Abstract Algebra

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Exercise 1. Let $H(F)$ be the Heisenberg group over the field F. Find and explicit formula for the commutator $[X, Y]$, where $X, Y \in H(F)$.

Solution. Let us remember that in the first homework we found that the inverse for a member of the Heisenberg group over F was: Observe that since the determinant of any element in H is one, then there exist a inverse matrix, *i.e* it has a matrix inverse, more explicitly:

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & ac - b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$$

so that let $x, y \in H(F)$ so that:

$$X = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$Y = \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}$$

by definition of the commutator $[X, Y] = X^{-1}Y^{-1}XY$ so that:

$$[X, Y] = \begin{pmatrix} 1 & -a & ac - b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -d & df - e \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix} XY$$

and putting this matrix into Mathematica (although I did remeber how to compute matrices but the laziness work), we found that:

$$[X, Y] = \begin{pmatrix} 1 & 0 & af - cd \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is a closed formula for the commutator

Exercise 2. Show that this operation on $\check{G}$ makes this set an abelian group. [Hint: show that the trivial homomorphism is the identity and that $\chi^{-1}(g) = \chi(g)^{-1}$.]

Solution. Consider A as an abelian group, and G a group, now consider the set of all homomorphism $G \to A$, so that we want to see and define a product on this set. We will do ut via pointwise multiplication, for identity observe that the trivial homomorphism is on the set described above, so that $(1\bar{A})(x) = (\bar{A}1)(x) = \bar{a}(x)$ and we have our identity, Now consider an arbitrary homomorphism, say ϕ

Exercise 3. Write definitions of the graph homomorphism and isomorphism $\gamma_1 \to \gamma_2$, and automorphism of a graph.

Solution.

Definition 0.1. A *graph* consists of a set of *vertices* $V(G)$ and a set of *edges* $E(G)$ represented by unordered pair of vertices. We define vertices x and y to be a adjacent of $(x, y) \in E(G)$

Definition 0.1. We define that a graph is *complete* if every vertex is connected to every other vertex

We define the notion of isomorphism as:

Definition 0.2. An *isomorphism* between two graphs G and H as a bijective map $f : G \to H$ with the property:

$$(x, y) \in E(G) \iff (f(x), f(y)) \in E(H)$$

we weakening the above definition for make sense of a graph homomorphism

Definition 0.3. A *homomorphism* from a graph G to a graph H is defined as a mapping $h : G \to H$ such that

$$(x, y) \in E(G) \implies (f(x), f(y)) \in E(H)$$

In the end we want that preserves edges. Which leads to:

Definition 0.4. A graph G with 2 independent vertex set is a bipartite graph.

Finally we want to discuss the transformations of a graph onto itself which leads to the concept of:

Definition 0.5. An isomorphism from a graph G to itself is called an **automorphism**