Complex Analysis
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Horario: Lu y Ju, 8:00 a 9:50

Nombre profesor(a) principal: Paul Bressler
Horario y lugar de atención: Mo y 17:00 a 18:00, Office H-409

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1 Organization of the course

- 3 Exams
- regular-weekly homework

Sometimes we will use the following alternative book: *Complex Analysis*, Gamelin

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2 Complex Numbers

We want to solve a quadratic equation:

$$ax^2 + bx + c = 0$$

so we first calculate the discriminant:

$$D = b^2 - 4ac$$

it is called that, because it differentiate solution types. We then compute:

$$r_{\pm} \frac{-b \pm \sqrt{D}}{2a}$$

In R squares are positives, but this poses a problem when $D < 0$. Meanwhile, let us observe that when $D > 0$ we get two solutions, and when we get $D = 0$

there is only one solution. So the main question is what to do when $D < 0$. Without loss of generality we can assume that $a = 1$

$$r_+ = -\frac{b}{2} + \sqrt{2}$$
$$r_- = -\frac{b}{2} - \sqrt{2}$$

the reason is that we want to be able write

$$p(x) = (x - r_+)(x - r_-)$$

expanding the product above

$$p(x) = x^2 - (r_+ + r_-)x + r_= r_-$$

so that D can be written as:

$$D = \mathrm{Sign}(D)|D|$$

Observe that we can take the definition of sign as above. But then, this give a suspect idea that there can exist a **not real number** that:

$$\pm(\sqrt{-1})^2 = -1$$

which is also a solution of the following equation:

$$x^2 + 1 = 0$$

This then motivate the following concept, does arithmetic follows natural with the inclusion:

$$z = a + b\sqrt{-1}$$

Observe the symmetry on choosing $\sqrt{-1}$ or $-\sqrt{-1}$. Which again motivates

$$\bar{z} = a - b\sqrt{-1}$$

As in the textbook i^n maps to $1, i, -1, -i$ with the trick $\mod 4$ Let us define then that addition comes naturally:

$$(a + b\sqrt{-1}) + (c + d\sqrt{-1}) = (a + d) + (b + d)\sqrt{-1}$$

Multiplication is a little bit tricky:

$$(a + b\sqrt{-1}) \times (c + d\sqrt{-1}) = (ac - bd) + (ad + bc)\sqrt{-1}$$

which comes natural by considering $(\sqrt{-1})^2$. Observe that the reals are embedded as:

$$a + 0\sqrt{-1}$$

The operation of taking $z \to \bar{z}$ is called *Complex conjugation*, observe that it complies $\bar{z} \to z$, which will allow us to extract a, b as:

$$\frac{1}{2}(z + \bar{z}) := \mathrm{Re}(z) = a$$

and also

$$\frac{1}{2\sqrt{-1}}(z - \bar{z}) := \mathrm{Im}(z) = b$$

our definition above can be then replaced by:

$$z = \mathrm{Re}(z) + \mathrm{Im}(z)\sqrt{-1}$$

The preceding equation proposes a big problem, since we have found another "strange" number:

$$\frac{1}{\sqrt{-1}} = x$$

or equivalently,

$$\sqrt{-1}x = 1$$

from which it can be solved by inspecting the table of exponentiation and we find:

$$x = -\sqrt{-1}$$

In literature we often define that $\sqrt{-1} = i$ A big question is that if $z \neq 0$, then there exist ω such that $w \cdot z = 1$, the answer is positive and it even has a stronger statement: with complex numbers we can solve any polynomial equation.

$$\begin{aligned} z &= a + bi \\ w &= x + iy \end{aligned}$$

and we are looking $z \cdot w = 1$ so we get the system of equations

$$\begin{aligned} ax - by &= 1 \\ bx + ay &= 0 \end{aligned}$$

and we can get information about the solution, via the determinant $\det = a^2 + b^2$, and this is only zero if only if $a = 0$ and $b = 0$. Solution can be found for any method

Observe that if we have a complex number and its conjugate, the following relation holds:

$$\begin{aligned} z \cdot \bar{z} &= a^2 + b^2 \\ \frac{1}{z} &= \frac{1}{a^2 + b^2} \cdot \bar{z} \end{aligned}$$

$$\frac{1}{2+3i} = \frac{1}{13}(2-3i)$$

Observe that this recall Cramer's rule, since it has the form of $\frac{1}{det}$

The following problem that we are facing is the existence of:

$$z = a + bi \quad \exists w \quad w^2 = z$$

if we let $w = x + iy$, and $w^2 = (x^2 - y^2) + i(2xy)$, we get the following system of equations.

$$a = x^2 - y^2$$
$$b = 2xy$$

Observe that 0 has a unique root. We denote the set of all complex numbers with $\mathbb{C}$ and observe that we can take bijection beetwen C and R^2

$$\mathbb{R} \leftrightarrow \mathbb{C}$$
$$z \longrightarrow (Re(z), Im(z))$$
$$a + ib \longleftarrow (a, b)$$

Observe then that $\mathbb{C}$ is a vector space over $\mathbb{R}$. Basis: $1, i$. Now observe that we can take a geometric interpretation, given this two bijections, to the multiplication of a complex numbers. Side note: A linear map is a map that plays nice with arithmetic. an important characterization of linear maps, is that:

$$f(a) = a \cdot f(1)$$

and also

$$f\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and it can be easy characterized via:

$$\begin{pmatrix} v_{11} & v_{22} \\ v_{21} & v_{22} \end{pmatrix}$$

Now consider the following map:

$$\mathbb{C} \to \mathbb{C}$$
$$w \rightarrowtail zw$$

where $z = a + ib$ Again observe that we are only taking care of the values on the basis.

Example Suppose $A^2 = -I$, and $\det > 0$, then there is an invertible matrix B such that $A = BJB^{-1}$. Returning to our previous point, we see that complex numbers are embedded nicely on 2×2 matrices with real coefficients. So that the span 1 and j is a 4 dimensional vector space over R. Finally, we conclude that multiplication on $\mathbb{C}$ is a matrix multiplication.

When we look a close view on $\mathbb{R}^2$ along with the normal dot product, we have the **Cauchy-Schwarz Inequality**: Let us then define:

$$|\vec{a}|^2 = \vec{a} \cdot \vec{a}$$

and proof is left to the reader:

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$$

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3 Remarks from last time

$$\mathbb{C} \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$$
$$A + IB \to a + bJ$$
$$J^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so that:

$$Z = a + ib \leftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Let us observe that in $\mathbb{R}2$ we have the dot product for which:

$$\vec{a} \cdot \vec{a} \geq 0$$

it is good as concept of measure of distance. Let us observe the analogue on complex numbers:

$$z = a + ib$$

$$w = c + id \implies \bar{w} = c - id$$

so that

$$z\bar{w} = (ac + bd) + (bc - ad)$$

for which the left parenthesis of the RHS is real and the other imaginary. Observe also:

$$|z|^2 = a^2 + b^2 \geq 0$$

and if $z \neq 0$ we define:

$$z^{"}-1 = \frac{1}{|z|}\bar{z}$$

Cauchy-Schawrz

Suppose we know the properties of the dot product:

$$\left|\vec{a} \cdot \vec{b}\right| \leq |\vec{a}| \left|\vec{b}\right|$$

Example
In $\mathbb{R}^n$:

$$< a_1, \ldots, a_n > \cdot < b_1, \ldots b_n$$
$$= \sum_{n=1}^{n} a_i b_i$$
$$\leq (\sum a_i)^{/frac12} (\sum b_j)^{\frac{1}{2}}$$

Example The functions that are square integrables on $I = [0, 1]$ for which we have:

$$\left|\int_0^1 fgdx\right| \leq \left|\int fdx\right| \left|\int gdx\right|$$

Proof Let recall that for parameterizing the line in beetwen $\vec{a}$ and $\vec{b}$ we could use the convex combination.

$$t\vec{b} + (1-t)\vec{a} \quad 0 \leq t \leq 1$$

$$\left|t\vec{b} + (1-t)\vec{a}\right|^2 > 0$$

We expand and we get: .... It is the same trick of consider a quadratic without root $a + \lambda b$. Geometrical Consideration:For two independent vectors, the line that pass through them do not pass on the origin.

4 C-S Consequences

The triangle inequality

$$\left|a \vec{+} \vec{b}\right| \leq |\vec{a}| + \left|\vec{b}\right|$$

the proof is given by considering the expansion of:

$$\left|\vec{a} + \vec{b}\right|$$

and the inequality: (Check!)
Definition of Angles

$$\cos(\theta) := \frac{\left|\vec{a} \cdot \vec{b}\right|}{|\vec{a}| \left|\vec{b}\right|}$$

5 Special (2×2) matrices

Scalar:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I$$

A - $2x2matricies$ A is orthogonal iff

$$\vec{a} \cdot \vec{b} = (A\vec{a}) \cdot (A\vec{b})$$

iff Preserves lengths, and angles.

Consider the matrix of rotation θ, and consider two rotations α y β, and the sum of these two, it will given then the following matrix. Observe that these shows explicitly the formulas of sum and sine cosine.

Consider the following computation $\cos(15\alpha)$

Definition 1. ***Conformal*** *preserves angles (but not conserve lengths)*

Remark: Orthogonal transformations preserve angles, and scalar conserve angles. Conformal are complex numbers. via matrices.

$$z = |z|$$

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6 Class 03

Notation

$$z, w \in \mathbb{C}$$

$$x, yin\mathbb{R} \quad \mathbb{C}$$

$$t \in \mathbb{R}$$

For this section all functions are well defined that is univalued and all functions are defined on an open set.

Definition 2. *U is an open set if:*

$$\forall x \in U \exists \epsilon > 0 \ s.t \ |y - x| < \epsilon \implies y \in U$$

Observe that for Ahlfors $\sqrt{x}$ is not a function.

Definition 3. *The function $f(x)$ "has a limit" A $asx \to a$ we write:*

$$\lim_{x \to a} f(x) = A$$

If $\forall \epsilon > 0 \exists \delta_\epsilon > 0$ *such that* $|x - a| < \delta_\epsilon \implies |f(x) - A| < \epsilon$

There are variance if A is an infinite the definitions will change accordingly.

Lemma 1. *$f(x) = \sin(x)$ Let show that* $\lim_{x\to 0} f(x) = 0$

Proof. Assuming given $\epsilon > 0$, need to find $\delta_\epsilon > 0$ s.t $x \in (-\delta_\epsilon, \delta_\epsilon)$ imples $\sin(x) \in (-\epsilon, \epsilon)$.

For any $\epsilon > 0$ take $\delta_\epsilon := \epsilon$ □

Definition 4. *A function $f(x)$ is continuous at $x = a$ if:*

$$\lim_{x\to a} f(x) = f(a)$$

Example $\sin(x)$ is continuous at $x = 0$

Proposition 1. *Let $f : \mathbb{C} \to \mathbb{C}$, $a, A \in \mathbb{C}$, then:*

$$\lim_{x\to a} f(x) = A$$
$$\lim_{x\to a} \bar{f(x)} = \bar{A}$$
$$\lim_{x\to a} \mathrm{Re}\,(f(x)) = \mathrm{Re}\,(A)$$
$$\lim_{x\to a} \mathrm{Im}\,(f(x)) = \mathrm{Im}\,(A)$$

A continuous function $f(x)$ is a function that is continuous at any x for which it is defined. So that for example $\frac{1}{x}$ is continuous.

Definition 5.

$$f'(a) := \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

, the difference quotient. $f(x)$ is differentiable at $x = a$ if $f'(a)$ exists.

If so,

$$f(x) \approx f(a) + (x - a)f'(a) + O((x - a))$$

Observe that the first two terms are the equation of the tangent. Observe that:

$$\frac{\epsilon(x-1)}{x-a} \to 0 \text{ as } x \to 0$$

Observe that if $f : \mathbb{R}^2 \to \mathbb{R}$, let us take $(a, b) \in \mathbb{R}^2$, if f is differentiable at $(a, b) \in \mathbb{R}^2$ then for (x, y) near (a, b), we have:

$$f(x, y) = f(a, b)) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) + o(||(x, y) - (a, b)||)$$

Let $f : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$ is differentiable at $z = a$ if:

$$f'(z) := \lim_{z\to a} \frac{f(z) - f(a)}{z - a}$$

exists. So observe:

$$f(z) = f(a) + f'(z)(z - a) + o()$$

Proposition 2. *Let us show that $f(z) = z^n$ is Complex differentiable for $z = a$, for all $a \in \mathbb{C}$*

Proof.

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

let $\xi = z - a$ so that

$$f'(a) = \lim_{\xi \to 0} \frac{(a + \xi)^n - a^n}{\xi}$$

and expanding using Newton binomial theorem. (and cancelling $\xi \neq 0$)

$$f'(a) = \lim_{\xi \to 0} (na^{n-1} + \text{ things that have xi})$$

so that :

$$f'(a) = na^{n-1}$$

implies

$$f'(z) = nz^{n-1}$$

□

Now for a pathological example. Let us take $f(z) = \operatorname{Re}(z)$ is not complex differentiable at any point. (Also imaginary part of Z is not complex differentiable)

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

again using $\xi = z - a$

$$f'(a) = \lim_{\xi \to 0} \frac{\operatorname{Re}(a + \xi) - \operatorname{Re}(a)}{\xi}$$

so that:

$$\lim_{\xi \to 0} \frac{\operatorname{Re}(\xi)}{\xi}$$

Observe that if $\xi = t \in \mathbb{R}$ then

$$\operatorname{Re}(\xi)/\xi = \frac{t}{t} = 1$$

Now, observe that if $\xi = it \in \mathbb{R}$ and

$$\frac{\operatorname{Re}(\xi)}{\xi} = \frac{0}{t} = 0$$

Definition 6. *A Complex function $f(z)$ and $f : \mathbb{C} \to \mathbb{C}$ is* ***analytic*** *or(holomorphic) at z if $f'(z)$ exists (the complex derivative)*

Theorem 1. ***Main thm*** *$f(z)$ is analytic at $z = a$ if and only if*

$$f(z) = \sum_{n=0}^{\inf} c_n (z - a)^n \text{ for } |z - a| < \epsilon$$

Cauchy-Riemann equations

Let us take $f : \mathbb{C} \to \mathbb{C}$ and rewritten as $f(z = x + iy) = u(x, y) + iv(x, y)$ $u, v \in RR$ and assume $f'(z)$ exist as a complex derivative. Equivalently,

$$f'(z) = \lim_{\eta \to 0} \frac{f(z+\eta) - f(z)}{\eta} \quad \eta \in \mathbb{C}$$

then the above is equivalently, to the two reformulations:

$$\lim_{\eta \in \mathbb{R} = h \to 0} \frac{f(z+h) - f(h)}{h}$$

and has to be equal to:

$$\lim_{\eta \in i\mathbb{R}, \eta = ik \to 0, \quad k \in \mathbb{R}} \frac{f(z+ik) - f(z)}{ik}$$

so that with u, v

$$\lim_{h \to 0} \frac{u(x+h, y) + iv(x+h, y) - u(x, y) - iv(x, y)}{h} = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y)$$

and for the second part:

$$\lim_{k \to 0} \frac{u(x, y+k) + iv(x, y+k) - u(x, y) - iv(x, y+k)}{ik} = \frac{\partial v}{\partial y}(x, y) - i\frac{\partial u}{\partial y}(x, y)$$

and since them are the manifestation if the same limit, we have the equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

For remembering, check that the function $f(z) = z$ is complex differentiable. If we are calculating the derivative (computing stuff):

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}$$

now let us observe:

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$$

and this is the Jacobian.

$$\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) - \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right)$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix}$$

Which is the Jacobian.

Assume u, v have continuous 2nd partial derivatives, so that the mix exist are equal

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

then $u.v$ are harmoni, that is:

$$\Delta u = 0$$
$$\Delta v = 0$$

where Δ is the laplace operator:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and that solves:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0$$

Observe that last time we conclude that:

Theorem 2. *if $f'(z)$ exists then it satisifies the Cauchy Riemman Equations, and if a function $f(x,y) = u(x,y)+iv(x,y)$, u, v has continous first order partial derivatives and Cuachy-Rieman equations holds then $f'(z)$ exists.*

Proof. Let $z = x + iy$ and and increment $deltaz = h + ik$, so that $u(z + \delta z) = u(x+h, x+h) = u(x,y) + \frac{\partial u}{\partial x}(x,y)k + \epsilon_1$, so that $\frac{\epsilon_{1,2}}{h+ik} \to 0$. As also, $v(x+\delta z) = v(x,y) + \frac{\partial u}{\partial v}$... same as above. So that we observe that $\square$

We take note that the above proof clearly demonstrate that no matter which curve we use to approach to 0 the limit is going to be the same.

Theorem 3. *If $f(z) = u(x,y)+iv(x,y)$ is analytic [and u, v has contious second order derivatives], then $\delta u = 0$ (the Laplacian) $\delta v = 0$*

Proof. let us observe $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = 0.$ $\square$

Definition 7. *Two harmonics functions $u(x,y), v(x,y)$ are said to be harmonic conjugate if $f(x,y) = u(x,y) + iv(x,y)$ is an analytic function.*

Exercise: Find an harmonic conjugate to $u(x,y) = x^2 - y^2$

Recall that we can use the Cauchy-Rieman Equations so that: we find:

$$\frac{\partial v}{\partial y} = 2x$$
$$\frac{\partial v}{\partial x} = 2y$$

Solve then we have:

$$f(x,y) = x^2 - y^2 + 2xiy$$

that obvious is that:

$$f(z) = z^2$$

For the nexxt section we use the following notation $f'(z)$ is the complex derivative if it exists, it can be written $\frac{df}{dz}$ and we introduce another different thing: $\frac{\partial f}{\partial z}$ and this somehow equals to:

$$" = " \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z}$$

and it is equal:

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

so that we define:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

and with this the Cauchy-Riemman can be written as:

$$\frac{\partial f}{\partial \bar{z}} = 0$$

and that:

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

As an example: with this notation this becomes:

$$f(x, y) \leq f(z, \bar{z})$$

which in humans terms is:

$$f(z, \bar{z}) = 2\,\mathrm{Re}(z^2)$$

and it has some sense.

Example Given $u(x, y)$, find a $f(z)$ analytic, such that $u(x, y)$ is the real part of $f(z)$, is the same part as $u(x, y) = \frac{1}{2}\left(f(x + iy) + f(x \bar{+} iy)\right)$

Leap of faith: $x = \frac{z}{2}, y = \frac{z}{2i}$ where z is a complex variable. This is motivated by the fact:

$$u(\frac{z}{2}, \frac{z}{2i}) = \frac{1}{2}\left(f(z)\right) = \frac{1}{2}\left(f(z) + f(\bar{0})\right)$$

Any hiw, it gives us the formula:

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - f(\bar{0})$$

And this solves the main problem, which is as mystical as we see.

Polynomial

A polynomial $f(z) = a_0 + a_1 z + \dots a_n z_n \quad a_j \in \mathbb{C}$, tenemos entonces $(z^n)'$ exists and z^n analytic, imples that: $f'(z)$ exists for all polynomials.

Fundamental Theorem of Algebra: For any polynomial $f(z)$ of deg ≥ 1 has at least 1 complex root. Suppose $p(z)$ is a polynomial of degree nn α any $\mathbb{C}$ number. So we observe that we have $P(z) = (z - \alpha)P_1(z) + (r \in \mathbb{C})$. By **Bezout's Theorem** we have that:

$$P(\alpha) = 0 \implies r = 0$$

So, step by step we have just shown that we decrease the degree of the polynomial minus 1.

Terminology let $P(z) = (z - \beta_1)^{h_1} \dots (z - \beta_m)^{h_m}$ h_j is the **multiplicity** of the root β_j

Theorem 4. ***Lucas*** *If all zeros of a polynomial $P(z)$ in a half-plane H, then all zeros of $P'(z)$ also line in H*

Corolary 1. *All zeros of $P'(z)$ are contained in the minimum convex polygon containing zeros of $P(z)$.*

Rational functions

We denote then by rational functions of the form:

$$\frac{P(z)}{Q(z)}$$

a typical member of the set of rational functions:

$$R(z) = \frac{a_0 + \dots + a_n z^n}{b_0 + \dots b_m z^m}$$

we can extend this function in the form of:

$$R(z) : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$$

and we consider $\mathbb{C} \cup \infty$ is called the extended complex plane. and we can extend also:

$$R(z) : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$$

So that:

$$R(\infty) = \lim_{|z| \to \infty} R(z)$$

observe that this has inside a theorem that no matter form where you closes to infinity so that is either: $0, \infty, \frac{a_n}{b_m}$ the last case holds if and only if: $n = m$

Neighborhood of zero & Neighborhoods of Infinite

Without loss of generality p, Q do not have common zeros.

Definition 8. *If α is a root of multiplicity h of $P(z)$, the α os said to be a zero of order h of $R(z)$*

Definition 9. *If β is a root of multiplicity k of $Q(z)$ then β is said to be a pole of order h of $R(z)$*

In the extended complex plane the total order of all poles is equal to the total order of roots.

Observe that we meant that f (for the real case variable) that f admits a linear approxiamtion at (near) a.

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + \text{ a linear function of } (\vec{h})$$

more formally, we define the function of h as an error term. $\epsilon\vec{h}$, Important conditions on ϵ: " decreases faster than linear as $\vec{h} \to \vec{0}$"

so that we want:

$$\frac{\left|\epsilon(\vec{h})\right|}{\left|\vec{h}\right|} \to 0$$

Linear functions are continuous. Take $f(x) = 3x$, say $a \in \mathbb{R}$, then $f(x)$ is continuous at a

Prove continuity of common functions. x

Review from the last time. Observe that :

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$$

and that:

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)}$$

Suppose that we want to try this with this function:

$$f(x,y) = f(\frac{1}{2}(z+\bar{z}, \frac{1}{2i}), (z-\bar{z})) = g(z,\bar{z})$$

we have the following chracterizaton that if f satisfy cauchy riemman if and only if $\partial\bar{z} = 0$ that is, is independent of $\bar{z}$.

Suppose $f = u + iv$, and we have that

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv)$$

so that: is an homogenoeus system:

$$\frac{1}{2}((u_x - v_y)) + i(u_y + v_x)$$

We now consider the following rational functions:

P, Q polynomial, we consider now:

$$R(z) = \frac{P(z)}{Q(z)}$$

Assume reduced fraction (no common root). "Extended complex plane" The Riemman Sphere.

we read "a is closed to ∞'' to be $\frac{1}{a}$ is closed to 0.

we know to send $\mathbb{C}$ is send to extended bar Riemman Sphere.

Observe that since the same rules for differentiation holds:

$$R'(z) = \frac{P'(z)Q(z) - P(z)Q'(z)}{Q^2(z)}$$

not reduced also a natural forme, same poles as $\mathbb{R}$.

order of Q is the same order of the same.

so observe that we can express that:

$$R(z) = \frac{a_0 + a_1 + \dots a_n z^n}{b_0 + b_1 \dots b_m z^m}$$

and we can change in the following form:

$$R_1(\omega = \frac{1}{z})\frac{a_0 + a_1 w}{\text{do it algebraically}}$$

Polling out the following

$$= \omega^{k-m}\frac{a_0 \cdots}{a_1 \dots}$$

do for this cases:

we conclude that

Linear fractional transformations, of the whole sphere,

We take polynomial and plug /

Partial fractions

$$R = \frac{P}{Q}$$

with no common factors. let us denote $\beta_1, \dots, \beta_q \in \mathbb{C}$ are distinct zeros of Q so that:

$$R(z) = G^{\text{new}}(z) + \sum_{i=1}^{q} G_i(\frac{1}{z - \beta_i})$$

G^{new}, G_i are polynomials.

Observe that at ∞, what we do is to apply the division algorithm, so that we have: (Part 1)

$$P = A \cdot z \cdot Q + B$$

wehre $\deg(B) \leq \deg(Q)$, and we rewrite that $R = \frac{P}{Q} = G + H$, where G is a polynomial such that $G(0) = 0$, and $H(\infty) \in \mathbb{C}$ which is the limit as $z \to \infty$.

Part 2. For each β_i, observe that we can rewrite,

$$R_i(\zeta) := R(\beta_i + \frac{1}{\zeta})$$

so that $R_i(\zeta)$ is a rational function of ζ with a pole at $\zeta = \infty$. So by the first part:

$$R_i(\zeta) = G_i(\zeta) + H_i(\zeta)$$

if now we write:

$$z = \beta_i + \frac{1}{\zeta} \quad \text{so that} \quad \zeta = \frac{1}{z - \beta_i}$$

and we can rewritte:

$$R(z) = G_i(\frac{1}{z - \beta_i}) + H_i(\frac{1}{z\beta_i})$$

with $G_i(\frac{1}{z-\beta_i})$ is a polynomial in $\frac{1}{z-\beta_i}$ and $H_i(\frac{1}{z-\beta_i})$ goes to something that is a complex number. $z \to \beta_i$

$$R(z) = G(z) - \sum_{n=1}^{q} G_i(\frac{1}{z - \beta_i})$$

has no pole in $\mathbb{C}$, for which we conclude it its a constant function. So that $G^{\text{new}}(z)G^{\text{old}}(z)$

Review of Sequences

Consider $\mathbb{N} : 0, 1, ...$ a sequence.

Definition 10. *$\forall \epsilon > 0 \exists N \in \mathbb{N} s.t. \forall n \geq N$ it happens that $|a_n - A| < \epsilon$*

Definition 11. *(a_n) a sequence $\iff \forall \epsilon > 0 \exists N$ s.t $\forall n \geq N \; |a_n - A| < \epsilon$*

Definition 12. *(a_n) is **Cauchy** $\iff \forall \epsilon > 0 \exists N$ s.t $m, n \geq N \; |a_m - a_n| < \epsilon$*

Series

We discuss the meaning of $\sum_{n=0}^{\infty} a_n = S$

Cauchy Since it is a sequence we can use the Cauchy criterion to estimate a sum

Condition $p = 0 \implies |a_m| < \epsilon$ which is equivalent to $\lim_{n \to \infty} a_n$

Remark $R_n = \sum_{i+n+1} a_i$

Observe that the following converges:

$$\sum^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots$$

Absolute Convergence

$$\sum_{n=0}^{\infty} a_n$$

converges absolutely if and only if (def) $\sum_{n=0}^{\infty} |a_n|$ converges.

Lemma Absolute converges implies convergence

Proof Check Cauchy.

The Comparison test If $0 \leq a_k \leq b_k$, then $\sum b_k \text{conv} \implies \sum a_k \text{conv}$

Arithmetic Let us suppose we have a series on the following way: $\sum^{\infty}$ converges or converges absolutely if and only if $\sum^{\infty} \bar{a_n}$ and $\sum^{\bar{\infty}} a_n = \sum^{\infty} \bar{a_n}$.

2. Suppose we have two series that conv. or absolutely converges. $\sum^{\infty} a_n \quad \sum^{\infty} b_n$ implies that $\sum^{\infty}(a_n + b_n)$ converges or absolutely, and $\sum(a_n + b_n) = \sum a_n + \sum b_n$.

3. Given a complex number $\lambda \in \mathbb{C}$, and $\sum^{\infty} a_n$ converges or converges absolutely, then $\sum^{\infty} \lambda a_n = \lambda \sum^{\infty} a_n$.

4.(Product) We define the Cauchy product $(a_n), (b_n)$ the Cauchy product is: $c_n = \sum_{i=0}^{n} a_i b_{n-i}$

Proposition $\sum^{\infty} a_n, \sum^{\infty}$ is absolutely convergent, then $\sum c_n$ is also absol. convergent, and $\sum c_n = \sum a_n \cdot \sum b_n$.

Using a proof with draw.

Sequence of functions $\mathbb{C} \supset E \to \mathbb{C}$ $(f_n) : f_0, f_1, \ldots \to \mathbb{C}$ Suppose that for all $a \in E(f_n(a))$ converges.

Study pointwise convergence to f. Study also uniform convergence.

Taylor

$$F(z) = \frac{f(z) - f(a)}{z - a} \quad \text{for } z \neq a$$

$$\lim_{z \to a}(z - a)F(z) = 0$$

$$\implies \exists! f_1(z) \quad \text{on } \Omega$$

$$f_1(z) = F(z) \text{ for } a \neq z$$

So that you get a sequence of equations:

$$\begin{aligned} f(z) &= f(a) + (z - a)f_1(z) \\ f_1(z) &= f_1(a) + (z - a)f_2(z) \\ &\vdots \\ f_{n-1} &= f_{n-1}(a) + (z - a)f_n(z) \end{aligned}$$

and this produces

$$f(z) = f(a) + f_1(a)(z-a) + \dots f_{n-1}(a)(z-a)^{n-1} + (z-a)^n f_n(z)$$

And we observe that:

$$\frac{d^k}{dz^k}|_{z=a}(\dots) = f^k(a) = k! f_k(a)$$

so that we have contructed f as:

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \dots \frac{f^{n-1}(a)}{(n-1)!}(z-a)^{n-1} + (z-a)^n f_n(z)$$

Lema: for $|z-a| < R$

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-a)^n(\zeta-z)}$$

Zeroes & Poles Suppose that happens the following:

$$f(a) = f'(a) = f^{n-1}(a) = 0$$

So it is interesting to see that if the following holds:

$$f(z) = (z-a)^n f_n(z)$$

Recall that the strange function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 \iff & x = 0 \end{cases}$$

for $|z-a| < R$

$$M := \sup_{|z-a|=R} |f_n(z)|$$

so that:

$$|f_n(z)| \leq \frac{1}{2\pi}\frac{M \cdot 2\pi R}{R^n(R-|z-a|)}$$

if we multiply both sides by $(z-a)^n$ we see that:

$$|f(z)| \leq \left(\frac{|z-a|}{R}\right)^n \frac{M \cdot R}{R - |z-a|}$$

if $f^k(a) = 0$ for all k

$$\left|\frac{(z-a)}{R}\right) < 1 \implies |f(z)| \leq \lim_{k\to\infty}(\dots) = 0$$

now consider

$$f : \omega \underset{\text{Holomorphic}}{\to} \mathbb{C}$$

$$f^k(a) = 0 \forall h = 0, \ldots$$
$$\implies f(z) = 0 \text{ on } |z-a| < R$$
$$f(z) = 0 \implies f(z)^k(z) = 0 \forall k$$
$$\Omega = \left\{ a \in \Omega | \forall k = 0, 1, \ldots f^{(k)} = 0 \right\} \text{ Open } = E_1$$
$$E_2 = \left\{ a \in \Omega | \exists k : f^(k) \right\}$$