Proceedings of International Joint Conference on Neural Networks, Atlanta, Georgia, USA, June 14-19, 2009

Complex-valued neural networks: The merits and their origins

Akira Hirose

Abstract— **This paper discusses what the merits of complex-valued neural networks (CVNNs) arise from. First we look back the mathematical history to elucidate the features of complex numbers, in particular to confirm the importance of the phase-and-amplitude viewpoint for designing and constructing CVNNs to enhance the features. The viewpoint is essential in general to deal with waves such as electromagnetic-wave and lightwave. Then we point out that, although we represent a complex number as an ordered pair of real numbers for example, we can reduce ineffective degree of freedom in learning or self-organization in CVNNs to achieve better generalization characteristics. This wave-oriented merit is useful widely for general signal processing with Fourier synthesis or in frequency-domain treatment through Fourier transform.**

I. Introduction

Complex-valued neural networks (CVNNs) extend the application fields steadily [1] . We have various application systems employing CVNNs in the field of, for example, ultrasonic fault detection to find defects in metals and other materials [2] , blind separation based on principal component analysis (PCA) in sonar [3] and voice processing [4] , radars including ground penetrating radars to visualize plastic landmines [5] [6] [7] [8] [9] [10] and satellite radars to estimate landscape information [11] and / or land-use classification [12] , blur-compensation image processing [13] , filtering and other time-sequential signal processing [14] [15], frequency-domain multiplexed microwave signal processing [16] and pulse beamforming in ultra-wideband (UWB) communications [17], frequency-domain multiplexed neural networks and learning logic circuits using lightwave [1] [18] [19] and fast adaptive three-dimensional holographic movie generation for optical tweezers [20] [21] , and developmental learning of motion control in combination with reinforcement learning [22] . In parallel, general associative memories [23] and independent component analysis (ICA) neural networks [24] [25] are also making progress in their improvement.

In the case of linear processing with a simple network structure, we often use the complex-valued least mean square (LMS) algorithm [26] . Neural networks in general conduct nonlinear processing. Regarding the nonlinearity to be employed, we have a series of discussions including several milestone papers [27] . The pros and cons of respective nonlinearities basically depend on the nature of the signal to be treated. We often deal with wave-related complex signals [28] . When we observe a wave signal by using coherent detection, or a baseband complex signal generated through Hilbert transform, we obtain the complex-amplitude, i.e., the phasor, inevitably. The CVNNs are compatible with such wave phenomena. This is the most significant feature of the CVNNs. Actually, in the very early stage of the CVNN research, a pioneering idea and a basic experiment was reported concerning this important feature. That is, in 1992, M.Takeda & T.Kishigami pointed out the fact that the electromagnetic field in a phase-conjugate resonator is formulated in the same manner as that of an associative memory, and that the resonant system realizes a quite fast recall [29] . In this case, the limitation in the energy supply causes amplitude saturation, which realizes the neural nonliearity in the signal amplitude in a natural way.

In such wave-information processing or wave control, it is essentially important to deal directly with phase (or phase difference) and amplitude. The reason lies in the facts that the amplitude corresponds to the wave energy (e.g., number of photons of lightwave), and that the phase difference represents time course and/or position change. From this viewpoint, the so-called amplitude-phase-type nonlinearity is consistent with the wave [27] [1] [28] , as is often the case in signal processing widely in electronics.

In this paper, with such an application background, we examine what gives rise to the merits of the CVNN. As a result, we find that the weight multiplication at synapses yields the phase rotation as well as the amplitude amplification or attenuation. This type of multiplication reduces ineffective degree of freedom in the learning or self-organization to enhance the generalization characteristics in comparison with double-dimensional real-number networks, in spite of the fact that a complex number can be represented as an ordered pair of real numbers. The network dynamics consisting of this elemental rotation and amplification / attenuation leads to significant merits in total, originating from the consistency with the wave-related phenomena and information.

II. What is the complex number?

A. Geometric and intuitive definition

In the old days history, the definition of the complex number changed gradually [30]. In the 16th century, Cardano tried to work with imaginary roots in dealing with quadratic equations. Afterward, Euler used complex numbers in his calculations intuitively and correctly. It is said that by 1728 he knows the transcendental relationship $i \log i = -\pi/2$. The Euler formulae appear in his book as

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2} \qquad (1)$$

In 1798, Wessel described representation of the points of a plane by complex numbers to deal with directed line

A. Hirose is with the Department of Electrical Engineering and Information Systems, The University of Tokyo,7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, Japan (email: ahirose_at_ee.t.u-tokyo.ac.jp).

This work was partly supported by the Grant-in-Aid for Exploratory Research, Japan Society for the Promotion of Science (JSPS), No.20651045, and also by the Assistance Grant of the Hoso Bunka Foundation.

978-1-4244-3553-1/09/$25.00 ©2009 IEEE

segments. Argand also interpreted $\sqrt{-1}$ as a rotation through a right angle in the plane, and justified this idea on the ground that two $\sqrt{-1}$ rotations yields a reflection, i.e., -1. It is also believed that, in early 1749, Euler already had a visual concept of complex numbers as points of plane. He described a number x on a unit circle as $x = \cos g + i \sin g$ where g is an arc of the circle. Gauss was in full possession of the geometrical theory by 1815. He proposed to call $+1$, -1, and $\sqrt{-1}$ as direct, inverse, and lateral unity, instead of positive, negative, and imaginary or "impossible" elements.

B. Definition as ordered pair of real numbers

The geometrical representation is intuitively simple and visually understandable, but may be weak in strictness. In 1835, Hamilton presented the formal definition of the complex number as an "ordered pair of real numbers," which also led to the discovery of quaternions, in his article entitled "Theory of conjugate functions, or algebra as the science of pure time." He defined addition and multiplication in such a manner that the distributive, associative, and commutative laws hold. The definition as the ordered pair of real numbers is algebraic, and can be stricter than the intuitive rotation interpretation.

At the same time, the fact that a complex number is defined by two real numbers may lead present-day neural-network researchers to consider a complex network equivalent to just a doubled-dimension real-number network effectively. However, in this paper, the authors would like to clarify the merit by focusing on the rotational function even with this definition.

Based on the definition of the complex number as an ordered pair of real numbers, we represent a complex number z as

$$z \equiv (x, y) \tag{2}$$

where x and y are real numbers. Then the addition and multiplication of z_1 and z_2 are defined in *complex domain* as

$$(x_1, y_1) + (x_2, y_2) \equiv (x_1 + x_2, y_1 + y_2) \tag{3}$$

$$(x_1, y_1) \cdot (x_2, y_2) \equiv (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2) \tag{4}$$

As a reference, the addition and multiplication (as a step in correlation calculation, for example) of *two-dimensional real values* is expressed as

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \tag{5}$$

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, y_1 y_2) \tag{6}$$

In the comparison, the addition process is identical. Contrarily, the complex multiplication seems quite artificial, but this definition (4) brings the complex number with its unique function, that is, the angle rotation, as well as amplitude amplification / attenuation, which are the result of the intermixture of the real and imaginary components.

It is easily verified that the commutative, associative, and distributive laws hold. We have the unit element $(1, 0)$ and the inverse of z ($\neq 0$) which is

$$\begin{aligned} z^{-1} &\equiv \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) \\ &= \left(\frac{x}{|z|^2}, \frac{-y}{|z|^2} \right) \end{aligned} \tag{7}$$

where $|z| \equiv \sqrt{x^2 + y^2}$.

C. Real 2×2 matrix representation

We can also use real 2×2 matrices, instead of the ordered pairs of real numbers, to represent complex numbers [30] [31] . With every complex number $c = a + ib$, we associate the $\boldsymbol{C}$-linear transformation

$$T_c : \boldsymbol{C} \rightarrow \boldsymbol{C}, \qquad z \mapsto cz = ax - by + i(bx + ay) \tag{8}$$

which includes a special case of $z \rightarrow iz$ that maps 1 into i, i into -1, ..., with a rotation with right angle each. In this sense, this definition is a more precise and general version of Argand's interpretation of complex numbers. If we identify $\boldsymbol{C}$ with $\boldsymbol{R}^2$ by

$$z = x + iy = \begin{pmatrix} x \\ y \end{pmatrix} \tag{9}$$

it follows that

$$\begin{aligned} T_c \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} \\ &= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned} \tag{10}$$

In other words, the linear transformation T_c determined by $c = a + ib$ is described by the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Generally a mapping represented by a 2×2 matrix is non-commutative. However, in the present case, it becomes *commutative*. By this real matrix representation, the imaginary unit i in $\boldsymbol{C}$ is given as

$$I \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad I^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -E \tag{11}$$

In the days of Hamilton, we did not have matrices yet. Even present, it is very rare to define complex numbers in terms of real 2×2 matrices [30] (Chapter 3, §2, 5.), [31] . The introduction of complex numbers through 2×2 matrices has the advantage, over introducing them through ordered pairs of real numbers, that it is unnecessary to define an ad hoc multiplication. What is most important is that this matrix representation clearly expresses the function specific to the complex numbers. That is, the rotation and amplification or attenuation as

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{12}$$

where r and θ denote amplification / attenuation of amplitude and rotation angle applied to signals, respectively, in the multiplication calculation. On the other hand, addition is rather plain. The complex addition function is identical to that in the case of doubled-dimension real numbers.

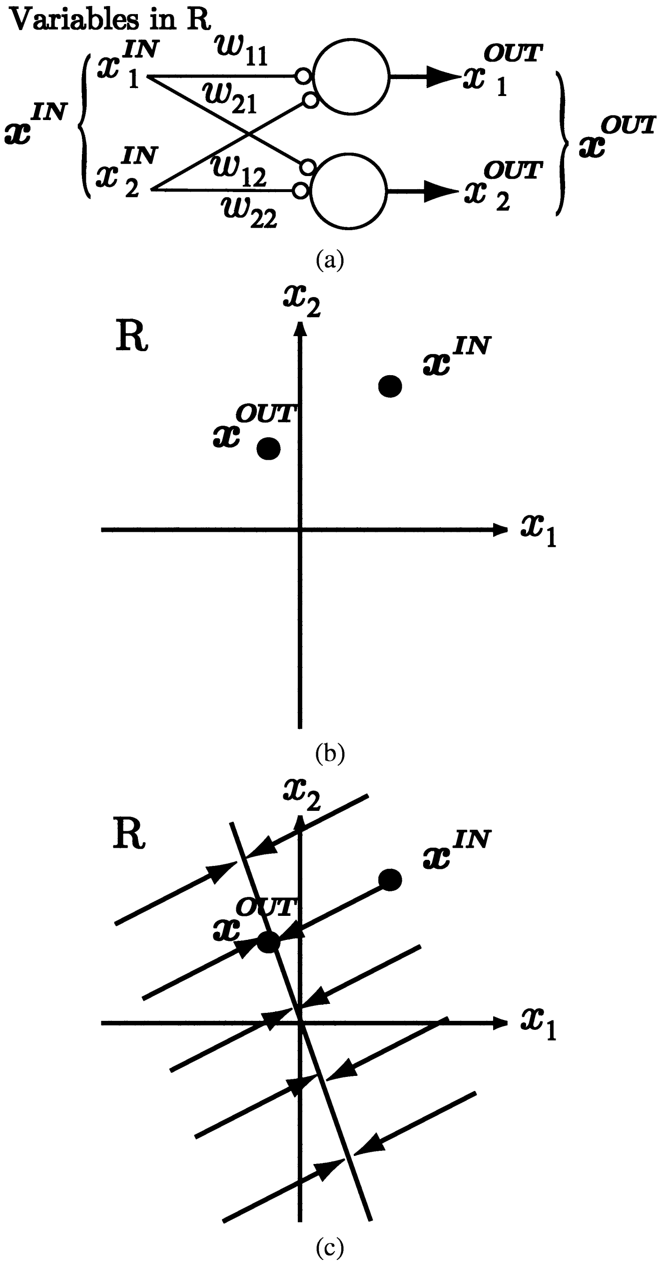


Fig. 1. A simple linear feedforward network to learn a mapping: (a)a real-valued single-layered two-input two-output feedforward network, (b)a task to learn a mapping that maps $\boldsymbol{x}^{IN}$ to $\boldsymbol{x}^{OUT}$, and (c)a possible but degenerate solution that is often unuseful.

In summary, the phase rotation and amplitude amplification / attenuation are the most important features of complex numbers.

III. Complex-valued neural networks

A. Synapse and network function

In wave-related adaptive processing, we often obtain excellent performance with learning or self-organization based on the CVNNs. As already mentioned, the reason depends on situations. However, the discussion in Section II suggests that the origin lies in the complex rule of arithmetics. That

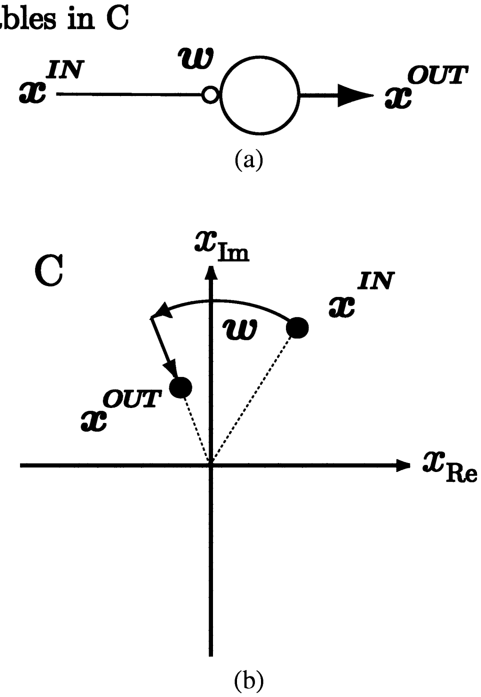


Fig. 2. Another simple linear feedforward network to learn the same task given in Fig.1(b): (a)complex-valued neural network seemingly identical to Fig.1(a), and (b)a solution obtained in this small degree-of-freedom case.

is to say, the merit arises from the functions of the four fundamental rules of arithmetics of complex numbers, in particular the multiplication, rather than the representation of the complex numbers, which can be geometric, algebraic, or in matrices. Moreover, the essence of the complex numbers also lies in the characteristic multiplication function, the phase rotation, as overviewed in Section II [28].

Let us consider a very simple case shown in Fig.1(a) where we have a single layer 2-input 2-output feedforward neural network in real number. For simplicity, we omit the possible nonlinearity at the neurons, i.e., the activation function is the identity function, where the neurons have no threshold. We assume that the network should realize a mapping that transforms an input $\boldsymbol{x}^{IN}$ to an output $\boldsymbol{x}^{OUT}$ in Fig.1(b) through supervised learning that adjusts the synaptic weights w_{ji}. Simply we have only a single teacher pair of input and output signals. Then we can describe a general input-output relationship as

$$\begin{pmatrix} x_1^{OUT} \\ x_2^{OUT} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1^{IN} \\ x_2^{IN} \end{pmatrix} \tag{13}$$

We have a variety of possible mapping obtained by the learning because the number of parameters to be determined is larger than the condition, i.e., the learning task is an ill-posed problem. The functional difference emerges as the difference in the generalization characteristics. For example, learning can result in a degenerate mapping shown in Fig.1(c), which is often unuseful in practice.

Next, let us consider the mapping learning task in the one-dimensional complex domain, which transforms a complex value $\boldsymbol{x}^{\boldsymbol{IN}} = (x_1^{\boldsymbol{IN}}, x_2^{\boldsymbol{IN}})$ to another complex value $\boldsymbol{x}^{\boldsymbol{OUT}} = (x_1^{\boldsymbol{OUT}}, x_2^{\boldsymbol{OUT}})$. Figure 2(a) shows the complex-valued network, where the weight is a single complex value. The situation is expressed just like in (13) as

$$\begin{pmatrix} x_1^{\boldsymbol{OUT}} \\ x_2^{\boldsymbol{OUT}} \end{pmatrix} = \begin{pmatrix} |w|\cos\theta & -|w|\sin\theta \\ |w|\sin\theta & |w|\cos\theta \end{pmatrix} \begin{pmatrix} x_1^{\boldsymbol{IN}} \\ x_2^{\boldsymbol{IN}} \end{pmatrix} \tag{14}$$

where $\theta \equiv \arg(w)$. The degree of freedom is reduced, and the arbitrariness of the solution is also reduced. Figure 2(b) illustrates the result of the learning. The mapping is a combination of phase rotation and amplitude attenuation. This example is truly an extreme. The dynamics of a neural network is determined by various parameters such as network structure, input–output data dimensions, and teacher signal numbers. However, the above characteristics of phase rotation and amplitude modulation are embedded in the complex-valued network as a universal elemental process of weighting.

The essential merit of neural networks in general lies in the high degree of freedom in learning and self-organization. However, if we know *a priori* that the objective quantities include "phase" and/or "amplitude," we can reduce possibly harmful portion of the freedom by employing a complex-valued neural network, resulting in a more meaningful generalization characteristics. The "rotation" in the complex multiplication works as an elemental process at the synapse, and realizes the advantageous reduction of the degree of freedom. This feature corresponds not only to the geometrical intuitive definition of complex numbers but also to the Hamilton's definition by ordered pairs of real numbers, or the real 2×2 matrix representation.

Though we considered a small feedforward network in this section, the conclusion is applicable also to other CVNNs such as complex-valued Hebbian-rule based network and complex correlation learning networks, where the weight is updated by the multiplication results. The elemental process of phase rotation and amplitude modulation results in the network behavior consistent with phase rotation and amplitude modulation in total. The nature is a great advantage when we deal with not only waves such as electromagnetic wave and lightwave, but also arbitrary signals with the Fourier synthesis principle, or in the frequency domain through the Fourier transform.

B. Nonlinearity of the neuron activation function

The rotation at the synapses is the most fundamental specific nature in CVNNs. The neuron nonlinearity can be another issue.

The complex LMS is the most widely-used basis of adaptive processing of complex signals [26]. The introduction of nonlinearity into the neuron activation function once seemed to have a serious problem in the differentiability in the complex domain. Liouville's theorem in complex analysis states that every entire (holomorphic) function must be constant. It follows that, if we introduce some nonlinearity, we have to abandon the differentiability. This fact was considered to be a big problem at around 1990 because some researchers believed that the indifferentiability should lead directly to the impossibility to obtain and/or analyze the dynamics of the CVNNs.

However, the concern was found to be a trifle because neural dynamics are generally described by partial differentiation in terms of a number of variables associated with the neurons. Actually, nowadays we calculate partial differentials in terms of real and imaginary parts, or phase and amplitude, to determine neural dynamics in CVNNs. This manner is practically effective.

At the same time, it is true that we discard the conformal mapping nature of the holomorphic function. However, when we utilize a conformal mapping function, we often concentrate upon the mapping structure itself, rather than a combination with some nonlinearity. Additional nonlinearity should rather be hindrance. Accordingly, the non-holomorphy is not a big problem again.

In complex-valued associative memories, researchers investigated the requirements on the nonlinearity to determine an effective energy function [32]. As a result, we have two types of possibility. One is to apply nonlinearity to real and imaginary parts respectively and to combine them to yield a complex output [33][34]. Another is to employ nonlinear functions for the phase and amplitude respectively [1].

In other CVNNs, we may have possibilities to employ other nonlinearity depending on the objects, i.e., what type of processing we aim at. Even in such cases, the above-mentioned two types of nonlinearity will be the most promising candidates since we normally consider that a direct extension of the real sigmoid function works well also widely in complex domain.

C. Amplitude & phase or real & imaginary in nonlinearity

When we deal with wave information or wave itself, the real and imaginary axes are essentially less meaningful than amplitude and phase (or phase difference) because the real and imaginary axes are determined relatively to an arbitrarily determined phase reference. An example is the coherent detection in communications receiver, where we prepare a local oscillator (LO) with a phase-locked loop (PLL) locked to some reference to be used for demodulation, that is, extraction of real and imaginary signals. The receiver determines the real and imaginary parts, which never exist beforehand [1] [28]. Instead, the difference of two phase values are meaningful itself, which corresponds to time course and/or position difference. In this sense, the phase difference represents certain information directly. The amplitude, orthogonal to phase, is also meaningful, signifying energy or power of the wave. Accordingly, the amplitude–phase nonlinearity is more suitable for wave related processing. Actually based on the amplitude–phase nonlinearity, we have proposed new adaptive systems such as optical learning logic circuits realizing frequency-multiplexed operation [19] and a fast method to yield computer-generated hologram (CGH) for three-dimensional movies [20] [21]. Figure 3 shows the basic concept of the frequency-multiplexed optical learning

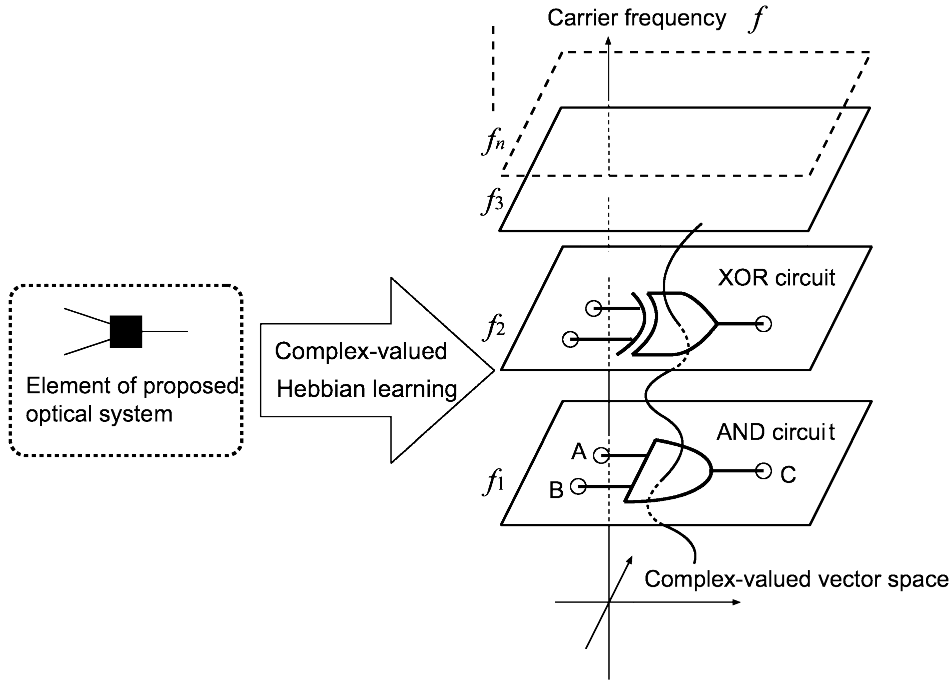


Fig. 3. Basic idea of frequency-multiplexed logic gate where f_n denotes frequency points at which the neuron learns a logic function [19].

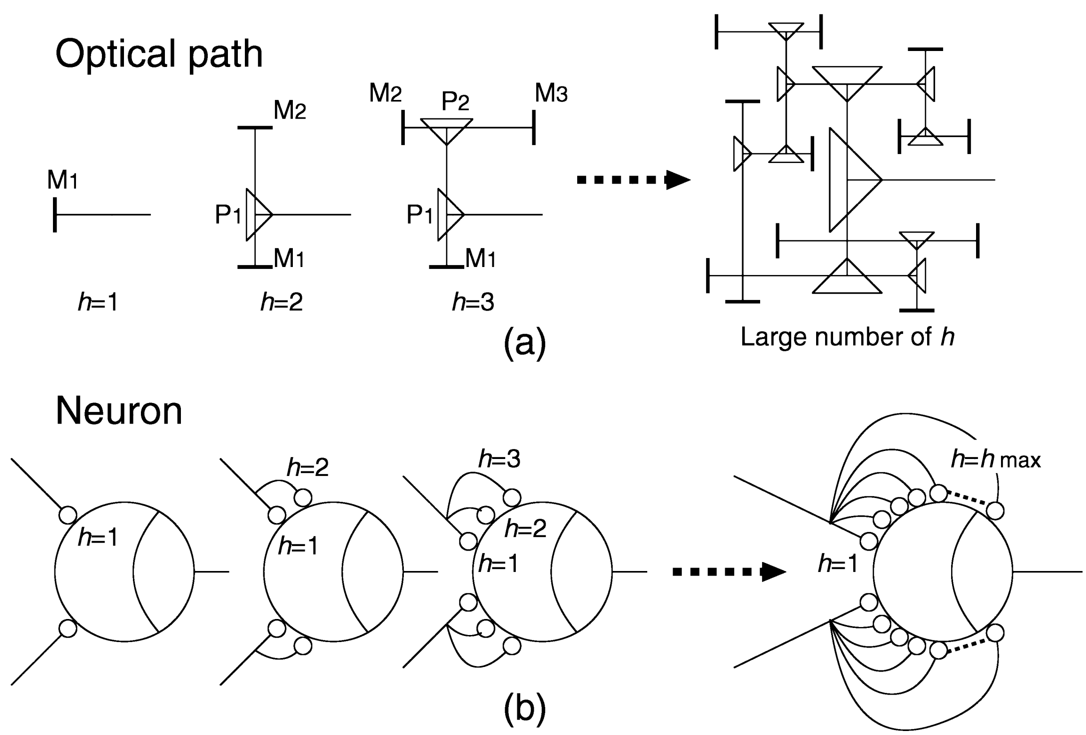


Fig. 4. (a)Conceptual illustration of optical circuits to realize optical-path differences (time delay) and (b)corresponding neurons with amplitude-modulating synapses consisting of attenuators and / or amplifiers utilizing power-saturating characteristics after summation. M_m: mirror, P_p: beam splitter, and h is the index for parallel connections per synapse [19].

logic gates, while Fig.4 presents the schematic structure of the parallel variable delays used in the optical circuits. The superiority of the amplitude–phase nonlinearity was reported also in blind separation task to treat voice signals [4].

Among various neurodynamics in the complex domain, the complex-valued self-organizing maps (CSOMs) may possess

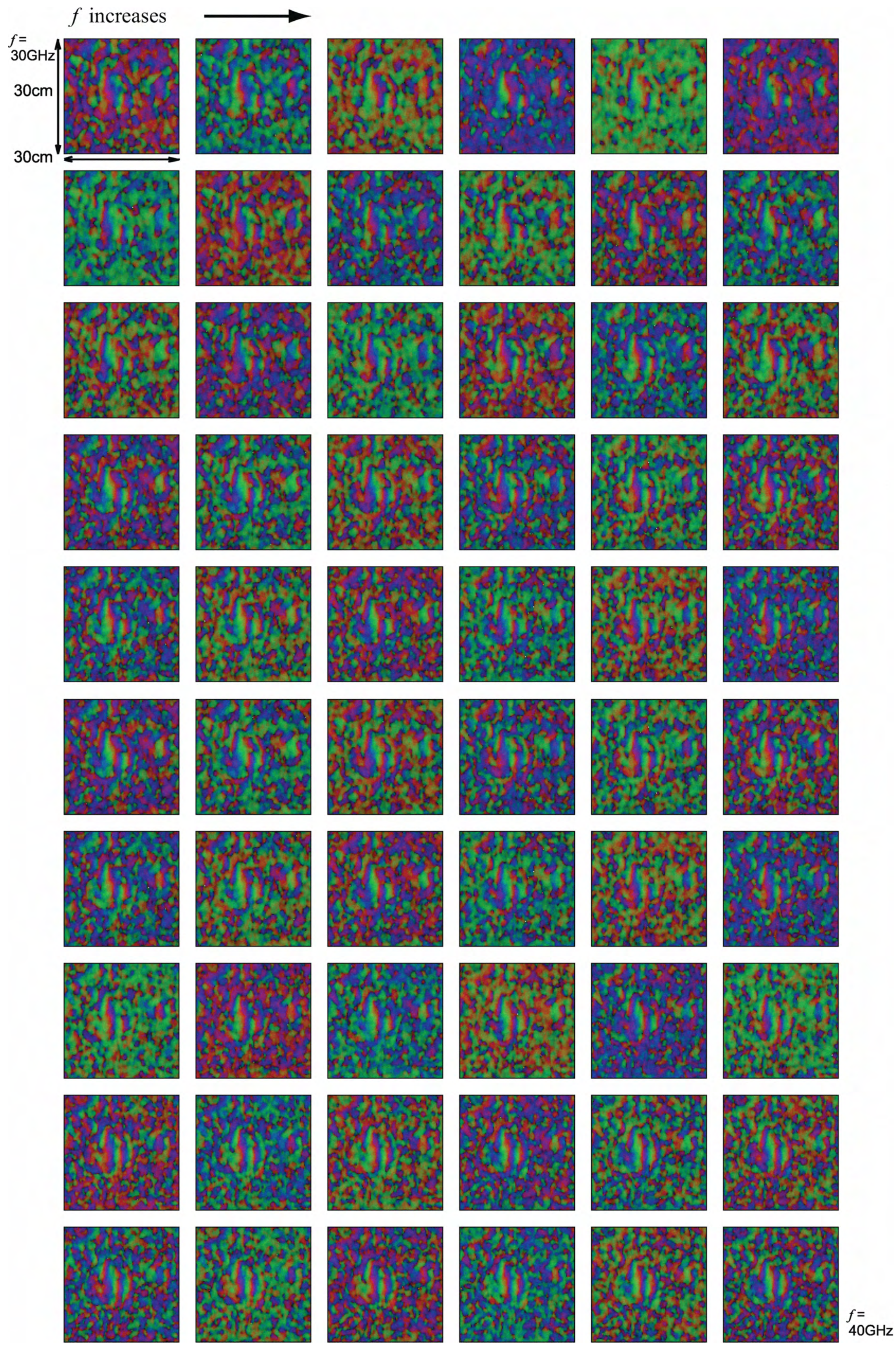
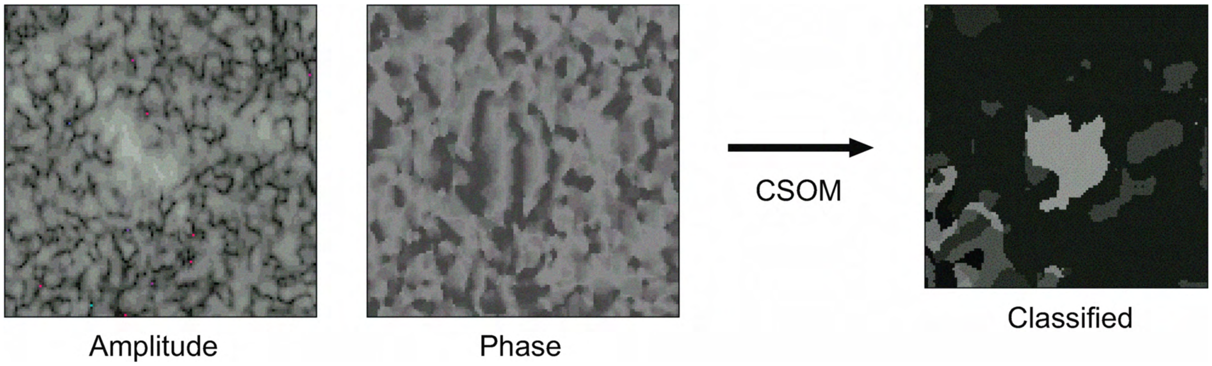
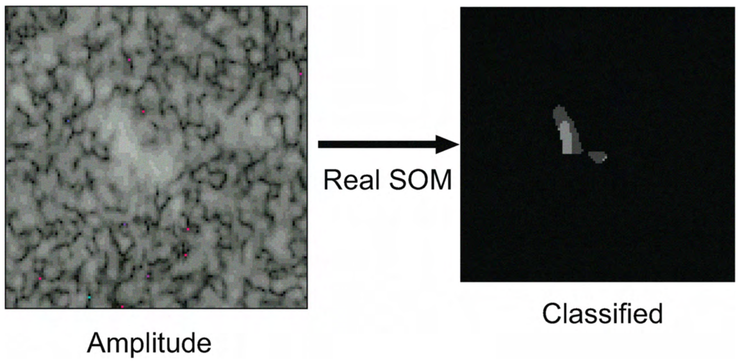


Fig. 5. Multi-frequency observation data for the plastic landmine buried in the ground at the center of 30×30 cm^2 area. In this early-stage experiment to find a plastic landmine buried very shallowly, the radar frequency f was stepped from 30GHz (top left) to 40GHz (bottom right) with a constant interval of about 0.16GHz. Brightness shows intensity, while hue presents phase [5].



(a) (b)

Fig. 6. (a)Sample amplitude and phase data at a frequency out of ten frequency-point data fed to the CSOM, shown in gray scale separately, and (b)CSOM classification result of the plastic mine buried near the ground surface at the center of the area [5].



(a) (b)

Fig. 7. (a)Sample amplitude data at a frequency out of ten frequency-point data fed to the real-SOM, shown in gray scale, and (b)real-SOM classification result of the plastic mine buried near the ground surface at the center of the area, for comparison with the CSOM case [5].

less features which reflect the complex multiplication, since SOM in general have two sub-processes in the operation, i.e., winner determination and weight update, both of which sub-processes consist of only addition and subtraction in its arithmetics without any multiplication that utilizes the complex nature of phase rotation. The fact is applicable also to the case of the adaptive ground penetrating radar (GPR) system to visualize plastic landmines buried in the ground by distinguishing them from metal fragments and stones [5] [8].

However, note that if we consider a CSOM with a complex-valued feature extractor included, the situation becomes different. The landmine visualization system reported in Ref.[8], for example, focuses on the complex-amplitude texture of scattered / reflected electromagnetic-wave images in three dimension (two spatial and one frequency) by calculating local correlation in the complex domain for the feature extraction to realize sufficient distinction between landmines and other clutter. We make a product of a pixel value with a conjugate of another pixel value to estimate the correlation, which includes the multiplication (4), which is equivalent to (10) and (12). The self-organization follows the result of the extraction. Therefore, the CSOM dynamics, consisting of the feature extraction and self-organization, depends largely on the phase rotation and amplitude modulation again.

The improvement realized by the CSOM is remarkable. Figure 5 shows an example set of scattering / reflection complex-amplitude image data [5]. In this experiment, we used a very high frequency band, 30–40GHz, for visualization of landmines buried very shallowly in the ground. We have to distinguish also the rough ground surface scattering. In Fig.5, the brightness represents amplitude, while the hue shows the phase. The fluctuation range in the amplitude is rather small. The phase changes much largely with random texture. A landmine is buried at about the center of the observation area. When we glahm the images in whole without concentration, then we can *see* something at the center because of the slight difference in the local texture. It is a great ability of human vision. By implementing this human function using the CSOM, we can visualize a landmine as shown by the white part at the center of the observation area in Fig.6. Contrarily, if we feed only the local correlation of amplitude information to a SOM so that it works as a real-valued SOM, the obtained image in 7 is very different from the previous one. It cannot visualize the target. In this way, we can see the superiority of the CSOM dealing with the complex-amplitude.

IV. Summary

In this paper, we first looked back the history of the complex number to elucidate and discuss its feature. We found that the phase rotation in the complex multiplication is the most important function. It follows that, in the neural network construction, we have to focus upon the phase and amplitude of the signals to be treated to emphasize the merit of the complex-valued neural networks. This nature is a great merit in dealing with wave-related information or wave itself such as electromagnetic wave, lightwave, sound wave, and ultrasonic wave. The advantage is useful not only for pure sinusoidal wave but also for arbitrary signals in combination with the concept of Fourier synthesis and/or in the treatment in the frequency domain through the Fourier transform. We reviewed our adaptive ground penetrating radar system to visualize plastic landmines which has been developed on the basis of the ideas discussed above.

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