Homework 1

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February 12, 2015

1. Compute

$$\sqrt{i}, \qquad \sqrt{-i}, \qquad \sqrt{1+i}, \qquad \sqrt{\frac{1-i\sqrt{3}}{2}}$$

For $\sqrt{i}$, we are looking for x and y such that

$$\sqrt{i} = x + iy$$
$$i = x^2 - y^2 + 2xyi$$
$$x^2 - y^2 = 0 \tag{1}$$
$$2xy = 1 \tag{2}$$

From (1), we see that $x^2 = y^2$ or $\pm x = \pm y$. Also, note that i is the upper half plane (UHP). That is, the angle is positive so $x = y$ and $2x^2 = 1$ from (1) Therefore, $\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$. We also could have done this problem using the polar form of z. Let $z = i$. Then $z = e^{i\pi/2}$ so $\sqrt{z} = e^{i\pi/4}$ which is exactly what we obtained. For $\sqrt{-i}$, let $z = -i$. Then z in polar form is $z = e^{-i\pi/2}$ so $\sqrt{z} = e^{-i\pi/4} = \frac{1}{\sqrt{2}}(1-i)$. For $\sqrt{1+i}$, let $z = 1+i$. Then $z = \sqrt{2}e^{i\pi/4}$ so $\sqrt{z} = 2^{1/4}e^{i\pi/8}$. Finally, for $\sqrt{\frac{1-i\sqrt{3}}{2}}$, let $z = \frac{1-i\sqrt{3}}{2}$. Then $z = e^{-i\pi/3}$ so $\sqrt{z} = e^{-i\pi/6} = \frac{1}{2}(\sqrt{3} - i)$.

2. Find the four values of $\sqrt[4]{-1}$.

Let $z = \sqrt[4]{-1}$ so $z^4 = -1$. Let $z = re^{i\theta}$ so $r^4e^{4i\theta} = -1 = e^{i\pi(1+2k)}$.

$$r^4 = 1$$
$$\theta = \frac{\pi}{4}(1+2k)$$

where $k = 0, 1, 2, 3$. Since when $k = 4$, we have $k = 0$. Then $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

$$z = e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}$$

3. Compute $\sqrt[4]{i}$ and $\sqrt[4]{-i}$.

Let $z = \sqrt[4]{i}$ and $z = re^{i\theta}$. Then $r^4 e^{4i\theta} = i = e^{i\pi/2}$.

$$r^4 = 1$$
$$\theta = \frac{\pi}{8}$$

so $z = e^{i\pi/8}$. Now, let $z = \sqrt[4]{-i}$. Then $r^4 e^{4i\theta} = e^{-i\pi/2}$ so $z = e^{-i\pi/8}$.

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1. Solve the quadratic equation

$$z^2 + (\alpha + i\beta)z + \gamma + i\delta = 0.$$

The quadratic equation is $x = \frac{-b \pm \sqrt{b^2 - ac}}{2}$. For the complex polynomial, we have

$$z = \frac{-\alpha - \beta i \pm \sqrt{\alpha^2 - \beta^2 - 4\gamma + i(2\alpha\beta - 4\delta)}}{2}$$

Let $a + bi = \sqrt{\alpha^2 - \beta^2 - 4\gamma + i(2\alpha\beta - 4\delta)}$. Then

$$z = \frac{-\alpha - \beta \pm (a + bi)}{2}$$

Prove that

$$\left| \frac{a - b}{1 - \bar{a}b} \right| = 1$$

if either $|a| = 1$ or $|b| = 1$. What exception must be made if $|a| = |b| = 1$?

Recall that $|z|^2 = z\bar{z}$.

$$\begin{aligned}
1^2 &= \left| \frac{a - b}{1 - \bar{a}b} \right|^2 \\
1 &= \left(\frac{a - b}{1 - \bar{a}b} \right) \left(\overline{\frac{a - b}{1 - \bar{a}b}} \right) \\
&= \left(\frac{a - b}{1 - \bar{a}b} \right) \left(\frac{\bar{a} - \bar{b}}{1 - a\bar{b}} \right) \\
&= \frac{a\bar{a} - a\bar{b} - \bar{a}b + b\bar{b}}{1 - \bar{a}b - a\bar{b} + a\bar{a}b\bar{b}}
\end{aligned} \tag{3}$$

If $|a| = 1$, then $|a|^2 = a\bar{a} = 1$ and similarly for $|b|^2 = 1$. Then (3) becomes

$$\frac{1 - a\bar{b} - \bar{a}b + b\bar{b}}{1 - \bar{a}b - a\bar{b} + b\bar{b}} \qquad \text{and} \qquad \frac{1 - a\bar{b} - \bar{a}b + a\bar{a}}{1 - \bar{a}b - a\bar{b} + a\bar{a}}$$

respectively which is one. If $|a| = |b| = 1$, then $|a|^2 = |b|^2 = 1$ so (3) can be written as

$$\frac{2 - a\bar{b} - \bar{a}b}{2 - \bar{a}b - a\bar{b}}.$$

Therefore, we must have that $a\bar{b} + \bar{a}b \neq 2$.

Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1$$

if $|a| < 1$ and $|b| < 1$.

From the properties of the modulus, we have that

$$\begin{aligned}
\left|\frac{a-b}{1-\bar{a}b}\right| &= \frac{|a-b|}{|1-\bar{a}b|} \\
&= \frac{|a-b|^2}{|1-\bar{a}b|^2} \qquad (4) \\
&= \frac{(a-b)(\bar{a}-\bar{b})}{(1-\bar{a}b)(1-a\bar{b})} \\
&= \frac{|a|^2+|b|^2-a\bar{b}-\bar{a}b}{1+|a|^2|b|^2-\bar{a}b-a\bar{b}} \\
&< \frac{2-a\bar{b}-\bar{a}b}{2-\bar{a}b-a\bar{b}} \\
&= 1 \qquad (5)
\end{aligned}$$

From (5), we have

$$\begin{aligned}
\frac{|a-b|^2}{|1-\bar{a}b|^2} &< 1 \\
\frac{|a-b|}{|1-\bar{a}b|} &< 1
\end{aligned}$$

If $|a_i| < 1$, $\lambda_i \geq 0$ for $i = 1, \ldots, n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, show that

$$|\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n| < 1.$$

Since $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geq 0$, $0 \leq \lambda_i < 1$. By the triangle inequality,

$$\begin{aligned}
|\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n| &\leq |\lambda_1||a_1| + \cdots + |a_n||\lambda_n| \\
&< \sum_{i=1}^{n} \lambda_i \\
&= 1
\end{aligned}$$

Show that there are complex numbers z satisfying

$$|z-a| + |z+a| = 2|c|$$

if and only if $|a| \leq |c|$. If this condition is fulfilled, what are the smallest and largest values $|z|$?

By the triangle inequality,

$$|z-a| + |z+a| \geq |(z-a)-(z+a)| = 2|a|$$

so

$$\begin{aligned} 2|c| &= |z-a| + |z+a| \\ &\geq |(z-a)-(z+a)| \\ &= 2|a| \end{aligned}$$

Thus, $|c| \geq |a|$. If $|a| \leq |c|$, then let $z = |c|\frac{a}{|a|}$.

$$\begin{aligned} |z-a| + |z+a| &\geq |(z-a)+(z+a)| \\ &= 2||c|| \end{aligned}$$

since $\frac{a}{|a|}$ is a unit vector.

$$= 2|c|$$

Thus, $2|c| \geq 2|c|$ which is equality.

$$\begin{aligned} 2|c| &= |z+a| + |z-a| \\ 4|c|^2 &= \big(|z+a| + |z-a|\big)^2 \\ &= 2(|z|^2 + |a|^2) \\ &\leq 4(|z|^2 + |a|^2) \\ |c|^2 &\leq |z|^2 + |a|^2 \\ \sqrt{|c|^2 - |a|^2} &\leq |z| \end{aligned}$$

If ω is given by $\omega = \cos\big(\frac{2\pi}{n}\big) + i\sin\big(\frac{2\pi}{n}\big)$, prove that

$$1 + \omega^h + \omega^{2h} + \cdots + \omega^{(n-1)h} = 0$$

for any integer h which is not a multiple of n.

Let $\omega = \cos\big(\frac{2\pi}{n}\big) + i\sin\big(\frac{2\pi}{n}\big)$ be written in exonential form as $\omega = e^{2\pi i/n}$. Then the series can be written as

$$\sum_{k=0}^{n-1}\big(e^{2\pi ih/n}\big)^k = \frac{e^{2ih\pi} - 1}{e^{2hi\pi/n} - 1}.$$

Since h is an integer, $e^{2ih\pi} = 1$; therefore, the series zero.