Review of Differential Equations

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These notes contains the main methods for solutions of differential equations which were covered in the course *Ecuaciones Diferenciales* at Universidad de los Andes.

1 First Order Differential Equations

Definition 1. *A first order differential equation of the form:*

$$\frac{dy}{dx} = g(x)h(y) \tag{1}$$

is said to be separable or to have separable variables.

Theorem 1. *The solution for a differential equation that is separable can be found by the following method:*

$$\frac{dy}{dx} = g(x)h(y)$$
$$\frac{dy}{h(y)} = g(x)dx$$

So that the general solution can be found as:

$$\int \frac{dy}{h(y)} = \int g(x)dx \tag{2}$$

Definition 2. *A first order differential equation of the form:*

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(y) \tag{3}$$

and $a_1 \neq 0$ is said to be a linear equation in the variable y.

Remark Observe that the linear equation (3) can be transformed into a more known form of the linear equation:

$$\frac{dy}{dx} + P(x)y = f(x) \tag{4}$$

by dividing (3) by $a_1(x)$ since $a_1(x) \neq 0$

Theorem 2. *A linear differential equation of the form:*

$$\frac{dy}{dx} + P(x)y = f(x)$$

has a solution of the form:

$$y(x) = \frac{\int \mu(x) f(x) dx}{\mu(x)} \tag{5}$$

where $\mu(x)$ is defined as:

$$\mu(x) = e^{\int P(x)dx}$$

Remark In conclusion, observe that for solving a linear first order differential equations we take the following steps:

- Put linear equation into the form (4).
- Identify correctly $P(x)$ and find the integrating factor $\mu(x) = e^{\int P(x)dx}$. For easiness choose the integration constant to be zero.
- Use (5) to find the solution.

So before we define of what it is a exact differential equation, we need the definition of an exact differential from vector calculus.

Definition 3. *A differential expression $M(x,y)dx + N(x,y)dy$ is an* ***exact differential*** *if it corresponds to the differential of some function $f(x,y)$, that is, given a function of two variables, say $z = f(x,y)$, it happens that $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x,y)dx + N(x,y)dy$.*

Remark Observe that in the special case that we take a function of the form $f(x,y) = c$ where c is any real number, the differential of $f(x,y)$ is:

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

Remember that the derivative of a number is zero.

Example Let $f(x,y) = 5x^3 + 12xy + 5y^2 = c$, where c is any real number, the differential of f is $(15x^2 + 12y)dx + (12x + 10y) = 0$. Since $\frac{\partial f}{\partial x} = 15x^2 + 12y$, $\frac{\partial f}{\partial y} = 12x + 10y$, and $df = 0$ since $f(x,y) = c$ and the derivative of a number (in this case c) is zero. We say $(15x^2 + 12y)dx + (12x + 10y)$ is an exact differential, since is the differential of $f(x,y) = 5x^3 + 12xy + 5y^2 = c$

Definition 4. *A first order differential equation of the form*

$$M(x,y)dx + N(x,y)dy = 0 \tag{6}$$

is said to be an ***exact equation*** *if the expression on the left-hand side is an exact differential.*

So the question at this moment is: given an expression of the form $M(x,y)dx + N(x,y)dy$. How we know is an exact differential? The following theorem provides an answer.

Theorem 3. *Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives. Then a necessary and sufficient condition that $M(x,y)dx + N(x,y)dy$ be an exact differential is:*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{7}$$

So that to check a first order differential equation is an exact equation, put the differential equation into the form (6) and check if the partial derivatives are equal.

Now if the differential equation is an exact equation the solution is found with the following method:

Method of solution for an exact equation Given an equation in the form:

$$M(x, y)dx + N(x, y)dy = 0$$

First determine if the expression on the left hand side is an exact differential, *i.e* the left hand side complies with the equality on (7).

If it does, Observe that there exist a function f such that:

$$\frac{\partial f}{\partial x} = M(x, y)$$

So that, we can find f by integrating $M(x, y)$ with respect to x while holding y constant:

$$f(x, y) = \int M(x, y)dx + g(y)$$

Where $g(y)$ appears since it is the constant of integration (Observe that since we are integrating with respect to x the constant of integration can be any function that depends only on y).

Now, if we differentiate the previous equation with respect to y, and assume that $\frac{\partial f}{\partial y} = N(x, y)$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int M(x, y)dx + g'(y) = N(x, y)$$

Isolating $g'(y)$ and integrating we find $g(y)$

$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)dx \tag{8}$$

So that:

$$g(y) = \int (N(x, y) - \frac{\partial}{\partial y} \int M(x, y)dx)dy$$

And finally, we have the solution of the equation given by $f(x, y) = c$.

Before going to do some examples, let us remark two things in this method of solution. First, observe that the expression on (8) is independent of x, that because

$$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial}{\partial y} \int M(x, y)dx\right] = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} \int M(x, y)dx\right) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

So observe that the hypothesis that the differential expression is an exact differential is been used (that is why is so important to check for the exact differential criterion, that is, the equality (7) holds). Second, observe that we could have also started assuming that $\frac{\partial f}{\partial y} = N(x, y)$. After integrating N with respect to y and then differentiating that result, we would have found:

$$f(x, y) = \int N(x, y)dy + h(x) \quad \text{and} \quad h'(x) = M(x, y) - \frac{\partial}{\partial x} \int N(x, y)dy$$

which are analogous to the equations we discussed in the method of solution for exact equations; usually you usd the last two relations when if integrating with respect to x is rather difficult or impossible, whereas integrating to y is much more easy.

It may seem to the reader that the previous discussion is rather complex and complicated, but once she sees the examples, he would find exact equations easy:

Examples Insert examples here

Definition 5. *The differential equation*

$$\frac{dy}{dx} + P(x)y = f(x)y^n \tag{9}$$

*where n is any real number, is called **Bernoulli's equation.***

Observe that for the values $n = 0$ or $n = 1$, (9) is a linear equation for which we know how to solve. The following method of solution is used for when $n \neq 0, 1$. **Method of Solution for Bernoulli Equation**

Assume we have the following differential equation:

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Since $y^n \neq 0$ for $n \neq 0, 1$ we will divide y^n and rewrite as:

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = f(x) \tag{10}$$

Observe $\dfrac{y}{y^n} = y^{1-n}$.

The previous equation suggest that we should do the substitution $z = y^{1-n}$ (Also called the Bernoulli substitution), which implies:

$$\frac{dz}{dx} = (1-n) \cdot y^{-n}\frac{dy}{dx}$$

or in a more suggestive way

$$\left(\frac{1}{1-n}\right)\frac{dz}{dx} = y^{-n}\frac{dy}{dx}$$

so that (10) transforms into:

$$\left(\frac{1}{1-n}\right)\frac{dz}{dx} + P(x)z = f(x)$$

or

$$\frac{dz}{dx} + (1-n)\,P(x)z = (1-n)\,f(x)$$

Which is a linear equation of the form (4), for which we can find z via (5). Observe that work is not finished there, at the end we need to go our original variables with the relation $z = y^{1-n}$, that is $y = z^{1/1-n}$

2 Second Order Differential Equations

2.1 Classification of Differential Equations

With the same classification we use for first order differential equations, we classify the most general differential equation, that is, the subject of this section is the study of the differential equation of the form:

$$\frac{dx^2}{dt^2} = f(x, t, \frac{dx}{dt})$$

Where x is a function of t, that is $x(t)$ and f is an arbitrary function.

In general is rather impossible to solve analytically this equation, but there exist some cases to which is possible to find a solution using paper and pencil, the characterization of these equations is that f, in the equation above is a linear function on $\frac{dx}{dt}$, in other words:

$$f(x, t, \frac{dx}{dt}) = A(t) + B(t)\frac{dx}{dt} + C(t)x$$

Observe then, that A, B, and C are function of t only, the independent variable. The second linear order differential equations is most commonly discussed as:

$$\frac{dx^2}{dt^2} + p(t)\frac{dx}{dt} + q(t)x = g(t)$$

This is obtained above with $p(t) = -B(t), q(t) = -C(t)$, and $g(t) = A(t)$.
Observe that if you encounter the equation:

$$P(t)\frac{dx^2}{dt^2} + Q(t)\frac{dx}{dt} + R(t)x = G(t)$$

it can be put in the standard form by dividing the whole equation by $P(t)$.
with this discussion, we can start with a definition of second order linear differential equation.

Definition 6. *A* ***second order linear differential equation*** *is a differential relationship that can be put into the standard form:*

$$\frac{dx^2}{dt^2} + p(t)\frac{dx}{dt} + q(t)x = g(t)$$

An ***initial value problem*** *is an specification of the above equation given by:*

$$x(t_0) = x_o \quad and \quad \left.\frac{dx}{dt}\right|_{x=t_0} = x_1$$

We say that the second order linear differential equation is ***homogeneous*** *if* $g(t) = 0$ *and* ***nonhomgenous*** *if it is otherwise.*

Remark We often abbreviate the differential operator $\frac{d}{dx}$ with the use of the primer mark $'$, if the independent variable is clear from context, that is, the standard form of the second order differential equation is:

$$x'' + p(t)x' + q(t)x = g(t)$$

3 The Laplace Transform

Many problems encountered in physics and engineering involves forces that are represented by discontinuous functions, observe that many of the methods described above for solving differential equations are rather complicated or awkward to use, in this section, we will define and get acquainted with a tool that will allow us to solve a complicated differential equation, by translating the problem into an algebraic problem, which is more easy to solve.

3.1 Mathematical Background

The following section contains a small discussion on improper integrals and the existence theorems of the Laplace transform, as such, it should be glimpsed.

Definition 7. *An improper integral over an bounded interval is defined as a limit of integrals over finite intervals; thus*

$$\int_0^\infty f(t)dt = \lim_{A\to\infty} \int_a^A f(t)dt$$

if the limit exist, we say that the improper integral ***converges****. Otherwise we say that the integral* ***diverges****.*

The following definitions will be important for studying the existence of improper integrals.

Definition 8. *A function f is said to be* ***piecewise continuous*** *on an interval $\alpha \leq t \leq \beta$. If there exist a partition $P = \{\alpha = t_0 < t_1 < \dots t_n = \beta\}$ so that:*

- *f is continuous in each open subinterval (t_{i-1}, t_i)*
- *f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval*

In different words, we say that f is piecewise continuous on an interval, if it is continuous there except for a finite number of jump discontinuities (not easy to show). Now since we are studying the existence of the improper integral, observe that if the integral has a closed form solution, i.e, you can "evaluate" the integral, it is usually simple to see if the limit exist. Observe then, that we cannot say much of the existence of the limit, if we do not have a closed form solution. For circumventing this problem, we will make use of the following theorem that will allow us to compare and test improper integrals:

Theorem 4. *If f is piecewise continuous for $t \geq a$, if $|f(t)| \leq g(t)$ when $t \geq M$ for some positive constant M, and if $\int_M^\infty g(t)dt$ converges, then $\int_a^\infty f(t)dt$ also converges. On the other hand, if $f(t) \geq g(t) \geq 0$ for $t \geq M$, and if $\int_M^\infty g(t)dt$ diverges, then $\int_a^\infty f(t)dt$ also diverges*

The proof of this theorem can be found in any Calculus book, or it can be thoroughly studied in a real analysis class. However, observe that our geometric intuition can give us an image of why the preceding theorem is true, in the way of comparing areas of the functions. Now, we will see that the Laplace transform is an special transform of a larger set of integral transforms.

Definition 9. *An* ***integral transform*** *is a relation of the form:*

$$F(s) = \int_\alpha^\beta K(s,t)f(t)dt$$

Where K is a given function known as the ***kernel*** *of the transformation. In all cases, α and β are given, and are elements of the extended real line ($\mathbb{R} \cup \{-\infty, \infty\}$)*

Observe that the previous relation is acting on f, that is, we are giving an arbitrary function f and we are getting a different function F.

There are many useful integral transformation in applied mathematics, but for this section we will use the Laplace transform, which is defined by:

Definition 10. *The **Laplace transform** of a function f is given by:*

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

The general idea of using the Laplace transform is:

- Use the previous relation to transform a differential equation on f in the t domain and then transforming into an algebraic problem for the transform F in the variable s
- Solve the algebraic problem to find F
- Recover the function f from F, this last step is usually "inverting" the transform

The problem now is to investigate, when does the Laplace transform exist, the following theorem provides us an answer:

Theorem 5. *Suppose that*

- *f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A*
- *$|f(t)| \leq Ke^{at}$ where $t \geq M$. In this inequality, $K, a,$ and M are real constants, K and M necessarily positive*

Then the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$, defined by Definition 12, exist for $s > a$.

Proof. We want to see that the integral in Definition 12 converges for $s > a$. So observe that we can write the integral as:

$$\int_0^\infty e^{-st} f(t)dt = \int_0^M e^{-st} f(t)dt + \int_M^\infty e^{-st} f(t)dt \tag{11}$$

and we see that the first integral of the right hand side exists with the first of the hypotheses of the theorem. and now for the second integral, we observe that the second of the hypotheses we have, for $t \geq M$,

$$|e^{-st}| \leq Ke^{-st}e^{at} = Ke^{a-s}t$$

And thus by comparison (the comparison theorem in this section), we see that $F(s)$ exists provided that $\int_M^\infty e^{(a-s)t}dt$ converges. We see then that this integral only converges when $(a-s) < 0$, or equivalently $a < s$. Which establish the theorem. □

A few remarks are in order. First, as in the book, class, and this notes, we deal exclusively with functions that satisfy the conditions of the above theorem, although this may seem that we are reducing our library of functions, we can say that these functions will be sufficient for physics and applied mathematics (some obscure functions may appear, but they are the *pathological* examples created by pure mathematicians). Second, the functions that satisfy the above theorem are called as piecewise continuous and of **exponential order** as $t \to \infty$. (For more information on this subjects look up Big-O notation).

Now, we have just shown that the Laplace transform is an application from the set of functions that are of **exponential order** to the set of function of s-domain. Now we want to be able to define a function from functions of s-domain to the functions of exponential order. The following theorem provides us with a tool:

Theorem 6. ***(Informal Lerch's Theorem*** *- Uniqueness of Inverese Laplace Transforms). Suppose $f(t)$ and $g(t)$ are continuous on $[,\infty)$ and of exponential order γ. If $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$ for all $s > \gamma$, then $f(t) = g(t)$ for all $t \geq 0$*

Strictly speaking this theorem is false in general, but for many (mostly all of them) functions in applied mathematics, the theorem above is true. The formal version of the above theorem is:

Theorem 7. ***Lerch's Theorem.*** *If there are two functions $F_1(t)$ and $F_2(t)$ with the same integral transform:*

$$\mathcal{L}\{F_1(t)\} = \mathcal{L}\{F_2(t)\} \equiv f(s)$$

then a ***null function*** *can be defined by:*

$$\delta_0(t) = F_1(t) - F_2(t)$$

so that the integral

$$\int_0^a \delta_0(t)dt = 0$$

vanishes for all $a > 0$

With these theorems we can then define an **inverse Laplace transform**:

Definition 11. *Given a function f that has a Laplace transform, that is:*

$$\mathcal{L}\{f(t)\} = F(s)$$

We define the ***Inverse Laplace transform*** *as:*

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

Although this seem a rather empty definition, we do not provide a direct method to get Laplace inverse transforms, rather we just spent doing a lot of Laplace transforms and defining the inverse Laplace transform with the above definition

3.2 Using the Laplace Transform

We define the Laplace Transform as:

Definition 12. *The* ***Laplace transform*** *of a function f is given by:*

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

But as you will see, it is rather cumbersome to work with the definition, let us see that with an example. Suppose that we want to know the Laplace transform of $f(t) = \sin(at)$

Example Compute the Laplace transform of $f(t) = \sin(at)$ for $t \geq 0$.

$$\mathcal{L}\{\sin(at)\} = F(s) = \int_0^\infty e^{-st}\sin(at)dt, \quad s > 0$$

Equivalently,

$$F(s) = \lim_{A\to\infty} \int_0^A e^{-st}\sin(at)dt$$

We integrate by parts. Using $u = e^{-st}$ and $dv = \sin(at)$, which implies $du = -se^{-st}$ and $v = -\frac{\cos(at)}{a}$

$$
\begin{aligned}
F(s) = \lim_{A\to\infty} &\left[-\frac{e^{-st}\cos(at)}{a}\bigg|_0^A - \frac{s}{a}\int_0^A e^{-st}\cos(at)dt \right] \\
&= \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos(at)dt
\end{aligned}
$$

A second integration on the integral, with $u = e^{-st}$ and $dv = \cos(at)$, which implies $du = -se^{-st}$ and $v = \sin(at)$, yields

$$
\begin{aligned}
F(s) = \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin(at)dt \\
F(s) = \frac{1}{a} - \frac{s^2}{a^2}F(s)
\end{aligned}
$$

Finally, solving for $F(s)$

$$
F(s) = \frac{a}{s^2+a^2} \quad s > 0
$$

So observe that wasn't a very useful or easy work.Before we embark on our study of Laplace transform of functions, we will need the following theorem.

Theorem 8. *The Laplace transform is a* ***linear operator****, that is, the Laplace transform complies with:*

$$
\mathcal{L}\left\{c_1 f_1(t) + c_2 f_2(t)\right\} = c_1\mathcal{L}\left\{f_1(t)\right\} + c_2\mathcal{L}\left\{f_2(t)\right\}
$$

Where c_1, c_2 are real numbers (it even holds for complex numbers, which allows for neat tricks), and f_1, f_2 are functions whose Laplace transforms exist.

Proof. From the definition, Suppose c_1, c_2 are real numbers, and f_1, f_2 are functions such that the Laplace transform of these functions exists. Observe

$$
\begin{aligned}
\mathcal{L}\left\{c_1 f_1(t) + c_2 f_2(t)\right\} &= \int_0^\infty e^{-st}\left[c_1 f_1(t) + c_2 f_2(t)\right] dt \\
&= c_1\int_0^\infty e^{-st} f_1(t) + c_2\int_0^\infty e^{-st} f_2(t)
\end{aligned}
$$

Therefore, we get

$$
\mathcal{L}\left\{c_1 f_1(t) + c_2 f_2(t)\right\} = c_1\mathcal{L}\left\{f_1(t)\right\} + c_2\mathcal{L}\left\{f_2(t)\right\}
$$

Which is the desired result. Observe that this proof can be generalized to a arbitrary number of terms, that is:

$$
\mathcal{L}\left\{\sum_{i=1}^n f_i(t)\right\} = \sum_{i=1}^n \mathcal{L}\left\{f_i(t)\right\}
$$

□

Now, since this a course on differential equation, the main objective is to solve differential equations using the Laplace transform, the bridge will be given by the following theorem.
Remark From now on, we will assume that the functions given are of exponential order and piecewise continuous, this is made in order to simplify the theorems.

Theorem 9. *The Laplace transform of the nth-derivative of a function f, that is, $\mathcal{L}\left\{f^{(n)}(t)\right\}$ is given by:*

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\left\{f(t)\right\} - s^{n-1}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Remark Observe that for case $n = 2$ (the most encountered case in this course, worth memorizing) the theorem is telling us that:

$$\mathcal{L}\left\{f^{(2)}(t)\right\} = s^2\mathcal{L}\left\{f(t)\right\} - sf(0) - f(0)$$

Let us observe this in action with an example.

Consider the following differential equation:

$$\frac{d^2f}{dt^2} - \frac{df}{dt} - 2f(t) = 0$$

with the initial value conditions:

$$f(0) = 1, \quad \left.\frac{df}{dt}\right|_{t=0} = 0$$

Although, we already know a procedure to solve this particular differential equation, let us use the tool of Laplace transform. First, let us take the Laplace transform to both sides.

$$\mathcal{L}\left\{\frac{d^2f}{dt^2} - \frac{df}{dt} - 2f(t) = 0\right\} = \mathcal{L}\left\{0\right\}$$
$$\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} - \mathcal{L}\left\{\frac{df}{dt}\right\} - \mathcal{L}\left\{2f(t)\right\} = 0$$
$$\mathcal{L}\left\{f^{(2)}(t)\right\} - \mathcal{L}\left\{f^{(1)}(t)\right\} - 2\mathcal{L}\left\{f(t)\right\} = 0$$

And using our previous theorem we have:

$$s^2\mathcal{L}\left\{f(t)\right\} - sf(0) - \left.\frac{df}{dt}\right|_{t=0} - \left[s\mathcal{L}\left\{f(t)\right\} - f(0)\right] - 2\mathcal{L}\left\{f(t)\right\} = 0$$
$$s^2\mathcal{L}\left\{f(t)\right\} - sf(0) - f^{(1)}(0) - \left[s\mathcal{L}\left\{f(t)\right\} - f(0)\right] - 2\mathcal{L}\left\{f(t)\right\} = 0$$

Now using the initial value conditions and letting $\mathcal{L}\left\{f(t)\right\} = F(s)$ we have

$$\left(s^2 - s - 2\right)F(s) + (1 - s)f(0) - f^{(1)}(0) = 0$$

equivalently,

$$F(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}$$

Let's use partial fractions on the last inequality to get:

$$F(s) = \frac{s-1}{(s-2)(s+1)} = \frac{1/3}{s-2} + \frac{2/3}{s+1}$$

Now observe that we evaluated that: $\mathcal{L}\left\{\frac{1}{3}e^{2t}\right\} = \frac{1/3}{s-2}$. Similarly, $\mathcal{L}\left\{\frac{2}{3}e^{-t}\right\} = \frac{2/3}{s+1}$. so that:

$$\mathcal{L}\left\{\frac{1}{3}e^{2t}\right\} + \mathcal{L}\left\{\frac{2}{3}e^{-t}\right\} = F(s)$$
$$\mathcal{L}\left\{\frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}\right\} = F(s) = \mathcal{L}\left\{f(t)\right\}$$

Which implies then

$$f(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

So observe that, we have just transformed a differential equation problem into an algebraic one, and we just work out the "reverse" Laplace transform at the end. This is the power of the Laplace transform. This last part allows us to define and introduce then the **inverse Laplace transform**.

Definition 13. *Given a function f that has a Laplace transform, that is:*

$$\mathcal{L}\left\{f(t)\right\} = F(s)$$

*We define the **Inverse Laplace transform** as:*

$$\mathcal{L}^{-1}\left\{F(s)\right\} = f(t)$$

The existence and *well definedness* of the inverse is discussed above. Although this seem a rather empty definition, we do not provide a direct method to get Laplace inverse transforms, rather we just spent doing a lot of Laplace transforms and defining the inverse Laplace transform with the above definition. The following provides a table with the most encountered Laplace transform and (Laplace transform inverses).

Table of Laplace Transforms

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	
1	$\dfrac{1}{s}$	(1)
$e^{at}f(t)$	$F(s-a)$	(2)
$\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$	(3)
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	(4)
$\delta(t)$	1	(5)
$\delta(t-t_0)$	e^{-st_0}	(6)
$t^n f(t)$	$(-1)^n \dfrac{d^n F(s)}{ds^n}$	(7)
$f'(t)$	$sF(s) - f(0)$	(8)
$f^n(t)$	$s^n F(s) - s^{(n-1)} f(0) -$	
	$\cdots - f^{(n-1)}(0)$	(9)
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$	(10)
$t^n \ (n = 0, 1, 2, \dots)$	$\dfrac{n!}{s^{n+1}}$	(11)
$t^x \ (x \geq -1 \in \mathbb{R})$	$\dfrac{\Gamma(x+1)}{s^{x+1}}$	(12)
$\sin kt$	$\dfrac{k}{s^2+k^2}$	(13)
$\cos kt$	$\dfrac{s}{s^2+k^2}$	(14)
e^{at}	$\dfrac{1}{s-a}$	(15)
$\sinh kt$	$\dfrac{k}{s^2-k^2}$	(16)
$\cosh kt$	$\dfrac{s}{s^2-k^2}$	(17)
$\dfrac{e^{at}-e^{bt}}{a-b}$	$\dfrac{1}{(s-a)(s-b)}$	(18)

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	
$\dfrac{ae^{at}-be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$	(19)
te^{at}	$\dfrac{1}{(s-a)^2}$	(20)
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$	(21)
$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$	(22)
$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$	(23)
$e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$	(24)
$e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$	(25)
$t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$	(26)
$t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$	(27)
$t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$	(28)
$t\cosh kt$	$\dfrac{s^2-k^2}{(s^2-k^2)^2}$	(29)
$\dfrac{\sin at}{t}$	$\arctan \dfrac{a}{s}$	(30)
$\dfrac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$	(31)
$\dfrac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$	(32)
$\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$	(33)