Informatics

Random Numbers Generation

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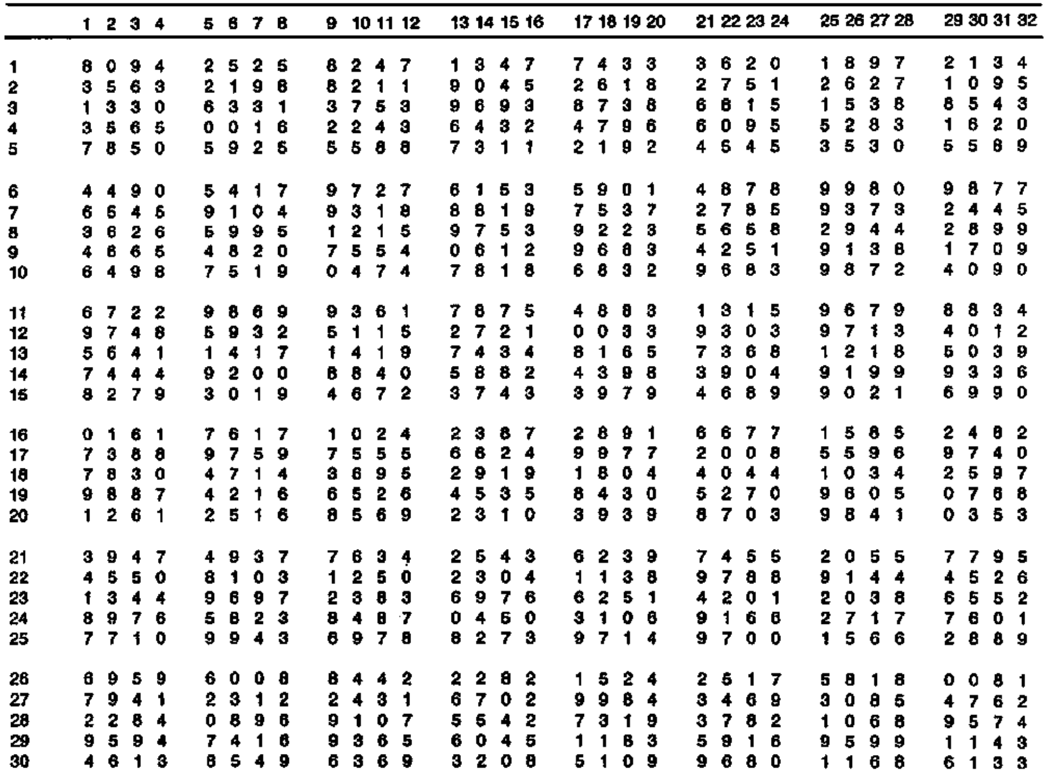
Learning Objectives

- Randomness: what's for?
- Randomness and computers?
- The principles
- Random generation in R
- Examples

Random numbers

- used in
  - statistics
  - programming
    - simulation
    - games
    - program testing
- tools
  - *tables of random numbers*
  - hardware generation
  - software generation



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Random numbers in statistics

- *sampling*
  - select a sample of items from a larger population
  - either for impossibility to access the entire population or for faster computation
  - the sample must be representative of the population
- **simulation**
  - of complex systems or processes when formal modelling is not viable
- **Monte Carlo methods**
  - random sampling to solve complex problems in mathematics, physics, chemistry, engineering, finance

Problem description

- a software using random numbers requires a software generator
- a computer running any softwares is a *deterministic* machine
  - the output is *functionally determined* by *the input and the status*
- an algorithm can generate numbers that are *seemingly random*
- *pseudo-random* number generators
- *we deal only with pseudo-random generation, therefore the word "pseudo" will be omitted*

Requirements for a pseudo-random generator

- the perfect generator should be able to generate an infinite sequence of numbers, drawn from a given interval, that are statistically independent
- a generator should be
  - efficient
    - eg a simulation could require the generation of millions of numbers
  - repeatable
    - we want to be always able to repeat a scientific experiment

Lehmer generator (i)

- an example of algorithm for generating pseudo-random numbers
  - It is absolutely not the best one, simply it is one of the simplest, and a good example of implementation of random number generation
- proposed in 1951
- parametric algorithm
- generates a **permutation** of the natural numbers up to a given **m**
  - scanning the sequence of numbers of the permutation we obtain the effect of the single number generation
  - **m** is one of the parameters

Lehmer generator (ii)

- there are several choices of the parameters that guarantee a ***seemingly random sequence***
  - statistical tests give results *compatible with the hypothesis of randomness* of the generation
  - each number of the sequence *seems to be independent* from the preceding portion of the sequence, i.e. observing a sequence of generated numbers it is hard to guess next number

Lehmer generator - description

Given

1. modulus: m integer, prime, *big*
2. multiplier: a integer, $1 < a < m$
3. generator *f(z):* $z_{n+1} = a * z_n \mod m$
4. seed: z_1 integer, $1 \leq z_1 \leq m-1$

Lehmer generator - discussion

- since *m* is prime, the generator does never generate 0, for any $1 \leq z \leq m-1$, therefore the sequence does never collapse to 0
  - otherwise there would exist a1 * m1 * z1 * m2 mod m1 * m2 = 0
- linear transformations of the sequence do not influence the apparent randomness
- the sequence is fully deterministic, but there are many choices for *a* and *m* giving sequences that seem perfectly random
- the values of *a* and *m,* determine the length of the period *p (p<=m),* such that $z_p = z_1$
- a *complete period sequence* is a *permutation* of the numbers 1,…,m-1
- there are several pairs *a* and *m* giving complete period sequences
- each number has probability 1/m
- the seed determines the starting point of the sequence
  - Changing the seed we simulate the effect of a different sequence, due to the apparent independence of the numbers in the sequence

Example: a=6, m=13

f(z) = 6z mod 13 1,6,10,8,9,2,12,7,3,5,4,11,1…

period

Parameter choice

- a long period is obviously preferred
- with $m = 2^{31}-1$ there exist 534 of good values for *a*
- an efficient implementation of f(z) is needed
- $a=16807$ and $m = 2^{31}-1$ is a good choice
  - it requires to manage integers with 46 bit, to contain the maximum value of $a * z$
- the seed can be chosen freely for each experiment

Some of the methods available

- Lehmer
- Middle Square
- Linear Congruential (LCG)
- Quadratic Congruential
- Inverse Congruential (ICG)
- Inverse Congruential Explicit (EICG)
- ICG and EICG composed
- Fibonacci delayed
- Shift register with linear feedback
- Mersenne Twister (*default generator in R*)
  https://en.wikipedia.org/wiki/Mersenne_Twister
- …

ony for general information

Verification of randomness

- uniformity of the distribution in the interval
    - easy to obtain and verify
- independence
    - difficult to obtain and verify
- verification criteria
    - statistical tests
    - theoretical analysis of the algorithm

Verification of randomness (ii)

- uniformity
  - Chi-Square uniformity test
  - Kolmogoroff-Smirnoff

- independence
  - Chi-Square independence test
  - "gap" test
  - …

Random numbers in R

- integer numbers
- real numbers
- uniform
- standard probability distributions
- sampling among given values
- …

The seed

- setting the seed allow to reproduce exactly the random sequence
- it is a good habit to set the seed, and to keep track of the seed used, at the beginning of an experiment
- in this way the experiment can be repeated with constant results

```
set.seed(integer)
```

Uniform: continuous

```
runif(n number of generated double values
    , min = 0
    , max = 1 interval of generated values
)
```

Discrete: integers

```
sample.int(n generates values from 1 to n
          , size = n #number of generated values
          , replace = FALSE when generating
            more than one value controls
            repetition of values
          , prob = NULL if not set the
            generation is uniform, otherwise
            values probabilities are given as
            a vector of n weights
          )
```

Discrete: general

```
sample (x vector of generated values
          , size #number of generated values
          , replace = FALSE when generating
            more than one value controls
            repetition of values
          , prob = NULL if not set the
            generation is uniform, otherwise
            values probabilities are given as
            a vector of n weights
          )
```

Some random distributions in R

beta: dbeta

binomial (including Bernoulli): rbinom

Cauchy: dcauchy

chi-squared: dchisq

exponential: rexp

F: df

gamma: dgamma

geometric: dgeom

This is also a special case of the negative binomial

hypergeometric: dhyper

log-normal: dlnorm

multinomial: dmultinom

negative binomial: dnbinom

normal: dnorm

Poisson: dpois

Student's t: dt

uniform: dunif

Weibull: dweibull

Quiz

- random integers in two separate intervals
- the hidden mines of a Minesweeper schema
- random points in a given rectangle
- random points inside a circle, given center and radius
- random elements of a data frame
- random letters with uniform distribution and replacement
- a Ruzzle schema

Montecarlo methods (a few basics)

(Source: wikipedia)

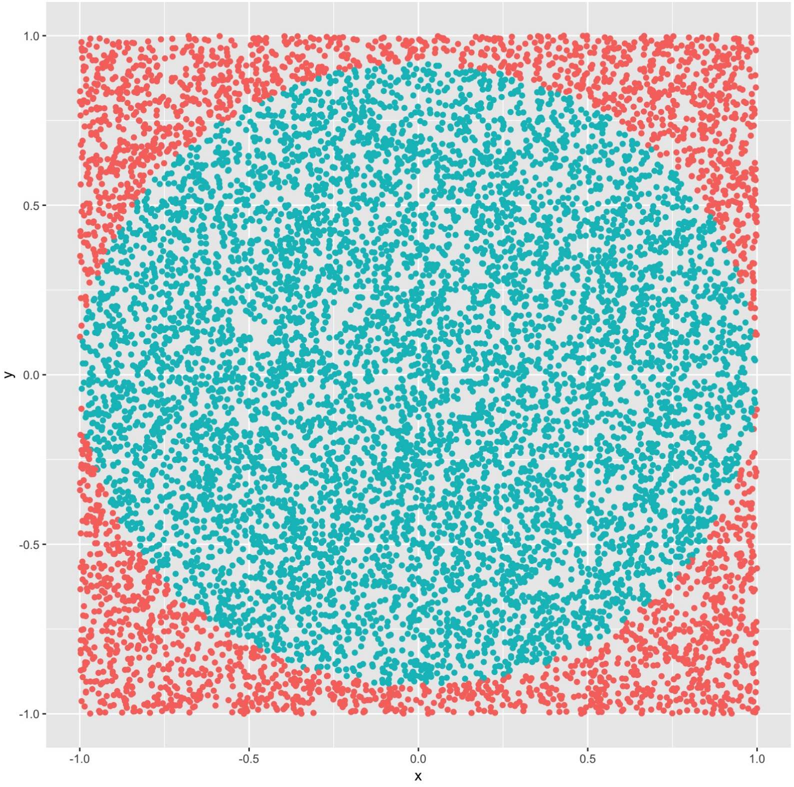
- Computational algorithms that rely on repeated random sampling to obtain numerical results
- Useful for physical and mathematical problems for which an analytic solution is difficult for any reason
- Non-deterministic approach
  - approximation with error
  - several trials
- Examples:
  - optimization
  - numerical integration
  - generating draws from probability distribution

General method (naive explanation)

- Find a method for simulating some situation
- Simulate with a high number of repetitions
  - more repetitions ➔ better precision
- Count the fraction of *success*
- Derive from that fraction the result

A toy example: computation of **PI**

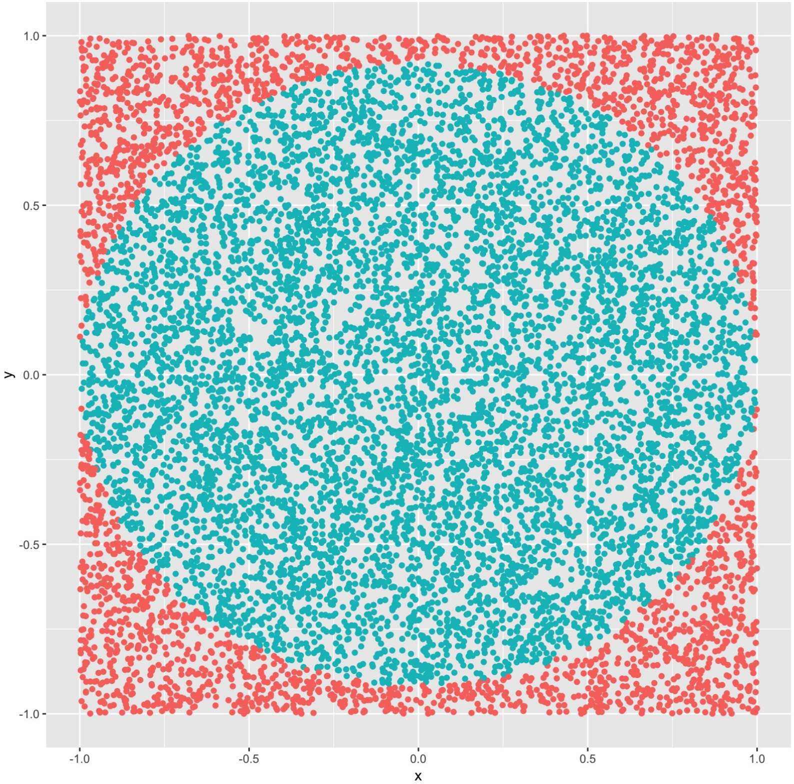
- Generate random points in a square bounding a circle
- Compute the frequency of points that are inside the circle
- PI is four times the ratio between the total number of points and the number of points inside the circle



A toy example: computation of **PI**

- repeat several times with different number of points
  - repeat several times without setting the seed and with the same parameters
  - compute the average of the results and store it
  - this repetition tries to compensate the (pseudo) *non-determinism*

See the example "`montecarlo_pi`"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0.0
-0.5
-1.0
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-1.0
-0.5
0.0
0.5
1.0
x