Complex Analysis

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Nombre del curso: Complex Analysis
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Horario: Lu y Ju, 8:00 a 9:50

Nombre profesor(a) principal: Paul Bressler
Horario y lugar de atención: Mo y 17:00 a 18:00, Office H-409

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1 Organization of the course

- 3 Exams
- regular-weekly homework

Sometimes we will use the following alternative book: *Complex Analysis*, Gamelin

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2 Complex Numbers

We want to solve a quadratic equation:

$$ax^2 + bx + c = 0$$

so we first calculate the discriminant:

$$D = b^2 - 4ac$$

it is called that, because it differentiate solution types. We then compute:

$$r_{\pm} \frac{-b \pm \sqrt{D}}{2a}$$

In R squares are positives, but this poses a problem when $D < 0$. Meanwhile, let us observe that when $D > 0$ we get two solutions, and when we get $D = 0$

there is only one solution. So the main question is what to do when $D < 0$. Without loss of generality we can assume that $a = 1$

$$r_+ = -\frac{b}{2} + \sqrt{2}$$
$$r_- = -\frac{b}{2} - \sqrt{2}$$

the reason is that we want to be able write

$$p(x) = (x - r_+)(x - r_-)$$

expanding the product above

$$p(x) = x^2 - (r_+ + r_-)x + r_= r_-$$

so that D can be written as:

$$D = \text{Sign}(D)|D|$$

Observe that we can take the definition of sign as above. But then, this give a suspect idea that there can exist a **not real number** that:

$$\pm(\sqrt{-1})^2 = -1$$

which is also a solution of the following equation:

$$x^2 + 1 = 0$$

This then motivate the following concept, does arithmetic follows natural with the inclusion:

$$z = a + b\sqrt{-1}$$

Observe the symmetry on choosing $\sqrt{-1}$ or $-\sqrt{-1}$. Which again motivates

$$\bar{z} = a - b\sqrt{-1}$$

As in the textbook i^n maps to $1, i, -1, -i$ with the trick $\mod 4$ Let us define then that addition comes naturally:

$$(a + b\sqrt{-1}) + (c + d\sqrt{-1}) = (a + d) + (b + d)\sqrt{-1}$$

Multiplication is a little bit tricky:

$$(a + b\sqrt{-1}) \times (c + d\sqrt{-1}) = (ac - bd) + (ad + bc)\sqrt{-1}$$

which comes natural by considering $(\sqrt{-1})^2$. Observe that the reals are embedded as:

$$a + 0\sqrt{-1}$$

The operation of taking $z \to \bar{z}$ is called *Complex conjugation*, observe that it complies $\bar{z} \to z$, which will allow us to extract a, b as:

$$\frac{1}{2}(z + \bar{z}) := \mathrm{Re}(z) = a$$

and also

$$\frac{1}{2\sqrt{-1}}(z - \bar{z}) := \mathrm{Im}(z) = b$$

our definition above can be then replaced by:

$$z = \mathrm{Re}(z) + \mathrm{Im}(z)\sqrt{-1}$$

The preceding equation proposes a big problem, since we have found another "strange" number:

$$\frac{1}{\sqrt{-1}} = x$$

or equivalently,

$$\sqrt{-1}x = 1$$

from which it can be solved by inspecting the table of exponentiation and we find:

$$x = -\sqrt{-1}$$

In literature we often define that $\sqrt{-1} = i$ A big question is that if $z \neq 0$, then there exist ω such that $w \cdot z = 1$, the answer is positive and it even has a stronger statement: with complex numbers we can solve any polynomial equation.

$$z = a + bi$$
$$w = x + iy$$

and we are looking $z \cdot w = 1$ so we get the system of equations

$$ax - by = 1$$
$$bx + ay = 0$$

and we can get information about the solution, via the determinant $\det = a^2 + b^2$, and this is only zero if only if $a = 0$ and $b = 0$. Solution can be found for any method

Observe that if we have a complex number and its conjugate, the following relation holds:

$$z \cdot \bar{z} = a^2 + b^2$$
$$\frac{1}{z} = \frac{1}{a^2 + b^2} \cdot \bar{z}$$

$$\frac{1}{2+3i} = \frac{1}{13}(2-3i)$$

Observe that this recall Cramer's rule, since it has the form of $\frac{1}{det}$

The following problem that we are facing is the existence of:

$$z = a + bi \quad \exists w \quad w^2 = z$$

if we let $w = x + iy$, and $w^2 = (x^2 - y^2) + i(2xy)$, we get the following system of equations.

$$a = x^2 - y^2$$
$$b = 2xy$$

Observe that 0 has a unique root. We denote the set of all complex numbers with $\mathbb{C}$ and observe that we can take bijection beetwen C and R^2

$$\mathbb{R} \leftrightarrow \mathbb{C}$$
$$z \longrightarrow (Re(z), Im(z))$$
$$a + ib \longleftarrow (a, b)$$

Observe then that $\mathbb{C}$ is a vector space over $\mathbb{R}$. Basis: $1, i$. Now observe that we can take a geometric interpretation, given this two bijections, to the multiplication of a complex numbers. Side note: A linear map is a map that plays nice with arithmetic. an important characterization of linear maps, is that:

$$f(a) = a \cdot f(1)$$

and also

$$f\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and it can be easy characterized via:

$$\begin{pmatrix} v_{11} & v_{22} \\ v_{21} & v_{22} \end{pmatrix}$$

Now consider the following map:

$$\mathbb{C} \to \mathbb{C}$$
$$w \rightarrowtail zw$$

where $z = a + ib$ Again observe that we are only taking care of the values on the basis.

Example Suppose $A^2 = -I$, and $\det > 0$, then there is an invertible matrix B such that $A = BJB^{-1}$. Returning to our previous point, we see that complex numbers are embedded nicely on 2×2 matrices with real coefficients. So that the span 1 and j is a 4 dimensional vector space over R. Finally, we conclude that multiplication on $\mathbb{C}$ is a matrix multiplication.

When we look a close view on $\mathbb{R}^2$ along with the normal dot product, we have the **Cauchy-Schwarz Inequality**: Let us then define:

$$|\vec{a}|^2 = \vec{a} \cdot \vec{a}$$

and proof is left to the reader:

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$$

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3 Remarks from last time

$$\mathbb{C} \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$$
$$A + IB \to a + bJ$$
$$J^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so that:

$$Z = a + ib \leftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Let us observe that in $\mathbb{R}2$ we have the dot product for which:

$$\vec{a} \cdot \vec{a} \geq 0$$

it is good as concept of measure of distance. Let us observe the analogue on complex numbers:

$$z = a + ib$$

$$w = c + id \implies \bar{w} = c - id$$

so that

$$z\bar{w} = (ac + bd) + (bc - ad)$$

for which the left parenthesis of the RHS is real and the other imaginary. Observe also:

$$|z|^2 = a^2 + b^2 \geq 0$$

and if $z \neq 0$ we define:

$$z^{"}-1 = \frac{1}{|z|}\bar{z}$$

Cauchy-Schawrz

Suppose we know the properties of the dot product:

$$\left|\vec{a}\cdot\vec{b}\right| \leq |\vec{a}|\left|\vec{b}\right|$$

Example
In $\mathbb{R}^n$:

$$< a_1,\ldots,a_n > \cdot < b_1,\ldots b_n$$
$$= \sum_{n=1}^{n} a_i b_i$$
$$\leq (\sum a_i)^{/frac12}(\sum b_j)^{\frac{1}{2}}$$

Example The functions that are square integrables on $I = [0,1]$ for which we have:

$$\left|\int_0^1 fgdx\right| \leq \left|\int fdx\right|\left|\int gdx\right|$$

Proof Let recall that for parameterizing the line in beetwen $\vec{a}$ and $\vec{b}$ we could use the convex combination.

$$t\vec{b} + (1-t)\vec{a} \quad 0 \leq t \leq 1$$
$$\left|t\vec{b} + (1-t)\vec{a}\right|^2 > 0$$

We expand and we get: .... It is the same trick of consider a quadratic without root $a + \lambda b$. Geometrical Consideration:For two independent vectors, the line that pass through them do not pass on the origin.

4 C-S Consequences

The triangle inequality

$$\left|a \vec{+} \vec{b}\right| \leq |\vec{a}| + \left|\vec{b}\right|$$

the proof is given by considering the expansion of:

$$\left|\vec{a} + \vec{b}\right|$$

and the inequality: (Check!)
Definition of Angles

$$\cos(\theta) := \frac{\left|\vec{a}\cdot\vec{b}\right|}{|\vec{a}|\left|\vec{b}\right|}$$

5 Special (2×2) matrices