

## COMPLEX ANALYSIS

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HORARIO Y LUGAR DE ATENCIÓN: Mo y 17:00 a 18:00, Office H-409

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## 1 Organization of the course

- 3 Exams
- regular-weekly homework

Sometimes we will use the following alternative book: *Complex Analysis*, Gamelin

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## 2 Complex Numbers

We want to solve a quadratic equation:

$$ax^2 + bx + c = 0$$

so we first calculate the discriminant:

$$D = b^2 - 4ac$$

it is called that, because it differentiate solution types. We then compute:

$$r_{\pm} = \frac{-b \pm \sqrt{D}}{2a}$$

In  $R$  squares are positives, but this poses a problem when  $D < 0$ . Meanwhile, let us observe that when  $D > 0$  we get two solutions, and when we get  $D = 0$

there is only one solution. So the main question is what to do when  $D < 0$ . Without loss of generality we can assume that  $a = 1$

$$\begin{aligned} r_+ &= -\frac{b}{2} + \sqrt{2} \\ r_- &= -\frac{b}{2} - \sqrt{2} \end{aligned}$$

the reason is that we want to be able write

$$p(x) = (x - r_+)(x - r_-)$$

expanding the product above

$$p(x) = x^2 - (r_+ + r_-)x + r_+r_-$$

so that  $D$  can be written as:

$$D = \text{Sign}(D)|D|$$

Observe that we can take the definition of sign as above. But then, this give a suspect idea that there can exist a **not real number** that:

$$\pm(\sqrt{-1})^2 = -1$$

which is also a solution of the following equation:

$$x^2 + 1 = 0$$

This then motivate the following concept, does arithmetic follows natural with the inclusion:

$$z = a + b\sqrt{-1}$$

Observe the symmetry on choosing  $\sqrt{-1}$  or  $-\sqrt{-1}$ . Which again motivates

$$\bar{z} = a - b\sqrt{-1}$$

As in the textbook  $i^n$  maps to  $1, i, -1, -i$  with the trick  $\pmod{4}$  Let us define then that addition comes naturally:

$$(a + b\sqrt{-1}) + (c + d\sqrt{-1}) = (a + c) + (b + d)\sqrt{-1}$$

Multiplication is a little bit tricky:

$$(a + b\sqrt{-1}) \times (c + d\sqrt{-1}) = (ac - bd) + (ad + bc)\sqrt{-1}$$

which comes natural by considering  $(\sqrt{-1})^2$ . Observe that the reals are embedded as:

$$a + 0\sqrt{-1}$$

The operation of taking  $z \rightarrow \bar{z}$  is called *Complex conjugation*, observe that it complies  $\bar{\bar{z}} \rightarrow z$ , which will allow us to extract  $a, b$  as:

$$\frac{1}{2}(z + \bar{z}) := \operatorname{Re}(z) = a$$

and also

$$\frac{1}{2\sqrt{-1}}(z - \bar{z}) := \operatorname{Im}(z) = b$$

our definition above can be then replaced by:

$$z = \operatorname{Re}(z) + \operatorname{Im}(z)\sqrt{-1}$$

The preceding equation proposes a big problem, since we have found another "strange" number:

$$\frac{1}{\sqrt{-1}} = x$$

or equivalently,

$$\sqrt{-1}x = 1$$

from which it can be solved by inspecting the table of exponentiation and we find:

$$x = -\sqrt{-1}$$

In literature we often define that  $\sqrt{-1} = i$ . A big question is that if  $z \neq 0$ , then there exist  $w$  such that  $w \cdot z = 1$ , the answer is positive and it even has a stronger statement: with complex numbers we can solve any polynomial equation.

$$z = a + bi$$

$$w = x + iy$$

and we are looking  $z \cdot w = 1$  so we get the system of equations

$$ax - by = 1$$

$$bx + ay = 0$$

and we can get information about the solution, via the determinant  $\det = a^2 + b^2$ , and this is only zero if only if  $a = 0$  and  $b = 0$ . Solution can be found for any method

Observe that if we have a complex number and its conjugate, the following relation holds:

$$z \cdot \bar{z} = a^2 + b^2$$

$$\frac{1}{z} = \frac{1}{a^2 + b^2} \cdot \bar{z}$$

$$\frac{1}{2+3i} = \frac{1}{13}(2-3i)$$

Observe that this recall Cramer's rule, since it has the form of  $\frac{1}{\det}$

The following problem that we are facing is the existence of:

$$z = a + bi \quad \exists w \quad w^2 = z$$

if we let  $w = x + iy$ , and  $w^2 = (x^2 - y^2) + i(2xy)$ , we get the following system of equations.

$$\begin{aligned} a &= x^2 - y^2 \\ b &= 2xy \end{aligned}$$

Observe that 0 has a unique root. We denote the set of all complex numbers with  $\mathbb{C}$  and observe that we can take bijection between  $\mathbb{C}$  and  $\mathbb{R}^2$

$$\begin{aligned} \mathbb{R} &\leftrightarrow \mathbb{C} \\ z &\longrightarrow (Re(z), Im(z)) \\ a + ib &\longleftarrow (a, b) \end{aligned}$$

Observe then that  $\mathbb{C}$  is a vector space over  $\mathbb{R}$ . Basis:  $1, i$ . Now observe that we can take a geometric interpretation, given this two bijections, to the multiplication of a complex numbers. Side note: A linear map is a map that plays nice with arithmetic. an important characterization of linear maps, is that:

$$f(a) = a \cdot f(1)$$

and also

$$f\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and it can be easy characterized via:

$$\begin{pmatrix} v_{11} & v_{22} \\ v_{21} & v_{22} \end{pmatrix}$$

Now consider the following map:

$$\begin{aligned} \mathbb{C} &\rightarrow \mathbb{C} \\ w &\mapsto zw \end{aligned}$$

where  $z = a + ib$  Again observe that we are only taking care of the values on the basis.

**Example** Suppose  $A^2 = -I$ , and  $\det > 0$ , then there is an invertible matrix  $B$  such that  $A = BJB^{-1}$ . Returning to our previous point, we see that complex

numbers are embedded nicely on  $2 \times 2$  matrices with real coefficients. So that the span 1 and  $j$  is a 4 dimensional vector space over  $\mathbb{R}$ . Finally, we conclude that multiplication on  $\mathbb{C}$  is a matrix multiplication.

When we look a close view on  $\mathbb{R}^2$  along with the normal dot product, we have the **Cauchy-Schwarz Inequality**: Let us then define:

$$|\vec{a}|^2 = \vec{a} \cdot \vec{a}$$

and proof is left to the reader:

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}| |\vec{b}|$$


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### 3 Remarks from last time

$$\begin{aligned}\mathbb{C} &\rightarrow \text{Mat}_{2 \times 2}(\mathbb{R}) \\ A + IB &\rightarrow a + bJ \\ J^2 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\end{aligned}$$

so that:

$$Z = a + ib \leftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Let us observe that in  $\mathbb{R}^2$  we have the dot product for which:

$$\vec{a} \cdot \vec{a} \geq 0$$

it is good as concept of measure of distance. Let us observe the analogue on complex numbers:

$$z = a + ib$$

$$w = c + id \implies \bar{w} = c - id$$

so that

$$z\bar{w} = (ac + bd) + (bc - ad)i$$

for which the left parenthesis of the RHS is real and the other imaginary. Observe also:

$$|z|^2 = a^2 + b^2 \geq 0$$

and if  $z \neq 0$  we define:

$$z^{-1} = \frac{1}{|z|} \bar{z}$$

## Cauchy-Schwarz

Suppose we know the properties of the dot product:

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}| |\vec{b}|$$

### Example

In  $\mathbb{R}^n$ :

$$\begin{aligned} \langle a_1, \dots, a_n \rangle \cdot \langle b_1, \dots, b_n \rangle \\ &= \sum_{i=1}^n a_i b_i \\ &\leq (\sum a_i^2)^{\frac{1}{2}} (\sum b_j^2)^{\frac{1}{2}} \end{aligned}$$

**Example** The functions that are square integrable on  $I = [0, 1]$  for which we have:

$$\left| \int_0^1 f g dx \right| \leq \left| \int_0^1 f^2 dx \right|^{\frac{1}{2}} \left| \int_0^1 g^2 dx \right|^{\frac{1}{2}}$$

**Proof** Let recall that for parameterizing the line in between  $\vec{a}$  and  $\vec{b}$  we could use the convex combination.

$$t\vec{b} + (1-t)\vec{a} \quad 0 \leq t \leq 1$$

$$\left| t\vec{b} + (1-t)\vec{a} \right|^2 \geq 0$$

We expand and we get: .... It is the same trick of consider a quadratic without root  $a + \lambda b$ . Geometrical Consideration: For two independent vectors, the line that pass through them do not pass on the origin.

## 4 C-S Consequences

The triangle inequality

$$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$$

the proof is given by considering the expansion of:

$$|\vec{a} + \vec{b}|^2$$

and the inequality: (Check!)

**Definition of Angles**

$$\cos(\theta) := \frac{|\vec{a} \cdot \vec{b}|}{|\vec{a}| |\vec{b}|}$$

## 5 Special $(2 \times 2)$ matrices

Scalar:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I$$

A -  $2 \times 2$  matrices A is orthogonal iff

$$\vec{a} \cdot \vec{b} = (A\vec{a}) \cdot (A\vec{b})$$

iff Preserves lengths, and angles.

Consider the matrix of rotation  $\theta$ , and consider two rotations  $\alpha$  y  $\beta$ , and the sum of these two, it will given then the following matrix. Observe that these shows explicitly the formulas of sum and sine cosine.

**Consider the following computation**  $\cos(15\alpha)$

**Definition 1.** *Conformal* preserves angles ( but not conserve lengths)

Remark: Orthogonal transformations preserve angles, and scalar conserve angles. Conformal are complex numbers. via matrices.

$$z = |z|$$


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## 6 Class 03

Notation

$$z, w \in \mathbb{C}$$

$$x, y \in \mathbb{R} \quad \mathbb{C}$$

$$t \in \mathbb{R}$$

For this section all functions are well defined that is univalued and all functions are defined on an open set.

**Definition 2.**  $U$  is an open set if:

$$\forall x \in U \exists \epsilon > 0 \text{ s.t } |y - x| < \epsilon \implies y \in U$$

Observe that for Ahlfors  $\sqrt{x}$  is not a function.

**Definition 3.** The function  $f(x)$  "has a limit"  $A$  as  $x \rightarrow a$  we write:

$$\lim_{x \rightarrow a} f(x) = A$$

If  $\forall \epsilon > 0 \exists \delta_\epsilon > 0$  such that  $|x - a| < \delta_\epsilon \implies |f(x) - A| < \epsilon$

There are variance if  $A$  is an infinite the definitions will change accordingly.

**Lemma 1.**  $f(x) = \sin(x)$  Let show that  $\lim_{x \rightarrow 0} f(x) = 0$

*Proof.* Assuming given  $\epsilon > 0$ , need to find  $\delta_\epsilon > 0$  s.t  $x \in (-\delta_\epsilon, \delta_\epsilon)$  implies  $\sin(x) \in (-\epsilon, \epsilon)$ .

For any  $\epsilon > 0$  take  $\delta_\epsilon := \epsilon$  □

**Definition 4.** A function  $f(x)$  is continuous at  $x = a$  if:

$$\lim_{x \rightarrow a} f(x) = f(a)$$

**Example**  $\sin(x)$  is continuous at  $x = 0$

**Proposition 1.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$ ,  $a, A \in \mathbb{C}$ , then:

$$\lim_{x \rightarrow a} f(x) = A$$

$$\lim_{x \rightarrow a} \bar{f(x)} = \bar{A}$$

$$\lim_{x \rightarrow a} \operatorname{Re}(f(x)) = \operatorname{Re}(A)$$

$$\lim_{x \rightarrow a} \operatorname{Im}(f(x)) = \operatorname{Im}(A)$$

A continuous function  $f(x)$  is a function that is continuous at any  $x$  for which it is defined. So that for example  $\frac{1}{x}$  is continuous.

**Definition 5.**

$$f'(a) := \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

, the difference quotient.  $f(x)$  is differentiable at  $x = a$  if  $f'(a)$  exists.

If so,

$$f(x) \approx f(a) + (x - a)f'(a) + O((x - a))$$

Observe that the first two terms are the equation of the tangent. Observe that:

$$\frac{\epsilon(x - 1)}{x - a} \rightarrow 0 \text{ as } x \rightarrow 0$$

Observe that if  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ , let us take  $(a, b) \in \mathbb{R}^2$ , if  $f$  is differentiable at  $(a, b) \in \mathbb{R}^2$  then for  $(x, y)$  near  $(a, b)$ , we have:

$$f(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) + o(\|(x, y) - (a, b)\|)$$

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  is  $\mathbb{C}$  is differentiable at  $z = a$  if:

$$f'(z) := \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a}$$

exists. So observe:

$$f(z) = f(a) + f'(z)(z - a) + o()$$



**Proposition 2.** *Let us show that  $f(z) = z^n$  is Complex differentiable for  $z = a$ , for all  $a \in \mathbb{C}$*

*Proof.*

$$f'(a) = \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a}$$

let  $\xi = z - a$  so that

$$f'(a) = \lim_{\xi \rightarrow 0} \frac{(a + \xi)^n - a^n}{\xi}$$

and expanding using Newton binomial theorem. (and cancelling  $\xi \neq 0$ )

$$f'(a) = \lim_{\xi \rightarrow 0} (na^{n-1} + \text{things that have xi})$$

so that :

$$f'(a) = na^{n-1}$$

implies

$$f'(z) = nz^{n-1}$$

□

Now for a pathological example. Let us take  $f(z) = \text{Re}(z)$  is not complex differentiable at any point. (Also imaginary part of  $Z$  is not complex differentiable)

$$f'(a) = \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a}$$

again using  $\xi = z - a$

$$f'(a) = \lim_{\xi \rightarrow 0} \frac{\text{Re}(a + \xi) - \text{Re}(a)}{\xi}$$

so that:

$$\lim_{\xi \rightarrow 0} \frac{\text{Re}(\xi)}{\xi}$$

Observe that if  $\xi = t \in \mathbb{R}$  then

$$\text{Re}(\xi) / \xi = \frac{t}{t} = 1$$

Now, observe that if  $\xi = it \in \mathbb{R}$  and

$$\frac{\text{Re}(\xi)}{\xi} = \frac{0}{t} = 0$$

**Definition 6.** *A Complex function  $f(z)$  and  $f : \mathbb{C} \rightarrow \mathbb{C}$  is **analytic** or(holomorphic) at  $z$  if  $f'(z)$  exists (the complex derivative)*

**Theorem 1. Main thm**  $f(z)$  is analytic at  $z = a$  if and only if

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \text{ for } |z - a| < \epsilon$$

## Cauchy-Riemann equations

Let us take  $f : \mathbb{C} \rightarrow \mathbb{C}$  and rewritten as  $f(z = x + iy) = u(x, y) + iv(x, y)$   
 $u, v \in \mathbb{R}$  and assume  $f'(z)$  exist as a complex derivative. Equivalently,

$$f'(z) = \lim_{\eta \rightarrow 0} \frac{f(z + \eta) - f(z)}{\eta} \quad \eta \in \mathbb{C}$$

then the above is equivalently, to the two reformulations:

$$\lim_{\eta \in \mathbb{R}, \eta \rightarrow 0} \frac{f(z + h) - f(z)}{h}$$

and has to be equal to:

$$\lim_{\eta \in i\mathbb{R}, \eta \rightarrow 0, \quad k \in \mathbb{R}} \frac{f(z + ik) - f(z)}{ik}$$

so that with  $u, v$

$$\lim_{h \rightarrow 0} \frac{u(x + h, y) + iv(x + h, y) - u(x, y) - iv(x, y)}{h} = \frac{\partial u}{\partial x}(x, y) + i \frac{\partial v}{\partial x}(x, y)$$

and for the second part:

$$\lim_{k \rightarrow 0} \frac{u(x, y + k) + iv(x, y + k) - u(x, y) - iv(x, y + k)}{ik} = \frac{\partial v}{\partial y}(x, y) - i \frac{\partial u}{\partial y}(x, y)$$

and since them are the manifestation if the same limit, we have the equation

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}$$

For remembering, check that the function  $f(z) = z$  is complex differentiable. If we are calculating the derivative (computing stuff):

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}$$

now let us observe:

$$|f'(z)|^2 = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2$$

and this is the Jacobian.

$$\left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial v}{\partial y} \right) - \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial v}{\partial x} \right)$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix}$$

Which is the Jacobian.

Assume  $u, v$  have continuous 2nd partial derivatives, so that the mix exist are equal

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

then  $u, v$  are harmoni, that is:

$$\Delta u = 0$$

$$\Delta v = 0$$

where  $\Delta$  is the laplace operator:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and that solves:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

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Observe that last time we conclude that:

**Theorem 2.** *if  $f'(z)$  exists then it satisfies the Cauchy Riemman Equations, and if a function  $f(x, y) = u(x, y) + iv(x, y)$ ,  $u, v$  has continous first order partial derivatives and Cuachy-Rieman equations holds then  $f'(z)$  exists.*

*Proof.* Let  $z = x + iy$  and and increment  $deltaz = h + ik$ , so that  $u(z + \delta z) = u(x + h, x + h) = u(x, y) + \frac{\partial u}{\partial x}(x, y)k + \epsilon_1$ , so that  $\frac{\epsilon_{1,2}}{h+ik} \rightarrow 0$ . As also,  $v(x + \delta z) = v(x, y) + \frac{\partial v}{\partial x} \dots$  same as above. So that we observe that  $\square$

We take note that the above proof clearly demonstrate that no matter which curve we use to approach to 0 the limit is going to be the same.

**Theorem 3.** *If  $f(z) = u(x, y) + iv(x, y)$  is analytic [and  $u, v$  has contious second order derivatives], then  $\delta u = 0$  (the Laplacian)  $\delta v = 0$*

*Proof.* let us observe  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} \right) = 0$ .  $\square$

**Definition 7.** *Two harmonics functions  $u(x, y), v(x, y)$  are said to be harmonic conjugate if  $f(x, y) = u(x, y) + iv(x, y)$  is an analytic function.*

Exercise: Find an harmonic conjugate to  $u(x, y) = x^2 - y^2$

Recall that we can use the Cauchy-Rieman Equations so that: we find:

$$\frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial v}{\partial x} = 2y$$

Solve then we have:

$$f(x, y) = x^2 - y^2 + 2xiy$$

that obvious is that:

$$f(z) = z^2$$

For the next section we use the following notation  $f'(z)$  is the complex derivative if it exists, it can be written  $\frac{df}{dz}$  and we introduce another different thing:  $\frac{\partial f}{\partial \bar{z}}$  and this somehow equals to:

$$= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{z}}$$

and it is equal:

$$\frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right)$$

so that we define:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right)$$

and with this the Cauchy-Riemann can be written as:

$$\frac{\partial f}{\partial \bar{z}} = 0$$

and that:

$$\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0$$

As an example: with this notation this becomes:

$$f(x, y) \leq f(z, \bar{z})$$

which in humans terms is:

$$f(z, \bar{z}) = 2 \operatorname{Re}(z^2)$$

and it has some sense.

**Example** Given  $u(x, y)$ , find a  $f(z)$  analytic, such that  $u(x, y)$  is the real part of  $f(z)$ , is the same part as  $u(x, y) = \frac{1}{2} (f(x + iy) + \bar{f}(x + iy))$

Leap of faith:  $x = \frac{z}{2}, y = \frac{\bar{z}}{2i}$  where  $z$  is a complex variable. This is motivated by the fact:

$$u\left(\frac{z}{2}, \frac{\bar{z}}{2i}\right) = \frac{1}{2} (f(z)) = \frac{1}{2} (f(z) + \bar{f}(0))$$

Any hiw, it gives us the formula:

$$f(z) = 2u\left(\frac{z}{2}, \frac{\bar{z}}{2i}\right) - \bar{f}(0)$$

And this solves the main problem, which is as mystical as we see.

## Polynomial

A polynomial  $f(z) = a_0 + a_1z + \dots + a_nz^n$   $a_j \in \mathbb{C}$ , tenemos entonces  $(z^n)'$  exists and  $z^n$  analytic, implies that:  $f'(z)$  exists for all polynomials.

**Fundamental Theorem of Algebra:** For any polynomial  $f(z)$  of  $\deg \geq 1$  has at least 1 complex root. Suppose  $p(z)$  is a polynomial of degree  $n$   $\alpha$  any  $\mathbb{C}$  number. So we observe that we have  $P(z) = (z - \alpha)P_1(z) + r$  ( $r \in \mathbb{C}$ ). By **Bezout's Theorem** we have that:

$$P(\alpha) = 0 \implies r = 0$$

So, step by step we have just shown that we decrease the degree of the polynomial minus 1.

**Terminology** let  $P(z) = (z - \beta_1)^{h_1} \dots (z - \beta_m)^{h_m}$   $h_j$  is the **multiplicity** of the root  $\beta_j$

**Theorem 4. Lucas** *If all zeros of a polynomial  $P(z)$  in a half-plane  $H$ , then all zeros of  $P'(z)$  also lie in  $H$*

**Corolary 1.** *All zeros of  $P'(z)$  are contained in the minimum convex polygon containing zeros of  $P(z)$ .*

## Rational functions

We denote then by rational functions of the form:

$$\frac{P(z)}{Q(z)}$$

a typical member of the set of rational functions:

$$R(z) = \frac{a_0 + \dots + a_n z^n}{b_0 + \dots + b_m z^m}$$

we can extend this function in the form of:

$$R(z) : \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$$

and we consider  $\mathbb{C} \cup \infty$  is called the extended complex plane. and we can extend also:

$$R(z) : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$$

So that:

$$R(\infty) = \lim_{|z| \rightarrow \infty} R(z)$$

observe that this has inside a theorem that no matter form where you closes to infinity so that is either:  $0, \infty, \frac{a_n}{b_m}$  the last case holds if and only if:  $n = m$

**Neighborhood of zero & Neighborhoods of Infinite**

Without loss of generality  $p, Q$  do not have common zeros.

**Definition 8.** If  $\alpha$  is a root of multiplicity  $h$  of  $P(z)$ , the  $\alpha$  is said to be a zero of order  $h$  of  $R(z)$

**Definition 9.** If  $\beta$  is a root of multiplicity  $k$  of  $Q(z)$  then  $\beta$  is said to be a pole of order  $h$  of  $R(z)$

In the extended complex plane the total order of all poles is equal to the total order of roots.

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Observe that we meant that  $f$  (for the real case variable) that  $f$  admits a linear approximation at (near)  $a$ .

$$f(\vec{a} + \vec{h}) = f(\vec{a}) + \text{a linear function of } (\vec{h})$$

more formally, we define the function of  $h$  as an error term.  $\epsilon \vec{h}$ , Important conditions on  $\epsilon$ : “decreases faster than linear as  $\vec{h} \rightarrow \vec{0}$ ”

so that we want:

$$\frac{|\epsilon(\vec{h})|}{|\vec{h}|} \rightarrow 0$$

Linear functions are continuous. Take  $f(x) = 3x$ , say  $a \in \mathbb{R}$ , then  $f(x)$  is continuous at  $a$

Prove continuity of common functions. x

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Review from the last time. Observe that :

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$

and that:

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

Suppose that we want to try this with this function:

$$f(x, y) = f\left(\frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z})\right) = g(z, \bar{z})$$

we have the following characterization that if  $f$  satisfy Cauchy-Riemann if and only if  $\partial \bar{z} = 0$  that is, is independent of  $\bar{z}$ .

Suppose  $f = u + iv$ , and we have that

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (u + iv)$$

so that: is an homogeneous system:

$$\frac{1}{2}((u_x - v_y)) + i(u_y + v_x)$$

We now consider the following rational functions:  
 $P, Q$  polynomial, we consider now:

$$R(z) = \frac{P(z)}{Q(z)}$$

Assume reduced fraction (no common root). "Extended complex plane" The Riemann Sphere.

we read "a is closed to  $\infty$ " to be  $\frac{1}{a}$  is closed to 0.

we know to send  $\mathbb{C}$  is send to extended bar Riemann Sphere.

Observe that since the same rules for differentiation holds:

$$R'(z) = \frac{P'(z)Q(z) - P(z)Q'(z)}{Q^2(z)}$$

not reduced also a natural form, same poles as  $\mathbb{R}$ .

order of  $Q$  is the same order of the same.

so observe that we can express that:

$$R(z) = \frac{a_0 + a_1 + \dots a_n z^n}{b_0 + b_1 \dots b_m z^m}$$

and we can change in the following form:

$$R_1(\omega = \frac{1}{z}) \frac{a_0 + a_1 \omega}{\text{do it algebraically}}$$

Polling out the following

$$= \omega^{k-m} \frac{a_0 \dots}{a_1 \dots}$$

do for this cases:

we conclude that

Linear fractional transformations, of the whole sphere,

We take polynomial and plug /

## Partial fractions

$$R = \frac{P}{Q}$$

with no common factors. let us denote  $\beta_1, \dots, \beta_q \in \mathbb{C}$  are distinct zeros of  $Q$  so that:

$$R(z) = G^{\text{new}}(z) + \sum_{i=1}^q G_i \left( \frac{1}{z - \beta_i} \right)$$

$G^{\text{new}}, G_i$  are polynomials.

Observe that at  $\infty$ , what we do is to apply the division algorithm, so that we have: (Part 1)

$$P = A \cdot z \cdot Q + B$$

where  $\deg(B) \leq \deg(Q)$ , and we rewrite that  $R = \frac{P}{Q} = G + H$ , where  $G$  is a polynomial such that  $G(0) = 0$ , and  $H(\infty) \in \mathbb{C}$  which is the limit as  $z \rightarrow \infty$ .

Part 2. For each  $\beta_i$ , observe that we can rewrite,

$$R_i(\zeta) := R(\beta_i + \frac{1}{\zeta})$$

so that  $R_i(\zeta)$  is a rational function of  $\zeta$  with a pole at  $\zeta = \infty$ . So by the first part:

$$R_i(\zeta) = G_i(\zeta) + H_i(\zeta)$$

if now we write:

$$z = \beta_i + \frac{1}{\zeta} \quad \text{so that} \quad \zeta = \frac{1}{z - \beta_i}$$

and we can rewrite:

$$R(z) = G_i(\frac{1}{z - \beta_i}) + H_i(\frac{1}{z - \beta_i})$$

with  $G_i(\frac{1}{z - \beta_i})$  is a polynomial in  $\frac{1}{z - \beta_i}$  and  $H_i(\frac{1}{z - \beta_i})$  goes to something that is a complex number.  $z \rightarrow \beta_i$

$$R(z) = G(z) - \sum_{n=1}^q G_i(\frac{1}{z - \beta_i})$$

has no pole in  $\mathbb{C}$ , for which we conclude it is a constant function. So that  $G^{\text{new}}(z)G^{\text{old}}(z)$

### Review of Sequences

Consider  $\mathbb{N} : 0, 1, \dots$  a sequence.

**Definition 10.**  $\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N \text{ it happens that } |a_n - A| < \epsilon$

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**Definition 11.**  $(a_n)$  a sequence  $\iff \forall \epsilon > 0 \exists N \text{ s.t. } \forall n \geq N |a_n - A| < \epsilon$

**Definition 12.**  $(a_n)$  is **Cauchy**  $\iff \forall \epsilon > 0 \exists N \text{ s.t. } m, n \geq N |a_m - a_n| < \epsilon$

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### Series

We discuss the meaning of  $\sum_{n=0}^{\infty} a_n = S$

**Cauchy** Since it is a sequence we can use the Cauchy criterion to estimate a sum

**Condition**  $p = 0 \implies |a_m| < \epsilon$  which is equivalent to  $\lim_{n \rightarrow \infty} a_n$

**Remark**  $R_n = \sum_{i=n+1}^{\infty} a_i$

Observe that the following converges:



$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots$$

### Absolute Convergence

$$\sum_{n=0}^{\infty} a_n$$

converges absolutely if and only if (def)  $\sum_{n=0}^{\infty} |a_n|$  converges.

**Lemma** Absolute converges implies convergence

*Proof* Check Cauchy.

**The Comparison test** If  $0 \leq a_k \leq b_k$ , then  $\sum b_k \text{ conv} \implies \sum a_k \text{ conv}$

**Arithmetic** Let us suppose we have a series on the following way:  $\sum_{n=0}^{\infty} a_n$  converges or converges absolutely if and only if  $\sum_{n=0}^{\infty} \bar{a}_n$  and  $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \bar{a}_n$ .

**2.** Suppose we have two series that conv. or absolutely converges.  $\sum_{n=0}^{\infty} a_n$   $\sum_{n=0}^{\infty} b_n$  implies that  $\sum_{n=0}^{\infty} (a_n + b_n)$  converges or absolutely, and  $\sum_{n=0}^{\infty} (a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$ .