BASICS OF LORENTZ SYMMETRY ON SPACETIME AND FIELDS

In preparation of the course *Quantum Field Theory 1*

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Contents

1 Lorentz symmetry on spacetime

Let x^μ denote the vector $x^\mu = (t, \boldsymbol{x})$ in a four-dimensional spacetime with metric tensor $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. The Lorentz symmetry is defined as the group of linear coordinate transformations,

$$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu \,, \tag{1}$$

which leave invariant the quantity

$$x \cdot x = \eta_{\mu\nu} x^\mu x^\nu = t^2 - x^2 - y^2 - z^2 \,. \tag{2}$$

A group that acts on a space with coordinates $(t_1, \ldots, t_m; x_1, \ldots, x_n)$ and leaves invariant the quadratic form $(t_1 + \ldots + t_m)^2 - (x_1 + \ldots + x_n)^2$ is called the *orthogonal group* $\mathrm{O}(1,3)$, so the Lorentz group corresponds to $\mathrm{O}(m,n)$. The quadratic condition of Eq. (2) requires the matrix Λ to satisfy the identity

$$x' \cdot x' = \eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\sigma x^\sigma = \eta_{\rho\sigma} x^\rho x^\sigma = x \cdot x \tag{3}$$

for a generic x, so the relation

$$\eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \qquad \rightarrow \qquad \eta = \Lambda^T \eta \Lambda \tag{4}$$

must hold. Taking the determinant on both sides of the above equation, we obtain

$$|\det(\Lambda)|^2 = 1 \qquad \Rightarrow \qquad \det(\Lambda) = \pm 1 \,. \tag{5}$$

Without loss of generality, we can assume the condition $\det(\Lambda) = +1$. In fact, a general Lorentz transformation with $\det(\Lambda) = -1$ can always be written as the product of a transformation with $\det(\Lambda) = +1$ and a discrete transformation that reverses the sign of an odd number of coordinates (e.g. the *parity transformation*). Therefore, from now on we will consider that subgroup of $\mathrm{O}(1,3)$ characterized by the condition $\det(\Lambda) = +1$, i.e., the *special orthogonal group* $\mathrm{SO}(1,3)$. A Lorentz transformation with $\det(\Lambda) = +1$ is called a *proper Lorentz transformation.*

Consider the 00 component of the Eq. (4):

$$1 = \eta_{00} = \eta_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = \eta_{00} \Lambda^0{}_0 \Lambda^0{}_0 + \eta_{ij} \Lambda^i{}_0 \Lambda^j{}_0 = (\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^i{}_0)^2. \tag{6}$$

Since Λ is a real matrix, then $(\Lambda^i{}_0)^2 \geq 0$ for $i = 1, 2, 3$. It follows that

$$(\Lambda^0{}_0)^2 \geq 1 \qquad \longrightarrow \qquad \Lambda^0{}_0 \leq -1 \text{ or } \Lambda^0{}_0 \geq +1. \tag{7}$$

Hence, the proper Lorentz group $\mathrm{SO}(1,3)$ exhibits two disconnected components: one with $\Lambda^0{}_0 \leq -1$, referred to as *non-orthochronous*, and the other with $\Lambda^0{}_0 \geq +1$, known as *orthochronous*. Any non-orthochronous transformation can be decomposed into the product of an orthochronous transformation and a specific discrete inversion, such as $(t, x, y, z) \mapsto (-t, -x, -y, -z)$ or $(t, x, y, z) \mapsto (-t, -x, y, z)$, among others. Therefore, we can focus on proper orthochronous Lorentz transformations, denoted by $\Lambda \in \mathrm{SO}(1,3)_+$ (where the subscript $+$ means $\Lambda^0{}_0 \geq 1$).

As the Lorentz group constitutes a Lie group, a general Lorentz matrix Λ can be expressed as

$$\Lambda = e^\lambda \,, \qquad \lambda \in \mathfrak{so}(1,3) \,, \tag{8}$$

where λ denotes a generic component of the Lorentz algebra $\mathfrak{so}(1,3)$.

NB

Let $\mathfrak{g}$ any Lie algebra, and let $\mathcal{G}$ be its Lie group. The exponential map

$$\exp\colon\ \mathfrak{g} \to \mathcal{G} \,, \qquad \exp\colon\ \lambda \mapsto e^\lambda \,. \tag{9}$$

serves as a *diffeomorphism* between a neighborhood of $0 \in \mathfrak{g}$ (the identity element of the Lie algebra) and a neighborhood of $\mathbb{1} \in \mathcal{G}$ (the identity element of the Lie group). Put differently, any element $\lambda \in \mathfrak{g}$ generates a one-parameter subgroup of $\mathcal{G}$ through the exponential map

$$\exp_\lambda \colon \ \mathbb{R} \to \mathcal{G}\,, \qquad\qquad \exp_\lambda \colon \ t \mapsto e^{t\,\lambda}\,, \tag{10}$$

such that $\exp(\lambda) = \exp_\lambda(1)$. Conversely, any $\Lambda \in \mathcal{G}$ in the neighborhood of the identity element $\mathbb{1}$ of the group belongs to a one-parameter subgroup of $\mathcal{G}$, meaning that Λ can be expressed as $\Lambda = \exp_\lambda(1)$ within a proper neighborhood of $\mathbb{1}$ and for some $\lambda \in \mathfrak{g}$.

Why is this relevant for us? As previously discussed, $\mathrm{SO}(1,3)$ is not a connected group, given that ${\Lambda^0}_0$ is not defined within the interval $[-1,1]$. Consequently, there cannot exist a continuous path linking an element of the subset ${\Lambda^0}_0 \geq +1$ with another component of the subset ${\Lambda^0}_0 \leq -1$. Since the Lorentz matrix Λ must be in a neighborhood of $\mathbb{1}$, the subset ${\Lambda^0}_0 \geq +1$, i.e., $\mathrm{SO}(1,3)_+$, is the subgroup we need to focus on.

Having introduced the Lorentz algebra $\mathfrak{so}(1,3)$, our objective is to obtain a detailed description of it. We understand that by fully characterizing the Lorentz algebra, we can effectively utilize the exponential map to provide an explicit representation of the Lorentz group. To this end, let λ represent an infinitesimal transformation, defined as

$${\Lambda^\mu}_\nu = {\delta^\mu}_\nu + {\lambda^\mu}_\nu + \mathcal{O}(\lambda^2)\,. \tag{11}$$

Substituting this expansion into Eq.(4), we obtain

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\left[{\delta^\mu}_\rho + {\lambda^\mu}_\rho + \mathcal{O}(\lambda^2)\right]\left[{\delta^\nu}_\sigma + {\lambda^\nu}_\sigma + \mathcal{O}(\lambda^2)\right] = \eta_{\rho\sigma} + \lambda_{\rho\sigma} + \lambda_{\sigma\rho} + \mathcal{O}(\lambda^2)\,, \tag{12}$$

implying

$$\lambda_{\rho\sigma} = -\lambda_{\sigma\rho}\,. \tag{13}$$

Hence, $\lambda_{\mu\nu}$ is 4×4 real *antisymmetric* matrix, so it must depend on six independent parameters, i.e.

$$\lambda_{\mu\nu} = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ -b_1 & 0 & r_3 & -r_2 \\ -b_2 & -r_3 & 0 & +r_1 \\ -b_3 & r_2 & -r_1 & 0 \end{pmatrix}. \tag{14}$$

Therefore, the Lorentz algebra has dimension 6, requiring six generators. However, when Λ acts on a vector x^μ as in Eq.(1), it appears in the form ${\Lambda^\mu}_\nu$. Hence, it is more convenient to have a general expression of ${\lambda^\mu}_\nu = \eta^{\mu\sigma}\lambda_{\sigma\nu}$, which is not symmetric and corresponds to

$${\lambda^\mu}_\nu = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -r_3 & r_2 \\ b_2 & r_3 & 0 & -r_1 \\ b_3 & -r_2 & r_1 & 0 \end{pmatrix}. \tag{15}$$

Now we are free to fix the six parameters to define a basis of the Lorentz algebra:

$$\begin{aligned} {(J^1)^\mu}_\nu &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \qquad & {(K^1)^\mu}_\nu &= \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ {(J^2)^\mu}_\nu &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad & {(K^2)^\mu}_\nu &= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{16}$$

$$(J^3)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (K^3)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

These matrices are referred to as the *generators* of the Lorentz group. The commutation rules of the algebra are given by

$$[J^i, J^j] = i\epsilon^{ijk} J^k\,, \qquad [K^i, K^j] = -i\epsilon^{ijk} J^k\,, \qquad [J^i, K^j] = i\epsilon^{ijk} K^k\,, \tag{17}$$

where ϵ^{ijk} denotes the totally antisymmetric tensor with $\epsilon^{123} = +1$. It might seem unconventional to have six complex matrices rather than real ones. However, this is merely a matter of convention. In reality, we can express $\lambda^\mu{}_\nu$ as a linear combination of the aforementioned matrices as

$$\lambda^\mu{}_\nu = i\sum_{i=1}^{3} \alpha^i \left(J^i\right)^\mu{}_\nu + i\sum_{i=1}^{3} \beta^i \left(K^i\right)^\mu{}_\nu\,, \qquad \alpha_i, \beta_i \in \mathbb{R}\,, \tag{18}$$

thus ensuring $\lambda^\mu{}_\nu$ is indeed real. Now, what is the physical significance of these six generators? As for J_i, they represent the generators of *spatial rotations*, i.e., $J^i \in \mathfrak{so}(3)$, akin to the three components of angular momentum. Concerning K^i, they stand for the generators of *boosts*. A couple of examples can elucidate this further.

Example:

Let's explore the implications of fixing $\lambda = i\eta K^1$, meaning we set the parameters of Eq. (18) to $\boldsymbol{\alpha} = (0,0,0)$ and $\boldsymbol{\beta} = (\eta, 0, 0)$. In this scenario, Λ takes the form (we leave μ, ν indices understood)

$$\Lambda(\eta) = e^\lambda = e^{i\eta K^1} = \sum_{n=0}^{\infty} \frac{\eta^n (iK^1)^n}{n!} = \mathbb{1}_4 + \sum_{n=1}^{\infty} \frac{\eta^{2n}(iK_1)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\eta^{2n+1}(iK^1)^{2n+1}}{(2n+1)!}\,. \tag{19}$$

Given that

$$(iK^1)^{2n} = \begin{pmatrix} \mathbb{1}_2 & \mathbb{0}_2 \\ \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix}, \qquad (iK^1)^{2n+1} = iK^1\,, \tag{20}$$

we derive

$$\begin{aligned} \Lambda(\eta) &= \mathbb{1}_4 + \begin{pmatrix} \mathbb{1}_2 & \mathbb{0}_2 \\ \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} \sum_{n=1}^{\infty} \frac{\eta^{2n}}{(2n)!} + iK^1 \sum_{n=0}^{\infty} \frac{\eta^{2n+1}}{(2n+1)!} \\ &= \mathbb{1}_4 + \begin{pmatrix} \mathbb{1}_2 & \mathbb{0}_2 \\ \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} (\cosh\eta - 1) + iK^1 \cdot \sinh\eta \\ &= \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{21}$$

The matrix $\Lambda(\eta)$ thus represents a boost along the x-axis, where the parameter η corresponds to the *rapidity* of the boost. Recalling that $v = \tanh\eta$ and $\gamma = (1 - v^2)^{-1/2} = \cosh\eta$, we can explicitly rewrite Λ as

$$\Lambda(\eta) = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{22}$$

Since $v < c = 1$, then $\Lambda^0{}_0 = \gamma \geq 1$ as expected. Additionally, note that $\det(\Lambda) = 1$.

Next, let's choose $\lambda = -i\theta J^3$, which corresponds to $\boldsymbol{\alpha} = (0, 0, -\theta)$ and $\boldsymbol{\beta} = (0, 0, 0)$. Following similar steps as before:

$$\Lambda(\theta) = e^\lambda = e^{-i\theta J_3} = \sum_{n=0}^{\infty} \frac{(-i\theta)^n (J^3)^n}{n!} = \mathbb{1}_4 + \sum_{n=1}^{\infty} \frac{(-i\theta)^{2n}(J^3)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-i\theta)^{2n+1}(J^3)^{2n+1}}{(2n+1)!}\,. \tag{23}$$

Considering that

$$(J^3)^{2n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (J^3)^{2n+1} = J_3, \tag{24}$$

we obtain

$$
\begin{aligned}
\Lambda(\theta) &= \mathbb{1}_4 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sum_{n=1}^{\infty} \frac{(-)^n \theta^{2n}}{(2n)!} - iJ^3 \sum_{n=0}^{\infty} \frac{(-)^n \theta^{2n+1}}{(2n+1)!} \\
&= \mathbb{1}_4 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (\cos\theta - 1) - iJ^3 \sin\theta \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .
\end{aligned}
\tag{25}
$$

This matrix represents a counterclockwise rotation by an angle θ around the z-axis. Again, the conditions ${\Lambda^0}_0 \geq 1$ and $\det(\Lambda) = 1$ hold.

Another insight we can draw from these examples pertains to the space of real $N \times N$ matrices, denoted $M_N(\mathbb{R})$, which can be thought as the Euclidean space $\mathbb{R}^{N^2}$. In this representation, each matrix $A \in M_N(\mathbb{R})$ corresponds to a point $\boldsymbol{x} \in \mathbb{R}^{N^2}$. For instance, for a matrix $A \in M_2(\mathbb{R})$ given by

$$
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} , \tag{26}
$$

we can view A as a point $\boldsymbol{x} \in \mathbb{R}^{N^2}$, such that $\boldsymbol{x} = (A_{11}, A_{12}, A_{21}, A_{22})$. This isomorphism allows us to define a *distance* between two matrices $A, B \in M_N(\mathbb{R})$ as

$$
\mathrm{d}(A,B)\colon M_N(\mathbb{R}) \times M_N(\mathbb{R}) \mapsto \mathbb{R} , \qquad \mathrm{d}(A,B) \overset{\text{def}}{=} \sqrt{\sum_{i,j=1}^{N} |A_{ij} - B_{ij}|^2} . \tag{27}
$$

Using this definition, we can compute the distance between the $\Lambda(\eta)$ matrix of Eq. (21) and the identity:[1]

$$
\mathrm{d}(\Lambda(\eta), \mathbb{1}) = \sqrt{\sum_{i,j=1}^{4} |\Lambda_{ij}(\eta) - \mathbb{1}_{ij}|^2} = \sqrt{2(\cosh\eta - 1)^2 + 2\sinh^2\eta} = 2\sqrt{2} \left|\sinh\frac{\eta}{2}\right| \sqrt{\cosh\eta} \underset{\eta\to\infty}{\to} \infty . \tag{28}
$$

This outcome implies that a Lorentz matrix Λ can be arbitrarily distant from the identity, rendering $\mathrm{SO}(1,3)_+$ a *non-compact group.* This non-compactness arises solely from the generators of the boosts K^i. Indeed, computing the distance from the identity of the rotation matrix in Eq. (25), we find

$$
\mathrm{d}(\Lambda(\theta), \mathbb{1}) = \sqrt{\sum_{i,j=1}^{N} |\Lambda_{ij}(\theta) - \mathbb{1}_{ij}|^2} = \sqrt{2(\cos\theta - 1)^2 + 2\sin^2\theta} = 2\sqrt{1 - \cos\theta} \leq 4 , \tag{29}
$$

which aligns with expectations since J^i are the generators of $\mathrm{SO}(1,3)$, a compact group.

These two examples also help clarify the optimal way to express the linear expansion of Eq.(18). From the first example, we observe that the coefficients β^i in Eq.(18) correspond to the rapidities η^i along the three axes. Similarly, in the second example, each α^i represents the angle of rotation $-\theta^i$ around the i-axis, with a negative sign to denote counterclockwise rotation. Hence, the most general expression for a matrix $\lambda \in \mathfrak{so}(1,3)$ is given by

$$
{\lambda^\mu}_\nu = i \sum_{i=1}^{3} \alpha^i {(J^i)^\mu}_\nu + i \sum_{i=1}^{3} \beta^i {(K^i)^\mu}_\nu = {\big(-i\boldsymbol{\theta}\cdot\boldsymbol{J} + i\,\boldsymbol{\eta}\cdot\boldsymbol{K} \big)^\mu}_\nu . \tag{30}
$$

Consequently, the most general expression for a Lorentz transformation $\Lambda \in \mathrm{SO}(1,3)_+$ is represented by

$$
{\Lambda^\mu}_\nu = {\left[e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J} + i\boldsymbol{\eta}\cdot\boldsymbol{K}} \right]^\mu}_\nu . \tag{31}
$$

It is worth noting that $(J^i)^\dagger = J^i$, while $(K^i)^\dagger = -K^i$, indicating that not all generators of $\mathrm{SO}(1,3)_+$ are Hermitian. This accounts for the *non-unitary* nature of our representation (of dimension 4) of the Lorentz group.

[1] Here, we are considering $\Lambda(\eta) \in M_4(\mathbb{R})$, so i and j represent indices ranging from 1 to 4. It's important not to confuse them with the spatial components of Lorentz indices μ and ν, that run from 1 to 3.

2 Lorentz symmetry on fields and the Poincarè group

2.1 General properties

In the preceding section, we explored the action of the Lorentz group on spacetime, introduced the Lorentz algebra $\mathfrak{so}(1,3)$, and established a basis in Eq. (16) along with the corresponding commutation rules in Eq. (17). Our next focus is on introducing *fields*, which is the subject of this subsection. A field, denoted as $\phi = \phi(x)$, is defined at every point in spacetime, making it a function of $x^\mu = (t, \boldsymbol{x})$. It represents a system with an infinite number of degrees of freedom, at least one for each point $\boldsymbol{x}$ in space.

In particle physics, we demand that the Lagrangian density $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$ remains invariant under Lorentz transformations. Consequently, under a Lorentz transformation, the field must undergo a specific transformation dictated by a representation of the Lorentz group. Specifically, let ϕ be an N-component multiplet, i.e., $\phi(x) = (\phi^1(x), \ldots, \phi^N(x))$. In accordance with our requirements, under a Lorentz transformation $x' = \Lambda \cdot x$, ϕ must transform in a representation $\mathcal{R}$ of dimension N of the Lorentz group, as shown by

$$\phi'^{\,i}(x') = [\Lambda_\mathcal{R}]^i{}_j \, \phi^j(x) = \left[e^{-i\,\boldsymbol{\theta}\cdot\boldsymbol{J}_\mathcal{R} + i\,\boldsymbol{\eta}\cdot\boldsymbol{K}_\mathcal{R}}\right]^i{}_j \phi^j(x)\,. \tag{32}$$

Here, $\boldsymbol{J}_\mathcal{R}$ and $\boldsymbol{K}_\mathcal{R}$ are the generators of the rotations $\boldsymbol{J}$ and boosts $\boldsymbol{K}$, respectively, in the representation $\mathcal{R}$.

In the previous subsection, we introduced a specific representation of the Lorentz algebra $\mathfrak{so}(1,3)$, known as the *vector representation*, wherein a general component $\lambda \in \mathfrak{so}(1,3)$ manifests as a 4×4 matrix. From a formal standpoint, λ is an abstract element of $\mathfrak{so}(1,3)$, while its representation $\lambda_\mathcal{R}$ takes the form of an $N \times N$ matrix. Importantly, the description of the algebra derived from Eq. (17) is entirely general, as the commutation rules are inherently linked to the algebra itself and remain unaffected by the chosen representation. Consequently, even if we were to adopt a different representation of the basis (16) with $N \neq 4$, Eq. (17) would continue to hold.

The commutation rules presented in Eq. (17) may suggest the existence of two rotation algebras. However, while the J_i generators belong to the algebra $\mathfrak{so}(3)$, the same cannot be said for the K_i generators. If they did, we would expect commutation rules of the form $[K^i, K^j] = i\epsilon^{ijk}K^k$ and $[J^i, K^j] = 0$. One might imagine finding a new basis of matrices $(J')^i$ and $(K')^i$, which are real combinations of J^i and K^i, defined as

$$(J')^i = \sum_{k=1}^{3} \left(a^i{}_k \, J^k + b^i{}_k \, K^k\right), \qquad (K')^i = \sum_{k=1}^{3} \left(c^i{}_k \, J^k + d^i{}_k \, K^k\right), \tag{33}$$

such that

$$[(J')^i, (J')^j] = i\epsilon^{ijk}(J')^k\,, \qquad [(K')^i, (K')^j] = i\epsilon^{ijk}(K')^k\,, \qquad [(J')^i, (K')^j] = 0\,. \tag{34}$$

However, it is easy to see that this is not possible unless we allow the coefficients a, b, c, d to be complex numbers. In principle, this violates the requirement that the Lorentz group and its algebra be real. Yet, by admitting complex combinations of generators, we find that conditions like those in Eq. (34) become feasible. In this regard, we need to introduce a new concept known as the *complexification of the algebra* $\mathfrak{so}(1,3)$.

Definition. *Let $\mathfrak{g}$ be a real algebra generated by the basis $\{T^1, \ldots, T^N\}$. Suppose that the generators are still linearly independent even considering a complex linear combination, i.e.*

$$\sum_{i=1}^{N} \alpha^i \, T^i = 0 \qquad \Longrightarrow \qquad \mathbb{C} \ni \alpha^i = 0 \;\; \forall \, i \in [1, N]\,. \tag{35}$$

We define the complexification of the algebra $\mathfrak{g}$, that we call $\tilde{\mathfrak{g}}$, as

$$\tilde{\mathfrak{g}} \stackrel{\text{def}}{=} \left\{ T' = \sum_{i=1}^{N} \alpha^i \, T^i\,, \text{ with } \alpha^i \in \mathbb{C} \right\}. \tag{36}$$

In Eq. 30, we initially assumed $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ to be real. However, if we consider them as complex quantities, we find that the six generators of $\mathfrak{so}(1,3)$ remain linearly independent. Consequently, we can introduce the algebra $\widetilde{\mathfrak{so}}(1,3)$ as defined in Eq. (36). To construct $\widetilde{\mathfrak{so}}(1,3)$, we define six new generators:

$$J^{i,+} \stackrel{\text{def}}{=} \frac{J^i + iK^i}{2}\,, \qquad J^{i,-} \stackrel{\text{def}}{=} \frac{J^i - iK^i}{2}\,, \tag{37}$$

according to which the commutation rules of $\widetilde{\mathfrak{so}}(1,3)$ are given by:

$$[J^{i,+}, J^{j,+}] = i\epsilon^{ijk}J^{k,+}\,, \qquad [J^{i,-}, J^{j,-}] = i\epsilon^{ijk}J^{k,-}\,, \qquad [J^{i,+}, J^{j,-}] = 0\,. \tag{38}$$

This results in two independent copies of the angular momentum algebra, where generators within each copy commute. Hence, we can represent $\widetilde{\mathfrak{so}}(1,3)$ as the direct sum of two (complexified) rotation algebras, $\widetilde{\mathfrak{su}}(2)$, i.e.,

$$\widetilde{\mathfrak{so}}(1,3) = \widetilde{\mathfrak{su}}(2) \oplus \widetilde{\mathfrak{su}}(2). \tag{39}$$

It is worth noting that, at the algebraic level, $\widetilde{\mathfrak{su}}(2)$ is isomorphic to $\mathfrak{so}(3)$: the generators of $\widetilde{\mathfrak{su}}(2)$ and those of $\mathfrak{so}(3)$ are the same objects but in different representations (this is not true at the level of the groups). This insight makes it straightforward to identify the proper representation of dimension N of the Lorentz group in Eq. (32): we simply require $\boldsymbol{J}^+$ in its representation $\mathcal{R}^+$ and $\boldsymbol{J}^-$ in its representation $\mathcal{R}^-$. Before delving into the specifics of $\mathcal{R}^+$ and $\mathcal{R}^-$, it is beneficial to explore an example related to the $\widetilde{\mathfrak{su}}(2)$ algebra.

Example:

Consider the $\widetilde{\mathfrak{su}}(2)$ algebra, and let $\{J^1, J^2, J^3\}$ be the generators. The operator $\boldsymbol{J}^2$ commutes whit all the generators, i.e., $[\boldsymbol{J}^2, J^i] = 0\ \forall\, i$. Therefore, we can choose one of the generators, let's say J^3, and construct a basis in which both J^3 and $\boldsymbol{J}^2$ are diagonal. In this basis, denoted as $|j, m_j\rangle$, the action of the operators is given by

$$\begin{cases} \boldsymbol{J}^2\,|j,m_j\rangle = j(j+1)\,|j,m_j\rangle \\ J^3\,|j,m_j\rangle = m_j\,|j,m_j\rangle \end{cases} \tag{40}$$

with $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, ...\}$ and $m_j \in \{-j, -j-1, ..., j-1, j\}$.

Physically, $\boldsymbol{J}^2$ represents the square of the angular momentum $\boldsymbol{J}$, while J^3 denotes the projection of $\boldsymbol{J}$ along the z axis. Mathematically, considering that m_j can take $(2j+1)$ different values for any fixed j, we can view it as a $(2j+1)\times(2j+1)$ matrix, serving as a representation of dimension $(2j+1)$ for the generator J^3. Thus, j determines the dimension of the representation of $\widetilde{\mathfrak{su}}(2)$, which corresponds to $(2j+1)$.

Based on the example above, the dimension of the representation of $\widetilde{\mathfrak{so}}(1,3)$ is determined by the dimension of the representation $\mathcal{R}^-$ of the first $\widetilde{\mathfrak{su}}(2)$ algebra, denoted as $(2j^-+1)$, multiplied by the dimension of the representation $\mathcal{R}^+$ of the second $\widetilde{\mathfrak{su}}(2)$ algebra, denoted as $(2j^++1)$. In the literature, a representation of $\widetilde{\mathfrak{so}}(1,3)$ is often labeled as $(\boldsymbol{J}^-, \boldsymbol{J}^+)$, thus yielding

$$\dim[(\boldsymbol{J}^-, \boldsymbol{J}^+)] = (2j^-+1)(2j^++1). \tag{41}$$

Now, let's delve into some important representations.

(0,0)

In this scenario, we have $j^- = j^+ = 0$. Consequently, $\boldsymbol{J}^\pm = 0$, leading to $\dim[(\mathbf{0},\mathbf{0})] = 1$. This representation is commonly referred to as the *scalar representation*, as the field transforms according to

$$\phi'(x') = \phi(x)\,, \tag{42}$$

which signifies transformation akin to a scalar. The field ϕ corresponds to spin 0 particles.

Example:

Let's consider a scalar field ϕ representing the temperature T of a system at a given point P. Suppose there are two observers, A and B, aiming to measure the temperature at point P. Additionally, assume that observer B is moving with a constant velocity $\boldsymbol{v}$ relative to A.

In the frame of observer A, the point P is identified by the spacetime vector x^μ, and thus A measures the temperature as $\phi(x) = T_A(P)$. Similarly, in observer B's frame, the point P is identified by the vector x'^μ, leading to a temperature measurement of $\phi'(x') = T_B(P)$. Notice that observer A employs his own function ϕ to express the temperature, while observer B uses his function ϕ'. However, both must measure the same temperature, as it is an intrinsic property of point P and remains invariant across observers. Hence, we conclude that $T_A(P) \equiv T_B(P)$, or equivalently, $\phi'(x') = \phi(x)$.

Between observers A and B there is a Lorentz transformation involved, more properly a boost $\Lambda(\boldsymbol{v})$ such that $x'^\mu = [\Lambda(\boldsymbol{v})]^\mu{}_\nu x^\nu$. If observer A chooses to measure in the frame of B, he simply needs to express his field $\phi(x)$ in terms of x'^μ, i.e., $x^\mu = [\Lambda(\boldsymbol{v})^{-1}]^\mu{}_\nu x'^\nu$. This yields:

$$\phi(x) = \phi\left(\Lambda(\boldsymbol{v})^{-1} \cdot x'\right) = \phi'(x')\,. \tag{43}$$

Thus, we've derived the transformation of a scalar field as described in Eq. (42).

$(\tfrac{\mathbf{1}}{\mathbf{2}}, \mathbf{0})$

This representation has $j^- = 1/2$ and $j^+ = 0$, resulting in a dimension $\dim[(\tfrac{\mathbf{1}}{\mathbf{2}}, \mathbf{0})] = 2$, which gives rise to spin 1/2 particles. The field that transforms under a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ in this representation holds significant importance in theoretical physics and is termed the *left-handed Weyl spinor*, labeled as ψ_L. Denoting Λ_L as the 2×2 matrix belonging to $(\tfrac{\mathbf{1}}{\mathbf{2}}, \mathbf{0})$, we have:

$$\psi'_L(x') = \Lambda_L, \psi_L(x). \tag{44}$$

Now, let's explicitly express Λ_L in terms of the generators of $\widetilde{\mathfrak{so}}(1,3)$. Since $\boldsymbol{J}^-$ resides in the fundamental representation of $\widetilde{\mathfrak{su}}(2)$ ($\tfrac{\mathbf{1}}{\mathbf{2}}$ representation), it can be written as

$$\boldsymbol{J}^- = \frac{\boldsymbol{\sigma}}{2}. \tag{45}$$

On the other hand, $\boldsymbol{J}^+$ is in the trivial representation ($\mathbf{0}$ representation), hence

$$\boldsymbol{J}^+ = 0. \tag{46}$$

From these, we derive the expressions for $\boldsymbol{J}$ and $\boldsymbol{k}$ as follows:

$$\begin{aligned} \boldsymbol{J} &= \boldsymbol{J}^+ + \boldsymbol{J}^- = \frac{\boldsymbol{\sigma}}{2}, \\ \boldsymbol{k} &= -i(\boldsymbol{J}^+ - \boldsymbol{J}^-) = i\frac{\boldsymbol{\sigma}}{2}. \end{aligned} \tag{47}$$

Substituting these into Eq. (32), we obtain

$$\psi'_L(x') = \Lambda_L\, \psi_L(x) = \exp\left\{(-i\,\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right\}\, \psi_L(x). \tag{48}$$

$(\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}})$

This representation functions similarly to $(\tfrac{\mathbf{1}}{\mathbf{2}}, \mathbf{0})$, but with the roles of $\boldsymbol{J}^+$ and $\boldsymbol{J}^-$ reversed. Thus, we have $j^- = 0$, $j^+ = 1/2$, and $\dim[(\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}})] = 2$. A field of this type, which is the "right" version of ψ_L, is known as a *right-handed Weyl spinor* and is denoted as ψ_R. Let Λ_R be the 2×2 matrix belonging to $(\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}})$, then

$$\psi'_R(x') = \Lambda_R\, \psi_R(x)\,. \tag{49}$$

In this scenario,

$$\boldsymbol{J}^- = 0\,, \qquad \boldsymbol{J}^+ = \frac{\boldsymbol{\sigma}}{2}\,, \tag{50}$$

thus yielding

$$\begin{aligned} \boldsymbol{J} &= \boldsymbol{J}^+ + \boldsymbol{J}^- = \frac{\boldsymbol{\sigma}}{2}\,, \\ \boldsymbol{k} &= -\,i(\boldsymbol{J}^+ - \boldsymbol{J}^-) = -i\frac{\boldsymbol{\sigma}}{2}\,, \end{aligned} \tag{51}$$

which leads to the transformation equation for ψ_R:

$$\psi'_R(x') = \Lambda_R\, \psi_R(x) = \exp\left\{(-i\,\boldsymbol{\theta} + \boldsymbol{\eta})\cdot\frac{\boldsymbol{\sigma}}{2}\right\}\,\psi_R(x)\,. \tag{52}$$

Now, with left- and right-handed Weyl spinors defined, we can examine their relation. Firstly, note that $\sigma^2\Lambda_L^*\sigma^2 = \Lambda_R$, since[2]

$$\begin{aligned}\sigma^2\Lambda_L^*\sigma^2 &= \sigma^2\exp\left\{(i\,\boldsymbol{\theta} - \boldsymbol{\eta})\cdot\frac{\boldsymbol{\sigma}^*}{2}\right\}\sigma^2 = \sum_{n=0}^{\infty}(i\theta^i + \eta^i)^n\,\frac{\sigma^2[(\sigma^i)^*]^n\sigma^2}{2^n}\\ &= \sum_{n=0}^{\infty}(i\theta^i + \eta^i)^n\,\frac{[\sigma^2(\sigma^i)^*\sigma^2]^n}{2^n} = \sum_{n=0}^{\infty}(i\theta^i + \eta^i)^n\,\left(-\frac{\sigma^i}{2}\right)^n\\ &= \sum_{n=0}^{\infty}(-i\theta^i - \eta^i)^n\,\left(\frac{\sigma^i}{2}\right)^n = \exp\left\{(-i\,\boldsymbol{\theta} - \boldsymbol{\eta})\cdot\frac{\boldsymbol{\sigma}}{2}\right\}\\ &= \Lambda_R\,.\end{aligned} \tag{53}$$

We then utilize this property to observe that, for a spinor ψ_L, the field $\sigma^2\psi_L^*$ must transform as a right-handed Weyl spinor:

$$\sigma^2[\psi_L(x)]^* \mapsto \sigma^2[\psi'_L(x')]^* = \sigma^2[\Lambda_L\psi_L(x)]^* = \sigma^2\Lambda_L^*\sigma^2\sigma^2\psi_L^*(x) = \Lambda_R[\sigma^2\psi_L^*(x)]\,. \tag{54}$$

Thus, $\sigma^2\psi_L^*(x) \in (\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}})$. It is natural to introduce an operation that transforms ψ_L into its corresponding right-handed Weyl spinor, which we call ψ_R. This operation is *charge conjugation*, defined as:

$$\psi_L^c(x) \stackrel{\text{def}}{=} i\sigma^2\psi_L^*(x) \equiv \psi_R(x)\,. \tag{55}$$

We introduce an extra i to ensure $[\psi_L^c(x)]^c = \psi_L(x)$, which also requires that $\psi_R^c(x)$ transforms under the charge conjugation as

$$\psi_R^c(x) \stackrel{\text{def}}{=} -i\sigma^2\psi_R^*(x). \tag{56}$$

With these definitions, it is straightforward to verify the relation $[\psi_L^c(x)]^c = \psi_L(x)$:

$$[\psi_L^c(x)]^c = \psi_R^c(x) = -i\sigma^2\psi_R^*(x) = -i\sigma^2[i\sigma^2\psi_L^*(x)]^* = -i\sigma^2(i\sigma^2)\psi_L(x) = \psi_L(x)\,. \tag{57}$$

NB

In Eqs. (47) and (51), we explicitly wrote the sum and difference of $\boldsymbol{J}^+$ and $\boldsymbol{J}^-$ exactly as they appear in many reference texts. While the notation is not incorrect, it can be misleading if not contextualized. Generally, suppose we have two algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ represented by $\lambda_{1,\mathcal{R}}$ and $\lambda_{2,\mathcal{R}}$, respectively. We then ask what the representation of the algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is. To answer this, let's suppose that the exponentials of $\lambda_{1,\mathcal{R}}$ and $\lambda_{2,\mathcal{R}}$ give the representations of the groups $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, so that

$$\begin{aligned}|\psi\rangle &\mapsto e^{\lambda_{1,\mathcal{R}}}\,|\psi\rangle\,,\\ |\varphi\rangle &\mapsto e^{\lambda_{2,\mathcal{R}}}\,|\varphi\rangle\,.\end{aligned} \tag{58}$$

Therefore, we can naturally define the representation of the product group $\mathcal{G}_1 \otimes \mathcal{G}_2$ as

$$|\psi\rangle \otimes |\varphi\rangle \mapsto \left[e^{\lambda_{1,\mathcal{R}}}\,|\psi\rangle\right] \otimes \left[e^{\lambda_{2,\mathcal{R}}}\,|\varphi\rangle\right] \tag{59}$$

[2]Remeber that, given any Pauli matrix σ^i, the identity $(\sigma^i)^2 = \mathbb{1}_2$ holds $\forall\, i = 1, 2, 3$.

At the level of the algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, this becomes

$$\begin{aligned} \frac{d}{dt}\left[\left[e^{t\lambda_{1,\mathcal{R}}}\left|\psi\right\rangle\right] \otimes \left[e^{t\lambda_{2,\mathcal{R}}}\left|\varphi\right\rangle\right]\right]\Bigg|_{t=0} &= \left[\lambda_{1,\mathcal{R}}\left|\psi\right\rangle\right] \otimes \left|\varphi\right\rangle + \left|\psi\right\rangle \otimes \left[\lambda_{2,\mathcal{R}}\left|\varphi\right\rangle\right] \\ &\equiv \left[\lambda_{1,\mathcal{R}} + \lambda_{2,\mathcal{R}}\right]\left|\psi\right\rangle \otimes \left|\varphi\right\rangle . \end{aligned} \tag{60}$$

Hence, when we write the sum $\lambda_{1,\mathcal{R}} + \lambda_{2,\mathcal{R}}$, we implicitly assume that the two operators act on different subspaces. This specification appears to be only a technicality as long as at least one of the two representations $\lambda_{1,\mathcal{R}}$ or $\lambda_{2,\mathcal{R}}$ is the trivial representation, as in previous cases. However, when both $\lambda_{1,\mathcal{R}}$ and $\lambda_{2,\mathcal{R}}$ are not in the trivial representations (as in the case of the vector representation we will discuss shortly), this argument becomes fundamental.

$(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}})$

In the case where $j^- = j^+ = 1/2$, the representation has dimension $\dim[(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}})] = 4$. According to the composition law of angular momenta, the total angular momentum of a particle in this representation can be $|j^+ - j^-| \leq j \leq |j^+ + j^-|$, which means $j = 0$ or $j = 1$. Therefore, a field in this representation can represent either a spin-0 particle or a spin-1 particle. A generic element $\phi \in (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}})$ can be expressed as a pair of independent left and right Weyl spinors:

$$\phi(x) = \begin{pmatrix} \xi_L(x) \\ \psi_R(x) \end{pmatrix} \tag{61}$$

While this representation may appear quite different from the vector representation discussed earlier, it can be shown that they are actually equivalent. To see this, note that in this representation, we can use charge conjugation to obtain the fields ψ_L and ξ_R, since

$$\phi^c(x) = \begin{pmatrix} \xi_L^c(x) \\ \psi_R^c(x) \end{pmatrix} = \begin{pmatrix} \xi_R(x) \\ \psi_L(x) \end{pmatrix} . \tag{62}$$

Now, introducing the matrices $\sigma^\mu = (\mathbb{1}_2, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (\mathbb{1}_2, -\boldsymbol{\sigma})$, we define the objects V^μ and W^μ as

$$\begin{aligned} V^\mu &\stackrel{\text{def}}{=} [\phi^c(x)]^\dagger \begin{pmatrix} \mathbb{0}_2 & \sigma^\mu \\ \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} \phi(x) = \xi_R^\dagger(x)\sigma^\mu\psi_R(x) , \\ W^\mu &\stackrel{\text{def}}{=} [\phi^c(x)]^\dagger \begin{pmatrix} \mathbb{0}_2 & \mathbb{0}_2 \\ \bar{\sigma}^\mu & \mathbb{0}_2 \end{pmatrix} \phi(x) = \psi_L^\dagger(x)\bar{\sigma}^\mu\xi_L(x) . \end{aligned} \tag{63}$$

We aim to prove that V^μ transforms as a vector (the proof for W^μ follows a similar approach). We do it in the following *Example*, where we assume that the Lorentz transformation is a boost of rapidity $\boldsymbol{\eta}$.

Example:

We aim to demonstrate tha V^μ, defined as in Eq. (63), behaves as a vector under a Lorentz transformation. If this true, then under a boost transformation of rapidity $\boldsymbol{\eta}$, V^μ must transform as

$$\begin{pmatrix} V'^0 \\ \boldsymbol{V}' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\boldsymbol{v} \\ \gamma\boldsymbol{v}^T & \mathbb{1} + (\gamma - 1)\boldsymbol{v}\boldsymbol{v}^T/v^2 \end{pmatrix} \begin{pmatrix} V^0 \\ \boldsymbol{V} \end{pmatrix} , \tag{64}$$

where the rapidity $\boldsymbol{\eta}$ relates to the boost velocity v via

$$\boldsymbol{\eta} = \frac{\eta}{v} \cdot \boldsymbol{v} , \qquad \cosh\eta = \gamma , \qquad \tanh\eta = v . \tag{65}$$

Expanding the above equation, we obtain the transformed components V'^0 and V'^i as follows:

$$\begin{aligned} V'^0 &= \gamma V^0 + \gamma\boldsymbol{v} \cdot \boldsymbol{V} , \\ V'^i &= \gamma v^i V^0 + \left[\delta^{ij} + (\gamma - 1)\frac{v^i v^j}{v^2}\right] V^j . \end{aligned} \tag{66}$$

We verify these results by considering the boost transformation for both fields ψ_R and ξ_R as expressed in Eq. (63). The transformation of V^μ under this boost is given by

$$V'^\mu = [\Lambda_R \xi_R]^\dagger \sigma^\mu [\Lambda_R \psi_R] = \xi_R^\dagger \Lambda_R^\dagger \sigma^\mu \Lambda_R \psi_R \,, \tag{67}$$

We start nothing the following property:

$$\exp(\boldsymbol{k}\cdot\boldsymbol{\sigma}) = \sum_{n=0}^{\infty} \frac{(\boldsymbol{k}\cdot\boldsymbol{\sigma})^n}{n!} = \mathbb{1}_2 \sum_{n=0}^{\infty} \frac{k^{2n}}{(2n)!} + \frac{\boldsymbol{k}\cdot\boldsymbol{\sigma}}{k} \sum_{n=0}^{\infty} \frac{k^{2n+1}}{(2n+1)!} = \cosh k \cdot \mathbb{1}_2 + \sinh k \cdot \frac{\boldsymbol{k}\cdot\boldsymbol{\sigma}}{k} \,, \tag{68}$$

with $\boldsymbol{k}$ any vector. Therefore, regarding V'^0, we get

$$V'^0 = \xi_R^\dagger \Lambda_R^\dagger \Lambda_R \psi_R = \xi_R^\dagger \, e^{\boldsymbol{\eta}\cdot\boldsymbol{\sigma}} \, \psi_R = \cosh\eta \cdot \xi_R^\dagger \psi_R + \sinh\eta \cdot \xi_R^\dagger \frac{\boldsymbol{\eta}\cdot\boldsymbol{\sigma}}{\eta} \psi_R \,. \tag{69}$$

Using the relations in Eq. (65), we obtain

$$V'^0 = \gamma \cdot \underbrace{\xi_R^\dagger \psi_R}_{=V^0} + \gamma v^i \cdot \underbrace{\xi_R^\dagger \sigma^i \psi_R}_{=V^i} = \gamma V^0 + \gamma \boldsymbol{v}\cdot\boldsymbol{V}, \tag{70}$$

that is exactly the time-component of Eq. (66).

Consider now V'^i. We have

$$\begin{aligned} V'^i &= \xi_R^\dagger \Lambda_R^\dagger \sigma^i \Lambda_R \psi_R = \xi_R^\dagger \Big[e^{\boldsymbol{\eta}\cdot\boldsymbol{\sigma}/2} \sigma^i e^{\boldsymbol{\eta}\cdot\boldsymbol{\sigma}/2} \Big] \psi_R \\ &= \xi_R^\dagger \left[\cosh\left(\frac{\eta}{2}\right) \cdot \mathbb{1}_2 + \sinh\left(\frac{\eta}{2}\right) \cdot \frac{\boldsymbol{\eta}\cdot\boldsymbol{\sigma}}{\eta} \right] \sigma^i \left[\cosh\left(\frac{\eta}{2}\right) \cdot \mathbb{1}_2 + \sinh\left(\frac{\eta}{2}\right) \cdot \frac{\boldsymbol{\eta}\cdot\boldsymbol{\sigma}}{\eta} \right] \psi_R \\ &= \xi_R^\dagger \left[\cosh^2\left(\frac{\eta}{2}\right) \cdot \sigma^i + \cosh\left(\frac{\eta}{2}\right) \sinh\left(\frac{\eta}{2}\right) \cdot \frac{\eta^j}{\eta} \underbrace{\left[\sigma^i \sigma^j + \sigma^j \sigma^i\right]}_{=2\delta^{jk}\cdot\mathbb{1}_2} + \sinh^2\left(\frac{\eta}{2}\right) \cdot \frac{\eta^j \eta^k}{\eta^2} \sigma^j \sigma^i \sigma^k \right] \psi_R \\ &= \cosh\left(\frac{\eta}{2}\right) \sinh\left(\frac{\eta}{2}\right) \frac{v^i}{v} \cdot \xi_R^\dagger \psi_R + \cosh^2\left(\frac{\eta}{2}\right) \cdot \xi_R^\dagger \sigma^i \psi_R + \sinh^2\left(\frac{\eta}{2}\right) \frac{v^j v^k}{v^2} \cdot \xi_R^\dagger \sigma^j \sigma^i \sigma^k \psi_R \,. \end{aligned} \tag{71}$$

Then, according to the identity

$$\sigma^j \sigma^i \sigma^k = \delta^{ij} \sigma^k + \delta^{ik} \sigma^j - \delta^{kj} \sigma^i + i\epsilon^{ikj} \mathbb{1}_2 \,, \tag{72}$$

we rewrite V'^i as

$$\begin{aligned} V'^i &= \cosh\left(\frac{\eta}{2}\right) \sinh\left(\frac{\eta}{2}\right) \frac{v^i}{v} \cdot \xi_R^\dagger \psi_R + \cosh^2\left(\frac{\eta}{2}\right) \cdot \xi_R^\dagger \sigma^i \psi_R + \sinh^2\left(\frac{\eta}{2}\right) \left[2\frac{v^i v^j}{v^2} \cdot \xi_R^\dagger \sigma^j \psi_R - \xi_R^\dagger \sigma^i \psi_R \right] \\ &= \cosh\left(\frac{\eta}{2}\right) \sinh\left(\frac{\eta}{2}\right) \frac{v^i}{v} V^0 + V^i + 2\sinh^2\left(\frac{\eta}{2}\right) \frac{v^i v^j}{v^2} V^j \,. \end{aligned} \tag{73}$$

Finally, we observe that

$$\cosh\left(\frac{\eta}{2}\right) \sinh\left(\frac{\eta}{2}\right) = \frac{\sinh\eta}{2} = \frac{v\gamma}{2} \,, \qquad \sinh^2\left(\frac{\eta}{2}\right) = \frac{\cosh\eta - 1}{2} = \frac{\gamma - 1}{2} \,, \tag{74}$$

so we can conclude that

$$V'^i = \gamma v^i V^0 + \left[\delta^{ij} + (\gamma - 1) \frac{v^i v^j}{v^2} \right] V^j, \tag{75}$$

which is exactly the spatial-component of Eq. (66).

We leave to the reader as an exercise how to prove the analogous statement where $\Lambda_R = \Lambda_R(\boldsymbol{\theta})$ is the rotation matrix.

This proof ensures that both V^μ and W^μ, constructed from the field $\phi \in (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}})$, exhibit vector-like transformation properties. Nonetheless, to ensure physical consistency, we impose a reality condition on V^μ, requiring $V^\mu = [V^\mu]^*$. Once established in a particular frame, this condition holds across all Lorentz frames. Consequently, we assert that V^μ transforms in the vector representation.

$(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}) \oplus (\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}})$

In the preceding subsection, we introduced the Weyl spinors ψ_L and ψ_R, residing in the representations $(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0})$ and $(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}})$, respectively. Now, let's consider a field belonging to the 4-dimensional representation $(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}) \oplus (\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}})$. Such a field is represented as:

$$\Psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \tag{76}$$

and must transform as

$$\Psi'(x') = \Lambda_D \Psi(x)\,, \qquad \Lambda_D = \begin{pmatrix} \Lambda_L & \mathbb{0}_2 \\ \mathbb{0}_2 & \Lambda_R \end{pmatrix}\,, \tag{77}$$

where Λ_L and Λ_R are described by Eqs. (48) and (52), respectively. This field, denoted as Ψ, commonly known as the *Dirac field*, is frequently employed to characterize spin-$\frac{1}{2}$ particles.

Λ_D is usually represented by means of the *Clifford algebra*, defined by a set of 4×4 matrices γ^μ, satisfying the anti-commutation rules:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \stackrel{\text{def}}{=} 2\eta^{\mu\nu} \cdot \mathbb{1}_4\,. \tag{78}$$

By utilizing these matrices, we introduce the antisymmetric tensor

$$S^{\mu\nu} \stackrel{\text{def}}{=} \frac{i}{2}[\gamma^\mu, \gamma^\nu]\,, \tag{79}$$

which allows us to derive the 4-dimensional representation of Λ_D:

$$\Lambda_D = \exp\left\{-\frac{i}{2}\lambda_{\mu\nu}S^{\mu\nu}\right\}\,, \tag{80}$$

where $\lambda_{\mu\nu}$ is an antisymmetric object defined as

$$\eta^i = \lambda^{i0} = -\lambda_{i0}\,, \qquad \theta^i = \frac{1}{2}\epsilon^{ijk}\lambda^{jk}\,. \tag{81}$$

While $\lambda_{\mu\nu}$ parametrizes the rotational and boost parameters $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$, $S^{\mu\nu}$ defines the generators of the representation. Note that $S^{\mu\nu}$ is an antisymmetric tensor with two Lorentz indices, possessing six independent components, aligning precisely with the dimension of the Lorentz algebra. Various representations of γ^μ are possible. Here, we adopt the *chiral representation* (or *Weyl representation*), defined as

$$\gamma^\mu \stackrel{\text{def}}{=} \begin{pmatrix} \mathbb{0}_2 & \sigma^\mu \\ \bar{\sigma}^\mu & \mathbb{0}_2 \end{pmatrix}\,. \tag{82}$$

In this representation, $S^{\mu\nu}$ corresponds to

$$\begin{aligned} S^{0i} &= \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2}\begin{pmatrix} \sigma^i & \mathbb{0}_2 \\ \mathbb{0}_2 & \sigma^i \end{pmatrix}\,, \\ S^{ij} &= \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & \mathbb{0}_2 \\ \mathbb{0}_2 & -\sigma^k \end{pmatrix}\,. \end{aligned} \tag{83}$$

To demonstrate the equivalence between Eqs (80) and Eq.(77), let's begin with the Λ_D matrix of Eq.(77), expressed as follows:

$$\Lambda_D = \begin{pmatrix} \Lambda_L & \mathbb{0}_2 \\ \mathbb{0}_2 & \Lambda_R \end{pmatrix} = \begin{pmatrix} e^{(-i\boldsymbol{\theta}-\boldsymbol{\eta})\boldsymbol{\sigma}/2} & \mathbb{0}_2 \\ \mathbb{0}_2 & e^{(-i\boldsymbol{\theta}+\boldsymbol{\eta})\boldsymbol{\sigma}/2} \end{pmatrix} \equiv \begin{pmatrix} e^{\lambda_L} & \mathbb{0}_2 \\ \mathbb{0}_2 & e^{\lambda_R} \end{pmatrix}\,. \tag{84}$$

Notice that Λ_D can be rewritten as an exponential of a certain matrix λ_D:

$$\begin{aligned} \Lambda_D &= \begin{pmatrix} e^{\lambda_L} & \mathbb{0}_2 \\ \mathbb{0}_2 & e^{\lambda_R} \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^\infty \frac{\lambda_L^n}{n!} & \mathbb{0}_2 \\ \mathbb{0}_2 & \sum_{n=0}^\infty \frac{\lambda_R^n}{n!} \end{pmatrix} = \sum_{n=0}^\infty \frac{1}{n!}\begin{pmatrix} \lambda_L^n & \mathbb{0}_2 \\ \mathbb{0}_2 & \lambda_R^n \end{pmatrix} \\ &= \sum_{n=0}^\infty \frac{1}{n!}\begin{pmatrix} \lambda_L & \mathbb{0}_2 \\ \mathbb{0}_2 & \lambda_R \end{pmatrix}^n = \exp\begin{pmatrix} \lambda_L & \mathbb{0}_2 \\ \mathbb{0}_2 & \lambda_R \end{pmatrix} \\ &= \exp\begin{pmatrix} (-i\boldsymbol{\theta}-\boldsymbol{\eta})\frac{\boldsymbol{\sigma}}{2} & \mathbb{0}_2 \\ \mathbb{0}_2 & (-i\boldsymbol{\theta}+\boldsymbol{\eta})\frac{\boldsymbol{\sigma}}{2} \end{pmatrix} \equiv e^{\lambda_D}\,. \end{aligned} \tag{85}$$

According to this expression, we only need to prove that $\lambda_D = -\frac{i}{2}\lambda_{\mu\nu}S^{\mu\nu}$. To do this, let's separate the rotational and boost contributions in λ_D:

$$\lambda_D = -\frac{i}{2}\theta^i \begin{pmatrix} \sigma^i & \mathbb{0}_2 \\ \mathbb{0}_2 & \sigma^i \end{pmatrix} - \frac{\eta^i}{2} \begin{pmatrix} \sigma^i & \mathbb{0}_2 \\ \mathbb{0}_2 & -\sigma^i \end{pmatrix} . \tag{86}$$

Then, we use Eq. (83) to get

$$\begin{pmatrix} \sigma^i & \mathbb{0}_2 \\ \mathbb{0}_2 & \sigma^i \end{pmatrix} = \epsilon^{ijk} S^{jk} , \qquad \begin{pmatrix} \sigma^i & \mathbb{0}_2 \\ \mathbb{0}_2 & -\sigma^i \end{pmatrix} = 2i\, S^{0i} , \tag{87}$$

and also utilizing Eq. (83), we rewrite λ_D as[3]

$$\begin{aligned} \lambda_D &= -\frac{i}{2}\theta^i \epsilon^{ijk} S^{jk} - i\eta^i S^{0i} = -\frac{i}{4}\epsilon^{ilm}\epsilon^{ijk}\lambda^{lm} S^{jk} - i\lambda_{0i}S^{0i} \\ &= -\frac{i}{4}[\lambda_{jk} - \lambda_{kj}]S^{jk} - \frac{i}{2}[\lambda_{0i}S^{0i} + \lambda_{i0}S^{i0}] = -\frac{i}{2}[\lambda_{jk}S^{jk} + \lambda_{0i}S^{0i} + \lambda_{i0}S^{i0}] \\ &= -\frac{i}{2}\lambda_{\mu\nu}S^{\mu\nu}. \end{aligned} \tag{88}$$

Thus, we have obtained Eq. (80).

2.2 Poincarè group

The preceding sections have delved into the Lorentz symmetry both in spacetime and fields. However, an additional symmetry remains unexplored: the *translation symmetry*. We demand that all fields (scalars, spinors, vectors, etc.) remain invariant under translations. Formally, this translates to the requirement that under a translation $x^\mu \mapsto x'^\mu = x^\mu + \epsilon^\mu$, any field transforms as

$$\phi'(x') = \phi'(x+\epsilon) = \phi(x) . \tag{89}$$

Given our four-dimensional spacetime, we expect to have four generators, denoted as $P^\mu = (P^0, \boldsymbol{P})$. The action of the translation group on a field is thus expressed as[4]

$$\phi(x+\epsilon) = e^{-i\epsilon_\mu P^\mu}\phi(x) . \tag{90}$$

What happens when we consider ϵ^μ as infinitesimal parameters of translation?

$$\begin{aligned} \delta\phi(x) &= \phi'(x) - \phi(x) = \phi'(x'-\epsilon) - \phi(x) = \overbrace{\phi'(x')}^{=\phi(x)} - \epsilon_\mu \partial^\mu \overbrace{\phi'(x')}^{=\phi(x)} - \phi(x) + \mathcal{O}(\epsilon^2) \\ &= -\epsilon_\mu \partial^\mu \phi(x) + \mathcal{O}(\epsilon^2) . \end{aligned} \tag{91}$$

Alternatively, we can compute the variation $\delta\phi(x)$ as

$$\begin{aligned} \delta\phi(x) &= \phi'(x) - \phi(x) = \phi'(x'-\epsilon) - \phi(x) = e^{-i(-\epsilon_\mu)P^\mu}\phi'(x') - \phi(x) \\ &= \phi'(x') + i\epsilon_\mu P^\mu \phi'(x') - \phi(x) + \mathcal{O}(\epsilon^2) \\ &= i\epsilon_\mu P^\mu \phi(x) + \mathcal{O}(\epsilon^2) . \end{aligned} \tag{92}$$

Comparing the equations, we deduce that the momentum operator P^μ can be represented as

$$P^\mu = i\partial^\mu , \tag{93}$$

yielding[5]

$$P^0 = i\partial^0 = i\partial_0 = i\frac{\partial}{\partial t} \equiv H , \qquad P^i = i\partial^i = -i\partial_i = -i\frac{\partial}{\partial x^i} . \tag{94}$$

[3]In the following steps, we use the antisymmetric property of $\lambda_{\mu\nu}$ and $S^{\mu\nu}$.

[4]Note that the tranlsation group is non-compact.

[5]H is the hamiltonian of the system.

Since the derivatives commute, we recover the well-known commutation rules of the translation group:

$$[P^\mu, P^\nu] = 0\,. \tag{95}$$

Translations combined with Lorentz transformations form the *Poincaré group*, also known as the *inhomogeneous Lorentz group*, denoted by ISO(1,3), where "I" signifies "inhomogeneous". We have to determine its commutation rules, encompassing all commutators between H, $\boldsymbol{P}$, $\boldsymbol{J}$, and $\boldsymbol{K}$. We observe that H must commute with $\boldsymbol{J}$, as H acts on the timelike component and $\boldsymbol{J}$ on the spacelike components. Additionally, $P^\mu = i\partial^\mu$ being a four-vector necessitates its spacelike components to transform as a vector under rotations. These observations lead to

$$[J^i, H] = 0\,, \qquad [J^i, P^j] = i\epsilon^{ijk}P^k\,. \tag{96}$$

As for the the commutators between P^μ and $\boldsymbol{K}$, one can formally prove that

$$[K^i, H] = iP^i\,, \qquad [K^i, P^j] = iH\delta^{ij}\,. \tag{97}$$

Let's clarify it through an example.

Example:

Let $\phi(x)$ represent a scalar field, and consider applying a Lorentz boost along the x-axis, denoted as $\Lambda(\eta) = \exp i\eta K^1$. According to Eq. (42), we have

$$\phi'(x') = \phi(x) = \phi(\Lambda(\eta)^{-1}\cdot x')\,. \tag{98}$$

Since we previously computed $\Lambda(\eta) = \exp i\eta K^1$ in Eq. (21), obtaining its inverse as $\Lambda(\eta)^{-1} = \Lambda(-\eta)$, we rewrite the equation as

$$\phi'(x') = \phi(\Lambda(-\eta)\cdot x') = \phi\Big(\cosh\eta\cdot t' - \sinh\eta\cdot x'^1, -\sinh\eta\cdot t' + \cosh\eta\cdot x'^1, x'^2, x'^3\Big)\,. \tag{99}$$

Let η be an infinitesimal parameter and erase the apex on x'. It follows that

$$\begin{aligned}\delta\phi(x) = \phi'(x) - \phi(x) &= \phi\Big(t - \eta\cdot x^1 + \mathcal{O}(\eta^2), -\eta\cdot t + x^1 + \mathcal{O}(\eta^2), x^2, x^3\Big) - \phi(x)\\ &= \phi(x) - \eta\, x^1\partial_t\phi(x) - \eta\, t\,\partial_1\phi(x) - \phi(x) + \mathcal{O}(\eta^2)\\ &= i\eta\Big[i(x^1\partial_t + t\,\partial_1)\Big]\phi(x) + \mathcal{O}(\eta^2)\,.\end{aligned} \tag{100}$$

However, we can also rewrite the variation $\delta\phi$ as

$$\begin{aligned}\delta\phi(x) = \phi'(x) - \phi(x) = \phi'(\Lambda(\eta)^{-1}\cdot x') - \phi(x) &= e^{-i\eta K^1}\overbrace{\phi'(x')}^{=\phi(x)} - \phi(x)\\ &= \phi(x) - i\eta K^1\phi(x) - \phi(x) + \mathcal{O}(\eta^2)\\ &= -i\eta K^1\phi(x) + \mathcal{O}(\eta^2)\,,\end{aligned} \tag{101}$$

from which we deduce

$$K^1 = -i(x^1\partial_t + t\,\partial_1) = i(t\,\partial^1 - x^1\partial^0)\,. \tag{102}$$

With this, we can compute the commutator

$$[K^1, H] = [i(t\,\partial^1 - x^1\partial^0), i\partial^0] = \partial^1 = -iP^1\,. \tag{103}$$

Pay attention, you get the result with the wrong sign. FIX IT!!!! Likewise, the commutators with $K^{2,3}$ yield

$$[K^i, H] = iP^i\,. \tag{104}$$

Now, considering the commutators between $\boldsymbol{K}$ and $\boldsymbol{P}$, we find

$$\begin{aligned}[K^1, P^1] &= [i(t\,\partial^1 - x^1\partial^0), i\partial^1] = \partial^0 = -iH\,,\\ [K^1, P^{2,3}] &= [i(t\,\partial^1 - x^1\partial^0), i\partial^{2,3}] = 0\,,\end{aligned} \tag{105}$$

Pay attention, you get the result with the wrong sign. FIX IT!!!! and similarly for $K^{2,3}$, leading to

$$[K^i, P^j] = i\delta^{ij}H\,. \tag{106}$$

At this point, we can finally collect all the commutation rules of the Poincaré group, that are

$$
\begin{aligned}
[J^i, J^j] &= i\epsilon^{ijk} J^k \,, \\
[J^i, K^j] &= i\epsilon^{ijk} K^k \,, \\
[J^i, H] &= 0 \,, \\
[J^i, P^j] &= i\epsilon^{ijk} P^k \,, \\
[K^i, K^j] &= -i\epsilon^{ijk} J^k \,, \\
[K^i, H] &= iP^i \,, \\
[K^i, P^j] &= iH\delta^{ij} \,, \\
[P^i, H] &= 0 \,, \\
[P^i, P^j] &= 0 \,.
\end{aligned}
\tag{107}
$$

Note that $\boldsymbol{K}$ does not commute with H, meaning it is not a conserved quantity. This is why the eigenvalues of $\boldsymbol{K}$ are not used to label physical states.

3 Wigner Little Group

Up to this point, we have discussed the theoretical background of a theory in which the Lagrangian is invariant under the Poincaré group. At the classical level, the Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$ provides all the necessary dynamics for describing the system. One could derive the equations of motion from $\mathcal{L}(\phi, \partial_\mu \phi)$, solve them, and obtain explicit expressions for the fields. For instance, a photon is described by the field $A^\mu(x)$, which is essentially a function of x^μ. However, at the quantum level, the situation becomes more intricate: every field is now treated as an operator that acts on physical states within a proper Hilbert space. This implies that we require $A^\mu(x)$ to act on a physical state in order to create a photon. Since we are dealing with a quantum theory, we need to address two key questions:

(i) how to represent a physical state,

(ii) how a physical state transforms under a Poincaré transformation.

Consider x^μ and a^μ as two four-vectors. A general Poincaré transformation acting on x^μ is given by

$$(\Lambda, a) \colon x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \,, \tag{108}$$

so $(\Lambda, a) \sim (\mathbb{1}, a) \cdot (\Lambda, 0)$, where $(\mathbb{1}, a)$ denotes a pure translation and $(\Lambda, 0)$ a pure Lorentz transformation. Hence, if $|\Psi\rangle$ represents a physical state, it must transform under a Poincaré transformation as

$$|\Psi'\rangle = U(\Lambda, a)\, |\Psi\rangle \ , \tag{109}$$

where $U(\Lambda, a)$ denotes a unitary representation of (Λ, a). Let's tackle the aforementioned questions.

We know that H and $\boldsymbol{p}$ are conserved quantities (commuting with H), so it is natural to express a physical state in terms of eigenvectors of the four-momentum operator P^μ. In principle, a physical state may depend on additional quantum numbers, denoted by σ, which are currently unknown. Hence, we denote $|\Psi\rangle = |\boldsymbol{p}, \sigma\rangle$ as a generic one-particle physical state and impose the condition of being an eigenvector of P^μ, i.e.,

$$P^\mu\, |\boldsymbol{p}, \sigma\rangle = p^\mu\, |\boldsymbol{p}, \sigma\rangle \ . \tag{110}$$

Accordingly, the representation of translations is straightforward:

$$U(\mathbb{1}, a)\, |\boldsymbol{p}, \sigma\rangle = e^{-i a_\mu P^\mu}\, |\boldsymbol{p}, \sigma\rangle = e^{-i a_\mu p^\mu}\, |\boldsymbol{p}, \sigma\rangle \ . \tag{111}$$

However, it is less trivial to represent a Lorentz transformation of the form $U(\Lambda, 0)\, |\boldsymbol{p}, \sigma\rangle \equiv U(\Lambda)\, |\boldsymbol{p}, \sigma\rangle$. To do this, we partition the space based on all possible momenta p^μ, depending on the value of $p^2 = p_\mu p^\mu$ (bearing in mind that this quantity is Lorentz invariant):

(i) $p^2 > 0$,

(ii) $p^2 = 0$,

(iii) $p^2 < 0$.

We focus on the first two regions, namely those in which an on-shell physical state satisfies the equation $p^2 - m^2 = 0$. In these two regions, we fix a generic four-momentum k^μ and express p^μ as a Lorentz boost with respect to k^μ, i.e.,

$$p^\mu = L(p)^\mu{}_\nu k^\nu \,, \tag{112}$$

where $L(p)$ represents a suitable Lorentz transformation (we label it with $L(p)$ to distinguish it from the Lorentz transformation Λ). At this juncture, since $U(\Lambda, 0)$ is a representation of the Lorentz group, it must be a *homomorphism*, i.e.,

$$U(\Lambda_1) U(\Lambda_2) = U(\Lambda_1 \cdot \Lambda_2) 1, . \tag{113}$$

Utilizing this property, we write

$$
\begin{aligned}
U(\Lambda,0)\,|\boldsymbol{p},\sigma\rangle \equiv U(\Lambda)\,|\boldsymbol{p},\sigma\rangle &= U(\Lambda)U(L(p))\,|\boldsymbol{k},\sigma\rangle = U(\Lambda\cdot L(p))\,|\boldsymbol{k},\sigma\rangle \\
&= U\Big(\overbrace{[L(\Lambda p)\cdot L(\Lambda p)^{-1}]}^{=\mathbb{1}}\cdot\Lambda\cdot L(p)\Big)\,|\boldsymbol{k},\sigma\rangle \\
&= U(L(\Lambda p))\,U\Big(L(\Lambda p)^{-1}\cdot\Lambda\cdot L(p)\Big)\,|\boldsymbol{k},\sigma\rangle\ .
\end{aligned}
\tag{114}
$$

We now focus on the Lorentz transformation $L(\Lambda p)^{-1}\cdot\Lambda\cdot L(p)$: note that

$$
\begin{aligned}
L(p)\colon\, &k^\mu \mapsto p^\mu\,, \\
\Lambda\cdot L(p)\colon\, &k^\mu \mapsto p^\mu \mapsto (\Lambda p)^\mu\,, \\
L(\Lambda p)^{-1}\cdot\Lambda\cdot L(p)\colon\, &k^\mu \mapsto p^\mu \mapsto (\Lambda p)^\mu \mapsto k^\mu\,.
\end{aligned}
\tag{115}
$$

Hence, the operator $C = L(\Lambda p)^{-1}\cdot\Lambda\cdot L(p)$ represents a Lorentz transformation that leaves any four-momentum k^μ unchanged, i.e.,

$$
C^\mu{}_\nu k^\nu = k^\mu\,,\quad \forall\ k^\mu\,.
\tag{116}
$$

All Lorentz operators C satisfying the above condition form a subgroup of $\mathrm{SO}(1,3)_+$ known as the *Wigner little group*. Suppose we know the explicit expression of a representation of C, denoted as

$$
U(C)\,|\boldsymbol{k},\sigma\rangle = \sum_{\sigma'}[C_{\mathcal{R}}]_{\sigma\sigma'}\,|\boldsymbol{k},\sigma'\rangle\ .
\tag{117}
$$

Then, we conclude that

$$
\begin{aligned}
U(\Lambda,0)\,|\boldsymbol{p},\sigma\rangle = U(\Lambda)\,|\boldsymbol{p},\sigma\rangle = U(L(\Lambda p))U(C)\,|\boldsymbol{k},\sigma\rangle &= \sum_{\sigma'}[C_{\mathcal{R}}]_{\sigma\sigma'}\overbrace{U(L(\Lambda p))\,|\boldsymbol{k},\sigma'\rangle}^{=|\Lambda\cdot p,\sigma'\rangle} \\
&= \sum_{\sigma'}[C_{\mathcal{R}}]_{\sigma\sigma'}\,|\Lambda p,\sigma'\rangle\ .
\end{aligned}
\tag{118}
$$

We observe that the problem of finding the representation of $U(\Lambda,0)$ acting on $|\boldsymbol{p},\sigma\rangle$ has been reduced to finding the representation of the little group, specifically obtaining the coefficients $[C\mathcal{R}]_{\sigma\sigma'}$. In other words, understanding how a one-particle physical state transforms under a Lorentz transformation requires knowing its transformation under the little group.

The next objective is to identify the operators C of the little group. Given that they satisfy Eq. (116), they can be expressed as

$$
C^\mu{}_\nu p^\nu = (e^{\lambda_c})^\mu{}_\nu p^\nu = \sum_{n=0}^{\infty}\frac{1}{n!}(\lambda_c^n)^\mu{}_\nu p^\nu = p^\mu + \sum_{n=1}^{\infty}\frac{1}{n!}(\lambda_c^n)^\mu{}_\nu p^\nu = p^\mu\,,
\tag{119}
$$

which yields

$$
(\lambda_c)^\mu{}_\nu p^\nu = 0\,.
\tag{120}
$$

Consequently, we must determine the most general element of the Lorentz algebra that vanishes on a four-momentum p^μ. However, at this point, we need to distinguish between the cases where $p^2 > 0$ and $p^2 = 0$.

Before delving into the specifics of the two cases, it is noteworthy that the representations of physical states are invariably labeled by the Casimir operators. For instance, in the rotation group, the Casimir operator $\boldsymbol{L}^2$ dictates the dimension of the representation, whose eigenvalues $j(j+1)$ labels the dimension of the representation (that is $2j+1$). Similarly, for the Lorentz group, there exist two Casimir operators. The first one is $P_\mu P^\mu$, yielding (acting on a physical state)

$$
P_\mu P^\mu\,|\boldsymbol{p},\sigma\rangle = p_\mu p^\mu\,|\boldsymbol{p},\sigma\rangle = m^2\,|\boldsymbol{p},\sigma\rangle\,.
\tag{121}
$$

However, the second Casimir operator is more intricate to ascertain, necessitating the utilization of Eq. (120) for its determination.

$p^2 > 0$

We aim to determine the most general expression of an element of the Lorentz algebra that satisfies Eq.(120) in the massive case $p^2 = m^2 > 0$. To begin, let's consider k^μ in the rest frame, where $p^\mu = (m, 0, 0, 0)$. Using the general expression (15) of an element of the Lorentz algebra, we find

$$[\lambda_c(m > 0)]^\mu{}_\nu p^\nu = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & r_3 & -r_2 \\ b_2 & -r_3 & 0 & +r_1 \\ b_3 & r_2 & -r_1 & 0 \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{122}$$

implying

$$[\lambda_c(m > 0)]^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -r_3 & r_2 \\ 0 & r_3 & 0 & -r_1 \\ 0 & -r_2 & r_1 & 0 \end{pmatrix} = -r_1 \cdot iJ^1 - r_2 \cdot iJ^2 - r_3 \cdot iJ^3 \,. \tag{123}$$

This expression represents a combination of the angular momentum matrices J^i from Eq. (16). Consequently, the Wigner little group of the Lorentz group in the massive case is SU(2) (preferred over SO(3) for the inclusion of spinor representations). Given that SU(2) is represented by the Casimir operator $\boldsymbol{J}$, it is natural to introduce the operator

$$W^\mu \overset{\text{def}}{=} \begin{pmatrix} 0 \\ m\boldsymbol{J} \end{pmatrix}, \tag{124}$$

known as the *Pauli–Lubanski operator*, such that $W_\mu W^\mu = -m^2 \boldsymbol{J}^2$. Since $W_\mu W^\mu$ is invariant under boosts, the expression we have derived is universally applicable and valid in any frame. Consequently, the action of the Casimir on a physical state will be

$$W_\mu W^\mu \left|\boldsymbol{p}, j\right\rangle = -m^2 j(j+1) \left|\boldsymbol{p}, j\right\rangle \,. \tag{125}$$

Hence, we conclude that a massive one-particle physical state is characterized by the mass m and the spin j. It is important to note that a field with mass $m > 0$ possesses $2j + 1$ degrees of freedom, representing the dimension of the representation.

$p^2 = 0$

Conceptually, we simply need to repeat the steps of the massive case. However, we cannot choose the rest frame anymore, as it does not satisfy the condition $p^2 = 0$. A suitable choice is given by $p^\mu = (E, 0, 0, E)$. Eq. (120) yields

$$[\lambda_c(m = 0)]^\mu{}_\nu p^\nu = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & r_3 & -r_2 \\ b_2 & -r_3 & 0 & r_1 \\ b_3 & r_2 & -r_1 & 0 \end{pmatrix} \begin{pmatrix} E \\ 0 \\ 0 \\ E \end{pmatrix} = \begin{pmatrix} b_3 E \\ (b_1 - r_2)E \\ (b_2 + r_1)E \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{126}$$

implying

$$\begin{cases} b_3 = 0\,, \\ r_2 - b_1 = 0\,, \\ r_1 + b_2 = 0\,. \end{cases} \tag{127}$$

Thus, $\lambda_c(m = 0)$ can be written as

$$[\lambda_c(m = 0)]^\mu{}_\nu = \begin{pmatrix} 0 & b_1 & b_2 & 0 \\ b_1 & 0 & r_3 & -b_1 \\ b_2 & -r_3 & 0 & -b_2 \\ 0 & b_1 & b_2 & 0 \end{pmatrix} = -b_1 \cdot i(J^2 + K^1) - b_2 \cdot i(J^1 - K^2) - b_3 \cdot iJ^3 \,. \tag{128}$$

We can introduce the Pauli-Lubanski operator W^μ, defined as

$$W^0 = W^3 = EJ^3 \,, \qquad W^1 := E(J^1 - K^2) \,, \qquad W^2 = E(J^2 + K^1) \,, \tag{129}$$

and obtain the operator

$$W_\mu W^\mu = -E^2 \Big[(J^1 - K^2)^2 + (J^2 + K^1)^2 \Big] \,. \tag{130}$$

We leave it as an exercise for the reader to prove that $W_\mu W^\mu$ is a Casimir operator.

Now, what type of little group do we have in this massless case? To answer this question, it is useful to introduce the operators

$$A^\mu{}_\nu = i(J^2 + K^1)^\mu{}_\nu \,, \qquad B^\mu{}_\nu = i(-J^1 + K^2)^\mu{}_\nu \,, \tag{131}$$

according to which the Casimir operator becomes

$$W_\mu W^\mu = -E^2(A^2 + B^2) \,. \tag{132}$$

Note that the commutation relations between A, B, and J^3 are

$$[J^3, A] = iB \,, \qquad [J^3, B] = -iA \,, \qquad [A, B] = 0 \,, \tag{133}$$

so the algebra generated by these three operators is the same as that generated by p^x, p^y and $L^z = x\,p^y - y\,p^x$. Consequently, the Wigner little group of the Lorentz group in the massless case corresponds to the translations and rotations of a Euclidean plane, i.e. ISO(2). A and B play the role of translation operators. Although they are not Hermitian since they are 4×4 matrices, we can treat them as Hermitian in an infinite-dimensional representation. This representation corresponds to one-particle physical states with momentum $\boldsymbol{p}$. A and B commute, so we can define a set of physical states $|\boldsymbol{p}, a, b\rangle$ that are eigenstates of both at the same time:

$$A\,|\boldsymbol{p}, a, b\rangle = a\,|\boldsymbol{p}, a, b\rangle \ , \qquad B\,|\boldsymbol{p}, a, b\rangle = b\,|\boldsymbol{p}, a, b\rangle \ . \tag{134}$$

However, using this result, we can derive a surprising consequence. Consider the physical state

$$|\boldsymbol{p}, a, b, \theta\rangle := e^{-i\theta J^3}\,|\boldsymbol{p}, a, b\rangle \ , \tag{135}$$

where θ is an arbitrary angle. Then, we find

$$A\,|\boldsymbol{p}, a, b, \theta\rangle = (a\cos\theta - b\sin\theta)\,|\boldsymbol{p}, a, b, \theta\rangle \ , \tag{136}$$

and similarly,

$$B\,|\boldsymbol{p}, a, b, \theta\rangle = (a\sin\theta + b\cos\theta)\,|\boldsymbol{p}, a, b, \theta\rangle \ . \tag{137}$$

This implies that a massless particle has a continuous internal degree of freedom θ unless $a = b = 0$. However, since we do not observe such a degree of freedom in experiments, we must impose the condition $a = b = 0$, which translates to

$$W_\mu W^\mu = 0 \,. \tag{138}$$

This result is reasonable, as it corresponds to the limit $m \to 0$ of the massive Casimir operator $W_\mu W^\mu\,|\boldsymbol{p}, j\rangle = -m^2 j(j+1)\,|\boldsymbol{p}, j\rangle$.

Given that the only possible generator of the little group is J^3, the Wigner little group of the massless case is ultimately SO(2). The latter is an abelian group, and its dimension corresponds dim[SO(2)] = 1. Therefore, a massless particle can have only one degree of freedom. If we consider a one-dimensional representation of J^3, it corresponds to a number, commonly denoted as h in the literature: it represents the *helicity* of the particle. One can demonstrate that h is quantized, taking values in the set $h = \{0, \pm 1/2, \pm 1, ...\}$. However, delving into the technical details of this proof exceeds the scope of our discussion. What is more pertinent for our purposes is the realization that a massless particle has only one degree of freedom, which is precisely its helicity h.

In principle, two massless particles with opposite helicities $+h$ and $-h$ are distinct. However, in our previous argument, we found the Wigner little group to be generated by J^3 because we initially set the four-momentum p^μ along the z-axis. If we were to repeat this proof with p^μ along the x- or y-axis, we would find J^1 and J^2 to be the generators of the Wigner little group, respectively. Therefore, helicity can be interpreted as the projection of the angular momentum along the direction of motion:

$$\boldsymbol{h} = \hat{\boldsymbol{p}} \cdot \boldsymbol{J} \,. \tag{139}$$

Note that $\boldsymbol{h}$ is a pseudo-vector, so it changes sign under a parity transformation. For a quantum field theory to be invariant under parity transformations, every particle with helicity $+h$ must have a counterpart with helicity $-h$. This is why it is more natural to define a massless particle as a superposition of two physical states with opposite helicities.[6]

[6] It is worth noting that parity is not a fundamental symmetry of our universe, as it is broken by the weak force.