Cuervo es un bastardo

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Solution

Suppose that we are given $P(\alpha_k) = c_k \in \mathbb{C}$. In the same spirit of the above problem, we put:

$$Q(z) = (z - \alpha_1) \cdot \ldots \cdot (z - \alpha_n)$$

Since by hypothesis we know that $\deg P < n$, we can use the previous result:

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z - \alpha_k)} = \sum_{k=1}^{n} \frac{c_k}{Q'(\alpha_k)(z - \alpha_k)}$$

we conclude then:

$$\begin{aligned} P(z) &= Q(z) \cdot \sum_{k=1}^{n} \frac{c_k}{Q'(\alpha_k)(z - \alpha_k)} && (1) \\ &= \sum_{k=1}^{n} \frac{c_k}{Q'(\alpha_k)} \cdot \left(\frac{Q(z)}{z - \alpha_k} \right) && (2) \end{aligned}$$

Now if we suppose that $P(z)$ is given explicitly as (2). So that:

$$P(\alpha_1) = \frac{c_1(\alpha_1 - \alpha_2) \ldots (\alpha_1 - \alpha_n)}{(\alpha_1 - \alpha_2) \ldots (\alpha_1 - \alpha_n)} = c_1$$

Similarly, $P(\alpha_k) = c_k$ for every $k = 1, \ldots, n$. We conclude that $P(z)$ is uniquely determined by (2), that is:

$$P(z) = \sum_{k=1}^{n} c_k \prod_{j=1, j \neq k}^{n} \frac{z - \alpha_j}{\alpha_k - \alpha_j}$$

This is the famous Lagrange's interpolation polynomial.