PID Control Performance Assessment: The Single-Loop Case

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An iterative solution is developed for the calculation of the best achievable (minimum variance) PID control performance and the corresponding optimal PID setting in an existing control loop. An analytic expression is derived for the closed-loop output as an explicit function of PID setting. The resulting benchmark allows for realistic performance assessment of an existing PID control loop, especially when the control loop fails to meet the minimum variance performance. A PID performance index is then defined based on the PID performance bound, and its confidence interval is estimated. A series of simulated examples are used to demonstrate the utility of the proposed method. © 2004 American Institute of Chemical Engineers *AIChE J,* 50: 1211–1218, 2004
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Introduction

Many different approaches have been proposed in the literature for the performance assessment of existing control loops (see surveys by Harris and Seppala (2001), Harris et al. (1999), Huang and Shah (1999), and Qin (1998)). Considerable research efforts in this area have focused on two main topics. The first deals with the estimation of minimum variance (MV) performance bounds for various control systems, including single-loop control (Harris, 1989; Desborough and Harris, 1992), feedforward and feedback control (Desborough and Harris, 1993; Stanfelj et al., 1993), cascade control (Ko and Edgar, 2000), and multivariable control (Harris et al., 1996; Huang et al., 1997; Ko and Edgar, 2001a; b). The second research area involves the development of more realistic performance bounds (Kozub, 1997; Huang and Shah, 1998; Horch and Isaksson, 1999) that take performance limitations into account other than the process time delay.

The MV performance bound represents the best achievable controller capability and, thus, can be useful in separating problems associated with the process model vs controller issues. Also, this MV performance bound requires the minimum process information compared to other performance measures. These properties of the MV performance bound make it well-suited as a first-level performance measure in assessing control loop performance.

When the initial performance assessment indicates poor control relative to the MV performance, more realistic performance measures that take other performance limitations into account can be evaluated. In general, this step requires more process information than the MV-based approaches. Kozub (1997) proposed with a desired first-order exponential decay of autocorrelation function as an acceptable performance measure. This idea was later formalized by Horch and Isaksson (1999) in the framework of a user-defined closed-loop pole, and a modified performance index was calculated in terms of a nonzero closed-loop pole. A similar user-defined benchmark has also been proposed in Huang and Shah (1998), where the H_2 optimal control augmented by a first-order filter has been used as a performance bound. These user-defined benchmarks provide flexibility over the MV performance bound, and, as such, represent more realistic performance measures. There is no guarantee, however, that these user-defined benchmarks are indeed achievable under the existing control scheme, for example, PID control.

When controller structure is limited to a certain type, it becomes important to take the controller structure limitation into account in assessing the controller performance. The use

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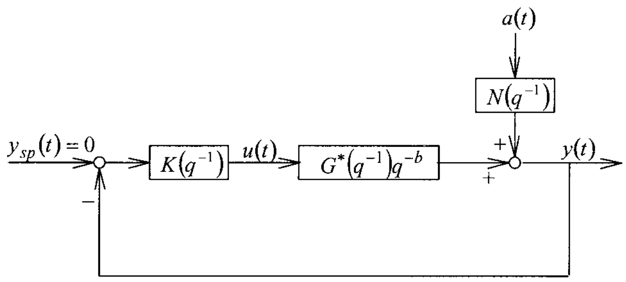

Figure 1. Discretized single-loop control system.

of a controller-specific performance bound ensures that the resulting performance bound is always achievable within a given control structure. Furthermore, it provides valuable information as to whether a more sophisticated control structure is needed to accomplish the desired control objective. Considering the fact that over 90% of controllers used in chemical process industries are of the PID type, it is sensible to develop methods that can be used specifically for PID loop performance analysis. In fact, several articles (Eriksson and Isaksson, 1994; Isaksson, 1996; Ko and Edgar, 1998) have addressed the usage of PID achievable performance bound for such a performance assessment.

In this article, we derive an iterative solution method for the calculation of PID achievable performance bound, where the iterative solution uses the process output data and the process model. This iterative solution is based on an analytic solution for closed-loop output, which is also derived in this article. A normalized PID performance index is then defined based on the best achievable PID performance, and its approximate confidence interval is estimated. A series of simulated examples are employed to illustrate the effectiveness of the proposed method in the PID loop performance assessment.

Problem formulation

Consider the discretized single-loop control system shown in Figure 1. The process output $y(t)$ is assumed to be represented by the discrete-time model

$$y(t) = G^*(q^{-1})q^{-b}u(t) + N(q^{-1})a(t) \tag{1}$$

where $G^*(q^{-1})q^{-b}$ is the process model with time delay b, and $N(q^{-1})a(t)$ represents the effects of all unmodeled and unmeasured disturbances. $N(q^{-1})$ is the disturbance model driven by a zero-mean white noise $a(t)$. The process output $y(t)$ and the manipulated variable $u(t)$ are expressed as a deviation from their nominal values. Finally, the feedback PID controller is expressed as

$$K(q^{-1}) = \frac{k_1 + k_2q^{-1} + k_3q^{-2}}{1 - q^{-1}} \tag{2}$$

where k_1, k_2, and k_3 are the PID settings. The derivation of this digital version of PID controller can be found in many textbooks on process control (see, for example, Seborg et al., 1989; Stephanopoulos, 1984). The backward shift operator $q^{-1}(q^{-1}y(t) = y(t-1))$ will be omitted for the rest of this article for simplicity of notation.

Referring to Figure 1, when there is no set point change, the output $y(t)$ under a closed-loop control K can be obtained as

$$y(t) = \frac{N}{1 + G^*Kq^{-b}} \cdot a(t) \equiv G_{cl}a(t) \tag{3}$$

where G_{cl} is defined as the closed-loop transfer function from $a(t)$ to $y(t)$. The output variance under V_{PID} under a PID control can then be evaluated as

$$V_{PID} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_y(w)dw = \left(\frac{1}{2\pi i}\oint G_{cl}(z)G_{cl}(z^{-1})\frac{dz}{z}\right)\cdot \sigma_a^2 \tag{4}$$

where $\Phi_y(w)$ represents the spectrum of output signal y (t), σ_a^2 represents the variance of random noise a (t) and $\oint$ denotes the counterclockwise integral along the unit circle in the complex plane. A numerical scheme is available in the literature (Åström, 1970) for the evaluation of the integral in Eq. 4 when G_{cl} has all its poles inside the unit circle. With this method, however, it is difficult to obtain the output variance as an explicit function of PID settings. Numerical optimization package is needed to obtain the best achievable PID performance.

In the next section, a more efficient iterative solution is developed that does not require any numerical optimization package.

An iterative solution for best achievable PID performance bound

In this section, an iterative solution is derived that gives the best achievable PID performance bound in an existing PID loop with the process output data and the nominal process model. The corresponding PID settings that give the minimum achievable variance are also obtained here.

The system is represented by Eq. 1, with its impulse response forms given by

$$y(t) = \sum_{i=1}^{m} g_iu(t-i) + \sum_{i=0}^{\infty} n_ia(t-i) \tag{5}$$

where g_i and n_i are impulse response coefficients of the process and the disturbance, respectively, and m is the largest number of time intervals for an output to reach a steady-state threshold. Under a discrete PID control given in Eq.2, the output $y(t)$ in Eq. 5 becomes

$$y(t) = -\sum_{i=1}^{m} s_i(k_1 + k_2q^{-1} + k_3q^{-2})\cdot y(t-i) + \sum_{i=0}^{\infty} n_ia(t-i) \tag{6}$$

where the $s_i(i=1,2,\ldots,m)$ represent step response coefficients of the process model.

To obtain an expression for the output variance as an explicit function of PID settings, we first consider closed-loop behavior for a single random shock, a_o, which is introduced to the system at $t=0$. In this case, from Eq. 6, the future closed-loop outputs over a finite-horizon p are related to the PID settings as follows

$$\begin{bmatrix} y_o \\ y_1 \\ \vdots \\ y_p \end{bmatrix} = -\begin{bmatrix} 0 & \cdots & & & \\ s_1 & 0 & & 0 & \\ s_2 & s_1 & 0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ s_p & s_{p-1} & \cdots & s_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_o \\ y_1 \\ \vdots \\ y_p \end{bmatrix} \cdot k_1 - \begin{bmatrix} 0 & \cdots & & & \\ s_1 & 0 & & 0 & \\ s_2 & s_1 & 0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ s_p & s_{p-1} & \cdots & s_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ y_o \\ \vdots \\ y_{p-1} \end{bmatrix} \cdot k_2 - \begin{bmatrix} 0 & \cdots & & & \\ s_1 & 0 & & 0 & \\ s_2 & s_1 & 0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ s_p & s_{p-1} & \cdots & s_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ y_o \\ \vdots \\ y_{p-2} \end{bmatrix} \cdot k_3 + \begin{bmatrix} n_o \\ n_1 \\ \vdots \\ n_p \end{bmatrix} \cdot a_o \quad (7)$$

with a "forward shift" matrix $F \in R^{(p+1)\times(p+1)}$ defined as

$$F = \begin{bmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix} \quad (8)$$

the future closed-loop outputs can be written in closed-form as

$$\begin{bmatrix} y_o \\ y_1 \\ \vdots \\ y_p \end{bmatrix} = (I + Sk_1 + FSk_2 + F^2Sk_3)^{-1}\bar{n}a_o \quad (9a)$$

where

$$S = \begin{bmatrix} 0 & \cdots & & & \\ s_1 & 0 & & 0 & \\ s_2 & s_1 & 0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ s_p & s_{p-1} & \cdots & s_1 & 0 \end{bmatrix} \text{ and } \bar{n} = \begin{bmatrix} n_o \\ n_1 \\ \vdots \\ n_p \end{bmatrix} \quad (9b)$$

Note that $FS = SF$. If we allow the random shocks to occur at every sampling instants, then the closed-loop output $y(t)$ at sampling time t can be obtained, by the principle of superposition, as

$$y(t) = \sum_{i=0}^{p} \psi_i a(t-i) \quad (10a)$$

where

$$\begin{bmatrix} \psi_o \\ \psi_1 \\ \vdots \\ \psi_p \end{bmatrix} = (I + Sk_1 + FSk_2 + F^2Sk_3)^{-1}\bar{n} \quad (10b)$$

In Eq. 10b, Ψ_i's define the closed-loop impulse response coefficients. Note that Eq. 10 gives exact results on the closed-loop output for any finite horizon p because of the special lower triangular structure of the matrix being converted; increasing the horizon length from p to $p+1$ does not affect the existing impulse response coefficients up to lag p. Also note that the inverse operation in Eq.10b is greatly simplified because of the lower triangular structure.

The output variance can now be obtained as

$$V_{PID} = \bar{n}^T(I + S^Tk_1 + (FS)^Tk_2 + (F^2S)^Tk_3)^{-1}(I + Sk_1 + FSk_2 + F^2Sk_3)^{-1}\bar{n}\sigma_a^2 \quad (11)$$

Equation 11 provides a "one-shot" solution for the closed-loop output variance as an explicit function of PID settings. It is important to note here that the process model takes the form of step response in Eq. 11. Hence, any higher order process models can be dealt with in this expression without approximating them as first-order or second-order process models. Also, any parametric model fitting procedures can be avoided with the proposed method when a system identification procedure results in step response coefficients directly.

The optimal PID settings that give minimal output variance satisfy the following first-order necessary condition

$$\frac{\partial V_{PID}}{\partial k_1} = -2\bar{n}^T(L^{-1})^TSL^{-2}\bar{n} = 0 \quad (12a)$$

$$\frac{\partial V_{PID}}{\partial k_2} = -2\bar{n}^T(L^{-1})^TFSL^{-2}\bar{n} = 0 \quad (12b)$$

$$\frac{\partial V_{PID}}{\partial k_3} = -2\bar{n}^T(L^{-1})^TF^2SL^{-2}\bar{n} = 0 \quad (12c)$$

where L is defined as $L \equiv I + Sk_1 + FSk^2 + F^2Sk_3$. Also, expressions for the second-order derivatives of the output variance can be obtained as

$$\frac{\partial^2 V_{PID}}{\partial k_1^2} = 2\bar{n}^T(L^{-2})^TS^TSL^{-2}\bar{n} + 4\bar{n}^T(L^{-1})^TS^2L^{-3}\bar{n} \quad (13a)$$

$$\frac{\partial^2 V_{PID}}{\partial k_1 \partial k_2} = 2\bar{n}^T(L^{-2})^T(FS)^TSL^{-2}\bar{n} + 4\bar{n}^T(L^{-1})^TFS^2L^{-3}\bar{n} \quad (13b)$$

$$\frac{\partial^2 V_{PID}}{\partial k_1 \partial k_3} = 2\bar{n}^T(L^{-2})^T(F^2S)^TSL^{-2}\bar{n} + 4\bar{n}^T(L^{-1})^TF^2S^2L^{-3}\bar{n} \quad (13c)$$

$$\frac{\partial^2 V_{PID}}{\partial k_2^2} = 2\bar{n}^T(L^{-2})^T(FS)^T(FS)L^{-2}\bar{n} + 4\bar{n}^T(L^{-1})^TF^2S^2L^{-3}\bar{n} \quad (13d)$$

$$\frac{\partial^2 V_{PID}}{\partial k_2 \partial k_3} = 2\bar{n}^T(L^{-2})^T(F^2S)^T(FS)L^{-2}\bar{n} + 4\bar{n}^T(L^{-1})^TF^3S^2L^{-3}\bar{n} \quad (13e)$$

$$\frac{\partial^2 V_{PID}}{\partial k_3^2} = 2\bar{n}^T(L^{-2})^T(F^2S)^T(F^2S)L^{-2}\bar{n} + 4\bar{n}^T(L^{-1})^T F^4 S^2 L^{-3}\bar{n} \tag{13f}$$

Using the Taylor series approximation and the expressions for the first-order and second-order derivatives of the output variance above, one can use Newton's iterative method to compute PID settings that satisfy Eq. 12.

$$\bar{k}_{new} = \bar{k}_{old} - \begin{bmatrix} \frac{\partial^2 V_{PID}}{\partial k_1^2} & \frac{\partial^2 V_{PID}}{\partial k_1 \partial k_2} & \frac{\partial^2 V_{PID}}{\partial k_1 \partial k_3} \\ \frac{\partial^2 V_{PID}}{\partial k_1 \partial k_2} & \frac{\partial^2 V_{PID}}{\partial k_2^2} & \frac{\partial^2 V_{PID}}{\partial k_2 \partial k_3} \\ \frac{\partial^2 V_{PID}}{\partial k_1 \partial k_3} & \frac{\partial^2 V_{PID}}{\partial k_2 \partial k_3} & \frac{\partial^2 V_{PID}}{\partial k_3^2} \end{bmatrix}^{-1}_{\bar{k}=\bar{k}_{old}} \cdot \begin{bmatrix} \frac{\partial V_{PID}}{\partial k_1} \\ \frac{\partial V_{PID}}{\partial k_2} \\ \frac{\partial V_{PID}}{\partial k_3} \end{bmatrix}_{\bar{k}=\bar{k}_{old}} \tag{14}$$

Once the convergence is obtained from Eq. 14, the Hessian matrix can be checked for positive-definiteness with the second-order derivatives in Eq. 13 to assure that the sufficient condition for optimality is met. Denoting such PID settings as $\bar{k}_{opt} = (k_{1,opt} \quad k_{2,opt} \quad k_{3,opt})^T$, the best achievable PID performance bound can be expressed as

$$(V_{PID})_{opt} = \bar{n}^T(L_{opt}^{-1})^T L_{opt}^{-1}\bar{n}\sigma_a^2 \tag{15}$$

where $L_{opt} = I + Sk_{1,opt} + FSk_{2,opt} + F^2Sk_{3,opt}$.

The best achievable PID performance bound in Eq. 15 is also valid for processes with noninvertible inverses, because the controller structure was already restricted to a *stable* PID form. Therefore, regardless of the process invertibility, the best achievable PID control performance can be obtained from the iterative scheme in Eqs. 12 through 15.

If there exists uncertainty in process model parameters, a range of PID performance bounds can be calculated for the uncertainty interval of the model parameter by calculating "optimal" PID settings for different values of model parameters and computing the corresponding PID performance bounds. Alternatively, sensitivity analyses can be performed with respect to each model parameter. For example, the sensitivity of the PID performance bound V_{PID} to an error in a model parameter θ can be estimated with the approximation below

$$\left(\frac{\partial V_{PID}}{\partial \theta}\right)_{\theta_0} \cong \frac{(V_{PID})_{\theta_0+\Delta\theta/2} - (V_{PID})_{\theta_0-\Delta\theta/2}}{\Delta\theta}$$

where θ_0 is the nominal value of θ.

Performance assessment

In the previous section, an iterative scheme was developed for the best achievable PID performance bound with the impulse responses of both process and disturbance models. In this section, it is described how PID performance bound can be estimated from a set of closed-loop operating data when the process impulse response is known *a priori*.

Case A: Stochastic disturbance regulation. It is noted from Eq. 10b that the disturbance model impulse response is related to the closed-loop impulse response as follows

$$\bar{n} = (I + Sk_1 + FSk_2 + F^2Sk_3) \cdot \begin{bmatrix} \psi_o \\ \psi_1 \\ \vdots \\ \psi_p \end{bmatrix} \tag{16}$$

Hence, the vector of disturbance model impulse response can be estimated with the closed-loop impulse response obtained after the time-series modeling of the closed-loop output data

$$\hat{\bar{n}} = (I + Sk_1 + FSk_2 + F^2Sk_3) \cdot \begin{bmatrix} \hat{\psi}_o \\ \hat{\psi}_1 \\ \vdots \\ \hat{\psi}_p \end{bmatrix} \tag{17}$$

where $(\hat{\cdot})$ represents the sample estimate of $(\cdot)$. If the PID controller used during the periods of data collection is either unavailable or nonlinear because of the presence of input constraints, the disturbance model N cannot be obtained from Eq. 17. For this case the disturbance sequence can still be reconstructed from $d(t)=y(t)-G^*u(t-b)$ and the disturbance impulse response estimated from the time-series modeling of the sequence. With the estimated disturbance impulse response, the iterative scheme described in previous section can be applied to obtain the PID achievable performance bound.

Case B: Deterministic set point tracking. Consideration has only been given so far to the performance assessment for a stochastic disturbance occurring at the process output with no set point change. If, however, deliberate deterministic set point changes are a predominant source of process upsets, and set point tracking is the primary measure of performance, the methodology used in this article can be extended to handle such a case.

It is noted in the literature (Åström and Wittenmark, 1990; Harris and MacGregor, 1987) that randomly occurring deterministic set point changes can be modeled with the same model structure used for stochastic disturbance. For example, a series of step changes in the set point can be modeled as $[1/(1 - q^{-1})]a(t)$, where the driving force $a(t)$ is zero except at the time of set point changes. Therefore, in this case, the vector of disturbance model impulse-response coefficients would simply be

$$\bar{n} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Note that it is not necessary for this case to obtain the closed-loop impulse response because the disturbance model is known *a priori*. Other types of set point changes can be treated similarly. For example, the ramp changes in the set point can be modeled as $[1/(1 - q^{-1})^2]a(t)$, and the corresponding vector of disturbance model impulse response would be

$$\bar{n} = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ p+1 \end{bmatrix}$$

Table 1. Details of the Iterative Process

Iteration Number	PI Control Setting k_1	PI Control Setting k_2	Sum of Squared Error
0	2.300	−2.100	4.807
1	3.230	−2.963	3.988
2	3.407	−3.050	3.756
3	3.408	−2.984	3.707
4	3.418	−2.976	3.704
5	3.418	−2.976	3.704

Again, the iterative scheme in Eqs. 12 through 15 can then be applied to those disturbance impulse responses to compute "optimal" PID settings and the achievable PID performance bound.

Example. This example considers the case where deterministic step set point changes are the predominant source of process upsets, and set point tracking is the primary objective of the control. The process model considered in this example is

$$G = \frac{0.1q^{-3}}{1 - 0.8q^{-1}}$$

A PI controller below is used to track step set point changes

$$K = \frac{2.3 - 2.1q^{-1}}{1 - q^{-1}}$$

To find "optimal" PI settings for a series of step changes in set point, the iterative in Eq. 14 is simplified for PI control as follows

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}_{new} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}_{old} - \begin{bmatrix} \dfrac{\partial^2 V_{PID}}{\partial k_1^2} & \dfrac{\partial^2 V_{PID}}{\partial k_1 \partial k_2} \\ \dfrac{\partial^2 V_{PID}}{\partial k_1 \partial k_2} & \dfrac{\partial^2 V_{PID}}{\partial k_2^2} \end{bmatrix}^{-1}_{\bar{k}=\bar{k}_{old}} \cdot \begin{bmatrix} \dfrac{\partial V_{PID}}{\partial k_1} \\ \dfrac{\partial V_{PID}}{\partial k_2} \end{bmatrix}_{\bar{k}=\bar{k}_{old}} \tag{18}$$

where the first and second-order derivatives of V_{PID} are defined in Eqs. 12 and 13 with $k_3=0$. Also note, the disturbance impulse response is $\bar{n} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ for the step set point changes. Therefore, starting with the original PI settings, the "optimal" PI settings can be found from the iterative formula in Eq. 18. In this example, the solution was found after five iterations, and the iterative process is summarized in Table 1. The closed-loop outputs with the initial and final PI settings are also compared in Figure 2 for a series of step changes in set point.

PID performance index and its approximate confidence interval

It is often convenient to define and monitor a single metric that measures the performance of feedback control loop. Various forms of performance indices have been defined and used in the literature (Desborough and Harris, 1992; Stanfelj et al., 1993; Kozub, 1997). In this section, a new PID performance index is defined based on the best achievable PID performance bound, and its approximate confidence interval is estimated. The PID performance index ζ defined in this article is the ratio of PID-achievable performance bound to the actual output variance

$$\zeta = \frac{(V_{PID})_{opt}}{V_{actual}} \tag{19}$$

This normalized performance index will have the range of $0 < \zeta \leq 1$, and $\zeta=1$ indicates the best performance under the PID control. With this definition of ζ, $1-\zeta$ will indicate the maximum fractional reduction in the output variance that can be achieved with PID control.

To characterize the distributional properties of the estimated PID performance index and, thus, obtain its approximate confidence interval, the approach used in Desborough and Harris (1992) will be adopted here. From the definition of the PID performance index in Eq. 19, the sample estimate of the PID performance index $\hat{\zeta}$ can be expressed in terms of the output $y(t)$ as follows

$$\hat{\zeta} = \frac{\bar{y}_{opt}^T \bar{y}_{opt}}{\bar{y}^T \bar{y}} \tag{20}$$

where $\bar{y} = \begin{bmatrix} y(t) \\ y(t-1) \\ \vdots \\ y(t-n+1) \end{bmatrix}$ is the vector of the actual output and $\bar{y}_{opt} = \begin{bmatrix} y_{opt}(t) \\ y_{opt}(t-1) \\ \vdots \\ y_{opt}(t-n+1) \end{bmatrix}$ is the "optimal" output vector that can be achieved with "optimal" PID setting. The "optimal" sequence of output is related to the actual output as $y_{opt}(t) = [(1 + G^*q^{-b}\tilde{K})/(1 + G^*q^{-b}K_{opt})]y(t)$, where $\tilde{K}$ is the PID controller used during the periods of data collection and k_{opt} is the PID controller with "optimal" PID setting. It is shown in the Appendix that, with a first-order Taylor series approxima-

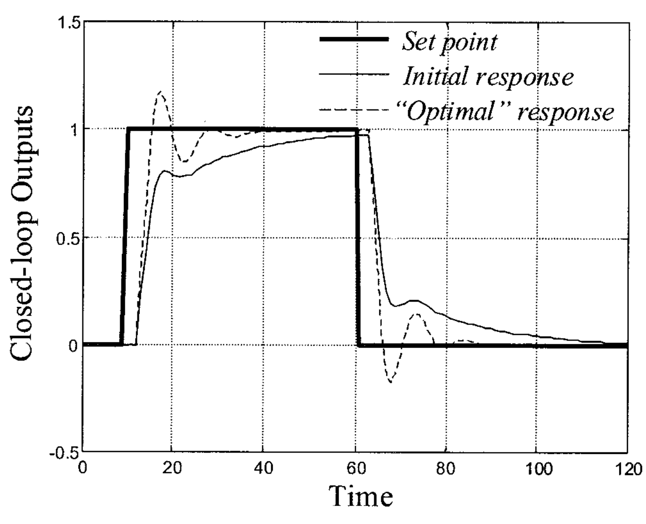

Figure 2. Comparison of closed-loop outputs with the initial and "optimal" PI settings for step set point changes.

tion of $\hat{\zeta}$, the mean and the variance of $\hat{\zeta}$ can be approximated for large n by

$$\text{mean}(\hat{\zeta}) = \zeta \tag{21}$$

$$\text{var}(\hat{\zeta}) = \frac{4}{n}\zeta^2 \sum_{k=1}^{\infty} (\rho_k - \rho_{k,opt})^2 \tag{22}$$

where ρ_k and $\rho_{k,opt}$ represent the true autocorrelations at lag k of the output $y(t)$ and the "optimal" output $y_{opt}(t)$, respectively. In practice, the population values of the autocorrelations and the performance index can be replaced by the sample values.

Example 1. It is well-known that for first-order processes with $b=1$, a PI controller achieves the minimum variance control performance for output disturbances represented by an integrated first-order moving average IMA (0,1,1) process. It will be shown in this example that the iterative solution derived earlier does actually result in the PI performance bound that is equal to the MV bound and the correct "optimal" PI settings by performing Monte-Carlo simulations. The following process and disturbance models were used for this example

$$G = \frac{q^{-1}}{1 - 0.8q^{-1}}, \quad N = \frac{1 - 0.2q^{-1}}{1 - q^{-1}}$$

For closed-loop simulations, a PI controller $K = [(0.24 - 0.2q^{-1})/(1 - q^{-1})]$ was chosen, and 1,500 sets of disturbance sequences were generated from different white noise sequences with 2,000 data points. The variance of the white noise sequence used was 0.01. For each simulated output, an autoregressive moving average ARMA (10,3) process was fitted to obtain the closed-loop impulse response. The disturbance model impulse response was then estimated from Eq. 17, and the best achievable PI performance and corresponding settings were obtained by applying the iterative solution scheme developed in the previous sections. The results of these simulations

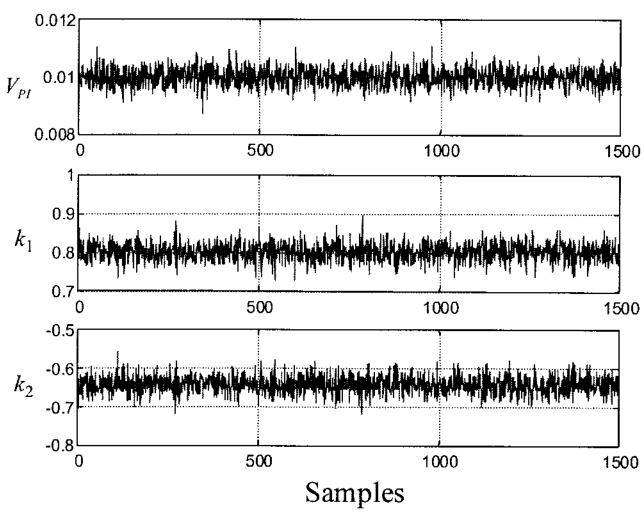

Figure 3. Simulation results for Example 1.

The top plot shows the PI-achievable performance bound, and the middle and the bottom plots show the corresponding PI settings.

Table 2. Summary of Simulation Results for Example 1

Theoretical minimum variance controller	$\frac{0.8 - 0.64q^{-1}}{1 - q^{-1}}$
Estimated "optimal" PI settings (average)	$\frac{0.801 - 0.641q^{-1}}{1 - q^{-1}}$
Output variance (average)	1.844×10^{-2}
Theoretical MV bound	1.000×10^{-2}
Estimated MV bound (average)	0.992×10^{-2}
Estimated PI performance bound (average)	1.002×10^{-2}

are plotted in Figure 3, and are also summarized in Table 2, where the average values of PI settings and the PI performance bound obtained show good agreement with the theoretical values.

Example 2. In this example, the following models for the process and the output disturbance are considered

$$G = \frac{0.1}{1 - 0.8q^{-1}} q^{-6}$$

$$N = \frac{1}{(1 - q^{-1})(1 - 0.7q^{-1})}, \quad \sigma_a^2 = 0.01$$

A PI controller below was used to regulate the disturbance in the simulation

$$K = \frac{2.3 - 2.1q^{-1}}{1 - q^{-1}}$$

The simulated closed-loop outputs and the disturbance sequence of length 2,000 are shown in Figure 4. An autoregressive moving average ARMA (10,3) process was then used to obtain closed-loop impulse response. With the obtained closed-loop impulse response, the disturbance impulse response was estimated from Eq. 17. With the estimated disturbance impulse response, the PI and PID performance bounds were estimated with Eqs. 12 through 15, along with the "optimal" PI/PID settings. The results are summarized in Table 3, where the actual output variance and the MV bound are also shown for comparison. The results in Table 3 indicate that the PI controller used in the simulation achieved nearly optimal PI controller performance, although it was far from achieving the theoretical minimum variance performance. Table 3 also indicates that substantial output variance reduction is warranted if PID control is used instead of PI control. On the basis of the PI/PID performance bounds obtained, the PI/PID performance indices were calculated from Eq. 19. The calculated PI/PID performance indices are also shown in Table 3 along with their $\pm 2\sigma$ confidence intervals that were obtained with Eqs. 21 and 22.

To account for the effects of the process modeling error, a range of PI performance indices are calculated for the uncertainty interval of the process time constant of [3.14 5.82] and for the process gain uncertainty interval of [0.35 0.65]. With the sampling interval of 1 min, these uncertainty intervals correspond to $\pm 30\%$ errors around the nominal process time constant of $\tau_0 = 4.48$ and the nominal process gain of $K_0 = 0.5$. The calculated PI performance indices around these nominal model parameters are plotted in Figure 5, along with their $\pm 2\sigma$ confidence intervals. It is seen from Figure 5 that the estimated PI performance index is rather insensitive to both model pa-

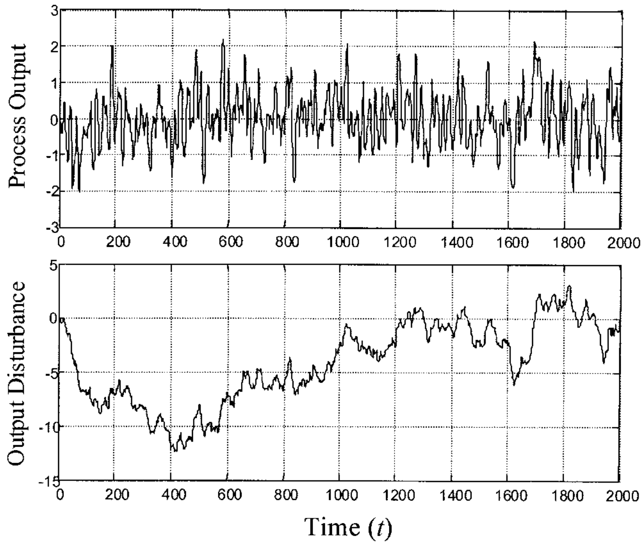

Figure 4. Simulation results for Example 2.

The process output is the top graph and the disturbance sequence is the bottom.

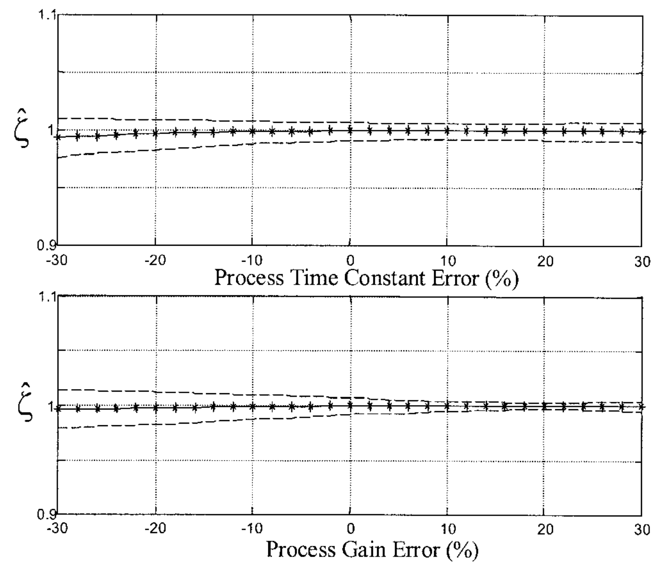

Figure 5. Sensitivity of the estimated PI performance index to modeling errors.

The dashed lines represent confidence intervals.

rameters for this example. In general, however, the sensitivities of the estimated PID performance index to the modeling errors depend on the PID settings used during the periods of data collection; the PID performance index tends to be less sensitive to the modeling errors when installed PID controller settings are close to the "optimal" PID settings.

Conclusions

In this article, the PID-achievable performance bound was proposed for use in assessing and monitoring single-loop PID control loop performance. For this, an iterative solution was derived that gives the best achievable PID control performance in terms of the closed-loop output data and the process model. An explicit "one-shot" solution for the closed-loop output was derived as a function of PID settings. A PID performance index was defined based on the best achievable PID performance for use as a realistic performance measure in the single-loop PID control systems. Expressions for the mean and the variance of this performance index were also obtained to compute its approximate confidence interval. A series of computer simulations showed the utility of the proposed method for the effective performance assessment of the single-loop PID control loops.

Table 3. Summary of Simulation Results for Example 2

Actual output variance	0.5480
Estimated MV bound	0.3131
Estimated PI control performance bound	0.5475
Corresponding optimal PI controller	$\frac{2.279 - 2.072q^{-1}}{1 - q^{-1}}$
Estimated PID control performance bound	0.4564
Corresponding optimal PID controller	$\frac{6.97 - 11.46q^{-1} + 4.84q^{-2}}{1 - q^{-1}}$
PI performance index	0.999 ± 0.007
PID performance index	0.833 ± 0.075

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Appendix: Approximate expressions for the mean and the variance of $\hat{\zeta}$

By defining the quadratic forms $T \equiv \bar{y}^T\bar{y}$ and $U \equiv \bar{y}_{opt}^T\bar{y}_{opt}$, the sample estimate of the PID performance index given in Eq. 19 can be rewritten as

$$\hat{\zeta} = \frac{U}{T} \tag{A1}$$

The quadratic forms T and U have the mean values as follows

$$E(T) = n\sigma_y^2 \text{ and } E(U) = n\sigma_{y_{opt}}^2 \tag{A2}$$

where $E(\cdot)$ denotes the expectation operator; σ_y^2 and $\sigma_{y_{opt}}^2$ are the true variances of the output $y(t)$ and the "optimal" output $y_{opt}(t)$, respectively. To obtain approximate expressions for the mean and the variance of $\hat{\zeta}$, a first-order Taylor series approximation can be applied to $\hat{\zeta}$ with the resulting expression below

$$\hat{\zeta} \approx \frac{E(U)}{E(T)} - \frac{E(U)}{E(T)^2}[T - E(T)] + \frac{1}{E(T)}[U - E(U)] \tag{A3}$$

It is clear that

$$\text{mean } (\hat{\zeta}) = \frac{E(U)}{E(T)} = \zeta \tag{A4}$$

For the approximate variance expression of $\hat{\zeta}$, we have

$$\begin{aligned}\text{var}(\hat{\zeta}) &= E[(\hat{\zeta} - \zeta)^2] = E\left[\left(\hat{\zeta} - \frac{E(U)}{E(T)}\right)^2\right] \\ &= \frac{E(U)^2}{E(T)^4}\text{var}(T) - \frac{2E(U)}{E(T)^3}\text{cov}(T, U) + \frac{1}{E(T)^2}\text{var}(U)\end{aligned} \tag{A5}$$

With the results from Desborough and Harris (1992), the expressions for the variance and the covariance in Eq. A5 can be obtained as

$$\begin{aligned}\text{var}(T) &= 2\sigma_y^4\left[n + 2\sum_{k=1}^{\infty}(n-k)\rho_k^2\right] \\ \text{var}(U) &= 2\sigma_{y_{opt}}^4\left[n + 2\sum_{k=1}^{\infty}(n-k)\rho_{k,opt}^2\right] \\ \text{cov}(T, U) &= 2\sigma_y^2\sigma_{y_{opt}}^2\left[n + 2\sum_{k=1}^{\infty}(n-k)\rho_k\rho_{k,opt}\right]\end{aligned} \tag{A6}$$

where ρ_k and $\rho_{k,opt}$ denote the true autocorrelations at lag k of the output $y(t)$ and the "optimal"output y_{opt} (t), respectively. When n is large, and the autocorrelations decay relatively quickly, the expressions above asymptotically approach to

$$\begin{aligned}\text{var}(T) &= 2n\sigma_y^4\left[1 + 2\sum_{k=1}^{\infty}\rho_k^2\right] \\ \text{var}(U) &= 2n\sigma_{y_{opt}}^4\left[1 + 2\sum_{k=1}^{\infty}\rho_{k,opt}^2\right] \\ \text{cov}(T, U) &= 2n\sigma_y^2\sigma_{y_{opt}}^2\left[1 + 2\sum_{k=1}^{\infty}\rho_k\rho_{k,opt}\right]\end{aligned} \tag{A7}$$

Substituting Eq. A7 into Eq. A5, it can be shown that the approximate expression for the variance of $\hat{\zeta}$ is given by

$$\text{var}(\hat{\zeta}) \approx \frac{4}{n}\zeta^2\sum_{k=1}^{\infty}(\rho_k - \rho_{k,opt})^2 \tag{A8}$$

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