Contents

Chapter 1

Some Basics

1.1 Lorentz symmetry on spacetime

Let x^μ be the vector $x^\mu = (t, \boldsymbol{x})$ of a fuor dimensional spacetime with metric tensor $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. The Lorentz symmetry is defined as the group of linear coordinate transformations,

$$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1.1}$$

that leave invariant the quantity

$$x \cdot x = \eta_{\mu\nu} x^\mu x^\nu = t^2 - x^2 - y^2 - z^2. \tag{1.2}$$

A group that acts on a space with coordinates $(t_1, ..., t_m; x_1, ..., x_n)$ and leaves invariant the quadratic form $(t_1 + ... + t_m)^2 - (x_1 + ... + x_n)^2$ is called *orthogonal group* $O(m, n)$, so the Lorentz group corresponds to $O(1, 3)$. The quadratic condition of Eq. (1.2) wants the matrix Λ to satisfy the identity

$$x' \cdot x' = \eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\sigma x^\sigma = \eta_{\rho\sigma} x^\rho x^\sigma = x \cdot x \tag{1.3}$$

for a generic x, so the relation

$$\eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \qquad \rightarrow \qquad \eta = \Lambda^T \eta \Lambda. \tag{1.4}$$

must hold. Taking the determinant on both the sides of the above equation, we get

$$|\det(\Lambda)|^2 = 1 \qquad \rightarrow \qquad \det(\Lambda) = \pm 1. \tag{1.5}$$

WLOG, we can assume the condition $\det(\Lambda) = +1$. In fact, a general Lorentz transformation with $\det(\Lambda) = -1$ can always be written as the product of a transformation with $\det(\Lambda) = +1$ and a discrete transformation that reverses the sign of an odd number of coordinates (e.g. the *parity transformation*). Therefore, from now on we will consider that subgroup of $O(1, 3)$ caracterized by the condition $\det(\Lambda) = +1$, i.e. the *special orthogonal group* SO(1,3). A Lorentz transformation with $\det(\Lambda) = +1$ is called *proper Lorentz transformation.*

Consider the 00 component of the Eq. (1.4):

$$1 = \eta_{00} = \eta_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = \eta_{00} \Lambda^0{}_0 \Lambda^0{}_0 + \eta_{ij} \Lambda^i{}_0 \Lambda^j{}_0 = (\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^i{}_0)^2. \tag{1.6}$$

Since Λ is a real matrix, then $(\Lambda^i{}_0)^2 \geq 0$ for $i = 1, 2, 3$. It follows that

$$(\Lambda^0{}_0)^2 \geq 1 \qquad \longrightarrow \qquad \Lambda^0{}_0 \leq -1 \text{ or } \Lambda^0{}_0 \geq +1. \tag{1.7}$$

Therefore, the propor Lorentz group $SO(1,3)$ has two disconnected components, one with ${\Lambda^0}_0 \leq -1$, called *non-orthochronous*, an the other with ${\Lambda^0}_0 \geq +1$, called *orthochronous*. Any non-orthochronous transformation can be written as the product of an orthochronous transformation and a specific discrete inversion of the type $(t,x,y,z) \to (-t,-x,-y,-z)$, or $(t,x,y,z) \to (-t,-x,y,z)$, etc. Therefore, from now on we can limit ourselves to consider proper orthochronous Lorentz transformations, i.e. a transformation $\Lambda \in SO(1,3)_+$ (the index $+$ stands for the condition ${\Lambda^0}_0 \geq 1$).

Since the Lorentz group is a Lie group, a general Lorentz matrix Λ can be written as

$$\Lambda = e^{\lambda}, \qquad \lambda \in \mathfrak{so}(1,3), \tag{1.8}$$

where λ is a general component of the Lorentz algebra $\mathfrak{so}(1,3)$.

NB: A quick observation that goes back to the previous paragraph: the exponential map is a diffeomorphism between a neighbourhood of $0 \in \mathfrak{g}$ (any Lie algebra) and a neighbourhood of $\mathbb{1} \in \mathcal{G}$ (its corresponding Lie group). In other words, any element $\lambda \in \mathfrak{g}$ generates a one-parameter subgroup of $\mathcal{G}$ by means of the exponetial map $t \mapsto \exp\{t \cdot \lambda\}$; viceversa, any $\Lambda \in \mathcal{G}$ in a neighbourhood of the identity $\mathbb{1}$ of the group belongs to a one-parameter subgroup of $\mathcal{G}$, namely Λ can be written as $\Lambda = \exp(\lambda)$ in a proper neighbourhood of $\mathbb{1}$ and for some $\lambda \in \mathfrak{g}$. Why is this interesting for us? As we saw, $SO(1,3)$ is not a connected group, since ${\Lambda^0}_0$ is not defined on the interaval $[-1,1]$. It follows that there cannot be a continuous path that links a component of the ${\Lambda^0}_0 \geq +1$ subset with an other component of the ${\Lambda^0}_0 \leq -1$ subset. Since our Lorentz matrix Λ must lay in a neighbourhood of $\mathbb{1}$, the ${\Lambda^0}_0 \geq +1$ subset, i.e. $SO(1,3)_+$, is the subgroup we need to work with.

Now that we have introduced the Lorentz algebra $\mathfrak{so}(1,3)$, we would like to find an explicit description of the latter. In fact, we have learnt that, if we have a proper description of the Lorentz algebra, we can use the exponential map to get an explicit representation of the Lorentz group as well. In this regard, let λ be an infinitesimal transformation, i.e.

$${\Lambda^\mu}_\nu = {\delta^\mu}_\nu + {\lambda^\mu}_\nu + \mathcal{O}(\lambda^2). \tag{1.9}$$

If we substitute this expansion into the Eq. (1.4), we obtain

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\left[{\delta^\mu}_\rho + {\lambda^\mu}_\rho + \mathcal{O}(\lambda^2)\right]\left[{\delta^\nu}_\sigma + {\lambda^\nu}_\sigma + \mathcal{O}(\lambda^2)\right] = \eta_{\rho\sigma} + \lambda_{\rho\sigma} + \lambda_{\sigma\rho} + \mathcal{O}(\lambda^2), \tag{1.10}$$

which implies

$$\lambda_{\rho\sigma} = -\lambda_{\sigma\rho}. \tag{1.11}$$

Therefore, $\lambda_{\mu\nu}$ is 4×4 real antisymmetric matrix, so it must have six independent parameters, i.e.

$$\lambda_{\mu\nu} = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ -b_1 & 0 & r_3 & -r_2 \\ -b_2 & -r_3 & 0 & +r_1 \\ -b_3 & r_2 & -r_1 & 0 \end{pmatrix}. \tag{1.12}$$

The Lorentz algebra has therefore dimension 6, that is we need to find six generators. However, when Λ acts on a vector x^μ as in Eq. (1.1), it appers in the form ${\Lambda^\mu}_\nu$. It is thus more convenient for us to have a general expression of ${\lambda^\mu}_\nu = \eta^{\mu\sigma}\lambda_{\sigma\nu}$, which obviously is not symmetric anymore and corresponds to

$${\lambda^\mu}_\nu = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -r_3 & r_2 \\ b_2 & r_3 & 0 & -r_1 \\ b_3 & -r_2 & r_1 & 0 \end{pmatrix}. \tag{1.13}$$

Now we are free to fix the six parameters in order to define a basis of the Lorentz algebra:

$$J^1 = i\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}, \qquad J^2 = i\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}, \qquad J^3 = i\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}$$
$$K^1 = -i\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \qquad K^2 = -i\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}, \qquad K^3 = -i\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}. \tag{1.14}$$

These matrices are called *generators* of the Lorentz group. The commutation rules of the algebra are

$$[J^i, J^j] = i\epsilon^{ijk}J^k, \qquad [K^i, K^j] = -i\epsilon^{ijk}J^k, \qquad [J^i, K^j] = i\epsilon^{ijk}K^k, \tag{1.15}$$

where ϵ_{ijk} is the totally antisymmetric tensor with $\epsilon_{123} = +1$. It could seem weird to have six complex matrices and not real. However, this is only a convention. In fact, we can write $\lambda^\mu{}_\nu$ as a linear combination of the above matrices as follows:

$$\lambda^\mu{}_\nu = \sum_{i=1}^{3} i\,\alpha^i \cdot J^i + \sum_{i=1}^{3} i\,\beta^i \cdot K^i, \qquad \alpha_i, \beta_i \in \mathbb{R}, \tag{1.16}$$

so $\lambda^\mu{}_\nu$ is actually real. Question: what is the physical meaning of these six generators? A for J_i, they are the generators of the *spatial rotations*, i.e. $J^i \in \mathfrak{so}(3)$, so they can be thought as the three components of the angular momentum. Regarding K^i, they are the generators of the *boosts*. A couple of examples can clarify it.

Example: Boost along the x-axis

Let's see what happens if we fix $\lambda^\mu{}_\nu = i\eta K^1$, where η is the *rapidity* (we are fixing to zero all the parameters of Eq. (1.16) except for $\beta^1 = \eta$). In this case, Λ corresponds to

$$\Lambda^\mu{}_\nu = e^{\lambda^\mu{}_\nu} = e^{i\eta K^1} = \sum_{n=0}^{\infty}\frac{\eta^n (iK^1)^n}{n!} = \mathbb{1}_4 + \sum_{n=1}^{\infty}\frac{\eta^{2n}(iK_1)^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{\eta^{2n+1}(iK^1)^{2n+1}}{(2n+1)!}. \tag{1.17}$$

Since

$$(iK^1)^{2n} = \begin{pmatrix}\mathbb{1}_2 & \mathbb{0}_2\\ \mathbb{0}_2 & \mathbb{0}_2\end{pmatrix}, \qquad (iK^1)^{2n+1} = iK^1, \tag{1.18}$$

then

$$\begin{aligned}\Lambda^\mu{}_\nu &= \mathbb{1}_4 + \begin{pmatrix}\mathbb{1}_2 & \mathbb{0}_2\\ \mathbb{0}_2 & \mathbb{0}_2\end{pmatrix}\sum_{n=1}^{\infty}\frac{\eta^{2n}}{(2n)!} + iK^1\sum_{n=0}^{\infty}\frac{\eta^{2n+1}}{(2n+1)!}\\ &= \mathbb{1}_4 + \begin{pmatrix}\mathbb{1}_2 & \mathbb{0}_2\\ \mathbb{0}_2 & \mathbb{0}_2\end{pmatrix}(\cosh\eta - 1) + iK^1\cdot\sinh\eta\\ &= \begin{pmatrix}\cosh\eta & \sinh\eta & 0 & 0\\ \sinh\eta & \cosh\eta & 0 & 0\\ 0&0&1&0\\0&0&0&1\end{pmatrix}.\end{aligned} \tag{1.19}$$

Remember that $v = \tanh\eta$ and $\gamma = (1-v^2)^{-1/2} = \cosh\eta$, so we can rewrite the above result as

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.20}$$

This is nothing but a boost along the x axis. Since $v < c = 1$, then $\Lambda^{0}{}_{0} \geq 1$ as expected. In addition, note that $\det(\Lambda) = 1$.

There is an other information we can grasp from this example. In general, the space of the real $N \times N$ matrices $M_N(\mathbb{R})$ can be thought as the euclidean space $\mathbb{R}^{N^2}$, namely each matrix $A \in M_N(\mathbb{R})$ corresponds to a point $\boldsymbol{x} \in \mathbb{R}^{N^2}$. For example, if you have $A \in M_2(\mathbb{R})$ such that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{1.21}$$

then A can be thought as a point $\boldsymbol{x} \in \mathbb{R}^{N^2}$ such that $\boldsymbol{x} = (A_{11}, A_{12}, A_{21}, A_{22})$. This isomorphism lets us to define a "distance" between two matrices $A, B \in M_N(\mathbb{R})$ as follows:

$$d(A,B) \colon M_N(\mathbb{R}) \times M_N(\mathbb{R}) \mapsto \mathbb{R} \qquad \text{such that} \qquad d(A,B) := \sqrt{\sum_{i,j=1}^{N} |A_{ij} - B_{ij}|^2}. \tag{1.22}$$

According to the above definition, we can compute the distance between the Λ matrix of Eq. (1.19) and the identity:

$$d(\Lambda, \mathbb{1}) = \sqrt{\sum_{i,j=1}^{N} |\Lambda_{ij} - \mathbb{1}_{ij}|^2} = \sqrt{2(\cosh\eta - 1)^2 + 2\sinh^2\eta} = 2\sqrt{2}\left|\sinh\frac{\eta}{2}\right|\sqrt{\cosh\eta} \underset{\eta\to\infty}{\to} \infty. \tag{1.23}$$

Therefore, a Lorentz matrix Λ can be arbitrarily distant from the identity, which makes $SO(1,3)_+$ a non-compact group.

Example: Rotation around the z-axis

In analogy with the previous example, let's now choose $\lambda^{\mu}{}_{\nu} = -i\theta J^3$, with θ generic angle (we are fixing to zero all the parameters of Eq. (1.16) except for $\alpha^3 = -\theta$). We go through the same steps,

$$\Lambda^{\mu}{}_{\nu} = e^{\lambda^{\mu}{}_{\nu}} = e^{-i\theta J_3} = \sum_{n=0}^{\infty} \frac{(-i\theta)^n (J^3)^n}{n!} = \mathbb{1}_4 + \sum_{n=1}^{\infty} \frac{(-i\theta)^{2n}(J^3)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-i\theta)^{2n+1}(J^3)^{2n+1}}{(2n+1)!}. \tag{1.24}$$

At this point we simply note that

$$(J^3)^{2n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (J^3)^{2n+1} = J_3, \tag{1.25}$$

so we obtain

$$\begin{aligned}
\Lambda^\mu{}_\nu =&\mathbb{1}_4 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sum_{n=1}^{\infty} \frac{(-)^n \theta^{2n}}{(2n)!} - iJ^3 \sum_{n=0}^{\infty} \frac{(-)^n \theta^{2n+1}}{(2n+1)!} \\
=&\mathbb{1}_4 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (\cos\theta - 1) - iJ^3 \cdot \sin\theta \qquad (1.26) \\
=& \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

This is nothing but a counterclockwise rotation of an angle θ around the z axis. Again the conditions $\Lambda^0{}_0 \geq 1$ and $\det(\Lambda) = 1$ hold. Finally, the distance between a rotation matrix and the identity is given by

$$d(\Lambda, \mathbb{1}) = \sqrt{\sum_{i,j=1}^{N} |\Lambda_{ij} - \mathbb{1}_{ij}|^2} = \sqrt{2(\cos\theta - 1)^2 + 2\sin^2\theta} = 2\sqrt{1-\cos\theta} \leq 4. \qquad (1.27)$$

It is thus clear the the non-compactness of $SO(1,3)_+$ derives from the generators of the boosts K^i.

These two examples are useful also to clarify which is the best way to write the linear expansion of Eq. (1.16). In fact, from the first example we see that the physical meaning of the coefficients β^i in Eq. (1.16) is nothing but being the rapidies η^i along the three axes. Similarly, from the second example we see that each α^i is the angle of rotation $-\theta^i$ around the i-axis, where a minus sign appears since we want θ^i to run counterclockwise. Therefore, the most general expression of a matrix $\lambda \in \mathfrak{so}(1,3)$ is

$$\lambda^\mu{}_\nu = -\sum_{i=1}^{3} i\,\theta_i \cdot J_i + \sum_{i=1}^{3} i\,\eta_i \cdot K_i = -i\,\boldsymbol{\theta} \cdot \boldsymbol{J} + i\,\boldsymbol{\eta} \cdot \boldsymbol{K}. \qquad (1.28)$$

It follows that the most general expression for a Lorentz transformation $\Lambda \in SO(1,3)_+$ corresponds to

$$\Lambda^\mu{}_\nu = \left[e^{-i\,\boldsymbol{\theta}\cdot\boldsymbol{J} + i\,\boldsymbol{\eta}\cdot\boldsymbol{K}} \right]^\mu{}_\nu. \qquad (1.29)$$

Note that $J^{i,\dagger} = J^i$, while $K^{i,\dagger} = -K^i$, so not all generators of $SO(1,3)_+$ are hermitian. This is why our representation (of dimension 4) of the Lorentz group is not unitary.

1.2 Lorentz symmetry on fields

1.2.1 General properties

In the previous section we showed the action of the Lorentz group on spacetime, we introduced the Lorentz algebra $\mathfrak{so}(1,3)$ and we provided a basis in Eq. (1.14) and the commutation rules in Eq. (1.15). The next step is the introduction of the *fields*, that is the topic of this subsection. A field is a quantity $\phi = \phi(x)$ defined at every point of spacetime, namely it is a function of $x^\mu = (t, \boldsymbol{x})$. It is a system with an infinite number of degrees of freedom, at least one for each point $\boldsymbol{x}$ in space. In particle physics, we always require the Lagrangia density $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$ to be invariant under Lorentz transformations. It follows that, under a Lorentz transformation, the field must transform in a certain representation of the Lorentz group. More specifically, let ϕ be an N component multiplet, i.e. $\phi(x) = (\phi^1(x), ..., \phi^N(x))$. According to what we have just said, under a Lorentz transformation ϕ must transform in a representation $\mathcal{R}$ of dimension N of the Lorentz group, that is

$$\phi'^{\,i}(x') = [\Lambda_{\mathcal{R}}]^i{}_j \, \phi^j(x) = \left[e^{-i\,\boldsymbol{\theta}\cdot\boldsymbol{J}_{\mathcal{R}}+i\,\boldsymbol{\eta}\cdot\boldsymbol{K}_{\mathcal{R}}}\right]^i{}_j \, \phi^j(x), \tag{1.30}$$

where $\boldsymbol{J}_{\mathcal{R}}$ and $\boldsymbol{K}_{\mathcal{R}}$ are the generators of the rotations $\boldsymbol{J}$ and boosts $\boldsymbol{K}$ respectively in the representation R. Note that in the previous subsection we provided a specifical representation of the Lorentz algebra $\mathfrak{so}(1,3)$, i.e. the *vector representation* in which a general component $\lambda \in \mathfrak{so}(1,3)$ is a 4×4 matrix. Formally speaking, λ is an abstract component of $\mathfrak{so}(1,3)$, while its representation $\lambda_{\mathcal{R}}$ is an $N \times N$ matrix. By the way, the description of the algebra that we got through the Eq. (1.15) is fully general, since the commutation rules are intrinsically related to the algebra itself and do not depend on the representation we choose. Therefore, if we change the representation of the basis (1.14) with another one with $N \neq 4$, then the Eq. (1.15) still hold.

The commutation rules Eq. (1.15) seem to suggest the presence of two rotation algebras. However, although the J_i generators are those of the algebra $\mathfrak{so}(3)$, the same cannot be said of the K_i generators. If so, in fact, we should have commutation rules of the type $[K_i, K_j] = i\epsilon_{ijk}K_k$ and $[J_i, K_j] = 0$. But there is no way to get such commutation rules as long as we consider real combinations, unless we introduce the *complexification of the algebra* $\mathfrak{so}(1,3)$. For the interested reader, the following NB clarifies what we are talking about.

NB: Let $\mathfrak{g}$ be a real algebra generated by the basis $\{T_1, ..., T_N\}$. Suppose we want to define a complexification $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$. Two different scenarios are possible:

(i) a combination of $\{T_1, ..., T_N\}$ is still *linearly independent* even if we consider a complex linear combination, i.e.

$$\alpha_i \cdot T_i = 0 \quad \Longrightarrow \quad \mathbb{C} \ni \alpha_i = 0 \;\; \forall\, i; \tag{1.31}$$

(ii) the basis $\{T_1, ..., T_N\}$ is *linearly independent* only if it is a real linear combination, but not a complex one.

In Eq.1.28 we assumed $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ to be real. What happens if we take them complex instead? We get the scenario (i), so the six generators of $\mathfrak{so}(1,3)$ are still linearly independent. Our next gol is to take a complex linear combination of the six generators of $\mathfrak{so}(1,3)$ and see what happens.

We have the freedom to consider a complex linear combination of the generators of $\mathfrak{so}(1,3)$, that is the complexified algebra $\widetilde{\mathfrak{so}}(1,3)$. We can define six new generators as follows:

$$J^{i,+} := \frac{J^i + iK^i}{2}, \qquad J^{i,-} := \frac{J^i - iK^i}{2}, \tag{1.32}$$

according to which the commutation rules of $\widetilde{\mathfrak{so}}(1,3)$ are given by:

$$[J^{i,+}, J^{j,+}] = i\epsilon^{ijk}J^{k,+}, \qquad [J^{i,-}, J^{j,-}] = i\epsilon^{ijk}J^{k,-}, \qquad [J^{i,+}, J^{j,-}] = 0. \tag{1.33}$$

It is clear that now we have two copies of the angular momentum algebra and that the generators of the first algebra do not "talk" at all with those of the second algebra, since they commute. Therefore, we can write $\widetilde{\mathfrak{so}}(1,3)$ as the direct sum of two (complexified) rotation algebras $\widetilde{\mathfrak{su}}(2)$, i.e.

$$\widetilde{\mathfrak{so}}(1,3) = \widetilde{\mathfrak{su}}(2) \oplus \widetilde{\mathfrak{su}}(2). \tag{1.34}$$

We point out that, at the level of the algebra, $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{so}(3)$: the generators of $\mathfrak{su}(2)$ and those of $\mathfrak{so}(3)$ are two different representations of the same objects (this is not true at the level of the groups). At this point, it is much easier to figure out what is in Eq. (1.30) the proper representation of dimension N of the Lorentz group. In fact, we simply need $\boldsymbol{J}^+$ in its representation $\mathcal{R}^+$ and $\boldsymbol{J}^-$ in its representation $\mathcal{R}^-$.

Remember that, if $\{J^1, J^2, J^3\}$ are the generators of $\mathfrak{su}(2)$, we can introduce the operator $\boldsymbol{J}^2$ that commutes whit all the generators, i.e. $[\boldsymbol{J}^2, J^i] = 0 \quad \forall\ i$. Then we can choose one of the generators, for example J_3, and define a basis with respect to which both J^3 and $\boldsymbol{J}^2$ are diagonals. It turns out that the action of two operators on the this basis (let's call it $|j, m_j\rangle$) is given by

$$\begin{cases} \boldsymbol{J}^2 \, |j, m_j\rangle = j(j+1) \, |j, m_j\rangle \\ J^3 \, |j, m_j\rangle = m_j \, |j, m_j\rangle \end{cases} \tag{1.35}$$

with $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, ...\}$ and $m_j \in \{-j, -j-1, ..., j-1, j\}$. From the physical point of view, we know that $\boldsymbol{J}^2$ is the square of the angular momentum $\boldsymbol{J}$ and J^3 the projection of $\boldsymbol{J}$ along the z axis. From the mathematical point of view, since m_j can assume $(2j+1)$ different values for any fixed j, then we can think about it as a $(2j+1) \times (2j+1)$ matrix, namely as a representation of dimension $(2j+1)$ of the generator J^3. This tells us that j fixes the dimension of the representation of $\mathfrak{su}(2)$, which corresponds to $(2j+1)$ as we already said.

In light of the above, we understand that the dimension of the representation of $\widetilde{\mathfrak{so}}(1,3)$ is given by the dimension of the representation $\mathcal{R}^-$ of the first $\widetilde{\mathfrak{su}}(2)$ algebra, that is $(2j^- + 1)$, times the dimension of the representation $\mathcal{R}^+$ of the second $\widetilde{\mathfrak{su}}(2)$ algebra, that is $(2j^+ + 1)$. In the literature, a representation of $\widetilde{\mathfrak{so}}(1,3)$ is often referred to as $(\boldsymbol{j}^-, \boldsymbol{j}^+)$. Therefore, we have

$$\dim[(\boldsymbol{j}^-, \boldsymbol{j}^+)] = (2j^- + 1)(2j^+ + 1). \tag{1.36}$$

At this point, it is useful to spend a few words on the most important representations.

(0,0)

In this case we simply have $j^- = j^+ = 0$. Therefore, $\boldsymbol{J}^\pm = 0$ and $\dim[(\mathbf{0}, \mathbf{0})] = 1$. This is called *scalar representation*, since the field transforms as

$$\phi'(x') = \phi(x), \tag{1.37}$$

i.e. as a scalar. ϕ gives rise to spin 0 particles.

Example: Scalar field under a Lorentz transformation

Let ϕ be a scalar field that gives the temperature T of a given system in any point. Suppose that two observers A and B want to measure the temperature at the point P of this system. Moreover, assume that B is in motion with constant velocity $\boldsymbol{v}$ with respect to A. Now, in the frame of A, the point P is identified by the vector of spacetime x^μ, therefore A will measure the temperature $\phi(x) = T_A(P)$. Similarly, in the frame of B the point P is identified by a vector x'^μ, therefore he will measure the temperature $\phi'(x') = T_B(P)$. Two important observations to point out. Firstly, observer A will use his own function to express the temperature, that is ϕ, and the same goes for B, who will use his function ϕ'. Secondly, both must measure the same temperature, as this is an intrinsic property of the point P of the system and does not depend in any way on the observer. Therefore we conclude that $T_A(P) \equiv T_B(P)$, i.e. $\phi'(x') = \phi(x)$. Note that between observers A and B there is a Lorentz transformation involved, more properly a boost $\Lambda(\boldsymbol{v})$ such that $x'^\mu = [\Lambda(\boldsymbol{v})]^\mu{}_\nu x^\nu$. So, if observer A decided to use the frame of B to make his own measurement, he would simply take his field $\phi(x)$ and express the position x^μ in terms of x'^μ, i.e. $x^\mu = [\Lambda(\boldsymbol{v})^{-1}]^\mu{}_\nu x'^\nu$, getting:

$$\phi(x) = \phi\left(\Lambda(\boldsymbol{v})^{-1} \cdot x'\right) = \phi'(x'). \tag{1.38}$$

We have thus obtained the transformation of a scalar field of Eq. (1.37).

$(\frac{1}{2}, \mathbf{0})$

This representation has $j^- = 1/2$ and $j^+ = 0$, so it has dimension $\dim[(\frac{1}{2}, \mathbf{0})] = 2$ and gives rise to spin 1/2 particles. The field that, under a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, transforms in this representation is of great importance in theoretical physics and it is called *left-handed Weyl spinor*; it is labellded by ψ_L. We call Λ_L the 2×2 matrix such that $\Lambda_L \in (\frac{1}{2}, \mathbf{0})$, so

$$\psi_L'(x') = \Lambda_L \, \psi_L(x). \tag{1.39}$$

Let's see how we can explicitly write Λ_L in terms of the generators of $\widetilde{\mathfrak{so}}(1,3)$. Since $\boldsymbol{J}^-$ lives in the foundamental representation of $\widetilde{\mathfrak{su}}(2)$ ($\frac{1}{2}$ representation), we can write is as

$$\boldsymbol{J}^- = \frac{\boldsymbol{\sigma}}{2}. \tag{1.40}$$

As for $\boldsymbol{J}^+$, it is in the *trivial representation* ($\mathbf{0}$ representation), so

$$\boldsymbol{J}^+ = 0. \tag{1.41}$$

We can thus easily derive the expressions of $\boldsymbol{J}$ and $\boldsymbol{K}$, that read

$$\begin{aligned} \boldsymbol{J} &= \boldsymbol{J}^+ + \boldsymbol{J}^- = \frac{\boldsymbol{\sigma}}{2}, \\ \boldsymbol{K} &= -\, i(\boldsymbol{J}^+ - \boldsymbol{J}^-) = -i\frac{\boldsymbol{\sigma}}{2}. \end{aligned} \tag{1.42}$$

Substituting this expression inside the Eq. (1.30) we obtain

$$\psi_L'(x') = \Lambda_L \, \psi_L(x) = \exp\Big\{(-i\,\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\Big\} \, \psi_L(x). \tag{1.43}$$

$(\mathbf{0}, \frac{1}{2})$

This representation works similarly to the $(\frac{1}{2}, \mathbf{0})$ but with the roles of $\boldsymbol{J}^+$ and $\boldsymbol{J}^-$ swapped. Therefore, we have $j^- = 0$, $j^+ = 1/2$ and $\dim[(\mathbf{0}, \frac{1}{2})] = 2$. Since a field of this type is the "right" version of ψ_L, it is known as *right-handed Weyl spinor* and it is labelled by ψ_R. We call Λ_R the 2×2 matrix such that $\Lambda_R \in (\mathbf{0}, \frac{1}{2})$, so

$$\psi_R'(x') = \Lambda_R \, \psi_R(x). \tag{1.44}$$

In this case we have

$$\boldsymbol{J}^- = 0, \qquad \boldsymbol{J}^+ = \frac{\boldsymbol{\sigma}}{2}, \tag{1.45}$$

that is

$$\begin{aligned} \boldsymbol{J} &= \boldsymbol{J}^+ + \boldsymbol{J}^- = \frac{\boldsymbol{\sigma}}{2}, \\ \boldsymbol{K} &= -\, i(\boldsymbol{J}^+ - \boldsymbol{J}^-) = -i\frac{\boldsymbol{\sigma}}{2}, \end{aligned} \tag{1.46}$$

so we can conclude that ψ_R transforms as

$$\psi_R'(x') = \Lambda_R \, \psi_R(x) = \exp\Big\{(-i\,\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\Big\} \, \psi_R(x). \tag{1.47}$$

Now that both left- and right-handed Weyl spinors have been defined, we can investigate what their relation is. First of all, we note that $\sigma^2\Lambda_L^*\sigma^2 = \Lambda_R$:[1]

$$\begin{aligned}
\sigma^2\Lambda_L^*\sigma^2 =& \sigma^2 \exp\Big\{(+i\,\boldsymbol{\theta}-\boldsymbol{\eta})\cdot\frac{\boldsymbol{\sigma}^*}{2}\Big\}\sigma^2 = \sum_{n=0}^{\infty}(+i\theta^i+\eta^i)\frac{\sigma^2[(\sigma^i)^*]^n\sigma^2}{2^n}\\
=& \sum_{n=0}^{\infty}(+i\theta^i+\eta^i)\frac{[\sigma^2(\sigma^i)^*\sigma^2]^n}{2^n} = \sum_{n=0}^{\infty}(+i\theta^i+\eta^i)\left(-\frac{\sigma^i}{2}\right)^n = \sum_{n=0}^{\infty}(-i\theta^i-\eta^i)\left(\frac{\sigma^i}{2}\right)^n\\
=& \exp\Big\{(-i\,\boldsymbol{\theta}-\boldsymbol{\eta})\cdot\frac{\boldsymbol{\sigma}}{2}\Big\} \equiv \Lambda_R.
\end{aligned} \tag{1.48}$$

Then, we use the above property to see that, given the spinor ψ_L, the field $\sigma^2\psi_L^*$ must transform as a right-handed Weyl spinor:

$$\sigma^2[\psi_L'(x')]^* \mapsto \sigma^2[\Lambda_L\psi_L(x)]^* = \sigma^2\Lambda_L^*\sigma^2\sigma^2\psi_L^*(x) = \Lambda_R[\sigma^2\psi_L^*(x)]. \tag{1.49}$$

Therefore, we can conclude that $\sigma^2\psi_L^*(x) \in (\mathbf{0},\mathbf{\tfrac{1}{2}})$. It is thus natural to introduce an operation that transforms ψ_L into its corresponding right-handed Weyl spinor, that we call ψ_R. We call this operation *charge conjugation* and we properly define it as

$$\psi_L^c(x) \coloneqq i\sigma^2\psi_L^*(x) \equiv \psi_R(x). \tag{1.50}$$

We have added an extra i in the above definition since we require the condition $[\psi_L^c(x)]^c = \psi_L(x)$, which also needs that the $\psi_R^c(x)$ transforms under the charge conjugation as

$$\psi_R^c(x) \coloneqq -i\sigma^2\psi_R^*(x). \tag{1.51}$$

Using these definitions, it is straightforward to verify that the relation $[\psi_L^c(x)]^c = \psi_L(x)$ actually holds:

$$[\psi_L^c(x)]^c = \psi_R^c(x) = -i\sigma^2\psi_R^*(x) = -i\sigma^2[i\sigma^2\psi_L^*(x)]^* = -i\sigma^2(i\sigma^2)\psi_L(x) = \psi_L(x). \tag{1.52}$$

NB: In Eqs. (1.42) and (1.46) we have explicitly written sum and difference of $\boldsymbol{J}^+$ and $\boldsymbol{J}^-$ exactly as they appear on many reference texts. The notation is not wrong, but it can easily be misleading if not contextualized. Generally speaking, suppose we have two algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ represented by $\lambda_{1,\mathcal{R}}$ and $\lambda_{2,\mathcal{R}}$ respectively. We then ask ourselves what is the representation of the algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. To answer this, suppose that the exponentials of $\lambda_{1,\mathcal{R}}$ and $\lambda_{2,\mathcal{R}}$ give the representations of the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, so that

$$\begin{aligned}
|\psi\rangle \mapsto& e^{\lambda_{1,\mathcal{R}}}\,|\psi\rangle\,,\\
|\varphi\rangle \mapsto& e^{\lambda_{2,\mathcal{R}}}\,|\varphi\rangle\,.
\end{aligned} \tag{1.53}$$

Therefore we can naturally define the representation of the product group $\mathcal{G}_1 \otimes \mathcal{G}_2$ as

$$|\psi\rangle \otimes |\varphi\rangle \mapsto \Big[e^{\lambda_{1,\mathcal{R}}}\,|\psi\rangle\Big] \otimes \Big[e^{\lambda_{2,\mathcal{R}}}\,|\varphi\rangle\Big], \tag{1.54}$$

that at the level of the algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ becomes

$$\begin{aligned}
\frac{d}{dt}\bigg[\Big[e^{t\lambda_{1,\mathcal{R}}}\,|\psi\rangle\Big] \otimes \Big[e^{t\lambda_{2,\mathcal{R}}}\,|\varphi\rangle\Big]\bigg]\bigg|_{t=0} =& \Big[\lambda_{1,\mathcal{R}}\,|\psi\rangle\Big] \otimes |\varphi\rangle + |\psi\rangle \otimes \Big[\lambda_{2,\mathcal{R}}\,|\varphi\rangle\Big]\\
\equiv& \Big[\lambda_{1,\mathcal{R}} + \lambda_{2,\mathcal{R}}\Big]\,|\psi\rangle \otimes |\varphi\rangle
\end{aligned} \tag{1.55}$$

Therefore, when we write the sum $\lambda_{1,\mathcal{R}} + \lambda_{2,\mathcal{R}}$, we are implicitly assuming that the two operators act on different subspaces. This specification seems only a technicality as long as at least one of the two representations $\lambda_{1,\mathcal{R}}$ or $\lambda_{2,\mathcal{R}}$ is the trivial representation, as in the previous

[1]Remeber that, given any Pauli matrix σ^i, the identity $(\sigma^i)^2 = \mathbb{1}_2$ holds $\forall\, i = 1, 2, 3$.

cases. However, as soon as both $\lambda_{1,\mathcal{R}}$ and $\lambda_{2,\mathcal{R}}$ are not in the trivial representations (as in the case of the vector representation that we will discuss shortly), this argument becomes fundamental.

$(\frac{1}{2}, \frac{1}{2})$

We are now considering the case in which $j^- = j^+ = 1/2$, so this representation has dimension $\dim[(\frac{1}{2}, \frac{1}{2})] = 4$. Considering the composition law of the angular momenta, the total angular momenta of a particle in this representation can be $|j^+ - j^-| \leq j \leq |j^+ + j^-|$, that is $j = 0$ or $j = 1$. Therefore, a field in this representation can be a spin-0 particle or a spin-1 particle. A generic element $\phi \in (\frac{1}{2}, \frac{1}{2})$ can be written as a pair of independent left and right Weyl spinors, i.e.

$$\phi(x) = \begin{pmatrix} \xi_L(x) \\ \psi_R(x) \end{pmatrix} \tag{1.56}$$

Even if this representation seems to be very different from that vector representation we illustrated in the previous section, it is possible to prove that they actually coincide. In order to see it, first of all we point out that in this representation we can use the charge conjugation to get the fields ψ_L and ξ_R, since

$$\phi^c(x) = \begin{pmatrix} \xi_L^c(x) \\ \psi_R^c(x) \end{pmatrix} = \begin{pmatrix} \xi_R(x) \\ \psi_L(x) \end{pmatrix}. \tag{1.57}$$

At this point, introducing the matrices $\sigma^\mu = (\mathbb{1}_2, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (\mathbb{1}_2, -\boldsymbol{\sigma})$, we can define the objects V^μ and W^μ as

$$\begin{aligned} V^\mu :=& [\phi^c(x)]^\dagger \begin{pmatrix} \mathbb{0}_2 & \sigma^\mu \\ \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} \phi(x) = \xi_R^\dagger(x) \sigma^\mu \psi_R(x), \\ W^\mu :=& [\phi^c(x)]^\dagger \begin{pmatrix} \mathbb{0}_2 & \mathbb{0}_2 \\ \bar{\sigma}^\mu & \mathbb{0}_2 \end{pmatrix} \phi(x) = \psi_L^\dagger(x) \bar{\sigma}^\mu \xi_L(x). \end{aligned} \tag{1.58}$$

Let's prove that V^μ transforms as a vector (the proof works in the same way for W^μ).

Proof. Let $\Lambda_R = \Lambda_R(\boldsymbol{\eta})$ be a boost transformation of rapidity $\boldsymbol{\eta}$ for both the fields ψ_R and ξ_R. Then the object V^μ must transform as

$$V'^\mu = [\Lambda_R \xi_R]^\dagger \sigma^\mu [\Lambda_R \psi_R] = \xi_R^\dagger \Lambda_R^\dagger \sigma^\mu \Lambda_R \psi_R, \tag{1.59}$$

with $\Lambda_R = \Lambda_R^\dagger = \exp\{\boldsymbol{\eta} \cdot \frac{\boldsymbol{\sigma}}{2}\}$. We need to prove that the above expression of V'^μ can be rewritten as[2]

$$\begin{pmatrix} V'^0 \\ \boldsymbol{V}' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma \boldsymbol{v} \\ \gamma \boldsymbol{v}^T & \mathbb{1} + (\gamma - 1) \boldsymbol{v}\boldsymbol{v}^T / v^2 \end{pmatrix} \begin{pmatrix} V^0 \\ \boldsymbol{V} \end{pmatrix}, \tag{1.60}$$

where the above writing has to be intended as

$$\begin{aligned} V'^0 =& \gamma V^0 + \gamma \boldsymbol{v} \cdot \boldsymbol{V}, \\ V'^i =& \gamma v^i V^0 + \left[\delta^{ij} + (\gamma - 1) \frac{v^i v^j}{v^2} \right] V^j. \end{aligned} \tag{1.61}$$

We start nothing the following property:

$$\exp(\boldsymbol{k} \cdot \boldsymbol{\sigma}) = \sum_{n=0}^{\infty} \frac{(\boldsymbol{k} \cdot \boldsymbol{\sigma})^n}{n!} = \mathbb{1}_2 \sum_{n=0}^{\infty} \frac{k^{2n}}{(2n)!} + \frac{\boldsymbol{k} \cdot \boldsymbol{\sigma}}{k} \sum_{n=0}^{\infty} \frac{k^{2n+1}}{(2n+1)!} = \cosh k \cdot \mathbb{1}_2 + \sinh k \cdot \frac{\boldsymbol{k} \cdot \boldsymbol{\sigma}}{k}, \tag{1.62}$$

[2] This is the general expression of a Lorentz boost, where $\boldsymbol{v}$ is the relative velocity between the frame of V'^μ and that of V^μ.

with $\boldsymbol{k}$ any vector. Therefore, regarding V'^0, we get

$$V'^0 = \xi_R^\dagger \Lambda_R^\dagger \Lambda_R \psi_R = \xi_R^\dagger \, e^{\boldsymbol{\eta}\cdot\boldsymbol{\sigma}} \, \psi_R = \cosh\eta \cdot \xi_R^\dagger \psi_R + \sinh\eta \cdot \xi_R^\dagger \frac{\boldsymbol{\eta}\cdot\boldsymbol{\sigma}}{\eta} \psi_R. \tag{1.63}$$

Now, since $\boldsymbol{\eta} = \frac{\eta}{v} \cdot \boldsymbol{v}$, $\cosh\eta = \gamma$ and $\tanh\eta = v$, we obtain

$$V'^0 = \gamma \cdot \underbrace{\xi_R^\dagger \psi_R}_{=V^0} + \gamma v^i \cdot \underbrace{\xi_R^\dagger \sigma^i \psi_R}_{=V^i} = \gamma V^0 + \gamma \boldsymbol{v}\cdot\boldsymbol{V}, \tag{1.64}$$

that is exactly the time-component of Eq. (1.61). As for V'^i, we have

$$\begin{aligned}
V'^i =& \xi_R^\dagger \Lambda_R^\dagger \sigma^i \Lambda_R \psi_R = \xi_R^\dagger \left[e^{\boldsymbol{\eta}\cdot\boldsymbol{\sigma}/2} \sigma^i e^{\boldsymbol{\eta}\cdot\boldsymbol{\sigma}/2} \right] \psi_R \\
=& \xi_R^\dagger \left[\cosh\left(\frac{\eta}{2}\right) \cdot \mathbb{1}_2 + \sinh\left(\frac{\eta}{2}\right) \cdot \frac{\boldsymbol{\eta}\cdot\boldsymbol{\sigma}}{\eta} \right] \sigma^i \left[\cosh\left(\frac{\eta}{2}\right) \cdot \mathbb{1}_2 + \sinh\left(\frac{\eta}{2}\right) \cdot \frac{\boldsymbol{\eta}\cdot\boldsymbol{\sigma}}{\eta} \right] \psi_R \\
=& \xi_R^\dagger \left[\cosh^2\left(\frac{\eta}{2}\right) \cdot \sigma^i + \cosh\left(\frac{\eta}{2}\right) \sinh\left(\frac{\eta}{2}\right) \cdot \frac{\eta^j}{\eta} \underbrace{\left[\sigma^i \sigma^j + \sigma^j \sigma^i \right]}_{=2\delta^{jk}\cdot\mathbb{1}_2} + \sinh^2\left(\frac{\eta}{2}\right) \cdot \frac{\eta^j \eta^k}{\eta^2} \sigma^j \sigma^i \sigma^k \right] \psi_R \\
=& \cosh\left(\frac{\eta}{2}\right) \sinh\left(\frac{\eta}{2}\right) \frac{v^i}{v} \cdot \xi_R^\dagger \psi_R + \cosh^2\left(\frac{\eta}{2}\right) \cdot \xi_R^\dagger \sigma^i \psi_R + \sinh^2\left(\frac{\eta}{2}\right) \frac{v^j v^k}{v^2} \cdot \xi_R^\dagger \sigma^j \sigma^i \sigma^k \psi_R.
\end{aligned} \tag{1.65}$$

Then, according to the identity

$$\sigma^j \sigma^i \sigma^k = \delta^{ij} \sigma^k + \delta^{ik} \sigma^j - \delta^{kj} \sigma^i + i\epsilon^{ikj} \mathbb{1}_2, \tag{1.66}$$

we rewrite V'^i as

$$\begin{aligned}
V'^i =& \cosh\left(\frac{\eta}{2}\right) \sinh\left(\frac{\eta}{2}\right) \frac{v^i}{v} \cdot \xi_R^\dagger \psi_R + \cosh^2\left(\frac{\eta}{2}\right) \cdot \xi_R^\dagger \sigma^i \psi_R + \sinh^2\left(\frac{\eta}{2}\right) \left[2\frac{v^i v^j}{v^2} \cdot \xi_R^\dagger \sigma^j \psi_R - \xi_R^\dagger \sigma^i \psi_R \right] \\
=& \cosh\left(\frac{\eta}{2}\right) \sinh\left(\frac{\eta}{2}\right) \frac{v^i}{v} V^0 + V^i + 2\sinh^2\left(\frac{\eta}{2}\right) \frac{v^i v^j}{v^2} V^j.
\end{aligned} \tag{1.67}$$

Finally, we observe that

$$\cosh\left(\frac{\eta}{2}\right) \sinh\left(\frac{\eta}{2}\right) = \frac{\sinh\eta}{2} = \frac{v\gamma}{2}, \qquad \sinh^2\left(\frac{\eta}{2}\right) = \frac{\cosh\eta - 1}{2} = \frac{\gamma - 1}{2}, \tag{1.68}$$

so we can conclude that

$$V'^i = \gamma v^i V^0 + \left[\delta^{ij} + (\gamma - 1)\frac{v^i v^j}{v^2} \right] V^j, \tag{1.69}$$

which is exactly the spatial-component of Eq. (1.61).

We leave to the reader as an exercise how to prove the analogous statement where $\Lambda_R = \Lambda_R(\boldsymbol{\theta})$ is the rotation matrix. □

This proof guarantees that V^μ (and W^μ as well), built through the field $\phi \in (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}})$, transforms as a vector. However, we should impose a condition of reality on V^μ, that is $V^\mu = [V^\mu]^*$. If we do it in a specific frame, it will remain true in all the Lorentz frame. At this point, we can state that V^μ transforms in the vector representation.

1.2.2 The Dirac field

In the previous subsection we introduced the Weyl spinors ψ_L and ψ_R that live in the representations $(\mathbf{\frac{1}{2}}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{\frac{1}{2}})$ respectively. What happens if we consider a field that transforms in the 4-dimensional representation $(\mathbf{\frac{1}{2}}, \mathbf{0}) \oplus (\mathbf{0}, \mathbf{\frac{1}{2}})$? A field like this is written as

$$\Psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \tag{1.70}$$

and must transform as

$$\Psi'(x') = \Lambda_D \Psi(x), \qquad \Lambda_D = \begin{pmatrix} \Lambda_L & \mathbb{0}_2 \\ \mathbb{0}_2 & \Lambda_R \end{pmatrix}, \tag{1.71}$$

where Λ_L and Λ_R are given by the Eqs. (1.43) and (1.47) respectively. The Ψ field is widely used to describe spin-1/2 particles and takes the name *Dirac field.* Λ_D is usually represented by means of the *Clifford algebra*. This algebra is defined through a set of 4×4 matrices γ^μ that satisfy the anti-commutation rules

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu := 2\eta^{\mu\nu} \cdot \mathbb{1}_4. \tag{1.72}$$

Using these γ^μ matrices, we can introduce the antisymmetric tensor

$$S^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{1.73}$$

and find the following 4-dimensional representation of Λ_D:

$$\Lambda_D = \exp\left\{ -\frac{i}{2}\lambda_{\mu\nu} S^{\mu\nu} \right\}, \tag{1.74}$$

where $\lambda_{\mu\nu}$ is an antisymmetric object defined as

$$\eta^i = \lambda^{i0} = -\lambda_{i0}, \qquad \theta^i = \frac{1}{2}\epsilon^{ijk}\lambda^{jk}. \tag{1.75}$$

While $\lambda_{\mu\nu}$ parametrize the rotational and boosts parameters $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$, $S^{\mu\nu}$ defines the generators of the representation. Note that $S^{\mu\nu}$ is an antisymmetric tensor with two Lorentz indices, so it has six independent components, that is exactly the dimension of the Lorentz algebra. There are different ways to write down the explicit expression of γ^μ. The one we choose (and that we will use throughout all these notes) is called *chiral representation* (or *Weyl representation*) and it is defined as

$$\gamma^\mu := \begin{pmatrix} \mathbb{0}_2 & \sigma^\mu \\ \bar{\sigma}^\mu & \mathbb{0}_2 \end{pmatrix}. \tag{1.76}$$

In this representation, $S^{\mu\nu}$ corresponds to

$$\begin{aligned} S^{0i} &= \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2}\begin{pmatrix} \sigma^i & \mathbb{0}_2 \\ \mathbb{0}_2 & \sigma^i \end{pmatrix}, \\ S^{ij} &= \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & \mathbb{0}_2 \\ \mathbb{0}_2 & -\sigma^k \end{pmatrix}. \end{aligned} \tag{1.77}$$

In the following, we prove how to get Eq. (1.74) starting from Eq. (1.71).

Proof. Let's start from Λ_D matrix of Eq. (1.71), that we write as follows:

$$\Lambda_D = \begin{pmatrix} \Lambda_L & \mathbb{0}_2 \\ \mathbb{0}_2 & \Lambda_R \end{pmatrix} = \begin{pmatrix} e^{(-i\boldsymbol{\theta}-\boldsymbol{\eta})\boldsymbol{\sigma}/2} & \mathbb{0}_2 \\ \mathbb{0}_2 & e^{(-i\boldsymbol{\theta}+\boldsymbol{\eta})\boldsymbol{\sigma}/2} \end{pmatrix} \equiv \begin{pmatrix} e^{\lambda_L} & \mathbb{0}_2 \\ \mathbb{0}_2 & e^{\lambda_R} \end{pmatrix}. \tag{1.78}$$

Note that Λ_D can be rewritten as an exponential of a certain matrix λ_D:

$$
\begin{aligned}
\Lambda_D =& \begin{pmatrix} e^{\lambda_L} & \mathbb{0}_2 \\ \mathbb{0}_2 & e^{\lambda_R} \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{\lambda_L^n}{n!} & \mathbb{0}_2 \\ \mathbb{0}_2 & \sum_{n=0}^{\infty} \frac{\lambda_R^n}{n!} \end{pmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} \lambda_L^n & \mathbb{0}_2 \\ \mathbb{0}_2 & \lambda_R^n \end{pmatrix} \\
=& \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} \lambda_L & \mathbb{0}_2 \\ \mathbb{0}_2 & \lambda_R \end{pmatrix}^n = \exp\left\{ \begin{pmatrix} \lambda_L & \mathbb{0}_2 \\ \mathbb{0}_2 & \lambda_R \end{pmatrix} \right\} \\
=& \exp\left\{ \begin{pmatrix} (-i\boldsymbol{\theta} - \boldsymbol{\eta})\frac{\boldsymbol{\sigma}}{2} & \mathbb{0}_2 \\ \mathbb{0}_2 & (-i\boldsymbol{\theta} + \boldsymbol{\eta})\frac{\boldsymbol{\sigma}}{2} \end{pmatrix} \right\} \equiv e^{\lambda_D}.
\end{aligned}
\tag{1.79}
$$

According to this writing, we only need to prove that $\lambda_D = -\frac{i}{2}\lambda_{\mu\nu}S^{\mu\nu}$. In order to do it, we separate the rotational and boost contributions in λ_D:

$$
\lambda_D = -\frac{i}{2}\theta^i \begin{pmatrix} \sigma^i & \mathbb{0}_2 \\ \mathbb{0}_2 & \sigma^i \end{pmatrix} - \frac{\eta^i}{2} \begin{pmatrix} \sigma^i & \mathbb{0}_2 \\ \mathbb{0}_2 & -\sigma^i \end{pmatrix}.
\tag{1.80}
$$

Then, we use Eq. (1.77) to get

$$
\begin{pmatrix} \sigma^i & \mathbb{0}_2 \\ \mathbb{0}_2 & \sigma^i \end{pmatrix} = \epsilon^{ijk}S^{jk}, \qquad \begin{pmatrix} \sigma^i & \mathbb{0}_2 \\ \mathbb{0}_2 & -\sigma^i \end{pmatrix} = 2i\, S^{0i},
\tag{1.81}
$$

so, also exploiting Eq. (1.75), we rewrite λ_D as[3]

$$
\begin{aligned}
\lambda_D =& -\frac{i}{2}\theta^i \epsilon^{ijk} S^{jk} - i\eta^i S^{0i} = -\frac{i}{4}\epsilon^{ilm}\epsilon^{ijk}\lambda^{lm}S^{jk} - i\lambda_{0i}S^{0i} \\
=& -\frac{i}{4}[\lambda_{jk} - \lambda_{kj}]S^{jk} - \frac{i}{2}[\lambda_{0i}S^{0i} + \lambda_{i0}S^{i0}] = -\frac{i}{2}[\lambda_{jk}S^{jk} + \lambda_{0i}S^{0i} + \lambda_{i0}S^{i0}] \\
=& -\frac{i}{2}\lambda_{\mu\nu}S^{\mu\nu}.
\end{aligned}
\tag{1.82}
$$

We have thus obtained Eq. (1.74). □

1.3 Poincarè group

We have dedicated the previous sections to the Lorentz symmetry on both spacetime and fields. However, there is an other symmetry that we have not taken into account yet, that is the *translation symmetry*. We require that all the fields (scalars, spinors, vectors, etc.) transform as scalar under translations. Formally, we are imposing that, under a translation $x^\mu \mapsto x'^\mu = x^\mu + \epsilon^\mu$, any field transforms as

$$
\phi'(x') = \phi'(x + \epsilon) = \phi(x).
\tag{1.83}
$$

Our spacetime has dimension four, so we expect to have four generators (one for the timelike component and the other three for the spacelike components): we label them with $P^\mu = (P^0, \boldsymbol{P})$. The action of the translation group on a field is thus[4]

$$
\phi(x + \epsilon) = e^{-i\epsilon_\mu P^\mu}\phi(x).
\tag{1.84}
$$

What happens if we take ϵ^μ as an infinitesimal paremeters of the translation?

$$
\begin{aligned}
\delta\phi(x) =& \phi'(x) - \phi(x) = \phi'(x' - \epsilon) - \phi(x) = \overbrace{\phi'(x')}^{=\phi(x)} - \epsilon_\mu\partial^\mu \overbrace{\phi'(x')}^{=\phi(x)} - \phi(x) + \mathcal{O}(\epsilon^2) \\
=& -\epsilon_\mu\partial^\mu\phi(x) + \mathcal{O}(\epsilon^2).
\end{aligned}
\tag{1.85}
$$

[3] In the following steps, we use the antisymmetric property of $\lambda_{\mu\nu}$ and $S^{\mu\nu}$.

[4] Note that the tranlsation group is non-compact.

However, we can also compute the variation $\delta\phi(x)$ as

$$\begin{aligned}\delta\phi(x) =&\phi'(x) - \phi(x) = \phi'(x' - \epsilon) - \phi(x) = e^{-i(-\epsilon_\mu)P^\mu}\phi'(x') - \phi(x)\\ =&\phi'(x') + -i\epsilon_\mu P^\mu \phi'(x') - \phi(x) + \mathcal{O}(\epsilon^2)\\ =&+i\epsilon_\mu P^\mu \phi(x) + \mathcal{O}(\epsilon^2).\end{aligned} \tag{1.86}$$

Comparing the above equations, we see that the memontum operator P^μ can be represented as

$$P^\mu = +i\partial^\mu, \tag{1.87}$$

that is[5]

$$P^0 = i\partial^0 = i\partial_0 = i\frac{\partial}{\partial t} \equiv H, \qquad P^i = i\partial^i = -i\partial_i = -i\frac{\partial}{\partial x^i}. \tag{1.88}$$

Since the derivatives commute, we find the well-known commutation rules of the translation group:

$$[P^\mu, P^\nu] = 0. \tag{1.89}$$

Translations plus Lorentz transformations form a group called *Poincaré group*, also known as the *inhomogeneous Lorentz group*, denoted by $ISO(1,3)$ (where I stands for inhomogeneous). We need to find its commutation rules, i.e. all the commutators between H, $\boldsymbol{P}$, $\boldsymbol{J}$ and $\boldsymbol{K}$. We start noting that H must commute with $\boldsymbol{J}$, since the former acts on the timelike component and the latter on the spacelike components. In addition, $P^\mu = i\partial^\mu$ is a four-vector, so its spacelike components must transform as a vector under rotations. These two observations imply

$$[J^i, H] = 0, \qquad [J^i, P^j] = i\epsilon^{ijk}P^k. \tag{1.90}$$

As for the the commutators between P^μ and $\boldsymbol{K}$, one can formally prove that

$$[K^i, H] = iP^i, \qquad [K^i, P^j] = iH\delta^{ij}. \tag{1.91}$$

Let's clarify it through an example.

Example: Computation of $[K^i, P^\mu]$ for a scalar field

Let $\phi(x)$ be a scalar field and suppose to take a Lorentz boost on it along the x-axis, i.e. $\Lambda(\eta) = \exp\{i\eta K^1\}$. According to Eq. (1.37) we have

$$\phi'(x') = \phi(x) = \phi(\Lambda(\eta)^{-1} \cdot x'). \tag{1.92}$$

We already computed $\Lambda(\eta) = \exp\{i\eta K^1\}$ in Eq. (1.19), so we can easily get its inverse as $\Lambda(\eta)^{-1} = \Lambda(-\eta)$. Therefore we rewrite the above equation as

$$\phi'(x') = \phi(\Lambda(-\eta)\cdot x') = \phi\Big(\cosh\eta\cdot t' - \sinh\eta\cdot x'^1, -\sinh\eta\cdot t' + \cosh\eta\cdot x'^1, x'^2, x'^3\Big). \tag{1.93}$$

Let η be an infinitesimal parameter and erase the apex on x'. It follows that

$$\begin{aligned}\delta\phi(x) =&\phi'(x) - \phi(x) = \phi\Big(t - \eta\cdot x^1 + \mathcal{O}(\eta^2), -\eta\cdot t + x^1 + \mathcal{O}(\eta^2), x^2, x^3\Big) - \phi(x)\\ =&\phi(x) - \eta\, x^1\partial_t\phi(x) - \eta\, t\,\partial_1\phi(x) - \phi(x) + \mathcal{O}(\eta^2)\\ =&+i\eta\Big[i(x^1\partial_t + t\,\partial_1)\Big]\phi(x) + \mathcal{O}(\eta^2).\end{aligned} \tag{1.94}$$

[5] H is the hamiltonian of the system.

However, we can also rewrite the variation $\delta\phi$ as

$$\begin{aligned}\delta\phi(x) =&\phi'(x) - \phi(x) = \phi'(\Lambda(\eta)^{-1}\cdot x') - \phi(x) = e^{-i\eta K^1}\overbrace{\phi'(x')}^{=\phi(x)} - \phi(x)\\ =&\phi(x) - i\eta K^1\phi(x) - \phi(x) + \mathcal{O}(\eta^2)\\ =&-i\eta K^1\phi(x) + \mathcal{O}(\eta^2),\end{aligned} \tag{1.95}$$

from which we get

$$K^1 = -i(x^1\partial_t + t\,\partial_1) = +i(t\,\partial^1 - x^1\partial^0). \tag{1.96}$$

Therefore, we can now compute the commutator

$$[K^1, H] = [i(t\,\partial^1 - x^1\partial^0), i\partial^0] = +\partial^1 = -iP^1. \tag{1.97}$$

Pay attention, you get the result with the wrong sign. FIX IT!!!! In a completely analogous way we compute the commutators with $K^{2,3}$, getting

$$[K^i, H] = iP^i. \tag{1.98}$$

As for the the commutators between $\boldsymbol{K}$ and $\boldsymbol{P}$, note that

$$\begin{aligned}[K^1, P^1] =&[i(t\,\partial^1 - x^1\partial^0), i\partial^1] = +\partial^0 = -iH,\\ [K^1, P^{2,3}] =&[i(t\,\partial^1 - x^1\partial^0), i\partial^{2,3}] = 0\end{aligned} \tag{1.99}$$

Pay attention, you get the result with the wrong sign. FIX IT!!!! and the same for $K^{2,3}$, so we get

$$[K^i, P^j] = i\delta^{ij}H. \tag{1.100}$$

At this point, we can finally collect all the commutation rules of the Poincaré group, that are

$$\begin{aligned}[J^i, J^j] =& i\epsilon^{ijk}J^k,\\ [J^i, K^j] =& i\epsilon^{ijk}K^k,\\ [J^i, H] =&0,\\ [J^i, P^j] =& i\epsilon^{ijk}P^k,\\ [K^i, K^j] =&-i\epsilon^{ijk}J^k,\\ [K^i, H] =& iP^i,\\ [K^i, P^j] =& iH\delta^{ij},\\ [P^i, H] =&0,\\ [P^i, P^j] =&0.\end{aligned} \tag{1.101}$$

Note that $\boldsymbol{K}$ does not commute with H, so it is not a conserved quantity. This is why the eigenvalues of $\boldsymbol{K}$ are not used for labeling physical states.

1.4 Wigner Little Group

Throughout this chapter we have built the background of a theory in which the Lagrangian is invariant under the Poincaré group. At the classical level, the Lagrangian $\mathcal{L}(\phi, \partial_\mu\phi)$ is all we need for the description of the dynamics of the system. We could "simply" derive the equations of motion from $\mathcal{L}(\phi, \partial_\mu\phi)$, solve them and get an explicit expression of the fields. For example, a photon is described by the field $A^\mu(x)$ that mathematically is nothing but a function of x^μ. However, at the quantum level the situation is a bit more subtle: any field is now an operator that acts on physical states of a proper Hilbert space. It means that now we need $A^\mu(x)$ acting on a physical state in order to create a photon.

Since we need a quantum theory, we have to figure out both

(i) how to represent a physical state,

(ii) how a physical state transforms under a Poincaré transformation.

In this regards, let x^μ and a^μ be two four-vectors. A general Poincaré transformation acting on x^μ is

$$(\Lambda, a):\ x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{1.102}$$

so $(\Lambda, a) \sim (\mathbb{1}, a)\cdot(\Lambda, 0)$, where $(\mathbb{1}, a)$ is a pure translation and $(\Lambda, 0)$ a pure Lorentz transformation. Therefore, if $|\Psi\rangle$ is a physical state, then it must transform under a Poincaré transformation as

$$|\Psi'\rangle = U(\Lambda, a)\,|\Psi\rangle\,, \tag{1.103}$$

where $U(\Lambda, a)$ is a unitary representation of (Λ, a). Let's try to face the above two open questions. We know that H and $\boldsymbol{P}$ are conserved quantities (they commute with H), so it is natural to write a physical state in terms of eigenvectors of the four-momentum operator P^μ. In principle, a physical state can depend on other quantum numbers that we still do not know at the moment and that we label with σ. Therefore, we call $|\Psi\rangle = |\boldsymbol{p}, \sigma\rangle$ a generic one-particle physical state and we impose the condition to be eigenvector of P^μ, i.e.

$$P^\mu\,|\boldsymbol{p}, \sigma\rangle = p^\mu\,|\boldsymbol{p}, \sigma\rangle\,. \tag{1.104}$$

According to this statement, the representation of the translations is straightforward:

$$U(\mathbb{1}, a)\,|\boldsymbol{p}, \sigma\rangle = e^{-ia_\mu P^\mu}\,|\boldsymbol{p}, \sigma\rangle = e^{-ia_\mu p^\mu}\,|\boldsymbol{p}, \sigma\rangle\,. \tag{1.105}$$

It is less trivial to represent a Lorentz transformation of the type $U(\Lambda, 0)\,|\boldsymbol{p}, \sigma\rangle \equiv U(\Lambda)\,|\boldsymbol{p}, \sigma\rangle$. To do so, we start partitioning the space with respect to all the possible momentum p^μ, depending on the value of $p^2 = p_\mu p^\mu$ (remember that this quantity is Lorentz invariant):

(i) $p^2 > 0$,

(ii) $p^2 = 0$,

(iii) $p^2 < 0$.

We focus on the first two regions, namely the ones in which an on shell physical state satisfies the equation $p^2 - m^2 = 0$. In these two regions, we fix a generic four-momentum k^μ and we write p^μ as a Lorentz boost with respect to k^μ, i.e.

$$p^\mu = L(p)^\mu{}_\nu k^\nu, \tag{1.106}$$

where $L(p)$ is a suitable Lorentz transformation (we label it with $L(p)$ to distinguish it from the Lorentz transformation Λ). The physical state $|\boldsymbol{p}, \sigma\rangle$ thus becomes

$$|\boldsymbol{p}, \sigma\rangle = U(L(p))\,|\boldsymbol{k}, \sigma\rangle\,. \tag{1.107}$$

At this point, since $U(\Lambda, 0)$ is a representation of the Lorentz group, it must be a *homomorphism*, that is

$$U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\cdot\Lambda_2). \tag{1.108}$$

We use this property to write

$$\begin{aligned}
U(\Lambda, 0)\,|\boldsymbol{p}, \sigma\rangle \equiv& U(\Lambda)\,|\boldsymbol{p}, \sigma\rangle = U(\Lambda)U(L(p))\,|\boldsymbol{k}, \sigma\rangle = U(\Lambda\cdot L(p))\,|\boldsymbol{k}, \sigma\rangle \\
=& U\Big(\overbrace{[L(\Lambda p)\cdot L(\Lambda p)^{-1}]}^{=\mathbb{1}}\cdot\Lambda\cdot L(p)\Big)\,|\boldsymbol{k}, \sigma\rangle \\
=& U(L(\Lambda p))\,U\Big(L(\Lambda p)^{-1}\cdot\Lambda\cdot L(p)\Big)\,|\boldsymbol{k}, \sigma\rangle\,.
\end{aligned} \tag{1.109}$$

Let's focus on the Lorentz transformation $L(\Lambda p)^{-1} \cdot \Lambda \cdot L(p)$: note that

$$\begin{aligned} L(p) : k^\mu &\mapsto p^\mu, \\ \Lambda \cdot L(p) : k^\mu &\mapsto p^\mu \mapsto (\Lambda p)^\mu, \\ L(\Lambda p)^{-1} \cdot \Lambda \cdot L(p) : k^\mu &\mapsto p^\mu \mapsto (\Lambda p)^\mu \mapsto k^\mu. \end{aligned} \tag{1.110}$$

Therefore the operator $C = L(\Lambda p)^{-1} \cdot \Lambda \cdot L(p)$ is a Lorentz transformation that leaves any four-momentum k^μ unchanged, i.e.

$$C^\mu{}_\nu k^\nu = k^\mu \qquad \forall\, k^\mu. \tag{1.111}$$

All the Lorentz operators C that satisfy the above condition form a subgrop of $SO(1,3)_+$ called *Wigner little group*. Suppose to know the explicit expression of a representation of C, namely

$$U(C)\,|\boldsymbol{k}, \sigma\rangle = \sum_{\sigma'} [C_{\mathcal{R}}]_{\sigma\sigma'}\, |\boldsymbol{k}, \sigma'\rangle\,. \tag{1.112}$$

Then we can conclude that

$$\begin{aligned} U(\Lambda, 0)\,|\boldsymbol{p}, \sigma\rangle \equiv & U(\Lambda)\,|\boldsymbol{p}, \sigma\rangle = U(L(\Lambda p))U(C)\,|\boldsymbol{k}, \sigma\rangle = \sum_{\sigma'} [C_{\mathcal{R}}]_{\sigma\sigma'} \overbrace{U(L(\Lambda p))\,|\boldsymbol{k}, \sigma'\rangle}^{=|\Lambda \cdot p, \sigma'\rangle} \\ = & \sum_{\sigma'} [C_{\mathcal{R}}]_{\sigma\sigma'}\, |\Lambda p, \sigma'\rangle\,. \end{aligned} \tag{1.113}$$

We see that the problem of finding the representation of $U(\Lambda, 0)$ acting on $|\boldsymbol{p}, \sigma\rangle$ has been reduced to the problem of findind the representation of the little group, namely to obtain the expression of the coefficients $[C_{\mathcal{R}}]_{\sigma\sigma'}$. In other words, in order to figure out how a one-particle physical state transforms under a Lorentz transformation it is sufficient to know how it transforms under the little group.

The next goal is to identify the operators C of the little group. Since they satisfy Eq. (1.111), they can be written as

$$C^\mu{}_\nu p^\nu = (e^{\lambda_c})^\mu{}_\nu p^\nu = \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda_c^n)^\mu{}_\nu p^\nu = p^\mu + \sum_{n=1}^{\infty} \frac{1}{n!} (\lambda_c^n)^\mu{}_\nu p^\nu = p^\mu, \tag{1.114}$$

which implies

$$(\lambda_c)^\mu{}_\nu p^\nu = 0. \tag{1.115}$$

Therefore, we have to find the most general element of the Lorentz algebra that vanishes on a four-momentum p^μ. But at this point we must separate the case with $p^2 > 0$ from that with $p^2 = 0$.

Before proceeding with the details of the two cases, we point out that the representations of the physical states are always labeled by the Casimir operators. For example, we saw that the Casimir operator[6] of the rotation group is given by $\boldsymbol{L}^2$, whose eigenvalues $j(j+1)$ labels the dimension of the representation (that is $2j+1$). In the case of the Lorentz group, there are two Casimir operator: the first one is $P_\mu P^\mu$, which gives (acting on a physical state)

$$P_\mu P^\mu\,|\boldsymbol{p}, \sigma\rangle = p_\mu p^\mu\,|\boldsymbol{p}, \sigma\rangle = m^2\,|\boldsymbol{p}, \sigma\rangle\,. \tag{1.116}$$

As for the second one, it is less trivial to find and we need to exploit Eq. (1.115) in order to do it.

[6]Remember that the Casimir operator commutes with all the generators of the group.

$p^2 > 0$

We want to find the most general expression of an element of the Lorentz algebra that satisfies the Eq. (1.115) in the massive case $p^2 = m^2 > 0$. In order to do it, let's start from considering k^μ in the rest frame, that is $p^\mu = (m, 0, 0, 0)$. Using the general expression (1.13) of an element of the Lorentz algebra, we find

$$[\lambda_c(m > 0)]^\mu{}_\nu p^\nu = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & r_3 & -r_2 \\ b_2 & -r_3 & 0 & +r_1 \\ b_3 & r_2 & -r_1 & 0 \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{1.117}$$

which implies

$$[\lambda_c(m > 0)]^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -r_3 & r_2 \\ 0 & r_3 & 0 & -r_1 \\ 0 & -r_2 & r_1 & 0 \end{pmatrix} = -r_1 \cdot iJ^1 - r_2 \cdot iJ^2 - r_3 \cdot iJ^3. \tag{1.118}$$

This is nothing but a combination of the angular momentum matrices J^i of Eq. (1.14). Therefore, the Wigner little group of the Lorentz group in the massive case is $SU(2)$ (we choose it instead of $SO(3)$ since we want to include spinor representations). We know that this group is represented by means of the Casimir operator $\boldsymbol{J}$, so it is natual to introduce the operator

$$W^\mu := \begin{pmatrix} 0 \\ m\boldsymbol{J} \end{pmatrix}, \tag{1.119}$$

known as *Pauli–Lubanski operator*, such that $W_\mu W^\mu = -m^2 \boldsymbol{J}^2$. $W_\mu W^\mu$ is the Casimir of the massive case. Note that $W_\mu W^\mu$ is invariant under boosts, so the expression we have just found is fully general and it holds in any frame. It follows that the action of the Casimir on a physical state will be

$$W_\mu W^\mu \left|\boldsymbol{p}, j\right\rangle = -m^2 j(j+1) \left|\boldsymbol{p}, j\right\rangle. \tag{1.120}$$

We have found that a massive one-particle physical state is labeled by the mass m and the spin j. Note that a field of mass $m > 0$ has $2j + 1$ degrees of freedom (namely the dimension of the representation).

$p^2 = 0$

Conceptually we simply have to repeat the steps of the massive case. However, we cannot choose the rest frame anymore, since it does not satisfies the condition $p^2 = 0$. A suitable choice is given by $p^\mu = (E, 0, 0, E)$. The Eq. (1.115) gives

$$[\lambda_c(m = 0)]^\mu{}_\nu p^\nu = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & r_3 & -r_2 \\ b_2 & -r_3 & 0 & r_1 \\ b_3 & r_2 & -r_1 & 0 \end{pmatrix} \begin{pmatrix} E \\ 0 \\ 0 \\ E \end{pmatrix} = \begin{pmatrix} b_3 E \\ (b_1 - r_2)E \\ (b_2 + r_1)E \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{1.121}$$

which implies

$$\begin{cases} b_3 = 0 \\ r_2 - b_1 = 0 \\ r_1 + b_2 = 0 \end{cases}. \tag{1.122}$$

$\lambda_c(m = 0)$ can thus be written as

$$[\lambda_c(m = 0)]^\mu{}_\nu = \begin{pmatrix} 0 & b_1 & b_2 & 0 \\ b_1 & 0 & r_3 & -b_1 \\ b_2 & -r_3 & 0 & -b_2 \\ 0 & b_1 & b_2 & 0 \end{pmatrix} = -b_1 \cdot i(J^2 + K^1) - b_2 \cdot i(J^1 - K^2) - b_3 \cdot iJ^3. \tag{1.123}$$

At this point, we can introduce the Pauli-Lubanski operator W^μ, defined as

$$W^0 = W^3 = EJ^3, \qquad W^1 := E(J^1 - K^2), \qquad W^2 = E(J^2 + K^1), \tag{1.124}$$

and get the operator

$$W_\mu W^\mu = -E^2\Big[(J^1 - K^2)^2 + (J^2 + K^1)^2\Big]. \tag{1.125}$$

We leave as an exercise to the reader to prove that $W_\mu W^\mu$ is actually a Casimir operator.

Here comes a question: what type of little group do we have in this massless case? To answer this question, it is useful to introduce the operators

$$A^\mu{}_\nu = i(J^2 + K^1)^\mu{}_\nu, \qquad B^\mu{}_\nu = i(-J^1 + K^2)^\mu{}_\nu, \tag{1.126}$$

according to which the Casimir operator becomes

$$W_\mu W^\mu = -E^2(A^2 + B^2). \tag{1.127}$$

Note that the commutation laws between A, B and J^3 are

$$[J^3, A] = iB, \qquad [J^3, B] = -iA, \qquad [A, B] = 0, \tag{1.128}$$

so the algebra generated by these three operators is the same that is generated by p^x, p^y and $L^z = xp^y - yp^x$. It follows that the Wigner little group of the Lorentz group in the massless case corresponds to the translations and rotations of a Euclidean plane, i.e. $I\,SO(2)$. A and B play the role of translation operators. In our case they are note hermitian, since they are 4×4 matrices, but we can take them hermitian as long as we consider an infinite-dimensional representation. The latter is the case of one-particle physical states with momentum $\boldsymbol{p}$. A and B commute, so we can define a set of physical states $|\boldsymbol{p}, a, b\rangle$ that are eigenstate of both at the same time:[7]

$$A\,|\boldsymbol{p}, a, b\rangle = a\,|\boldsymbol{p}, a, b\rangle\,, \qquad B\,|\boldsymbol{p}, a, b\rangle = b\,|\boldsymbol{p}, a, b\rangle\,. \tag{1.129}$$

Everything seems fine, but using this result we can derive a weird consequence. In fact, let's introduce the physical state

$$|\boldsymbol{p}, a, b, \theta\rangle := e^{-i\theta J^3}\,|\boldsymbol{p}, a, b\rangle\,, \tag{1.130}$$

where θ is an arbitrary angle. Then we see that

$$A\,|\boldsymbol{p}, a, b, \theta\rangle = A\,e^{-i\theta J^3}\,|\boldsymbol{p}, a, b\rangle = e^{-i\theta J^3}\Big(e^{i\theta J^3} A\, e^{-i\theta J^3}\Big)\,|\boldsymbol{p}, a, b\rangle \tag{1.131}$$

and, using the commutation laws, one can prove that

$$\Big(e^{i\theta J^3} A\, e^{-i\theta J^3}\Big) = A\cos\theta - B\sin\theta. \tag{1.132}$$

We thus obtain

$$A\,|\boldsymbol{p}, a, b, \theta\rangle = (a\cos\theta - b\sin\theta)\,|\boldsymbol{p}, a, b, \theta\rangle\,. \tag{1.133}$$

and, similarly,

$$B\,|\boldsymbol{p}, a, b, \theta\rangle = (a\sin\theta + b\cos\theta)\,|\boldsymbol{p}, a, b, \theta\rangle\,. \tag{1.134}$$

This is a weird result: we have found a representation of a one-particle massless physical state that has a continuous internal degree of freedom θ unless $a = b = 0$. However, since we physically do not see such a degree of freedom in experiments, we must impose the condition $a = b = 0$, that in term of the Casimir operator becomes

$$W_\mu W^\mu = 0. \tag{1.135}$$

[7] Note that on such a set physical states the Casimir operator is diagonal, since $W_\mu W^\mu\,|\boldsymbol{p}, a, b\rangle = -E^2(a^2 + b^2)\,|\boldsymbol{p}, a, b\rangle$. This is as it should be, since the Casimir operator defines the representation of the physical states.

This result seems reasonable, since it corresponds to the limit $m \to 0$ of the massive Casimir operator $W_\mu W^\mu |\boldsymbol{p}, j\rangle = -m^2 j(j+1) |\boldsymbol{p}, j\rangle$.

From the fact that the only possible generator of the little group is J^3, the Wigner little group of the massless case is at the end of the day $SO(2)$. The latter is an abelian group, so $\dim[SO(2)] = 1$. Therefore a massless particle can have only one degree of freedom. If we take a one-dimensional representation of J^3, this is nothing but a number, usually called h in literature: it is the *helicity* of the particle. One can prove that h is quantized, so $h = \{0, \pm 1/2, \pm 1, ...\}$, but the proof is quite technical and beyond our scope. What is interesting for us instead is that we have shown that a massless particle has only one degree of freedom and that this corresponds to helicity h.

In principle two massless particles with oppisite helicities $+h$ and $-h$ are two different particles. However, in the above argument we found the Wigner little gruop to be generated by J^3 simply because at the beginning we set the four-momentum p^μ to be along the z-axis. If we repeated this proof choosing p^μ along the x- or y-axis, we would find J^1 and J^2 to be the generators of the Wigner little group respectively. Therefore, the helicity can be thought as the projection of the angular momentum along the direction of the motion, i.e.

$$\boldsymbol{h} = \hat{\boldsymbol{p}} \cdot \boldsymbol{J}. \tag{1.136}$$

Note that $\boldsymbol{h}$ is a pseudo-vector, so it changes sign under a parity transformation. If we want a quantum field theory invariant under the parity transformation, for every particle with helicity $+h$ there must also be its counterpart with helicity $-h$. This is why it is more natural to define a massless particle as the superposition of two physical states with opposite helicities.[8]

[8]Note that parity is not a foundamental symmetry of our universe, since it is broken by weak force.