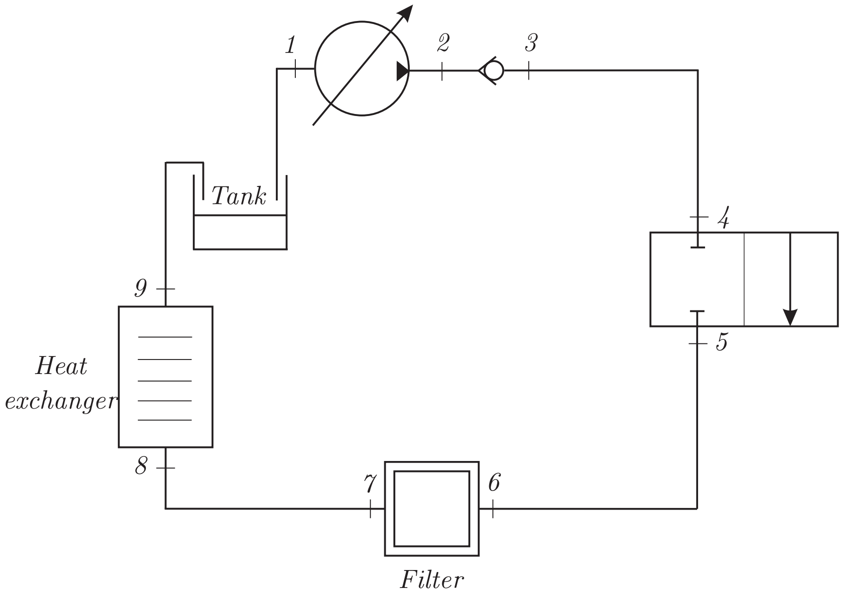
  Pilot piston:

  - Diameter: $D_k = 1$ cm
  - Control lever length: $l_c = 10$ cm
  - Maximum angle of the control plate: $\theta_{\max} = 20$ deg
  - Rotation speed: $n = 4000$ rpm
  - Equivalent mass: $m_k = 2$ kg
  - Pre-loaded force: $F_0 = 5$ N
  - Friction coefficient: $r_k = 1$ Ns/m
  - Diameter of the pipe: $d_k = 1$ mm
  - Pilot pipe head loss: $k_p = 2.5$
- Distributor: Coefficient of pressure drop across the distributor $k_d = 15$, diameter $d_o = 10$ mm. At $t_0 = 0$ s the valve is half open; it is fully open after $\Delta t = 2$ s.
- Cooler: Diameter of the pipe $D = 20$ mm; Heat flux $\dot{Q}_c = 100$ W.
- Filter: Coefficient of pressure drop across the filter $k_f = 35$, leaking coefficient $k_l = 2.5\%$[2];

[2] $Q_{leak} = k_l Q_{in}$



(a) Heat exchanger model close-up.



(b) Hydraulic system physical model

Figure 28: Thermo-hydraulic system. Assume any other missing data.

- Heat exchanger: Diameter of the pipe $D = 20$ mm, length of the pipe inside the exchanger: $L_e = 0.5$ m; planar exchanger with exchange area $A_e = 1000$ cm^2; First layer thermal properties: $k_1 = 395$ W/(m $\cdot$ K), $\ell_1 = 1$ cm, second layer thermal properties: $k_2 = 310$ W/(m $\cdot$ K), $\rho_2 = 8620$ kg/m^3, $c_2 = 100$ J/(kg $\cdot$ K), $\ell_2 = 2.5$ cm, third layer thermal properties: $k_3 = 125$ W/(m $\cdot$ K), $\ell_3 = 1$ cm, heat transfer coefficient with the fluid $h = 20$ W/(m$^2 \cdot$ K);
- $T_0 = 350$K, $k_T = 20$, $\omega = 5$ s^{-1}, $\kappa = 1000$ W;
- The temperatures are propagated instantaneously along the pipes;
- The heat exchanger layers have an initial temperature of 320 K.

It is asked to:

1) Derive a lumped-approach physical model for the heat exchanger;
2) Derive the mathematical model of the whole system;
3) Select and motivate the most appropriate integration scheme;
4) Show the system response until $t = 25$ s;
5) Discuss at least one possible way to modify the system in order to keep the fluid temperature as close as possible to 20 °C.

Heat exchanger physical model

The heat exchanger physical model has been done considering one single node per layer similarly to what has already been done in exercise 3. In accordance with the electrical similarity the physical model is represented in figure 29.

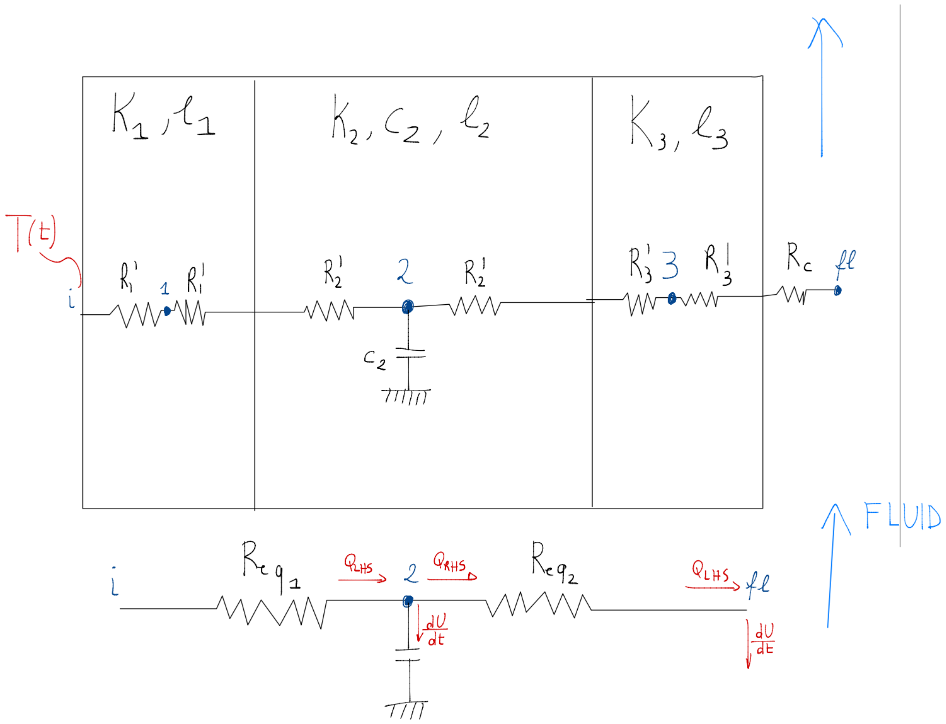


Figure 29: System eigenvalues

With respect to exercise 3, there is a convective heat exchange contribute to be considered. The equivalent resistances are:

$$R_{eq_1} = \frac{l_1}{k_1 A_e} + \frac{l_2}{2k_2 A_e}$$

$$R_{eq_2} = \frac{1}{hA_e} + \frac{l_3}{k_3 A} + \frac{l_2}{2k_2 A_e}$$

Main assumptions and approximations

In the modeling phase of the system, some approximations has been made to simplify, in a reasonable way, the equations. First of all the temperatures are propagated instantaneously along the pipes and the tank, since the variation over time is very slow. The pump has been modeled with unitary volumetric efficiency, which results in $Q_1 = Q_2$. The tank is open and the pressure remains constant with the atmospheric pressure. The fluid is incompressible. The fluid in the heat exchanger absorbs all the heat coming from the layers. The rotor is considered to be at steady state from the initial instant, with a rotational speed of 4000 rpm.

Mathematical model

In order to properly derive the mathematical model it has been analyzed which are the variables changing over time. With regard to the heat exchanger, the time-varying quantities are the temperature at node two and the fluid temperature, due to the capacitive material and the absorption of heat, coming from the layers, by the fluid. Instead, considering the displacement pump, and in particular its pilot piston which regulates the pressure cascade through the circuit, the other two variables will be the piston stroke and speed. Another quantity that varies over time is the volume of fluid in the tank due to the flow losses in the filter, but it will be computed after the integration. The resulting state variables are:

$$\mathbf{x} = \begin{pmatrix} T_2 \\ T_{fluid} \\ x_k \\ v_k \end{pmatrix} \tag{36}$$

Considering now the heat balances (as done in exercise 3) inside the heat exchanger and the mass-damper-spring equations of motion for the pilot piston, the derived mathematical model, expressed as a system of first-order differential equations is:

$$\begin{cases} \dot{T}_2 = \frac{T(t)-T_2}{R_{eq_1}\rho_2 A_e l_2 c_2} - \frac{T_2-T_{fluid}}{R_{eq_2}\rho_2 A_e l_2 c_2} \\ \dot{T}_{fluid} = \frac{T_2-T_{fluid}}{R_{eq_2} c_w \rho V} \\ \dot{x}_k = v_k \\ \dot{v}_k = \frac{1}{m_k}(P_k A_k - F_0 - K x_k - r_k v_k) \end{cases} \tag{37}$$

where V is the volume of the pipe inside the heat exchanger ($[m^3]$), K is the spring coefficient ($[\frac{N}{m}]$) computed in static condition as follow:

$$K = \frac{P_{nom}A_k - F_0}{c}$$

and P_k is the sensing pressure ([Pa]) inside the pilot piston and is calculated by computing the pressure cascade backward (which depends on the losses along the circuit), starting from P_T and ending at P_k, as follows:

$$\begin{cases} P_9 = P_T + \frac{1}{2} f_{9T}\rho \frac{L_{9T}}{D}\frac{Q_7^2}{A^2} \\ P_8 = \frac{P_9}{e^{\dot{Q}/k}} \\ P_7 = P_8 + \frac{1}{2} f_{78}\rho \frac{L_{78}}{D}\frac{Q_7^2}{A^2} \\ P_6 = P_7 + \frac{1}{2} k_f \rho \frac{Q_1^2}{A^2} \\ P_5 = P_6 + \frac{1}{2} f_{56}\rho \frac{L_{56}}{D}\frac{Q_1^2}{A^2} \\ P_4 = P_5 + \frac{1}{2} k_d \rho \frac{Q_1^2}{A_d(z)^2} \\ P_3 = P_4 + \frac{1}{2} f_{34}\rho \frac{L_{34}}{D}\frac{Q_1^2}{A^2} \\ P_2 = P_3 + \frac{1}{2} k_{cv} \rho \frac{Q_1^2}{A^2} \\ P_k = P_2 - \frac{1}{2} k_p \rho (\frac{v_k A_k}{A_{k_{pipe}}})^2 \end{cases} \tag{38}$$

where $\dot{Q}$ is the heat flow rate absorbed by the fluid and is equal to:

$$\frac{T_2 - T_{fluid}}{R_{eq_2}}$$

Q_1 is the flow rate until 6 (before entering the filter) and Q_7 is the flow rate from 7 (after the leakage) to the tank. They are computed as follows:

$$\begin{cases} Q_1 = nN_pA_ps \\ Q_7 = (1-k_l)Q_1 \end{cases} \tag{39}$$

where s is the stroke ([m]) and is computed as:

$$s = \frac{d_p}{l_c}(c_{max} - x_k)$$

The section of the distributor $A_d(z)$ has been modeled considering that after 2 seconds is fully open, instead between 0 and 2 seconds a radial variation of the parameter z is considered and modelled taking advantage of several trigonometric relationships resulting in:

$$\begin{cases} z = r_0\frac{t}{2} \\ \alpha = 2cos^{-1}(\frac{z}{r}) \\ A_d(t) = \pi r_d^2 - \frac{r_0^2}{2}(\alpha - sin(\alpha)) \end{cases} \tag{40}$$

One last thing to conclude the mathematical model is the change in fluid volume inside the tank which is related to the incoming and outgoing flows by the following balance:

$$\dot{V}_T = -Q_1(t) + Q_9(t) \tag{41}$$

Integrator selection
Analyzing the equations, it can be seen that these are non-homogeneous first-order differential equations. In order to simplify the selection of the integrator it has chosen to consider only the associated homogeneous ODEs and compute the eigenvalues of the matrix.

$$A = \begin{pmatrix} -\frac{1}{R_{eq_1}\rho_2 A_e l_2 c_2} - \frac{1}{R_{eq_2}\rho_2 A_e l_2 c_2} & \frac{1}{R_{eq_2}\rho_2 A_e l_2 c_2} & 0 & 0 \\ \frac{1}{R_{eq_2} c_w \rho V} & -\frac{1}{R_{eq_2} c_w \rho V} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{k}{m_k} & -\frac{r_k}{m_k} \end{pmatrix} \tag{42}$$

The associated eigenvalues are:

$$\lambda = \begin{pmatrix} -0.7079 \\ -0.0030 \\ -0.2500 + 21.6896i \\ -0.2500 - 21.6896i \end{pmatrix} \tag{43}$$

As we expected from a coupled thermal-mechanical (it's not hydraulic since the fluid is incompressible) system, we obtained two real eigenvalues for the thermal system and two complex conjugates eigenvalues for the mechanical system. It can be noticed that the order of magnitude is similar, thus the system is non stiff. An additional confirmation comes from experience: these two domains have relatively slow characteristic times. For these reasons the selected integrator is *ode45*.

System response
System response is obtained integrating the problem presented above and solving the algebraic equations. The initial state has been selected as:

$$\mathbf{x}_0 = [320, 293.15, 0, 0]^T$$

In particular, pilot piston dynamic is shown in figure 30 and 31. As can be easily noticed after less than five seconds, the piston stabilizes. In figure 32 is represented the the temperature profile over time of node 2 and of the fluid; the oscillatory behavior is explained by the outside temperature having an harmonic form; the fluid temperature increases over time but not linearly, since at a certain time the thermal equilibrium will be reached. In figure 33 and 34 are reported pressures over time for each node of the system. Pressures are oscillating since they are controlled by the piston dynamic which is oscillating too. In conclusion in figure 35 are reported are reported the volume flow rate variations: Q_7 is always smaller than Q_1 due to the leakage in the filter. After a few seconds it stabilizes at a certain value of flow rate. In figure 36 is reported how the volume of fluid inside the tank varies over time. There is no need to check if there could occur cavitation problems since the tank is open.

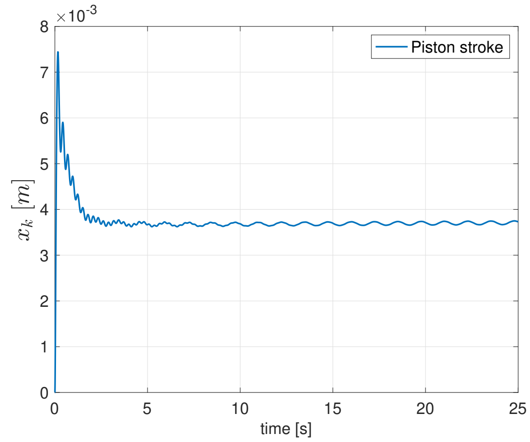


Figure 30: Piston stroke

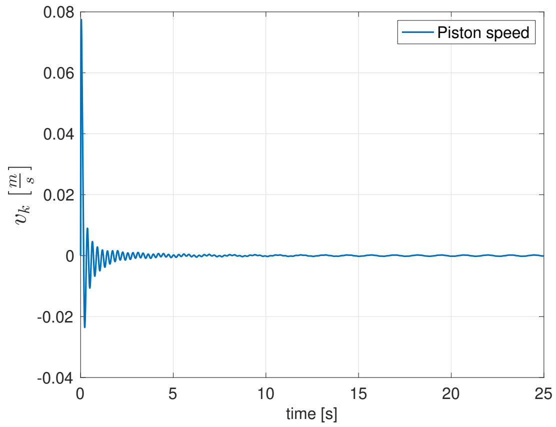


Figure 31: Piston speed

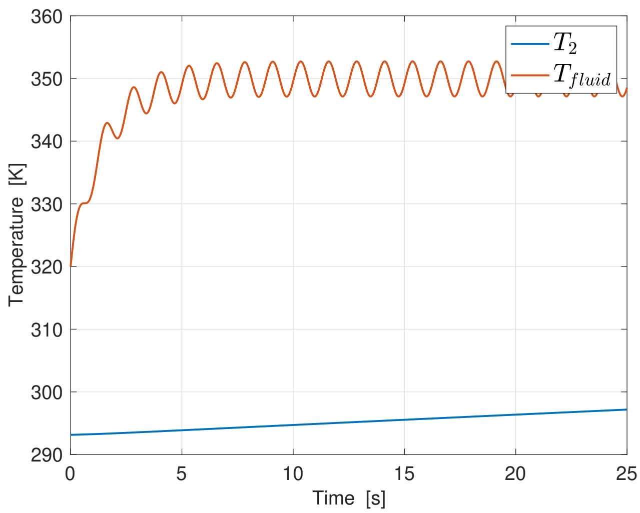


Figure 32: Heat exchanger temperature profiles

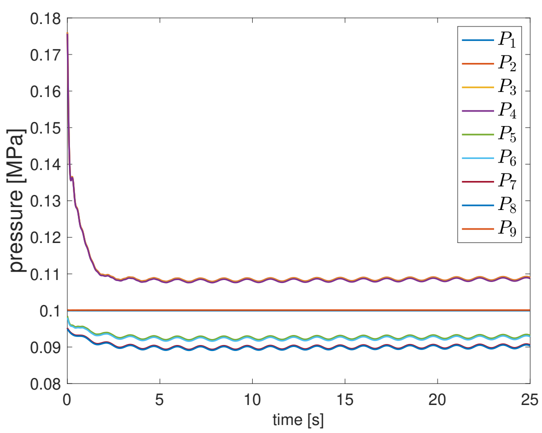


Figure 33: Pressures

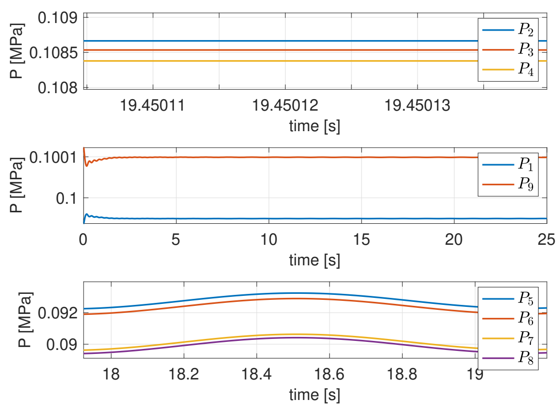


Figure 34: Pressures: zoom in

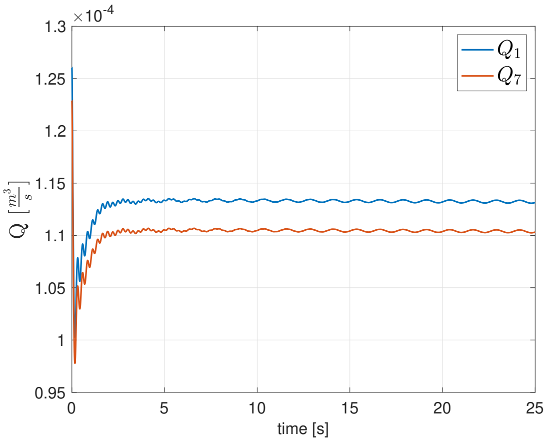


Figure 35: Volume flow rate

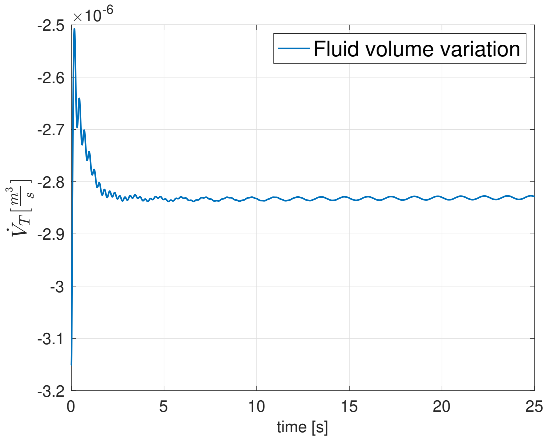


Figure 36: Tank volume variation

Cooling strategies

To keep the temperature around 20 degrees there are several strategies that can be applied, based on the context in which the system is used. For example a cooler can be installed so that an heat transfer is forced and the fluid temperature decreases. There are multiple cooling strategies that goes from a conductive dissipation to a convective dissipation.

Other changes that can be made to the system are the materials of the heat exchanger that can be can be replaced with more insulating materials, or even you can think about the change of a liquid with a higher specific heat so that it is more "resistant" to the increase in temperature. One possible change, although a bit extreme, concerns the tubes, trying to reduce the coefficients of friction.

Depending on what kind of needs you have, you can opt for one strategy over another, possibly reshape the system and study its behavior.

References

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[4] Francesco Topputo, lecture notes, PoliMi, 2022.

[5] Figure 1 and figure 2 taken from slides.

[6] Zirconia properties: https://www.matweb.com/search/DataSheet.aspx?MatGUID=05e29689151440d3904052baa788ae22&ckck=1

[7] A286 properties: https://www.hightempmetals.com/techdata/hitempA286data.php

[8] Inconel 718 properties: https://www.alloywire.it/products/inconel-718/