Complex Analysis

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Horario: Lu y Ju, 8:00 a 9:50

Nombre profesor(a) principal: Paul Bressler
Horario y lugar de atención: Mo y 17:00 a 18:00, Office H-409

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1 Organization of the course

- 3 Exams
- regular-weekly homework

Sometimes we will use the following alternative book: *Complex Analysis*, Gamelin

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2 Complex Numbers

We want to solve a quadratic equation:

$$ax^2 + bx + c = 0$$

so we first calculate the discriminant:

$$D = b^2 - 4ac$$

it is called that, because it differentiate solution types. We then compute:

$$r_{\pm}\frac{-b \pm \sqrt{D}}{2a}$$

In R squares are positives, but this poses a problem when $D < 0$. Meanwhile, let us observe that when $D > 0$ we get two solutions, and when we get $D = 0$

there is only one solution. So the main question is what to do when $D < 0$. Without loss of generality we can assume that $a = 1$

$$r_+ = -\frac{b}{2} + \sqrt{2}$$
$$r_- = -\frac{b}{2} - \sqrt{2}$$

the reason is that we want to be able write

$$p(x) = (x - r_+)(x - r_-)$$

expanding the product above

$$p(x) = x^2 - (r_+ + r_-)x + r_= r_-$$

so that D can be written as:

$$D = \mathrm{Sign}(D)|D|$$

Observe that we can take the definition of sign as above. But then, this give a suspect idea that there can exist a **not real number** that:

$$\pm(\sqrt{-1})^2 = -1$$

which is also a solution of the following equation:

$$x^2 + 1 = 0$$

This then motivate the following concept, does arithmetic follows natural with the inclusion:

$$z = a + b\sqrt{-1}$$

Observe the symmetry on choosing $\sqrt{-1}$ or $-\sqrt{-1}$. Which again motivates

$$\bar{z} = a - b\sqrt{-1}$$

As in the textbook i^n maps to $1, i, -1, -i$ with the trick $\mod 4$ Let us define then that addition comes naturally:

$$(a + b\sqrt{-1}) + (c + d\sqrt{-1}) = (a + d) + (b + d)\sqrt{-1}$$

Multiplication is a little bit tricky:

$$(a + b\sqrt{-1}) \times (c + d\sqrt{-1}) = (ac - bd) + (ad + bc)\sqrt{-1}$$

which comes natural by considering $(\sqrt{-1})^2$. Observe that the reals are embedded as:

$$a + 0\sqrt{-1}$$

The operation of taking $z \to \bar{z}$ is called *Complex conjugation*, observe that it complies $\bar{z} \to z$, which will allow us to extract a, b as:

$$\frac{1}{2}(z + \bar{z}) := \mathrm{Re}(z) = a$$

and also

$$\frac{1}{2\sqrt{-1}}(z - \bar{z}) := \mathrm{Im}(z) = b$$

our definition above can be then replaced by:

$$z = \mathrm{Re}(z) + \mathrm{Im}(z)\sqrt{-1}$$

The preceding equation proposes a big problem, since we have found another "strange" number:

$$\frac{1}{\sqrt{-1}} = x$$

or equivalently,

$$\sqrt{-1}x = 1$$

from which it can be solved by inspecting the table of exponentiation and we find:

$$x = -\sqrt{-1}$$

In literature we often define that $\sqrt{-1} = i$ A big question is that if $z \neq 0$, then there exist ω such that $w \cdot z = 1$, the answer is positive and it even has a stronger statement: with complex numbers we can solve any polynomial equation.

$$z = a + bi$$
$$w = x + iy$$

and we are looking $z \cdot w = 1$ so we get the system of equations

$$ax - by = 1$$
$$bx + ay = 0$$

and we can get information about the solution, via the determinant $\det = a^2 + b^2$, and this is only zero if only if $a = 0$ and $b = 0$. Solution can be found for any method

Observe that if we have a complex number and its conjugate, the following relation holds:

$$z \cdot \bar{z} = a^2 + b^2$$
$$\frac{1}{z} = \frac{1}{a^2 + b^2} \cdot \bar{z}$$

$$\frac{1}{2+3i} = \frac{1}{13}(2-3i)$$

Observe that this recall Cramer's rule, since it has the form of $\frac{1}{det}$

The following problem that we are facing is the existence of:

$$z = a + bi \quad \exists w \quad w^2 = z$$

if we let $w = x + iy$, and $w^2 = (x^2 - y^2) + i(2xy)$, we get the following system of equations.

$$\begin{aligned} a &= x^2 - y^2 \\ b &= 2xy \end{aligned}$$

Observe that 0 has a unique root. We denote the set of all complex numbers with $\mathbb{C}$ and observe that we can take bijection beetwen C and R^2

$$\begin{aligned} \mathbb{R} &\leftrightarrow \mathbb{C} \\ z &\longrightarrow (Re(z), Im(z)) \\ a + ib &\longleftarrow (a, b) \end{aligned}$$

Observe then that $\mathbb{C}$ is a vector space over $\mathbb{R}$. Basis: $1, i$. Now observe that we can take a geometric interpretation, given this two bijections, to the multiplication of a complex numbers. Side note: A linear map is a map that plays nice with arithmetic. an important characterization of linear maps, is that:

$$f(a) = a \cdot f(1)$$

and also

$$f\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and it can be easy characterized via:

$$\begin{pmatrix} v_{11} & v_{22} \\ v_{21} & v_{22} \end{pmatrix}$$

Now consider the following map:

$$\begin{aligned} \mathbb{C} &\to \mathbb{C} \\ w &\rightarrowtail zw \end{aligned}$$

where $z = a + ib$ Again observe that we are only taking care of the values on the basis.

Example Suppose $A^2 = -I$, and $\det > 0$, then there is an invertible matrix B such that $A = BJB^{-1}$. Returning to our previous point, we see that complex numbers are embedded nicely on 2×2 matrices with real coefficients. So that the span 1 and j is a 4 dimensional vector space over R. Finally, we conclude that multiplication on $\mathbb{C}$ is a matrix multiplication.

When we look a close view on $\mathbb{R}^2$ along with the normal dot product, we have the **Cauchy-Schwarz Inequality**: Let us then define:

$$|\vec{a}|^2 = \vec{a} \cdot \vec{a}$$

and proof is left to the reader:

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$$

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3 Remarks from last time

$$\mathbb{C} \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$$
$$A + IB \to a + bJ$$
$$J^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so that:

$$Z = a + ib \leftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Let us observe that in $\mathbb{R}2$ we have the dot product for which:

$$\vec{a} \cdot \vec{a} \geq 0$$

it is good as concept of measure of distance. Let us observe the analogue on complex numbers:

$$z = a + ib$$

$$w = c + id \implies \bar{w} = c - id$$

so that

$$z\bar{w} = (ac + bd) + (bc - ad)$$

for which the left parenthesis of the RHS is real and the other imaginary. Observe also:

$$|z|^2 = a^2 + b^2 \geq 0$$

and if $z \neq 0$ we define:

$$z\text{"}-1 = \frac{1}{|z|}\bar{z}$$

Cauchy-Schawrz

Suppose we know the properties of the dot product:

$$\left|\vec{a} \cdot \vec{b}\right| \leq |\vec{a}| \left|\vec{b}\right|$$

Example
In $\mathbb{R}^n$:

$$< a_1, \ldots, a_n > \cdot < b_1, \ldots b_n$$
$$= \sum_{n=1}^{n} a_i b_i$$
$$\leq (\sum a_i)^{/frac12} (\sum b_j)^{\frac{1}{2}}$$

Example The functions that are square integrables on $I = [0, 1]$ for which we have:

$$\left|\int_0^1 fgdx\right| \leq \left|\int fdx\right| \left|\int gdx\right|$$

Proof Let recall that for parameterizing the line in beetwen $\vec{a}$ and $\vec{b}$ we could use the convex combination.

$$t\vec{b} + (1-t)\vec{a} \quad 0 \leq t \leq 1$$

$$\left|t\vec{b} + (1-t)\vec{a}\right|^2 > 0$$

We expand and we get: .... It is the same trick of consider a quadratic without root $a + \lambda b$. Geometrical Consideration:For two independent vectors, the line that pass through them do not pass on the origin.

4 C-S Consequences

The triangle inequality

$$\left|a \vec{+} \vec{b}\right| \leq |\vec{a}| + \left|\vec{b}\right|$$

the proof is given by considering the expansion of:

$$\left|\vec{a} + \vec{b}\right|$$

and the inequality: (Check!)
Definition of Angles

$$\cos(\theta) := \frac{\left|\vec{a} \cdot \vec{b}\right|}{|\vec{a}| \left|\vec{b}\right|}$$

5 Special (2×2) matrices

Scalar:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I$$

A - $2x2matricies$ A is orthogonal iff

$$\vec{a} \cdot \vec{b} = (A\vec{a}) \cdot (A\vec{b})$$

iff Preserves lengths, and angles.

Consider the matrix of rotation θ, and consider two rotations α y β, and the sum of these two, it will given then the following matrix. Observe that these shows explicitly the formulas of sum and sine cosine.

Consider the following computation $\cos(15\alpha)$

Definition 1. ***Conformal*** *preserves angles (but not conserve lengths)*

Remark: Orthogonal transformations preserve angles, and scalar conserve angles. Conformal are complex numbers. via matrices.

$$z = |z|$$

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6 Class 03

Notation

$$z, w \in \mathbb{C}$$

$$x, yin\mathbb{R} \quad \mathbb{C}$$

$$t \in \mathbb{R}$$

For this section all functions are well defined that is univalued and all functions are defined on an open set.

Definition 2. *U is an open set if:*

$$\forall x \in U \exists \epsilon > 0 \ s.t \ |y - x| < \epsilon \implies y \in U$$

Observe that for Ahlfors $\sqrt{x}$ is not a function.

Definition 3. *The function $f(x)$ "has a limit" A $asx \to a$ we write:*

$$\lim_{x \to a} f(x) = A$$

If $\forall \epsilon > 0 \exists \delta_\epsilon > 0$ *such that* $|x - a| < \delta_\epsilon \implies |f(x) - A| < \epsilon$

There are variance if A is an infinite the definitions will change accordingly.

Lemma 1. $f(x) = \sin(x)$ *Let show that* $\lim_{x\to 0} f(x) = 0$

Proof. Assuming given $\epsilon > 0$, need to find $\delta_\epsilon > 0$ s.t $x \in (-\delta_\epsilon, \delta_\epsilon)$ imples $\sin(x) \in (-\epsilon, \epsilon)$.

For any $\epsilon > 0$ take $\delta_\epsilon := \epsilon$ □

Definition 4. *A function $f(x)$ is continuous at $x = a$ if:*

$$\lim_{x\to a} f(x) = f(a)$$

Example $\sin(x)$ is continuous at $x = 0$

Proposition 1. *Let $f : \mathbb{C} \to \mathbb{C}$, $a, A \in \mathbb{C}$, then:*

$$\lim_{x\to a} f(x) = A$$
$$\lim_{x\to a} f(\bar{x}) = \bar{A}$$
$$\lim_{x\to a} \mathrm{Re}\,(f(x)) = \mathrm{Re}\,(A)$$
$$\lim_{x\to a} \mathrm{Im}\,(f(x)) = \mathrm{Im}\,(A)$$

A continuous function $f(x)$ is a function that is continuous at any x for which it is defined. So that for example $\frac{1}{x}$ is continuous.

Definition 5.

$$f'(a) := \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

, the difference quotient. $f(x)$ is differentiable at $x = a$ if $f'(a)$ exists.

If so,

$$f(x) \approx f(a) + (x - a)f'(a) + O((x - a))$$

Observe that the first two terms are the equation of the tangent. Observe that:

$$\frac{\epsilon(x-1)}{x-a} \to 0 \ \textit{as}\ x \to 0$$

Observe that if $f : \mathbb{R}^2 \to \mathbb{R}$, let us take $(a, b) \in \mathbb{R}^2$, if f is differentiable at $(a, b) \in \mathbb{R}^2$ then for (x, y) near (a, b), we have:

$$f(x, y) = f(a, b)) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) + o(||(x, y) - (a, b)||)$$

Let $f : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$ is differentiable at $z = a$ if:

$$f'(z) := \lim_{z\to a} \frac{f(z) - f(a)}{z - a}$$

exists. So observe:

$$f(z) = f(a) + f'(z)(z - a) + o()$$

Proposition 2. *Let us show that $f(z) = z^n$ is Complex differentiable for $z = a$, for all $a \in \mathbb{C}$*

Proof.

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

let $\xi = z - a$ so that

$$f'(a) = \lim_{\xi \to 0} \frac{(a + \xi)^n - a^n}{\xi}$$

and expanding using Newton binomial theorem. (and cancelling $\xi \neq 0$)

$$f'(a) = \lim_{\xi \to 0} (na^{n-1} + \text{ things that have xi})$$

so that :

$$f'(a) = na^{n-1}$$

implies

$$f'(z) = nz^{n-1}$$

□

Now for a pathological example. Let us take $f(z) = \mathrm{Re}\,(z)$ is not complex differentiable at any point. (Also imaginary part of Z is not complex differentiable)

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

again using $\xi = z - a$

$$f'(a) = \lim_{\xi \to 0} \frac{\mathrm{Re}\,(a + \xi) - \mathrm{Re}\,(a)}{\xi}$$

so that:

$$\lim_{\xi \to 0} \frac{\mathrm{Re}\,(\xi)}{\xi}$$

Observe that if $\xi = t \in \mathbb{R}$ then

$$\mathrm{Re}\,(\xi) / \xi = \frac{t}{t} = 1$$

Now, observe that if $\xi = it \in \mathbb{R}$ and

$$\frac{\mathrm{Re}\,(\xi)}{\xi} = \frac{0}{t} = 0$$

Definition 6. *A Complex function $f(z)$ and $f : \mathbb{C} \to \mathbb{C}$ is* ***analytic*** *or(holomorphic) at z if $f'(z)$ exists (the complex derivative)*

Theorem 1. ***Main thm*** *$f(z)$ is analytic at $z = a$ if and only if*

$$f(z) = \sum_{n=0}^{\inf} c_n (z - a)^n \textit{ for } |z - a| < \epsilon$$

Cauchy-Riemann equations

Let us take $f : \mathbb{C} \to \mathbb{C}$ and rewritten as $f(z = x + iy) = u(x, y) + iv(x, y)$ $u, v \in RR$ and assume $f'(z)$ exist as a complex derivative. Equivalently,

$$f'(z) = \lim_{\eta \to 0} \frac{f(z + \eta) - f(z)}{\eta} \quad \eta \in \mathbb{C}$$

then the above is equivalently, to the two reformulations:

$$\lim_{\eta \in \mathbb{R} = h \to 0} \frac{f(z + h) - f(h)}{h}$$

and has to be equal to:

$$\lim_{\eta \in i\mathbb{R}, \eta = ik \to 0, \quad k \in \mathbb{R}} \frac{f(z + ik) - f(z)}{ik}$$

so that with u, v

$$\lim_{h \to 0} \frac{u(x + h, y) + iv(x + h, y) - u(x, y) - iv(x, y)}{h} = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y)$$

and for the second part:

$$\lim_{k \to 0} \frac{u(x, y + k) + iv(x, y + k) - u(x, y) - iv(x, y + k)}{ik} = \frac{\partial v}{\partial y}(x, y) - i\frac{\partial u}{\partial y}(x, y)$$

and since them are the manifestation if the same limit, we have the equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

For remembering, check that the function $f(z) = z$ is complex differentiable. If we are calculating the derivative (computing stuff):

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}$$

now let us observe:

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$$

and this is the Jacobian.

$$\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) - \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right)$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix}$$

Which is the Jacobian.

Assume u, v have continuous 2nd partial derivatives, so that the mix exist are equal

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

then $u.v$ are harmoni, that is:

$$\Delta u = 0$$
$$\Delta v = 0$$

where Δ is the laplace operator:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and that solves:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0$$