Complex Analysis
David Cardozo

Nombre del curso: Complex Analysis
Código del curso: MATE2211
Unidad académica: Departamento de Matemáticas
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Horario: Lu y Ju, 8:00 a 9:50

Nombre profesor(a) principal: Paul Bressler
Horario y lugar de atención: Mo y 17:00 a 18:00, Office H-409

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1 Organization of the course

- 3 Exams
- regular-weekly homework

Sometimes we will use the following alternative book: *Complex Analysis*, Gamelin

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2 Complex Numbers

We want to solve a quadratic equation:

$$ax^2 + bx + c = 0$$

so we first calculate the discriminant:

$$D = b^2 - 4ac$$

it is called that, because it differentiate solution types. We then compute:

$$r_{\pm} \frac{-b \pm \sqrt{D}}{2a}$$

In R squares are positives, but this poses a problem when $D < 0$. Meanwhile, let us observe that when $D > 0$ we get two solutions, and when we get $D = 0$

there is only one solution. So the main question is what to do when $D < 0$. Without loss of generality we can assume that $a = 1$

$$r_+ = -\frac{b}{2} + \sqrt{2}$$
$$r_- = -\frac{b}{2} - \sqrt{2}$$

the reason is that we want to be able write

$$p(x) = (x - r_+)(x - r_-)$$

expanding the product above

$$p(x) = x^2 - (r_+ + r_-)x + r_= r_-$$

so that D can be written as:

$$D = \mathrm{Sign}(D)|D|$$

Observe that we can take the definition of sign as above. But then, this give a suspect idea that there can exist a **not real number** that:

$$\pm(\sqrt{-1})^2 = -1$$

which is also a solution of the following equation:

$$x^2 + 1 = 0$$

This then motivate the following concept, does arithmetic follows natural with the inclusion:

$$z = a + b\sqrt{-1}$$

Observe the symmetry on choosing $\sqrt{-1}$ or $-\sqrt{-1}$. Which again motivates

$$\bar{z} = a - b\sqrt{-1}$$

As in the textbook i^n maps to $1, i, -1, -i$ with the trick $\mod 4$ Let us define then that addition comes naturally:

$$(a + b\sqrt{-1}) + (c + d\sqrt{-1}) = (a + d) + (b + d)\sqrt{-1}$$

Multiplication is a little bit tricky:

$$(a + b\sqrt{-1}) \times (c + d\sqrt{-1}) = (ac - bd) + (ad + bc)\sqrt{-1}$$

which comes natural by considering $(\sqrt{-1})^2$. Observe that the reals are embedded as:

$$a + 0\sqrt{-1}$$

The operation of taking $z \to \bar{z}$ is called *Complex conjugation*, observe that it complies $\bar{z} \to z$, which will allow us to extract a, b as:

$$\frac{1}{2}(z + \bar{z}) := \mathrm{Re}(z) = a$$

and also

$$\frac{1}{2\sqrt{-1}}(z - \bar{z}) := \mathrm{Im}(z) = b$$

our definition above can be then replaced by:

$$z = \mathrm{Re}(z) + \mathrm{Im}(z)\sqrt{-1}$$

The preceding equation proposes a big problem, since we have found another "strange" number:

$$\frac{1}{\sqrt{-1}} = x$$

or equivalently,

$$\sqrt{-1}x = 1$$

from which it can be solved by inspecting the table of exponentiation and we find:

$$x = -\sqrt{-1}$$

In literature we often define that $\sqrt{-1} = i$ A big question is that if $z \neq 0$, then there exist ω such that $w \cdot z = 1$, the answer is positive and it even has a stronger statement: with complex numbers we can solve any polynomial equation.

$$z = a + bi$$
$$w = x + iy$$

and we are looking $z \cdot w = 1$ so we get the system of equations

$$ax - by = 1$$
$$bx + ay = 0$$

and we can get information about the solution, via the determinant det $= a^2 + b^2$, and this is only zero if only if $a = 0$ and $b = 0$. Solution can be found for any method

Observe that if we have a complex number and its conjugate, the following relation holds:

$$z \cdot \bar{z} = a^2 + b^2$$
$$\frac{1}{z} = \frac{1}{a^2 + b^2} \cdot \bar{z}$$

$$\frac{1}{2+3i} = \frac{1}{13}(2-3i)$$

Observe that this recall Cramer's rule, since it has the form of $\frac{1}{det}$

The following problem that we are facing is the existence of:

$$z = a + bi \quad \exists w \quad w^2 = z$$

if we let $w = x + iy$, and $w^2 = (x^2 - y^2) + i(2xy)$, we get the following system of equations.

$$\begin{aligned} a &= x^2 - y^2 \\ b &= 2xy \end{aligned}$$

Observe that 0 has a unique root. We denote the set of all complex numbers with $\mathbb{C}$ and observe that we can take bijection beetwen C and R^2

$$\begin{aligned} \mathbb{R} &\leftrightarrow \mathbb{C} \\ z &\longrightarrow (Re(z), Im(z)) \\ a + ib &\longleftarrow (a, b) \end{aligned}$$

Observe then that $\mathbb{C}$ is a vector space over $\mathbb{R}$. Basis: $1, i$. Now observe that we can take a geometric interpretation, given this two bijections, to the multiplication of a complex numbers. Side note: A linear map is a map that plays nice with arithmetic. an important characterization of linear maps, is that:

$$f(a) = a \cdot f(1)$$

and also

$$f\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and it can be easy characterized via:

$$\begin{pmatrix} v_{11} & v_{22} \\ v_{21} & v_{22} \end{pmatrix}$$

Now consider the following map:

$$\begin{aligned} \mathbb{C} &\to \mathbb{C} \\ w &\rightarrowtail zw \end{aligned}$$

where $z = a + ib$ Again observe that we are only taking care of the values on the basis.

Example Suppose $A^2 = -I$, and det > 0, then there is an invertible matrix B such that $A = BJB^{-1}$. Returning to our previous point, we see that complex numbers are embedded nicely on 2×2 matrices with real coefficients. So that the span 1 and j is a 4 dimensional vector space over R. Finally, we conclude that multiplication on $\mathbb{C}$ is a matrix multiplication.

When we look a close view on $\mathbb{R}^2$ along with the normal dot product, we have the **Cauchy-Schwarz Inequality**: Let us then define:

$$|\vec{a}|^2 = \vec{a} \cdot \vec{a}$$

and proof is left to the reader:

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$$