Homework 2

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1. Find the most general harmonic polynomial of the form $ax^3+bx^2y+cxy^2+dy^3$. Determine the conjugate harmonic function and the corresponding analytic function by integration and by the formal method.

   In order to be harmonic, $u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$ has to satisfy $\nabla^2 u = 0$ so
   $$u_{xx} + u_{yy} = (3a + c)x + (3d + b)y = 0.$$
   Thus, $3a = -c$ and $3d = -b$ so
   $$u(x,y) = ax^3 - 3axy^2 - 3dx^2y + dy^3.$$
   To find the harmonic conjugate $v(x,y$, we need to look at the Cauchy-Riemann equations. By the Cauchy-Riemann equations,
   $$u_x = 3ax^2 - 3ay^2 - 6dxy = v_y.$$
   Then we can integrate with respect to y to find $v(x,y)$.
   $$v(x,y) = \int (3ax^2 - 3ay^2 - 6dxy)dy = 3ax^2y - ay^3 - 3dxy^2 + g(x)$$
   Using the second Cauchy-Riemann, we have
   $$v_x = 6axy - 3dy^2 + g'(x) = -u_y = 3dx^2 + 6axy - 3dy^2$$
   so $g'(x) = 3dx^2$. Then $g(x) = dx^3 + C$ and
   $$v(x,y) = 3ax^2y - ay^3 - 3dxy^2 + dx^3 + C.$$

2. Show that an analytic function cannot have a constant absolute value without reducing to a constant.

   Let $f = u(x,y) + iv(x,y)$. Then the modulus of f is $|f| = \sqrt{u^2 + v^2}$. If the modulus of f is constant, then $u^2 + v^2 = c$ for some constant c. If $c = 0$, then $f = 0$ which is constant. Suppose $c \neq 0$. By taking the derivative with respect to x and y, we have

$$\begin{aligned}
0 &= \frac{\partial}{\partial x}(u^2 + v^2) \\
&= 2uu_x + 2vv_x \\
&= uu_x + vv_x \\
0 &= \frac{\partial}{\partial y}(u^2 + v^2) \\
&= uu_y + vv_y
\end{aligned}$$

Since f is analytic, f satisfies the Cauchy-Riemann. That is, $u_x = v_y$ and $u_y = -v_x$.

$$uv_y + vv_x = 0 \tag{1a}$$

$$-uv_x + vv_y = 0 \tag{1b}$$

Setting eq. (1a) equal to eq. (1b), we have

$$v_x(u + v) + v_y(u - v) = 0.$$

Now, either v_x and v_y are zero, v_x and $u - v$ are zero, v_y and $u + v$ are zero, or $u + v$ and $u - v$ are zero. If $v_x = v_y = 0$, then f is constant. If $v_x = 0$ and $u - v = 0$, then $u_y = 0$ and $u = v$. Since $u = v$ and $v_x = 0$, then so does $u_x = 0$ and it also follows that $v_y = 0$; thus, f is a constant. By the same argument, f is a constant when $v_y = 0$ and $u + v = 0$. If $u + v = 0$ and $u - v = 0$, then $u = \pm v$ so $u = v = 0$ and f is a constant.

3. Prove rigorously that the functions $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously analytic.

   Let $g(z) = \overline{f(\bar{z})}$ and suppose f is analytic. Then $g'(z)$ is

$$\begin{aligned}
g'(z) &= \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\overline{f(\bar{z} + \overline{\Delta z})} - \overline{f(\bar{z})}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \left[\overline{\frac{f(\bar{z} + \overline{\Delta z}) - f(\bar{z})}{\overline{\Delta z}}} \right]
\end{aligned}$$

   Since conjugation is continuous, we can move the limit inside the conjugation.

$$\begin{aligned}
&= \overline{\lim_{\Delta z \to 0} \frac{f(\bar{z} + \overline{\Delta z}) - f(\bar{z})}{\overline{\Delta z}}} \\
&= \overline{f'(\bar{z})}
\end{aligned}$$

Thus, g is differentiable with derivative $\overline{f'(\bar{z})}$. Suppose $\overline{f(\bar{z})}$ is analytic and let $g(\bar{z}) = \overline{f(z)}$. Then by the same argument, f is differentiable with derivative $\overline{g'(\bar{z})}$. Therefore, $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously analytic.

We could also use the Cauchy-Riemann equations. Let $f(z) = u(x,y) + iv(x,y)$ where $z = x+iy$ so $\bar{z} = x - iy$. Then $\overline{f(\bar{z})} = \alpha(x,y) - i\beta(x,y)$ where $\alpha(x,y) = u(x,-y)$ and $\beta(x,y) = v(x,-y)$. In order for both to be analytic, they both need to satisfy the Cauchy-Riemann equations. That is, $u_x = v_y$, $u_y = -v_x$, $\alpha_x = \beta_y$ and $\alpha_y = -\beta_x$.

$$\begin{aligned}
u_x(x,y) &= v_y(x,y)\\
u_y(x,y) &= -v_x(x,y)\\
\alpha_x(x,y) &= u_x(x,-y)\\
\alpha_y(x,y) &= -u_y(x,-y)\\
-\beta_x(x,y) &= v_x(x,-y)\\
\beta_y(x,y) &= v_y(x,-y)
\end{aligned}$$

Suppose that $\overline{f(\bar{z})}$ satisfies the Cauchy-Riemann equations. Then $\alpha_x = u_x(x,-y) = v_y(x,-y) = \beta_y$ and $\alpha_y = -u_y(x,-y) = v_x(x,-y) = -\beta_x$. Therefore,

$$\begin{aligned}
u_x(x,-y) &= v_y(x,-y)\\
u_y(x,-y) &= -v_x(x,-y)
\end{aligned}$$

which means $f(\bar{z})$ satisfies the Cauchy-Riemann equations. Now, recall that $|z| = |\bar{z}|$. Since $f(\bar{z})$ satisfies the Cauchy-Riemann equations, for an $\epsilon > 0$ there exists a $\delta > 0$ such that when $0 < |\Delta z| < \delta$, $|f(\bar{z}) - \overline{z_0}| = |f(z) - z_0| < \epsilon$. Thus, $\lim_{\Delta z \to 0} f(z) = z_0$ so $f(z)$ is analytic if $\overline{f(\bar{z})}$ is analytic.

4. Prove that the functions $u(z)$ and $u(\bar{z})$ are simultaneously harmonic.

   Since u is the real part of $f(z)$, $u(z) = u(x,y)$ where $z = x+iy$. Suppose $u(z)$ is harmonic. Then $u(z)$ satisfies Laplace equation.

   $$\nabla^2 u(z) = u_{xx} + u_{yy} = 0$$

   Now, $u(\bar{z}) = u(x,-y)$ where $\frac{\partial^2}{\partial x^2}u(\bar{z}) = u_{xx}$ and $\frac{\partial^2}{\partial y^2}u(\bar{z}) = u_{yy}$ so

   $$\nabla^2 u(\bar{z}) = u_{xx} + u_{yy} = 0.$$

   Since $u(z)$ is harmonic, $u_{xx} + u_{yy} = 0$ so it follows that $u(\bar{z})$ is harmonic as well.

5. If Q is a polynomial with distinct roots $\alpha_1, \dots, \alpha_n$, and if P is a polynomial of degree $< n$, show that

   $$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z - \alpha_k)}.$$

Let's multiple by $Q(z)$. We then have

$$P(z) = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)} Q(z)$$

which are both polynomials of degree less than n and agreeing at $z = \alpha_k$.

6. Use the formula in the preceding exercise to prove that there exists a unique polynomial P or degree $< n$ with given values c_k at the points α_k (Lagrange's interpolation polynomial).

   Suppose that we are given $P(\alpha_k) = c_k \in \mathbb{C}$. In the same spirit of the above problem, we put:

$$Q(z) = (z-\alpha_1) \cdot \ldots \cdot (z-\alpha_n)$$

   Since by hypothesis we know that $\deg P < n$, we can use the previous result:

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)} = \sum_{k=1}^{n} \frac{c_k}{Q'(\alpha_k)(z-\alpha_k)}$$

   we conclude then:

$$\begin{aligned} P(z) &= Q(z) \cdot \sum_{k=1}^{n} \frac{c_k}{Q'(\alpha_k)(z-\alpha_k)} && (2) \\ &= \sum_{k=1}^{n} \frac{c_k}{Q'(\alpha_k)} \cdot \left(\frac{Q(z)}{z-\alpha_k}\right) && (3) \end{aligned}$$

   Now if we suppose that $P(z)$ is given explicitly as (3). So that:

$$P(\alpha_1) = \frac{c_1(\alpha_1-\alpha_2)\ldots(\alpha_1-\alpha_n)}{(\alpha_1-\alpha_2)\ldots(\alpha_1-\alpha_n)} = c_1$$

   Similarly, $P(\alpha_k) = c_k$ for every $k = 1, \ldots, n$. We conclude that $P(z)$ is uniquely determined by (3), that is:

$$P(z) = \sum_{k=1}^{n} c_k \prod_{j=1, j\neq k}^{n} \frac{z-\alpha_j}{\alpha_k-\alpha_j}$$

   This is the famous Lagrange's interpolation polynomial.