$SO(3)$ and $so(3)$ [1]

Lie group and Lie algebra for 3D rotation matrices

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1.1 Skew-symmetric matrices

Definition 1.1.1: Skew-symmetric matrix

A matrix $A \in \mathbb{R}^{n\times n}$ is called skew-symmetric or anti-symmetric if

$$A^T = -A \tag{1.1}$$

Given a vector $w \in \mathbb{R}^3$ the hat operator give the corresponding skew-symmetric matrix

$$\hat{w} = [w]_\times = \begin{pmatrix} 0 & -w_3 & -w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} \tag{1.2}$$

In particular

$$\hat{w}u = [w]_\times u = w\times u \tag{1.3}$$

1.2 $SO(3)$ and $so(3)$

Definition 1.2.1: Lie group

A Lie group (or infinitesimal group) is a smooth manifold that is also a group, such that the group operations multiplication and inversion are smooth maps.

We define as the special orthogonal group $SO(3)$ the Lie group of all 3D rotation matrices

$$SO(3) = \left\{R \in \mathbb{R}^{3x3} \mid R^T R = I, \det(R) = 1\right\} \tag{1.4}$$

Let's now consider a family of rotations $R(t)$ with $R(0) = I$ which continuously transform a point from its original location

$$X(t) = R(t)X_0 \quad R(t) \in SO(3) \tag{1.5}$$

since $R(t)R(t)^T = I$ then

$$\frac{d}{dt}(RR^T) = \dot{R}R^T + R\dot{R}^T = 0 \quad \rightarrow \quad \dot{R}R^T = -(\dot{R}R^T)^T \tag{1.6}$$

It follow that $\dot{R}R^T$ is a skew-symmetric matrix, so

$$\dot{R}R^T = \hat{w}(t) \quad \rightarrow \quad \dot{R} = \hat{w}(t)R \tag{1.7}$$

Since $R(0) = I$ then $\dot{R}(0) = \hat{w}(0)$, therefore from the first order Taylor expansion

$$R(dt) = R(0) + dR = I + \hat{w}(0)dt \tag{1.8}$$

We define the space $so(3)$ as

$$so(3) = \left\{\hat{w} \mid w \in \mathbb{R}^3\right\} \tag{1.9}$$

The $so(3)$ is the Lie algebra of the Lie group $SO(3)$.

Definition 1.2.2: Lie algebra

A Lie group gives rise to a Lie algebra, which is its tangent space at the identity.

1.2.1 Exponential map

Given the differential equation system

$$\begin{cases} \dot{R}(t) = \hat{w}R(t) \\ R(0) = I \end{cases} \tag{1.10}$$

the solution is

$$R(t) = e^{\hat{w}t} = \sum_{n=0}^{\infty} \frac{(\hat{w}t)^n}{n!} = I + \hat{w}t + \frac{(\hat{w}t)^2}{2} + \dots \tag{1.11}$$

This matrix exponential defines a map from the Lie algebra to the Lie group

$$\exp : \, so(3) \longrightarrow SO(3) \tag{1.12}$$

which is a rotation around the axis $w \in \mathbb{R}^3$ by an angle of t if $|w| = 1$.

1.2.2 Logarithm map

The logarithm map is the inverse of the exponential map

$$\log : \, SO(3) \longrightarrow so(3) \tag{1.13}$$

Given $R \in SO(3), R \neq I$

$$|w| = \cos^{-1}\left(\frac{\mathrm{tr}(R) - 1}{2}\right) \tag{1.14}$$

$$\frac{w}{|w|} = \frac{1}{2\sin(|w|)}\begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix} \tag{1.15}$$

Note 1.1

The above statement says any orthogonal transformation $R \in SO(3)$ can be realized by rotating by an angle $|w|$ around an axis $\frac{w}{|w|}$ as defined above.

The above representation is not unique because increasing the angle by multiples of 2π will give the same rotation.

1.2.3 Rodrigues' Formula

Given $w \in \mathbb{R}^3$, the exponential map can be computed in closed form as

$$e^{\hat{w}} = I + \frac{\hat{w}}{|w|}\sin(|w|) + \frac{\hat{w}^2}{|w|^2}(1 - \cos(|w|)) \tag{1.16}$$

1.3 $SE(3)$ and $se(3)$

In the same way it is possible to define a Lie group and a Lie algebra for the rigid-body transformation.

Definition 1.3.1: Special euclidian group $SE(3)$

$$SE(3) = \left\{ g = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} \mid R \in SO(3), T \in \mathbb{R}^3 \right\} \subset \mathbb{R}^{4\times 4} \tag{1.17}$$

Given $g(t) = \begin{pmatrix} R(t) & T(t) \\ 0 & 1 \end{pmatrix}$ we have

$$\dot{g}(t)g^{-1}(t) = \begin{pmatrix} \dot{R}R^T & \dot{T} - \dot{R}R^T T \\ 0 & 1 \end{pmatrix} \tag{1.18}$$

As in the case of $SO(3)$, the matrix $\dot{R}R^T$ corresponds to some skew-symmetric matrix $\hat{w} \in so(3)$

$$\dot{g}(t)g^{-1}(t) = \begin{pmatrix} \hat{w}(t) & v(t) \\ 0 & 1 \end{pmatrix} = \hat{\xi}(t) \tag{1.19}$$

So

$$\dot{g} = \dot{g}g^{-1}g = \hat{\xi}g \tag{1.20}$$

therefore $\hat{\xi}$ is the tangent space of $g(t)$ and it is called a twist.

Definition 1.3.2: $se(3)$

The Lie algebra of $SE(3)$ is

$$se(3) = \left\{ \hat{\xi} = \begin{pmatrix} \hat{w} & v \\ 0 & 0 \end{pmatrix} \mid \hat{w} \in so(3), v \in \mathbb{R}^3 \right\} \tag{1.21}$$

1.3.1 Exponential map

For $\hat{w} = 0$, $e^{\hat{\xi}t} = \begin{pmatrix} I & v \\ 0 & 1 \end{pmatrix}$, otherwise

$$e^{\hat{\xi}} = \begin{pmatrix} e^{\hat{w}} & \frac{(I - e^{\hat{w}})\hat{w}v + ww^T v}{|w|^2} \\ 0 & 1 \end{pmatrix} \tag{1.22}$$

1.3.2 Logarithm map

Given $g = (R, T)$, we know that there exists $w \in \mathbb{R}^3$ with $e^{\hat{w}} = R$.
If $|w| \neq 0$, the exponential form of g introduced above shows that we merely need to solve the equation

$$\frac{(I - e^{\hat{w}})\hat{w}v + ww^T v}{|w|^2} = T \tag{1.23}$$

for the vector $v \in \mathbb{R}^3$.
Just as in the case of $SO(3)$, this representation is generally not unique, because there exist many twists which represent the same rigid-body motion.

Bibliography

[1] Prof. Daniel Cremers. Multiple view geometry course: Representing a moving scene.