Towards a General Subtraction Formula for NNLO QCD Corrections to Processes At Hadron Colliders

Davide Maria Tagliabue

Based on the following Refs.
[Devoto, Melnikov, Röntsch, Signorile-Signorile, DMT, 2310.17598]
[Devoto, Melnikov, Röntsch, Signorile-Signorile, DMT, Tresoldi, 2503.15251]
[Devoto, Melnikov, Röntsch, Signorile-Signorile, DMT, Tresoldi, 25xx.xxxxx]

LOOPFEST XXIII

Karlsruher Institut für Technologie

1. The problem of subtracting infrared (IR) singularities at NLO is considered solved. By "solved", we mean the following:

   i) There exist well-defined prescriptions (Catani-Seymour and FKS subtractions) that allow for the cancellation of $\mathcal{O}(\epsilon^{-2})$ and $\mathcal{O}(\epsilon^{-1})$ IR singularities explicitly

   ii) A clear set of rules is available to obtain IR-finite results systematically

   iii) These methods can be applied to compute NLO cross sections for any process at the LHC

2. At NNLO we do not yet have a fully general framework comparable to the NLO case

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i) This is a very active field of research (see talk by T. Gehrmann)

- Analytic Sector Subtraction [Magnea et al. 1806.09570, ...]
- ColorfullNNLO [Del Duca et al. 1603.08927, ...]
- Geometric IR subtraction [Herzog 1804.07949, ...]
- Universal Factorization [Anastasiou et al. 2008.12293, ...]
- Nested Soft-Collinear Subtraction [Caola et al. 1702.01352, ...]
- Antenna [Gehrmann-De Ridder et al. 0505111, ...]
- STRIPPER [Czakon 1005.0274, ...]
- Unsubtraction [Sborlini et al. 1608.01584, ...]
- FDR [Pittau 1208.5457, ...]

ii) General NNLO corrections are currently feasible only for processes with color-singlet initial states [Bertolotti et al. 2212.11190]

iii) Impressive case-by-case results exist, such as the NNLO prediction for $pp \to X + 3$ Jets [Czakon et al. 2106.05331]

iv) However, no explicit demonstration of IR divergence cancellation at hadron colliders is currently available

3.

Although a fully general solution to IR subtraction at NLO was established over twenty years ago, extending it to NNLO remains a challenging task. The main reasons are:

i) <u>*Overlapping singularities:*</u> at NNLO, one encounters simultaneous soft and collinear divergences that overlap

ii) <u>*Phase-space partitioning:*</u> to disentangle these singularities, we must partition and sector the phase space

iii) <u>*Loss of transparency:*</u> while sectoring renders each integral finite and computable, it obscures the physical interpretation of the cancellation mechanisms

4.

We have to identify the building blocks of the class of QCD processes $pp \to X + N$ Jets. A good starting point is the process $\mathscr{A}_0 : \; q\bar{q} \to X + N g$

[Devoto, Melnikov, Röntsch, Signorile-Signorile, DMT, 2310.17598]

Main Features of the Nested Soft-Collinear at NNLO

$$\langle \Delta^{(\mathfrak{m}\mathfrak{n})} F_{\mathrm{LM}}^{ab}[\,\ldots\,|\,\mathfrak{m},\mathfrak{n}]\rangle$$

Main Features of the Nested Soft-Collinear at NNLO

Double-Soft Counterterm

Can be integrated over the phase spaces $[\mathrm{d}p_{\mathfrak{m}}]$ and $[\mathrm{d}p_{\mathfrak{n}}]$

[Caola, Delto, Frellesvig, Melnikov '18]

Soft-Regulated Term

It contains TRIPLE- and SINGLE-COLLINEAR singularities

$$\langle \Delta^{(\mathfrak{m}\mathfrak{n})} F_{\rm LM}^{ab}[\ldots\,|\,\mathfrak{m},\mathfrak{n}]\rangle = \langle S_{\mathfrak{m}\mathfrak{n}} F_{\rm LM}^{ab}[\ldots\,|\,\mathfrak{m},\mathfrak{n}]\rangle + \langle \bar{S}_{\mathfrak{m}\mathfrak{n}} S_{\mathfrak{n}} \Delta^{(\mathfrak{m}\mathfrak{n})} F_{\rm LM}^{ab}[\ldots\,|\,\mathfrak{m},\mathfrak{n}]\rangle + \langle \bar{S}_{\mathfrak{m}\mathfrak{n}} \bar{S}_{\mathfrak{n}}\, \Delta^{(\mathfrak{m}\mathfrak{n})} F_{\rm LM}^{ab}[\ldots\,|\,\mathfrak{m},\mathfrak{n}]\rangle$$

Single-Soft Counterterm

It contains SINGLE-COLLINEAR singularities

Main Features of the Nested Soft-Collinear at NNLO

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Can be integrated over the phase spaces $[dp_{\mathfrak{m}}]$ and $[dp_{\mathfrak{n}}]$

[Caola, Delto, Frellesvig, Melnikov '18]

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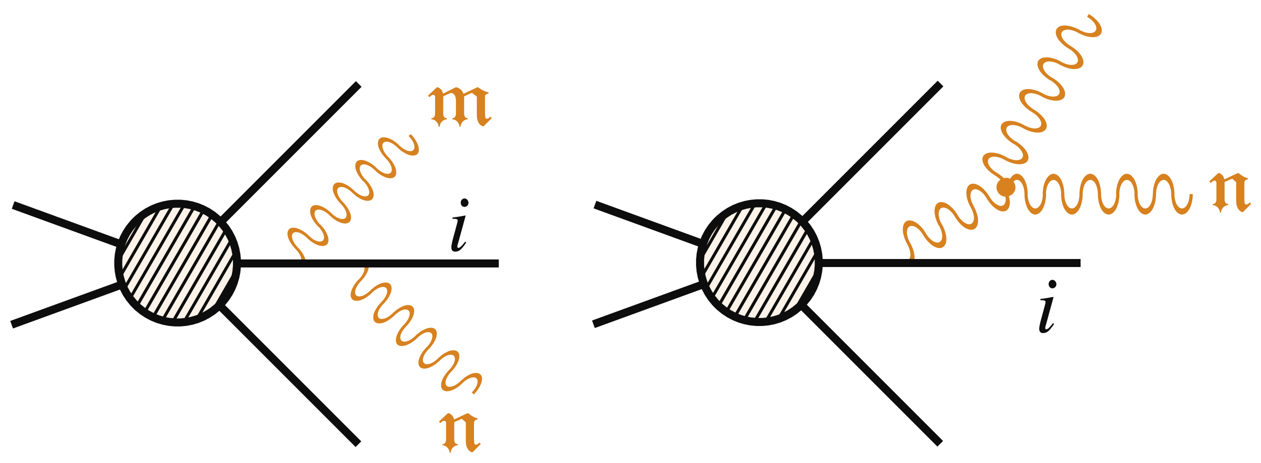
Problem of Overlapping Singularities

$$1 = \sum_{i,j\in\mathscr{H}_f} \omega^{\mathfrak{m}i,\mathfrak{n}j}$$

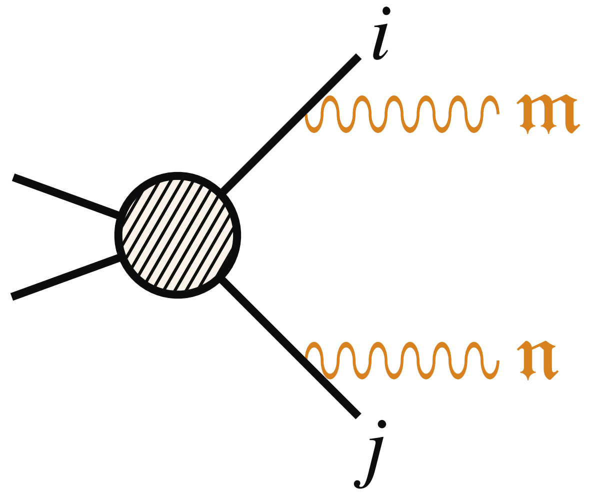
Triple-Collinear Sector

$$C_{i\mathfrak{m}} - C_{i\mathfrak{n}} - C_{\mathfrak{m}\mathfrak{n}} - C_{\mathfrak{m}\mathfrak{n},i}$$



Double-Collinear Sector

$$C_{i\mathfrak{m}} - C_{j\mathfrak{n}}$$



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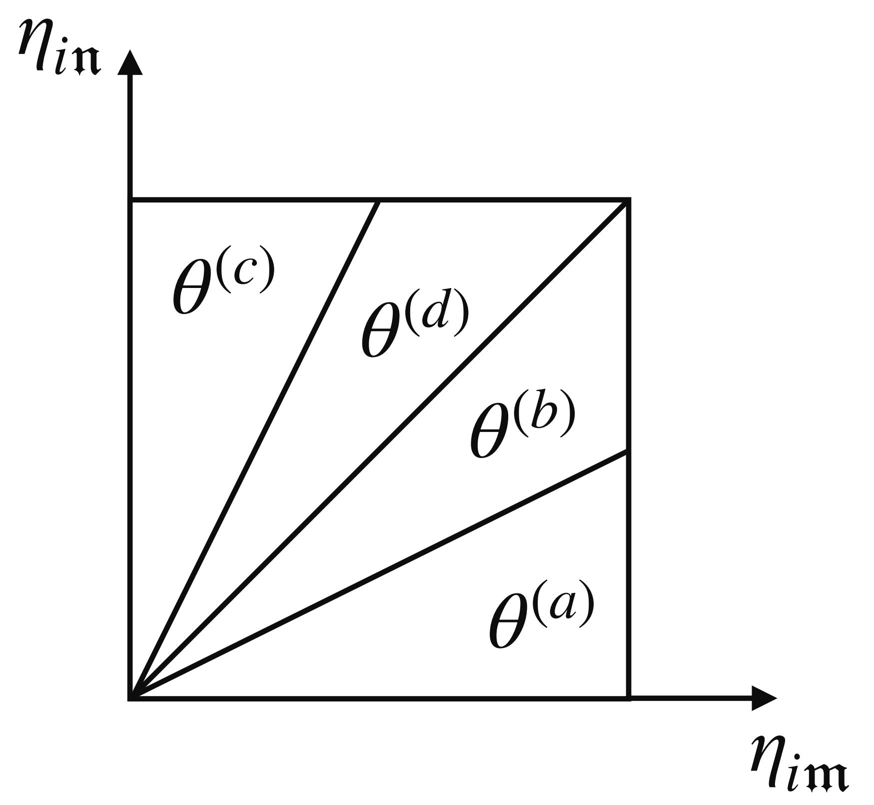
Sector Decomposition

[Czakon '10]

$$\eta_{ij} = (1 - \cos\theta_{ij})/2$$



Angular Ordering

$$\theta^{(a)} = \Theta\big(\eta_{i\mathfrak{n}} < \eta_{i\mathfrak{m}}/2\big)$$

$$\theta^{(c)} = \Theta\big(\eta_{i\mathfrak{m}} < \eta_{i\mathfrak{n}}/2\big)$$

$$\theta^{(b)} = \Theta\big(\eta_{i\mathfrak{m}}/2 < \eta_{i\mathfrak{n}}/2 < \eta_{i\mathfrak{m}}\big)$$

$$\theta^{(d)} = \Theta\big(\eta_{i\mathfrak{n}}/2 < \eta_{i\mathfrak{m}}/2 < \eta_{i\mathfrak{n}}\big)$$

Problem of Overlapping Singularities

$$1 = \sum_{i,j\in\mathcal{H}_f} \omega^{\mathfrak{m}i,\mathfrak{n}j}$$

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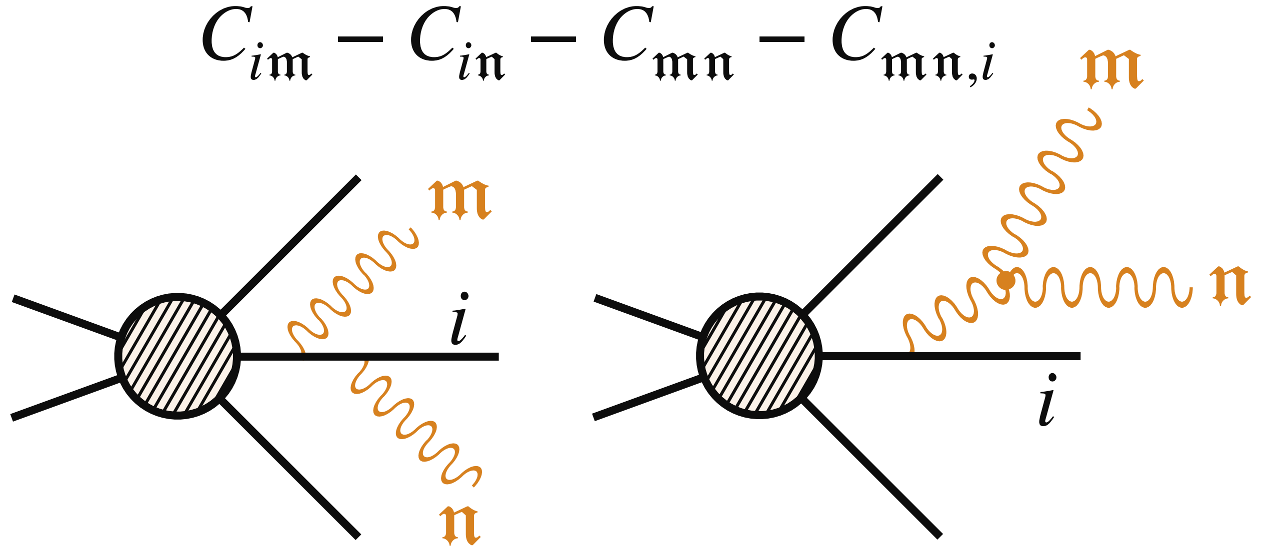
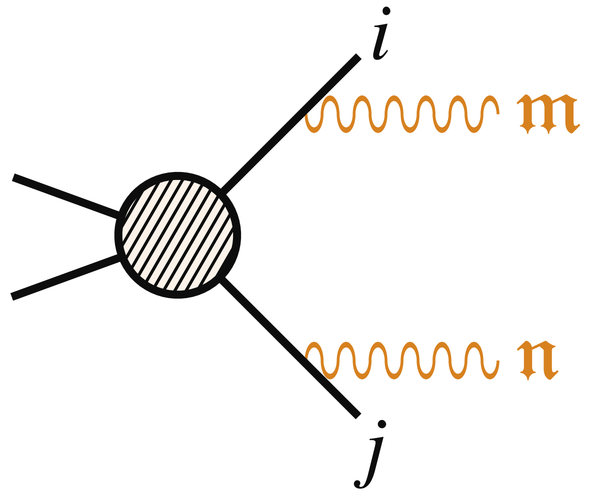


Double-Collinear Sector

$$C_{i\mathfrak{m}} - C_{j\mathfrak{n}}$$



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$$\langle \Delta^{(\mathfrak{mn})} F_{\mathrm{LM}}^{ab}[\ldots|\mathfrak{m},\mathfrak{n}]\rangle = \langle S_{\mathfrak{mn}} F_{\mathrm{LM}}^{ab}[\ldots|\mathfrak{m},\mathfrak{n}]\rangle + \langle \overline{S}_{\mathfrak{mn}} S_{\mathfrak{n}} \Delta^{(\mathfrak{mn})} F_{\mathrm{LM}}^{ab}[\ldots|\mathfrak{m},\mathfrak{n}]\rangle + \langle \overline{S}_{\mathfrak{mn}} \overline{S}_{\mathfrak{n}} \Delta^{(\mathfrak{mn})} F_{\mathrm{LM}}^{ab}[\ldots|\mathfrak{m},\mathfrak{n}]\rangle$$

Single-Soft Counterterm

It contains SINGLE-COLLINEAR singularities

In principle, this formula can be applied to any process at the LHC.

In practice, identifying structures that can be combined with the VV and RV contributions becomes nearly impossible, rendering the calculation heavily process-dependent.

$$\Omega_1 = \sum_{(ij)} \overline{C}_{i\mathfrak{m}} \overline{C}_{j\mathfrak{n}} [\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}]\, \omega^{\mathfrak{m}i,\mathfrak{n}j} + \sum_{i\in\mathcal{H}} \left[\overline{C}_{i\mathfrak{n}}\theta^{(a)} + \overline{C}_{\mathfrak{mn}}\theta^{(b)} + \overline{C}_{i\mathfrak{m}}\theta^{(c)} + \overline{C}_{\mathfrak{mn}}\theta^{(d)}\right] [\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}]\, \overline{C}_{\mathfrak{mn},i}\, \omega^{\mathfrak{m}i,\mathfrak{n}i}$$

$$\Omega_2 = \sum_{i\in\mathcal{H}} \left[\overline{C}_{i\mathfrak{n}}\theta^{(a)} + \overline{C}_{\mathfrak{mn}}\theta^{(b)} + \overline{C}_{i\mathfrak{m}}\theta^{(c)} + \overline{C}_{\mathfrak{mn}}\theta^{(d)}\right] [\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}]\, C_{\mathfrak{mn},i}\, \omega^{\mathfrak{m}i,\mathfrak{n}i}$$

$$\Omega_3 = -\sum_{(ij)} C_{j\mathfrak{n}} C_{i\mathfrak{m}} [\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}]\, \omega^{\mathfrak{m}i,\mathfrak{n}j}$$

$$\Omega_4 = \sum_{(ij)} \left[C_{i\mathfrak{m}}[\mathrm{d}p_{\mathfrak{m}}] + C_{j\mathfrak{n}}[\mathrm{d}p_{\mathfrak{n}}]\right] \omega^{\mathfrak{m}i,\mathfrak{n}j} + \sum_{i\in\mathcal{H}} \left[C_{i\mathfrak{n}}\theta^{(a)} + C_{\mathfrak{mn}}\theta^{(b)} + C_{i\mathfrak{m}}\theta^{(c)} + C_{\mathfrak{mn}}\theta^{(d)}\right] [\mathrm{d}p_{\mathfrak{m}}][\mathrm{d}p_{\mathfrak{n}}]\, \omega^{\mathfrak{m}i,\mathfrak{n}i}$$

$$\sum_{i=1}^{4} \langle \overline{S}_{\mathfrak{mn}} \overline{S}_{\mathfrak{n}}\, \Omega_i\, \Delta^{(\mathfrak{mn})} F_{\mathrm{LM}}^{ab}[\ldots|\mathfrak{m},\mathfrak{n}]\rangle$$

computable, it obscures the physical interpretation of the cancellation mechanisms

4. We have to identify the building blocks of the class of QCD processes $pp \to X + N$ Jets. A good starting point is the process $\mathscr{A}_0 : \; q\bar{q} \to X + N g$ [Devoto, Melnikov, Röntsch, Signorile-Signorile, DMT, 2310.17598]

i) From the subtraction point of view, it is the most complicated channel

ii) From a combinatorial perspective, it is the simplest channel, as it is symmetric in both the initial and final states

iii) For NLO and NNLO corrections, we add only gluons

5. Now, we want to see what happens if we add quarks to the process. Thus, the natural step consists of analyzing $\mathscr{B}_1 : \; gq \to X + (N-1)g + q$ [Devoto

NLO

Soft and Collinear Regularizations in FKS/NSC

$$
\langle \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_\mathfrak{g}] \rangle = \overbrace{\langle S_\mathfrak{m}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_\mathfrak{g}] \rangle}^{\text{Soft Counterterm}} + \overbrace{\sum_{i \in \mathscr{H}} \langle \bar{S}_\mathfrak{m} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_\mathfrak{g}] \rangle}^{\text{Hard-Collinear Counterterm}}
$$

$$
+ \langle \mathscr{O}_{\mathrm{NLO}}^{(\mathfrak{m})}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_\mathfrak{g}] \rangle
$$

Soft and Collinear Regularizations in FKS/NSC

Soft Counterterm

Hard-Collinear Counterterm

$$\langle \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle = \langle S_{\mathfrak{m}} \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle + \sum_{i \in \mathscr{H}} \langle \bar{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle$$
$$+ \langle \mathscr{O}_{\mathrm{NLO}}^{(\mathfrak{m})} \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle$$

The Soft Operator

$$\langle S_{\mathfrak{m}} \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle \sim \langle \boxed{I_{\mathrm{S}}(\epsilon)} \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \rangle$$

$$I_{\mathrm{S}}(\epsilon) = -\frac{(2E_{\max}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{i \neq j} \eta_{ij}^{-\epsilon} K_{ij} \boxed{(\boldsymbol{T}_i \cdot \boldsymbol{T}_j)}$$

Color - Correlations

Soft and Collinear Regularizations in FKS/NSC

Soft Counterterm

Hard-Collinear Counterterm

$$\langle \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_g] \rangle = \langle S_{\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_g] \rangle + \sum_{i \in \mathscr{H}} \langle \bar{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_g] \rangle$$
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Color - Correlations

+

The Virtual Operator

$$2 s_{ab}\, \mathrm{d}\hat{\sigma}_{\mathscr{A}_0}^{\mathrm{V}} \equiv \langle F_{\mathrm{LV}}^{\mathscr{A}_0} \rangle \sim \langle \boxed{I_{\mathrm{V}}(\epsilon)} \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \rangle$$

$$I_{\mathrm{V}}(\epsilon) = \bar{I}_1(\epsilon) + \bar{I}_1^{\dagger}(\epsilon)$$

$$\bar{I}_1(\epsilon) = \frac{1}{2} \sum_{i \neq j} \left(\frac{1}{\epsilon^2} + \frac{\gamma_i}{\epsilon\, \boldsymbol{T}_i^2} \right) \left(\frac{\mu^2 e^{i\pi\lambda_{ij}}}{s_{ij}} \right)^{\epsilon} \boxed{(\boldsymbol{T}_i \cdot \boldsymbol{T}_j)}$$

Color - Correlations

Soft and Collinear Regularizations in FKS/NSC

Soft Counterterm

Hard-Collinear Counterterm

$$\langle \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle = \langle S_{\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle + \sum_{i \in \mathscr{H}} \langle \bar{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle$$

$$+ \langle \mathcal{O}_{\mathrm{NLO}}^{(\mathfrak{m})}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle$$

$$\boxed{I_{\mathrm{S}}(\epsilon) + I_{\mathrm{V}}(\epsilon) = -\sum_{i \in \mathscr{H}} \frac{1}{\epsilon} \left(2\boldsymbol{T}_i^2 L_i + \gamma_i \right) + \mathcal{O}(\epsilon^0)}$$

- The pole of $\mathcal{O}(\epsilon^{-2})$ vanishes
- No color - correlations at $\mathcal{O}(\epsilon^{-1})$
- Trivially dependent on the number of hard partons N

Soft and Collinear Regularizations in FKS/NSC

Soft Counterterm

Hard-Collinear Counterterm

$$\langle \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle = \langle S_{\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle + \sum_{i \in \mathscr{H}} \langle \bar{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle$$

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- The pole of $\mathcal{O}(\epsilon^{-2})$ vanishes
- No color - correlations at $\mathcal{O}(\epsilon^{-1})$
- Trivially dependent on the number of hard partons N

$$\sum_{i \in \mathscr{H}} \langle \bar{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \sim \langle \boxed{I_{\mathrm{C}}(\epsilon)} \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \rangle$$

$$\boxed{I_{\mathrm{C}}(\epsilon) = \frac{\Gamma_{a,g} + \Gamma_{b,q}}{\epsilon} + \sum_{i \in \mathscr{H}_{\mathfrak{f}_{\mathfrak{g}}}} \frac{\Gamma_{i,g \to gg}}{\epsilon}}$$

Soft and Collinear Regularizations in FKS/NSC

$$\langle \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_\mathfrak{g}]\rangle = \underbrace{\langle S_\mathfrak{m}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_\mathfrak{g}]\rangle}_{\text{Soft Counterterm}} + \underbrace{\sum_{i\in\mathscr{H}} \langle \bar{S}_\mathfrak{m} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_\mathfrak{g}]\rangle}_{\text{Hard-Collinear Counterterm}}$$

$$+\langle \mathscr{O}_{\mathrm{NLO}}^{(\mathfrak{m})}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_\mathfrak{g}]\rangle$$

$$\boxed{I_{\mathrm{S}}(\epsilon) + I_{\mathrm{V}}(\epsilon) = -\sum_{i\in\mathscr{H}} \frac{1}{\epsilon}\left(2\boldsymbol{T}_i^2\, L_i + \gamma_i\right) + \mathscr{O}(\epsilon^0)}$$

- The pole of $\mathscr{O}(\epsilon^{-2})$ vanishes
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- Trivially dependent on the number of hard partons N

$$\boxed{I_{\mathrm{C}}(\epsilon) = +\sum_{i\in\mathscr{H}} \frac{1}{\epsilon}\left(2\boldsymbol{T}_i^2\, L_i + \gamma_i\right) + \mathscr{O}(\epsilon^0)}$$

Soft and Collinear Regularizations in FKS/NSC

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$$+ \langle \mathcal{O}_{\mathrm{NLO}}^{(\mathfrak{m})}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle$$

$$I_{\mathrm{T}}(\epsilon) = I_{\mathrm{S}}(\epsilon) + I_{\mathrm{V}}(\epsilon) + I_{\mathrm{C}}(\epsilon) = \mathcal{O}(\epsilon^0)$$

The Total Operator

- It does not contain poles
- General procedure for color - correlations
- Trivially dependent on the number of hard partons

Soft and Collinear Regularizations in FKS/NSC

$$\langle \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle = \underbrace{\langle S_{\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle}_{\text{Soft Counterterm}} + \underbrace{\sum_{i\in\mathscr{H}} \langle \bar{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle}_{\text{Hard-Collinear Counterterm}}$$

$$+ \langle \mathscr{O}_{\mathrm{NLO}}^{(\mathfrak{m})}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle$$

The Final Result at NLO

$$2s_{ab}\, \mathrm{d}\hat{\sigma}_{\mathscr{A}_0}^{\mathrm{NLO}} = [\alpha_{\mathrm{s}}] \langle \boxed{I_{\mathrm{T}}^{(0)}} \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \rangle + [\alpha_{\mathrm{s}}] \Big[\langle \mathscr{P}_{aa}^{\mathrm{NLO}} \otimes F_{\mathrm{LM}}^{\mathscr{A}_0} \rangle + \langle F_{\mathrm{LM}}^{\mathscr{A}_0} \otimes \mathscr{P}_{bb}^{\mathrm{NLO}} \rangle \Big] + \langle \mathscr{O}_{\mathrm{NLO}}^{(\mathfrak{m})}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle + \langle F_{\mathrm{LV,fin}}^{\mathscr{A}_0} \rangle$$

Total Operator at $\mathscr{O}(\epsilon^0)$

$$I_{\mathrm{T}}^{(0)} = -\sum_{i\neq j} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \Bigg[\left(2L_{\max} + \frac{1}{2}\log\eta_{ij} \right) \log\eta_{ij} - \frac{1}{2} L_{ij} \left(L_{ij} + \frac{2\gamma_i}{\boldsymbol{T}_i^2} \right)$$
$$+ \mathrm{Li}_2(1-\eta_{ij}) + \frac{\pi^2}{2}\lambda_{ij} \Bigg] + \sum_{i\in\mathscr{H}} \boldsymbol{T}_i^2 \Bigg[2L_{\max}^2 - \frac{\pi^2}{6} - \left(2\widetilde{L}_i \tilde{\gamma}_i^{(0)} - \tilde{\gamma}_i^{(1)}\right)\theta_{\mathscr{H}_{\mathrm{f}}}$$
$$-2\left(L_i^2 + 2L_i\widetilde{L}_i + \widetilde{L}_i \frac{\gamma_i}{\boldsymbol{T}_i^2} \right) \bar{\theta}_{\mathscr{H}_{\mathrm{f}}} \Bigg]$$

NB

$$\beta_0 \equiv \frac{11}{6} C_{\mathrm{A}}$$

NNLO

The Double-Virtual Contribution

Quartic Color-Correlations

$I_{\mathrm{V}}^2(\epsilon) \sim (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)(\boldsymbol{T}_k \cdot \boldsymbol{T}_l)$

$$\begin{aligned}\langle F_{\mathrm{VV}}^{\mathscr{A}_0} \rangle = [\alpha_{\mathrm{s}}]^2 \Big\langle & \boxed{\frac{1}{2} I_{\mathrm{V}}^2(\epsilon)} - \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_{\mathrm{E}}}} \left(\frac{\beta_0}{\epsilon} I_{\mathrm{V}}(\epsilon) - \left(\frac{\beta_0}{\epsilon} + K \right) I_{\mathrm{V}}(2\epsilon) \right) \Big] \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \Big\rangle \\ + [\alpha_{\mathrm{s}}]^2 \Big\langle & \boxed{\left[-\frac{1}{2} \left[\overline{I}_1(\epsilon), \overline{I}_1^{\dagger}(\epsilon) \right] + \left[\mathcal{H}_{2,\mathrm{cd}} + \mathcal{H}_{2,\mathrm{tc}} + \mathrm{h.c.} \right] \right]} \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \Big\rangle \\ + [\alpha_{\mathrm{s}}] & \langle I_{\mathrm{V}}(\epsilon) \cdot F_{\mathrm{LV,fin}}^{\mathscr{A}_0} \rangle + \langle F_{\mathrm{LV^2,fin}}^{\mathscr{A}_0} \rangle + \langle F_{\mathrm{VV,fin}}^{\mathscr{A}_0} \rangle \,.\end{aligned}$$

Triple Color-Correlations

$f_{abc} T_i^a T_j^b T_k^c$

Questions

- Does the presence of I_{V}^2 imply a structure with I_{T}^2 ?
- Who else contains triple color-correlaions?
- Are the I-operators enough to cancel all the color-correlated poles?

Soft Counterterm

Hard-Collinear Counterterm

$$\langle \Delta^{(\mathfrak{m})} F_{\mathrm{RV}}^{\mathscr{A}_1}[\mathfrak{m}_g] \rangle = \langle S_{\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{RV}}^{\mathscr{A}_1}[\mathfrak{m}_g] \rangle + \boxed{\sum_{i \in \mathscr{H}} \langle \bar{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{RV}}^{\mathscr{A}_1}[\mathfrak{m}_g] \rangle}$$

$$+ \langle \mathscr{O}_{\mathrm{NLO}}^{(\mathfrak{m})}\, \Delta^{(\mathfrak{m})} F_{\mathrm{RV}}^{\mathscr{A}_1}[\mathfrak{m}_g] \rangle$$

The Real-Virtual Contribution

IF COMBINED WITH THE VV

$$\frac{1}{2}\left[I_{\mathrm{V}}^2 + (I_{\mathrm{S}} + I_{\mathrm{C}})I_{\mathrm{V}} + I_{\mathrm{V}}(I_{\mathrm{S}} + I_{\mathrm{C}})\right]$$

$$\langle S_{\mathfrak{m}} F_{\mathrm{RV}}^{\mathscr{A}_1}[\mathfrak{m}_g] \rangle = [\alpha_s]^2 \left\langle \boxed{\frac{1}{2}\left[I_{\mathrm{S}}(\epsilon) I_{\mathrm{V}}(\epsilon) + I_{\mathrm{V}}(\epsilon) I_{\mathrm{S}}(\epsilon)\right]} \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle$$
$$- [\alpha_s]^2 \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} \frac{\beta_0}{\epsilon} \langle I_{\mathrm{S}}(\epsilon) \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \rangle - \frac{[\alpha_s]^2}{\epsilon^2} C_A A_K(\epsilon) \langle \widetilde{I}_{\mathrm{S}}(2\epsilon) \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \rangle$$
$$+ [\alpha_s]^2 \left\langle \boxed{[I_{\mathrm{S}}(\epsilon), I_-(\epsilon)] + I_{\mathrm{tri}}^{\mathrm{RV}}(\epsilon)} \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle .$$

IF COMBINED WITH THE VV

$$-[I_+, I_-] + [2I_+ + I_{\mathrm{S}}, I_-] + \mathscr{H}_{2,\mathrm{tc}} + \mathscr{H}_{2,\mathrm{tc}}^{\dagger} + I_{\mathrm{tri}}^{\mathrm{RV}}$$

These operators can describe the NNLO

$$Y_{\mathrm{RR}}^{(\mathrm{ss})} = \left\langle \frac{1}{2} I_{\mathrm{S}}^2 \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle + \cdots$$

$$Y_{\mathrm{VV}} = \left\langle \frac{1}{2} I_{\mathrm{V}}^2 \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle + \cdots$$

$$Y_{\mathrm{RR}}^{(\mathrm{cc})} = \left\langle \frac{1}{2} I_{\mathrm{C}}^2 \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle + \cdots$$

$$\mathrm{d}\sigma^{\mathrm{NNLO}} \sim \left\langle \frac{1}{2} I_{\mathrm{T}}^2 \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle + \cdots$$

$$Y_{\mathrm{RV}}^{(\mathrm{s})} = \left\langle I_{\mathrm{V}}\, I_{\mathrm{C}} \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle + \cdots$$

$$Y_{\mathrm{RR}}^{(\mathrm{shc})} = \left\langle I_{\mathrm{S}}\, I_{\mathrm{C}} \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle + \cdots$$

$$Y_{\mathrm{RV}}^{(\mathrm{shc})} = \left\langle \frac{1}{2} \left(I_{\mathrm{S}}\, I_{\mathrm{V}} + I_{\mathrm{V}}\, I_{\mathrm{S}} \right) \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle + \cdots$$

iii) For NLO and NNLO corrections, we add only gluons

5. Now, we want to see what happens if we add quarks to the process. Thus, the natural step consists of analyzing $\mathscr{B}_0: \; gq \to X + (N-1)g + q$ [Devoto, Melnikov, Röntsch, Signorile-Signorile, DMT, Tresoldi, 2503.15251]

i) It contains all the IR divergences of the process $\mathscr{A}_0$

ii) From a combinatorial perspective, it is more complex than $\mathscr{A}_0$. Indeed, if we add quarks, the collinear limit in the initial state can change the flavour of the incoming parton. We do not consider this problem now

iii) We can see how the final-state quark anomalous dimension arises

6. At this point, we have all the ingredients to approach the general problem,

We consider the following Born:

$$\mathscr{B}_0: \; a_g b_q \to X + (N-1)g + q$$

$$2s_{ab}\,\mathrm{d}\hat{\sigma}^{\mathrm{LO}}_{\mathscr{B}_0} = \langle F^{gq}_{\mathrm{LM}}[\{g\}_{N-1}, q]\rangle = \langle F^{\mathscr{B}_0}_{\mathrm{LM}}\rangle$$

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Damping Factors

They select the final-state parton that is potentially unresolved

$$2s_{ab}\,\mathrm{d}\hat{\sigma}^{\mathrm{R}}_{\mathscr{B}_1} = \langle F^{gq}_{\mathrm{LM}}[\{g\}_N, q]\rangle = \sum_{i\in\mathscr{H}_{\mathrm{f}}} \langle \Delta^{(i)} F^{gq}_{\mathrm{LM}}[\{g\}_N, q]\rangle$$
$$= \Big\langle \Delta^{(\mathfrak{m})} \underline{\Big(F^{gq}_{\mathrm{LM}}[\{g\}_{N-1}, q \,|\, \mathfrak{m}_{\mathfrak{g}}] + F^{gq}_{\mathrm{LM}}[\{g\}_N \,|\, \mathfrak{m}_{\mathfrak{q}}]\Big)} \Big\rangle$$

Rename the Damping Factors

Two contributions identified

- Unresolved gluon
- Unresolved quark

We consider the following Born:

$$\mathscr{B}_0 : \; a_g b_q \to X + (N-1)g + q$$

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Damping Factors

They select the final-state parton that is potentially unresolved

$$2s_{ab}\, \mathrm{d}\hat{\sigma}^{\mathrm{R}}_{\mathscr{B}_1} = \langle F^{gq}_{\mathrm{LM}}[\{g\}_N, q]\rangle = \sum_{i\in\mathscr{H}_f} \langle \Delta^{(i)}\, F^{gq}_{\mathrm{LM}}[\{g\}_N, q]\rangle$$

$$= \Big\langle \Delta^{(\mathfrak{m})} \underline{\Big(F^{gq}_{\mathrm{LM}}[\{g\}_{N-1}, q\,|\,\mathfrak{m}_{\mathfrak{g}}] + F^{gq}_{\mathrm{LM}}[\{g\}_N\,|\,\mathfrak{m}_{\mathfrak{q}}]\Big)} \Big\rangle$$

Rename the Damping Factors

Two contributions identified

- Unresolved gluon
- Unresolved quark

Soft Counterterm **Hard-Collinear Counterterm** **Hard-Collinear Counterterm**

$$\langle \Delta^{(\mathfrak{m})} F^{\mathscr{B}_1}_{\mathrm{LM}}[\mathfrak{m}_{\mathfrak{g}}]\rangle = \langle S_{\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F^{\mathscr{B}_1}_{\mathrm{LM}}[\mathfrak{m}_{\mathfrak{g}}]\rangle + \sum_{i\in\mathscr{H}} \langle \bar{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F^{\mathscr{B}_1}_{\mathrm{LM}}[\mathfrak{m}_{\mathfrak{g}}]\rangle + \langle \mathscr{O}^{(\mathfrak{m})}_{\mathrm{NLO}}\, \Delta^{(\mathfrak{m})} F^{\mathscr{B}_1}_{\mathrm{LM}}[\mathfrak{m}_{\mathfrak{g}}]\rangle$$

$$\langle \Delta^{(\mathfrak{m})} F^{\mathscr{B}_1}_{\mathrm{LM}}[\mathfrak{m}_{\mathfrak{q}}]\rangle = \sum_{i\in\mathscr{H}} \langle C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F^{\mathscr{B}_1}_{\mathrm{LM}}[\mathfrak{m}_{\mathfrak{q}}]\rangle + \langle \mathscr{O}^{(\mathfrak{m})}_{\mathrm{NLO}}\, \Delta^{(\mathfrak{m})} F^{\mathscr{B}_1}_{\mathrm{LM}}[\mathfrak{m}_{\mathfrak{q}}]\rangle$$

$$I_{\mathrm{C}}^{\mathrm{inc}}(\epsilon) = \frac{\Gamma_{a,g} + \Gamma_{b,q}}{\epsilon} + \sum_{i \in \mathcal{H}_{\mathrm{f_g}}} \frac{\Gamma_{i,g\to gg}}{\epsilon} + \frac{\Gamma_{q\to qg}}{\epsilon}$$

$$\sum_{i \in \mathcal{H}} \langle \bar{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathcal{B}_1}[\mathfrak{m}_{\mathfrak{g}}] \sim \langle \boxed{I_{\mathrm{C}}^{\mathrm{inc}}(\epsilon)} \cdot F_{\mathrm{LM}}^{\mathcal{B}_0} \rangle$$

$$\sum_{i \in \mathcal{H}} \langle C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathcal{B}_1}[\mathfrak{m}_{\mathfrak{q}}] \sim \left\langle \frac{\Gamma_{q\to gq}}{\epsilon} \cdot F_{\mathrm{LM}}^{\mathcal{B}_0} \right\rangle$$

Hard-Collinear Counterterm

Hard-Collinear Counterterm

$$\langle \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathcal{B}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle = \langle S_{\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathcal{B}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle + \sum_{i \in \mathcal{H}} \langle \bar{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathcal{B}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle + \langle \mathcal{O}_{\mathrm{NLO}}^{(\mathfrak{m})}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathcal{B}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle$$

$$\langle \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathcal{B}_1}[\mathfrak{m}_{\mathfrak{q}}] \rangle = \sum_{i \in \mathcal{H}} \langle C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathcal{B}_1}[\mathfrak{m}_{\mathfrak{q}}] \rangle + \langle \mathcal{O}_{\mathrm{NLO}}^{(\mathfrak{m})}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathcal{B}_1}[\mathfrak{m}_{\mathfrak{q}}] \rangle$$

$$\Gamma_{q\to qg} \propto -\int_0^1 \mathrm{d}z \left(1 - \lim_{z\to 1}\right) z \left[\frac{P_{qq}(z)}{[z(1-z)]^{2\epsilon}}\right]$$

$$\Gamma_{q\to gq} \propto -\int_0^1 \mathrm{d}z \left(1 - \lim_{z\to 1}\right) z \left[\frac{P_{qq}(1-z)}{[z(1-z)]^{2\epsilon}}\right]$$

$$\Gamma_{q} \propto -\int_0^1 \mathrm{d}z \left(1 - \lim_{z\to 1}\right) \left[\frac{P_{qq}(z)}{[z(1-z)]^{2\epsilon}}\right] \equiv \gamma_q + \mathscr{O}(\epsilon)$$

$$I_{\mathrm{C}}^{\mathrm{inc}}(\epsilon) = \frac{\Gamma_{a,g}+\Gamma_{b,q}}{\epsilon} + \sum_{i\in\mathscr{H}_{\mathfrak{f}_g}} \frac{\Gamma_{i,g\to gg}}{\epsilon} + \boxed{\frac{\Gamma_{q\to qg}}{\epsilon}}$$

$$\boxed{\frac{\Gamma_q}{\epsilon}}$$

$$\sum_{i\in\mathscr{H}} \langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{B}_1}[\mathfrak{m}_g] \sim \langle \boxed{I_{\mathrm{C}}^{\mathrm{inc}}(\epsilon)} \cdot F_{\mathrm{LM}}^{\mathscr{B}_0}\rangle$$

$$\sum_{i\in\mathscr{H}} \langle C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{B}_1}[\mathfrak{m}_q] \sim \left\langle \boxed{\frac{\Gamma_{q\to gq}}{\epsilon}} \cdot F_{\mathrm{LM}}^{\mathscr{B}_0}\right\rangle$$

Hard-Collinear Counterterm

Hard-Collinear Counterterm

$$\langle \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{B}_1}[\mathfrak{m}_g]\rangle = \langle S_{\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{B}_1}[\mathfrak{m}_g]\rangle + \sum_{i\in\mathscr{H}} \langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{B}_1}[\mathfrak{m}_g]\rangle + \langle \mathscr{O}_{\mathrm{NLO}}^{(\mathfrak{m})}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{B}_1}[\mathfrak{m}_g]\rangle$$

$$\langle \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{B}_1}[\mathfrak{m}_q]\rangle = \sum_{i\in\mathscr{H}} \langle C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{B}_1}[\mathfrak{m}_q]\rangle + \langle \mathscr{O}_{\mathrm{NLO}}^{(\mathfrak{m})}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{B}_1}[\mathfrak{m}_q]\rangle$$

$$\Gamma_{g\to gg} \propto -\int_0^1 \mathrm{d}z \left(1 - \lim_{z\to 1}\right) z \left[\frac{P_{gg}(z)}{[z(1-z)]^{2\epsilon}}\right]$$

$$\Gamma_{g\to q\bar{q}} \propto -\int_0^1 \mathrm{d}z \left(1 - \lim_{z\to 1}\right) z \left[\frac{P_{gq}(z)}{[z(1-z)]^{2\epsilon}}\right]$$

$$\Gamma_g \propto -\frac{1}{2}\int_0^1 \mathrm{d}z \left(1 - \lim_{z\to 1}\right) \left[\frac{P_{gg}(z) + P_{gq}(z)}{[z(1-z)]^{2\epsilon}}\right] \equiv \gamma_g + \mathcal{O}(\epsilon)$$

We can recreate the gluon anomalous dimension following a similar idea

iii) We can see how the final-state quark anomalous dimension arises

6. At this point, we have all the ingredients to approach the general problem. We now consider the process $pp \to X + N$ Jets

i) The generalization is mostly a matter of combinatorics (we need a smart parametrization of the channels)

ii) For the unboosted contribution, we need to recreate the I_{T} operator

iii) For the boosted contribution, our expression must match the renormalization of the PDFs

7. The final step in obtaining a final subtraction formula at NNLO for the process $pp \to X + N$ Jets consists of generalizing the proof from $n = 1$ to a

PRELIMINARY RESULTS

$n_{\mathrm{f}} = 1$

$$2s\,\mathrm{d}\sigma^{\mathrm{NLO}} = \sum_{\{a,b\}} (f_a \otimes f_b) \otimes \Bigg[\sum_n \big\langle \mathcal{O}^{(\mathfrak{m})}_{\mathrm{NLO}} \Delta^{(\mathfrak{m})} F^{ab}_{\mathrm{R},n}[\mathfrak{m}] \big\rangle + \sum_{\mathcal{B}^{ab}_{\mathrm{f}}} \Big[[\alpha_{\mathrm{s}}] \big\langle I^{(0)}_{\mathrm{T}} \cdot F^{ab}_{\mathrm{LM}}[\mathcal{B}^{ab}_{\mathrm{f}}] \big\rangle + \big\langle F^{ab}_{\mathrm{LV,fin}}[\mathcal{B}^{ab}_{\mathrm{f}}] \big\rangle \Big]$$

$$+ \sum_x [\alpha_{\mathrm{s}}] \Big[\sum_{\mathcal{B}^{xb}_{\mathrm{f}}} \big\langle \mathcal{P}^{\mathrm{NLO}}_{xa} \otimes F^{xb}_{\mathrm{LM}}[\mathcal{B}^{xb}_{\mathrm{f}}] \big\rangle + \sum_{\mathcal{B}^{ax}_{\mathrm{f}}} \big\langle F^{ax}_{\mathrm{LM}}[\mathcal{B}^{ax}_{\mathrm{f}}] \otimes \mathcal{P}^{\mathrm{NLO}}_{xb} \big\rangle \Big] \Bigg],$$

Parametrization of the Born processes

$\{a,b\}$	$\mathcal{B}^{ab}_{\mathrm{f}}$	N *even*	N *odd*
$\{q,\bar{q}\}, \{g,g\}$	$\big(\{g\}_{N-2n}, \{q\}_n, \{\bar{q}\}_n\big)$	$n \in [0, N/2]$	$n \in [0, (N-1)/2]$
$\{q,q\}$	$\big(\{g\}_{N-2-2n}, \{q\}_{n+2}, \{\bar{q}\}_n\big)$	$n \in [0, N/2-1]$	$n \in [0, (N-3)/2]$
$\{\bar{q},\bar{q}\}$	$\big(\{g\}_{N-2-2n}, \{q\}_n, \{\bar{q}\}_{n+2}\big)$	$n \in [0, N/2-1]$	$n \in [0, (N-3)/2]$
$\{g,q\}$	$\big(\{g\}_{N-1-2n}, \{q\}_{n+1}, \{\bar{q}\}_n\big)$	$n \in [0, N/2-1]$	$n \in [0, (N-1)/2]$
$\{g,\bar{q}\}$	$\big(\{g\}_{N-1-2n}, \{q\}_n, \{\bar{q}\}_{n+1}\big)$	$n \in [0, N/2-1]$	$n \in [0, (N-1)/2]$

PRELIMINARY RESULTS

$n_\mathrm{f} = 1$

$$
\begin{aligned}
2s\,\mathrm{d}\sigma^{\mathrm{NLO}} = \sum_{\{a,b\}} (f_a \otimes f_b) \otimes \Bigg[&\sum_{n} \big\langle \mathcal{O}^{(\mathfrak{m})}_{\mathrm{NLO}} \Delta^{(\mathfrak{m})} F^{ab}_{\mathrm{R},n}[\mathfrak{m}] \big\rangle + \sum_{\mathcal{B}^{ab}_{\mathrm{f}}} \Big[[\alpha_\mathrm{s}] \big\langle I^{(0)}_{\mathrm{T}} \cdot F^{ab}_{\mathrm{LM}}[\mathcal{B}^{ab}_{\mathrm{f}}] \big\rangle + \big\langle F^{ab}_{\mathrm{LV,fin}}[\mathcal{B}^{ab}_{\mathrm{f}}] \big\rangle \Big] \\
&+ \sum_{x} [\alpha_\mathrm{s}] \Big[\sum_{\mathcal{B}^{xb}_{\mathrm{f}}} \big\langle \mathcal{P}^{\mathrm{NLO}}_{xa} \otimes F^{xb}_{\mathrm{LM}}[\mathcal{B}^{xb}_{\mathrm{f}}] \big\rangle + \sum_{\mathcal{B}^{ax}_{\mathrm{f}}} \big\langle F^{ax}_{\mathrm{LM}}[\mathcal{B}^{ax}_{\mathrm{f}}] \otimes \mathcal{P}^{\mathrm{NLO}}_{xb} \big\rangle \Big] \Bigg] ,
\end{aligned}
$$

***N* EVEN**

$$F^{ab}_{\mathrm{R},n}[\mathfrak{m}] \stackrel{\mathrm{def}}{=} \Theta^{N/2}\Big[F^{\mathcal{S}^{\mathrm{NLO}}_{n,ab}}_{\mathrm{LM}}[\mathfrak{m}_g] + \Theta_1 F^{\mathcal{S}^{\mathrm{NLO}}_{n,ab}}_{\mathrm{LM}}[\mathfrak{m}_q] + \Theta_1 F^{\mathcal{S}^{\mathrm{NLO}}_{n,ab}}_{\mathrm{LM}}[\mathfrak{m}_{\bar{q}}] \Big] , \qquad \{a,b\} \in \{q,\bar{q}\}, \{g,g\} ,$$

$$F^{qq}_{\mathrm{R},n}[\mathfrak{m}] \stackrel{\mathrm{def}}{=} \Theta^{N/2-1}\Big[F^{\mathcal{Q}^{\mathrm{NLO}}_{n,qq}}_{\mathrm{LM}}[\mathfrak{m}_g] + F^{\mathcal{Q}^{\mathrm{NLO}}_{n,qq}}_{\mathrm{LM}}[\mathfrak{m}_q] + \Theta_1 F^{\mathcal{Q}^{\mathrm{NLO}}_{n,qq}}_{\mathrm{LM}}[\mathfrak{m}_{\bar{q}}] \Big] ,$$

$$F^{\bar{q}\bar{q}}_{\mathrm{R},n}[\mathfrak{m}] \stackrel{\mathrm{def}}{=} F^{qq}_{\mathrm{R},n}[\mathfrak{m}]|_{q\leftrightarrow\bar{q}} ,$$

$$F^{ab}_{\mathrm{R},n}[\mathfrak{m}] \stackrel{\mathrm{def}}{=} \Theta^{N/2}\Big[\Theta^{N/2-1} F^{\mathcal{A}^{\mathrm{NLO}}_{n,ab}}_{\mathrm{LM}}[\mathfrak{m}_g] + F^{\mathcal{A}^{\mathrm{NLO}}_{n,ab}}_{\mathrm{LM}}[\mathfrak{m}_q] + \Theta_1 F^{\mathcal{A}^{\mathrm{NLO}}_{n,ab}}_{\mathrm{LM}}[\mathfrak{m}_{\bar{q}}] \Big] , \qquad \{a,b\} \in \{g,q\} ,$$

$$F^{g\bar{q}}_{\mathrm{R},n}[\mathfrak{m}] \stackrel{\mathrm{def}}{=} F^{gq}_{\mathrm{R},n}[\mathfrak{m}]|_{q\leftrightarrow\bar{q}} ,$$

***N* ODD**

$$F^{ab}_{\mathrm{R},n}[\mathfrak{m}] \stackrel{\mathrm{def}}{=} \Theta^{(N+1)/2}\Big[\Theta_1^{(N-1)/2} F^{\mathcal{S}^{\mathrm{NLO}}_{n,ab}}_{\mathrm{LM}}[\mathfrak{m}_g] + \Theta_1 F^{\mathcal{S}^{\mathrm{NLO}}_{n,ab}}_{\mathrm{LM}}[\mathfrak{m}_q] + \Theta_1 F^{\mathcal{S}^{\mathrm{NLO}}_{n,ab}}_{\mathrm{LM}}[\mathfrak{m}_{\bar{q}}] \Big] , \qquad \{a,b\} \in \{q,\bar{q}\}, \{g,g\} ,$$

$$F^{qq}_{\mathrm{R},n}[\mathfrak{m}] \stackrel{\mathrm{def}}{=} \Theta^{(N-1)/2}\Big[\Theta_1^{(N-3)/2} F^{\mathcal{Q}^{\mathrm{NLO}}_{n,qq}}_{\mathrm{LM}}[\mathfrak{m}_g] + F^{\mathcal{Q}^{\mathrm{NLO}}_{n,qq}}_{\mathrm{LM}}[\mathfrak{m}_q] + \Theta_1 F^{\mathcal{Q}^{\mathrm{NLO}}_{n,qq}}_{\mathrm{LM}}[\mathfrak{m}_{\bar{q}}] \Big] ,$$

$$F^{\bar{q}\bar{q}}_{\mathrm{R},n}[\mathfrak{m}] \stackrel{\mathrm{def}}{=} F^{qq}_{\mathrm{R},n}[\mathfrak{m}]|_{q\leftrightarrow\bar{q}} ,$$

$$F^{ab}_{\mathrm{R},n}[\mathfrak{m}] \stackrel{\mathrm{def}}{=} \Theta^{(N-1)/2}\Big[F^{\mathcal{A}^{\mathrm{NLO}}_{n,ab}}_{\mathrm{LM}}[\mathfrak{m}_g] + F^{\mathcal{A}^{\mathrm{NLO}}_{n,ab}}_{\mathrm{LM}}[\mathfrak{m}_q] + \Theta_1 F^{\mathcal{A}^{\mathrm{NLO}}_{n,ab}}_{\mathrm{LM}}[\mathfrak{m}_{\bar{q}}] \Big] , \qquad \{a,b\} \in \{g,q\} ,$$

$$F^{g\bar{q}}_{\mathrm{R},n}[\mathfrak{m}] \stackrel{\mathrm{def}}{=} F^{gq}_{\mathrm{R},n}[\mathfrak{m}]|_{q\leftrightarrow\bar{q}} .$$

ii) For the unboosted contribution, we need to recreate the I_{T} operator

iii) For the boosted contribution, our expression must match the renormalization of the PDFs

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The final step in obtaining a final subtraction formula at NNLO for the process $pp \to X + N$ Jets consists of generalizing the proof from $n_{\mathrm{f}} = 1$ to a general n_{f}. I do not have this result yet, but it will be available soon!

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THANK YOU FOR YOUR ATTENTION

BACK UP

Cancellation of the Triple Color-Correlations

From the Real-Virtual Contribution: $\mathcal{O}(\epsilon^{-2})$

$$S_{\mathrm{m}}^{\mathrm{tri}}\,\mathrm{RV} \sim \sum_{(ijk)} \frac{s_{ij}}{s_{i\mathrm{m}}s_{j\mathrm{m}}} \left(\frac{s_{jk}}{s_{j\mathrm{m}}s_{k\mathrm{m}}}\right)^{\epsilon} T_i^a\, T_j^b\, T_k^c$$

If combined with the VV

$$\boxed{-[I_+, I_-] + [2I_+ + I_{\mathrm{S}}, I_-]} + \boxed{\mathscr{H}_{2,\mathrm{tc}} + \mathscr{H}^{\dagger}_{2,\mathrm{tc}}} + \boxed{I^{\mathrm{RV}}_{\mathrm{tri}}}$$

$$\left[I_+, I_-^{\dagger}\right] \neq 0$$

$$\left[I_\pm, I_{\mathrm{S}}\right] \neq 0 \;\to\; f_{abc}T_i^aT_j^bT_k^c$$

$\neq 0$ when $N \geq 2$

$$\begin{aligned}\mathscr{H}_{2,\mathrm{tc}} = {}& \frac{i f_{abc}}{384\epsilon}(\gamma_0^{\mathrm{cusp}})^2 \sum_{(ijk)} T_i^a\, T_j^b\, T_k^c \log\frac{-s_{ij}}{-s_{jk}} \log\frac{-s_{jk}}{-s_{ki}} \log\frac{-s_{ki}}{-s_{ij}} \\ & - \frac{i f_{abc}}{128\epsilon}\gamma_0^{\mathrm{cusp}} \sum_{(ijk)} T_i^a\, T_j^b\, T_k^c \left(\frac{\gamma_0^i}{C_{f_i}} - \frac{\gamma_0^j}{C_{f_j}}\right) \log\frac{-s_{ij}}{-s_{jk}} \log\frac{-s_{ki}}{-s_{ij}} \\ & + \frac{\mathbf{\Gamma}_1}{16\epsilon} - \frac{\gamma_1^{\mathrm{cusp}}\,\mathbf{\Gamma}_0}{64\epsilon} - \frac{\pi^2\beta_0\Gamma_0'}{128\epsilon}\end{aligned}$$

From the Double-Virtual Contribution: $\mathcal{O}(\epsilon^{-1})$

Cancellation of the Triple Color-Correlations

If combined with the VV

$$\boxed{-[I_+, I_-] + [2I_+ + I_S, I_-]} + \boxed{\mathscr{H}_{2,\mathrm{tc}} + \mathscr{H}^\dagger_{2,\mathrm{tc}}} + \boxed{I^{\mathrm{RV}}_{\mathrm{tri}}}$$

$$I^{\mathrm{RV}}_{\mathrm{tri}} = +\left\{\frac{2\pi}{\epsilon^2}\log\frac{\eta_{bk}}{\eta_{ak}} + \frac{2\pi}{\epsilon}\left[\log^2\eta_{ak} - \log^2\eta_{bk} + 2L^2_{\max}\log\left(\frac{\eta_{ak}}{\eta_{bk}}\right) + 2\mathrm{Li}_2(1-\eta_{ak}) - 2\mathrm{Li}_2(1-\eta_{bk})\right]\right\} + \mathcal{O}(\epsilon^0)$$

$$-[I_+, I_-] + [2I_+ + I_S, I_-] + \mathscr{H}_{2,\mathrm{tc}} + \mathscr{H}^\dagger_{2,\mathrm{tc}} = -\left\{\frac{2\pi}{\epsilon^2}\log\frac{\eta_{bk}}{\eta_{ak}} + \frac{2\pi}{\epsilon}\left[\log^2\eta_{ak} - \log^2\eta_{bk} + 2L^2_{\max}\log\left(\frac{\eta_{ak}}{\eta_{bk}}\right) + 2\mathrm{Li}_2(1-\eta_{ak}) - 2\mathrm{Li}_2(1-\eta_{bk})\right]\right\} + \mathcal{O}(\epsilon^0)$$