PROBLEM SET 1

Dynamic Economic Modelling

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Contents

1 Consumption with different generations

An economy is composed of identical individuals. Each individual lives for 2 periods (you may imagine them as adulthood and old age). Individuals may work during the first period of their life for a proportion L of the day, for an income equal to wL. In the second period they retire and consume their remaining lifetime savings. Their lifetime utility is given by:

$$U = lnC_1 + lnC_2 + ln(1 - L)$$

where C_i is consumption in period i and L is the fraction of time the individual spends working in period 1.

(a) If the rate of interest on savings is R, write down the individual's budget constraints for both periods, and then combine them in a lifetime (intertemporal) budget constraint.

In period 1 (adulthood) we face the following constraint:

$$Y_1 = C_1 + S_1$$

From the data, we know that Y_1 is equal to wL, thus the budget constraint in period 1 is:

$$wL = C_1 + S_1$$

In period 2 (old age) we face the following constraint:

$$Y_2 + (1 + r)S_1 = C_2 + S_2$$

Since the individual lives only two periods, $S_2 = 0$. In addition, the individual retires in old age, so there is no Y_2. The budget constraint in period 2 is then

$$(1 + r)S_1 = C_2$$

Knowing that $r = R$ we can combine the two constraints to get the intertemporal budget constraint by discounting at rate R what we have in period 2.

The intertemporal budget constraint is:

$$wL + (1 + R)\frac{S_1}{(1 + R)} = C_1 + S_1 + \frac{C_2}{(1 + R)}$$

We can rearrange this expression and get:

$$wL = C_1 + \frac{C_2}{(1 + R)}$$

This means that the stream of income evaluated at time t, which is the present value, needs to be equal to the stream of consumption evaluated at the same time.

(b) Solve for optimal consumption each period and the optimal work effort. Comment on what you find.

We obtained the budget constraint in point (a):

$$wL = C_1 + \frac{C_2}{(1+R)}$$

In addition, the exercise already provides the utility function:

$$U = lnC_1 + lnC_2 + ln(1-L)$$

In order to solve the problem, we need to maximize the utility function subject to the intertemporal budget constraint.

We can substitute C_2 in the utility function and get an unconstrained problem:

$$C_2 = (1+R)(wL - C_1)$$

$$U = lnC_1 + ln((1+R)(wL - C_1)) + ln(1-L)$$

By logarithm properties, we can rewrite it as:

$$U = lnC_1 + ln(1+R) + ln(wL - C_1) + ln(1-L)$$

Now we need to set the partial derivatives of U with respect to C_1 and L equal to 0:

$$\frac{\partial U}{\partial C_1} = \frac{1}{C_1} - \frac{1}{wL - C_1} = 0$$

$$\frac{\partial U}{\partial L} = \frac{w}{wL_1 - C_1} - \frac{1}{1-L} = 0$$

After some math in the first expression we get that:

$$C_1 = \frac{1}{2}wL$$

and by substituting it in the other derivative we get:

$$L = \frac{2}{3}$$

which leads to:

$$C_1 = \frac{1}{3}w$$

Finally we can substitute the values of L and C_1 in the expression for C_2 and we get:

$$C_2 = (1+R)\frac{1}{3}w$$

What we obtained is that the individual spends two thirds of her time in period 1 working, leading to an income of $\frac{2}{3}w$. Then she consumes one half of her income ($\frac{1}{3}w$) in period 1, whereas the other half is saved and capitalized at rate R, then consumed in period 2.

Coherently with the theory, the individual saves a part of her income for consumption in old age. In fact, when the income is zero, desired consumption is still a positive quantity. This can only be achieved through saving a positive quantity of income, since the individual only lives two periods and she is not allowed to die in debt.

Another (intuitivally obvious) result is the fact that the individual does not spend all of her time working. The choice of the number of hours dedicated to working can be thought as a choice between two good: wage (perceived for working hours) and free time (obtained while not working). Wage contributes to utility through consumption in both period 1 and 2, while free time provides utility by its own. Considering that the model assumes positive but diminishing marginal utilities, we expect to get $0 < L < 1$, and our results are coherent with that. This also explain why consumption is distributed between C_1 and C_2 in a quite homogeneous way.

(c) The government introduces a fixed pension paid to individuals in the second period of their lives, funded by a "lump sum tax" paid by those who work. Re-write the intertemporal budget constraint. What will be the impact of the pension on consumption and labour supply decisions of the young? What about the old when the pension is introduced? Give intuition for your answers.

We need to re-write the budget constraint in order to take the tax into account.
In period 1, now we have:

$$wL - T = C_1 + S_1$$

while in period 2, the budget constraint is:

$$(1 + R)S_1 + T = C_2$$

The new intertemporal budget constraint is:

$$wL - T + \frac{T}{(1 + R)} = C_1 + \frac{C_2}{(1 + R)}$$

In terms of C_2 this is equal to:

$$C_2 = (1 + R)(wL - C_1) - RT$$

The unconstrained problem is now to maximize:

$$U = lnC_1 + ln((1 + R)(wL - C_1) - RT) + ln(1 - L)$$

The new partial derivatives are:

$$\frac{\partial U}{\partial C_1} = \frac{1}{C_1} - \frac{(1+R)}{(wL - C_1)(1+R) - RT} = 0$$

$$\frac{\partial U}{\partial L} = \frac{w(1+R)}{(wL_1 - C_1)(1+R) - RT} - \frac{1}{1-L} = 0$$

The first expression can be rearranged as:

$$C_1 = \frac{1}{2}wL - \frac{1}{2}\frac{RT}{(1+R)}$$

and by substituting it in the other derivative we get:

$$L = \frac{2}{3} + \frac{1}{3}\frac{RT}{w(1+R)}$$

then:

$$C_1 = \frac{1}{3}w - \frac{1}{3}\frac{RT}{(1+R)}$$

Finally if we substitute the values of L and C_1 in the expression for C_2 we get:

$$C_2 = (1+R)\frac{1}{3}w - \frac{1}{3}RT$$

What is immediate to note is that the individual compensates the income reduction caused by the tax working a higher amount of hour. In fact, $\frac{1}{3}\frac{RT}{w(1+R)}$ is a positive quantity (and it is positively correlated with T and negatively with w), then the hour of work will increase, whereas the free time will be reduced. Consumption is reduced in both periods, higher the tax, higher the reduction.

We can say that the general picture is now worse for the individual, since she works more to get lower consumption levels. This is caused by the fact that she renounces to a quantity of income T at time 1 and have it again at time 2 without any capitalization. The only scenario where nothing change is when $R = 0$. In that case she is able to maintain the same levels of L, C_1 and C_2. If the amount perceived in time 2 was capitalized at rate R, this would be comparable to private saving, thus the individual would just reduce her saving, and keep the same L, assuming that $T \leq S_1$, where S_1 refers to the no-tax scenario.

(d) The pension is now funded by an income tax, i.e., a tax equal to τwL where τ is the tax rate. Will the behaviour of the young change in the case where $R = 0$? Interpret this result.

The budget constraint should be rewritten considering that the tax is now proportional to the income and that the interest rate is equal to zero.
In period 1, now we have:

$$(1-\tau)wL = C_1 + S_1$$

while in period 2, the budget constraint is:

$$S_1 + \tau w L = C_2$$

The intertemporal budget constraint becomes:

$$wL(1-\tau) - C_1 + \tau w L = C_2$$

that can be simplified to

$$C_2 = wL - C_1$$

The utility function becomes

$$U = lnC_1 + ln(wL - C_1) + ln(1-L)$$

Partial derivatives are:

$$\frac{\partial U}{\partial C_1} = \frac{1}{C_1} - \frac{1}{wL - C_1} = 0$$

$$\frac{\partial U}{\partial L} = \frac{w}{wL_1 - C_1} - \frac{1}{1-L} = 0$$

Since these are the same values we obtained in point (b) the solutions will be the same as well, except C_2 that is just $\frac{1}{3}w$, but since $R = 0$ it is the same as well:

$$L = \frac{2}{3}; \quad C_1 = \frac{1}{3}w; \quad C_2 = \frac{1}{3}w$$

The reason why we get the same solution of point (b) is that $R = 0$. As already stated in point (c), if the amount of income to which the consumer renounce in period 1 is regained in time 2 and it keep the same value (since with $R = 0$ €1 today is worth the same as €1 tomorrow) she just saves a lower amount of money in period 1. If $R \neq 0$, we would expect L, C_1 and C_2 to change.

2 Stylised facts of the business cycle

A business cycle is made of an expansion (boom) and a contraction (recession). During the expansion all good things (GDP, employment, productivity, and so on) tend to go up, or grow faster than "normal", and bad things (e.g. unemployment) tend to fall. During the contraction good things go down and bad things go up. Table 1 gives a more complete description of the volatilities and cross correlations of consumption and labour market variables. These results are from the US because most of the theoretical models we shall examine have been constructed with this data in mind. Consider also the study by King and Rebelo (1999), who also verified the findings of Cooley and Prescott.

(a) Replicate Table 1 and 2 for the US economy from – ideally – 1950 Q1 to the newest data you can find (either 2020 or 2023). You can use real GDP instead of GNP if you want.

First of all we will describe the data we used. It was not possible to find adequate useful data, so it has been necessary to do some manipulation. We will now describe every column of the initial tables on which the final tables have been built, providing the source. All variables have been transformed to logarithms and HP filtered. Additional details about how data have been treated, can be found at 'github.com/DavideMatta/DynamicEconomicModeling' along with the original and intermediate files used. The code has been developed in collaboration with Leonardo Acquaroli.

Table 1:

- *Date*: This is just the date to which data refers. When only monthly data (and not quarterly) were available, we used the average of the three months. In table 1 data cover from 1964 to 2019.
- *AveW_nom*: This is the nominal average hourly earnings for production and non-supervisory employees in the private sector. [1]
- *AveH*: This is the number of average weekly hours worked by production and non-supervisory employees in the private sector.[2]
- *CPI*: This is the consumer price index. It has been fixed equal to 100 in 2012. [3]
- *GDP_nom*: This is the nominal GDP. [4]
- *GNP_nom*: This is the nominal GNP. [5]
- *H*: This is the number of hours worked by all workers in the nonfarm business sector. It has been fixed equal to 100 in 2012. [6]
- *CD_nom*: This is the nominal total personal consumption expenditure for durable goods. [7]

- CND_nom: This is the nominal total personal consumption expenditure for non-durable goods. [8]

- $GNPL$: This is the total labor productivity. It has been fixed equal to 100 in 2012. [9]

- L: This is the employment-population ratio. [10]

- $AveW$: This is the real wage, obtained dividing the nominal wage by the CPI, and then multiplying by 100. The base year is 2012. The same proceeding has been followed for the following four variables as well.

- GDP: This is the real GDP.

- GNP: This is the real GNP.

- CD: This is the real total personal consumption expenditure for durable goods.

- CND: This is the real total personal consumption expenditure for nondurable goods.

Table 2:

- $Date$: This is just the date to which data refers. When only monthly data (and not quarterly) were available, we used the average of the three months. In table 1 data cover from 1964 to 2019.

- Y_nom: This is the nominal GDP per-capita.

- C_nom: This is the nominal personal consumption expenditure per-capita. [11]

- I_nom: This is the nominal gross domestic private investment divided by the population. [12][13]

- w_nom: This is what we have called AveW_nom in table 1.

- N: This is what we have called AveH in table 1

- CPI: This is the same series from table 1.

- A: This is the total factor productivity. It has been fixed equal to 100 in 2012. Since only annual data were available, we assume productivity stays constant during the year. [14]

- r_nom: This is the nominal discount rate charged by the Federal Reserve Bank. [15]

- Y: This is the real GDP per-capita, obtained dividing the nominal GDP per-capita by the CPI, and then multiplying by 100. The base year is 2012. The same proceeding has been followed for the following three variables as well.

- C: This is the real personal consumption expenditure per-capita.

- I: This is the real investment per-capita.

- w: This is the real average hourly wage.

- Y_N: This is the per-capita GDP divided by average hours worked.
- inf: This is the inflation rate, computed as percentage change of CPI.
- r: This is the real interest rate, computed as the Fed discount rate minus the inflation rate.

The reproduced tables are then:

Variable	SD%		Cross-Correlation of GNP with:								
		Shift	-4	-3	-2	-1	0	1	2	3	4
GNP	1.661		0.299	0.513	0.721	0.885	1.000	0.885	0.721	0.513	0.299
CND	1.407		0.457	0.573	0.657	0.686	0.637	0.509	0.345	0.135	-0.057
CD	4.407		0.094	0.298	0.510	0.696	0.860	0.842	0.749	0.619	0.465
H	1.824		0.597	0.732	0.810	0.821	0.751	0.578	0.370	0.152	-0.040
Ave H	0.418		0.069	0.274	0.456	0.629	0.710	0.643	0.510	0.336	0.180
L	0.971		0.678	0.787	0.838	0.815	0.706	0.520	0.305	0.097	-0.092
GNP/L	0.872		-0.071	0.114	0.342	0.553	0.754	0.687	0.607	0.462	0.303
Ave W	1.044		-0.038	0.114	0.264	0.414	0.559	0.599	0.582	0.524	0.426

Table 1: Cyclical Behaviour of the US Economy (1964Q1-2019Q4)

Variable	SD%	Relative SD	1st Order Auto-Corr.	Contemp. Corr. with Y
Y	1.431	1.000	0.853	1.000
C	1.180	0.825	0.848	0.908
I	5.959	4.165	0.813	0.870
N	0.384	0.269	0.749	0.710
Y/N	1.188	0.830	0.830	0.973
w	0.919	0.642	0.868	0.583
r	0.780	0.545	0.735	0.121
A	0.650	0.454	0.650	0.706

Table 2: Business Cycle Statistics for the US Economy (1964Q1-2019Q4)

(b) Verify whether or not the following business cycle facts from Cooley and Prescott (1995) still hold today:

- Consumption is smoother than output.

  We can see from table 2 that consumption has a relative standard deviation lower than output's, then it is smoother. However, form table 2 we can note that if we differentiate between durable and nondurable consumption, only nondurable goods' consumption is actually smoother.

- Volatility in GNP is similar in magnitude to volatility in total hours.

  This seems true, since percentage standard deviations are similar. However, H is slightly more volatile.

- Volatility in employment is greater than volatility in average hours. Therefore most labour market adjustments operate on the extensive rather than intensive margin.

  This is still true, according to percentage SDs.

- Productivity is slightly pro-cyclical.

  According to the two tables, both labor productivity (GNP/L and Y/N) and total factor productivity (A) are correlated with the output, highly than they were in the original tables.

- Wages are less variable than productivity.

  According to both tables this is no longer true, wages are more variable than productivity.

- There is no correlation between wages and output (nor with employment for that matter).

  This is no longer true since there is a significant correlation. However, it is the weaker one compared to all the others.

(c) Verify whether or not the following business cycle facts from King and Rebelo (1999) still hold today:

- Consumption of non-durables is less volatile than output.

  According to table 1 this is still true.

- Consumer durables are more volatile than output.

  This is true, CD is actually the most volatile variable.

- Investment is three times more volatile than output.

  Now investment seems to be four times more volatile than output.

- Government expenditures are less volatile than output.

  We have no data to compare.

- Total hours worked are about the same volatility as output.

  As already stated in point (b),this seems true, since percentage standard deviations are similar. However, H is slightly more volatile.

- Capital is much less volatile than output.

  We have no data to compare.

- Employment is as volatile as output, while hours per worker are much less volatile than output.

  Hours per worker is less volatile than output, but also employment is, though it is still more volatile than hours per worker.

- Labour productivity is less volatile than output.

  This is true.

- The real wage is much less volatile than output.

  The real wage is less volatile than output, although there are other less volatile variables.

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