Complex Analysis

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Exercise 1. If $f(z)$ and $g(z)$ have the algebraic orders h and k at $z = a$, show that:

- fg has the order $h + k$
- f/g the order $h - k$
- $f + g$ an order which does not exceed max (h, k).

  Solution. 1. To say that $f(z)$ and $g(z)$ have algebraic orders h and k respectively at $z = a$ is to say that

  $$\lim_{n\to\infty} |(z-a)|^{\alpha}|f(z)| = 0 \text{ for all } \alpha > r$$

  and

  $$\lim_{n\to\infty} |(z-a)|^{\beta}|g(z)| = 0 \text{ for all } \beta > k$$

  where h and k are the minimal integers satisfying these properties.

  2. Then we have that

  $$\begin{aligned} &\lim_{n\to\infty} |(z-a)|^{\alpha+\beta}|f(z)\cdot g(z)| = 0 \\ \iff &\lim_{n\to\infty} |(z-a)|^{\alpha}|(z-a)|^{\beta}|f(z)\cdot g(z)| = 0 \\ \iff &\alpha+\beta > h+k \end{aligned}$$

  as desired.

- First, we observe that if $g(z) \neq 0$, then

$$
\begin{aligned}
&\lim_{n\to\infty} |(z-a)|^{\beta}\, |g(z)| = 0 \\
\iff &\lim_{n\to\infty} |(z-a)|^{-\beta} \left|\frac{1}{g(z)}\right| = 0 \\
\iff &-\beta > -k
\end{aligned}
$$

2. Then if we continue to assume that $g(z) \neq 0$, we have that

$$
\begin{aligned}
&\lim_{n\to\infty} |(z-a)|^{\alpha-\beta} \left|\frac{f(z)}{g(z)}\right| = 0 \\
\iff &\lim_{n\to\infty} |(z-a)|^{\alpha}|(z-a)|^{-\beta} \left|\frac{f(z)}{g(z)}\right| = 0 \\
\iff &\alpha - \beta > h - k
\end{aligned}
$$

as desired.

- If $f(z) = z^n f'(z)$, and $g(z) = z^n g'(z)$, then $f(z)+g(z) = x^n(f'(z)+g'(z))$, which means that if both orders are at least n, then the sum is at least n. In other words, the order of the sum must be at least the minimum of the two orders.

Exercise 2. Show that a function which is analytic in the whole plane and has a nonessential singularity at ∞ reduces to a polynomial.

Solution. Let f be a function that is analytic in the whole plane and has a nonessential singularity at ∞, define $F(z) = f(\frac{1}{z})$. Then F has a nonessential singularity at $z = 0$, therefore we have two cases:

- *Case I* The singularity is removable. If the singularity is removable then F is a bounded function in a neighborhood of of zero and f is bounded at infinity, and with the analyticity of f we have then, that it is a constant.
- *Case II* The singularity is a pole. Suppose that the singularity is a pole, then:
$$F(z) = f(\frac{1}{z}) = \sum_{k=1}^{n} c_k z^k + g(z)$$
and g is analytic at 0. Which then can use as:
$$f(z) = g\left(\frac{1}{z}\right) = \sum_{k=1}^{n} c_k z^k + g(\frac{1}{z})$$
we observe that $g(\frac{1}{z})$ is bounded in a neighborhood of zero since it is f and the rest of a polynomial, and $g(\frac{1}{z})$ then is analytic on the entire complex plane and has a finite limit $g(0)$, which means that is just a constant.

Exercise 3. Show that the functions $e^z, \sin(z)$, and $\cos(z)$ have essential singularities at ∞

Solution. We observe that:

$$\lim_{z\to 0^+} \left|e^{1/z}\right| = \infty$$

and

$$\lim_{z\to 0^-} \left|e^{1/z}\right| = 0$$

so that

$$0 \neq \lim_{z\to 0} |z|^\alpha \left|e^{1/z}\right| \neq \infty$$

so we observe that e^z has an essential singularity at ∞.

We observe that $\cos(z)$ has essential singularity at ∞ iff $\cos(\frac{1}{z})$ has essential singularity 0. Now, we proceed by contradiction. Suppose that $\cos(1/z)$ has a pole or removable singularity at 0. Then the same is true for its derivative $z^{-2}\sin(1/z)$. Since

$$e^{i/z} = \cos(1/z) + i\sin(1/z)$$

we have a contradiction: essential singularity on the left but not on the right.

Argument is similar for $\sin(z)$

Exercise 4. Show that any function which is meromorphic in the extended plane is rational

Solution. Our main idea of thought will be to prove the following:

Theorem 1. *Suppose $f : C_\infty \to C_\infty$ is a meromorphic function in the extended complex plane. Then f is a rational function.*

Proof. Let $\{z_n\} \in \mathbb{C}$ be the set of the poles of the function f. The function $F(z) = f(\frac{1}{z})$ must be analytic in a deleted neighborhood of the origin, hence f is analytic in a deleted neighborhood of ∞, the rest of the complex plane can contain only finitely many singularities, which implies $\{z_n\}$ is finite. Now suppose that the orders of $z_1, \ldots, z_n$ with multiplicities $m_1, \ldots, m_k$ and let $b_1, \ldots, b_l$ the poles of f with orders $o_1, \ldots, o_l$. Now consider:

$$g(z) = \frac{\prod_{j=1}^{k}(z-z_j)^{m_j}}{\prod_{d=1}^{l}(z-b_d)^{o_d}}$$

Observe g has exactly the same zeros and poles of f with the same multiplicities, so $h : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by:

$$h(z) = \frac{f(z)}{g(z)}$$

is meromorphic function with no zeros or poles, then h extends to a nonzero bounded entire function, so by Liouville, $h(z) = c$ for $c \neq 0$. Then

$$f(z) = cg(z) = c\frac{\prod_{j=1}^{k}(z-z_j)^{m_j}}{\prod_{d=1}^{l}(z-b_d)^{o_d}}$$

Then f is rational function □

Problem 2 Find zeroes and orders of zeroes of the following functions

1. $\frac{z^2+1}{z^2-1}$

   The zeros are i and $-i$. Which both have order 1

2. $\frac{z^4+1}{z^5}$

   The zeros are $-\sqrt{i}, \sqrt{i}, \sqrt{-i}, -\sqrt{-i}, \infty$. Which all have order 1. For the infinite case, consider $F(z) = f(1/z) = z + z^5$ so that it has order 1.

3. $z^2 \sin(z)$ We see that the zeros are $n\pi n \in \mathbb{Z}$, and the order of zeros different from 0 is 1, lastly, the order of 0 is 2.

4. $\cos(z) - 1$ We see that the zeros of the function are of the form $2\pi n, n \in \mathbb{Z}$ and each of them are of order 2

5. $\frac{\cos(z)-1}{z}$

   By periodicity, the zeros are $2\pi n$ and each have order 2

6. $\frac{\cos(z)-1}{z^2}$

   Again the zeros are $2\pi n$ and each have order 2

7. $e^z - 1$ From periodicity of e^z the zeros are of the form $2\pi n$, and each have order 1

Problem 2

- Which of the functions in porblem 1 are holomorphic at ∞ ?

  Since we are looking for essential singularities, the only functions holomorphic at ∞ are 1 and 2 becuase are regular in ∞. The rest have essential singularities at ∞

- For functions in problem 1 which are holomorphic at ∞ determine the order of any zeros at ∞

  For (2), the order of ∞ is of order 1.