Review of Differential Equations

David Cardozo

December 19, 2014

These notes contains the main methods for solutions of differential equations which were covered in the course *Ecuaciones Diferenciales* at Universidad de los Andes.

1 First Order Differential Equations

Definition 1 *A first order differential equation of the form:*

$$\frac{dy}{dx} = g(x)h(y) \tag{1}$$

is said to be separable or to have separable variables.

Theorem 1 *The solution for a differential equation that is separable can be found by the following method:*

$$\frac{dy}{dx} = g(x)h(y)$$
$$\frac{dy}{h(y)} = g(x)dx$$

So that the general solution can be found as:

$$\int \frac{dy}{h(y)} = \int g(x)dx \tag{2}$$

Definition 2 *A first order differential equation of the form:*

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(y) \tag{3}$$

and $a_1 \neq 0$ is said to be a linear equation in the variable y.

Remark Observe that the linear equation (3) can be transformed into a more known form of the linear equation:

$$\frac{dy}{dx} + P(x)y = f(x) \tag{4}$$

by dividing (3) by $a_1(x)$ since $a_1(x) \neq 0$

Theorem 2 *A linear differential equation of the form:*

$$\frac{dy}{dx} + P(x)y = f(x)$$

has a solution of the form:

$$y(x) = \frac{\int \mu(x) f(x) dx}{\mu(x)} \tag{5}$$

where $\mu(x)$ is defined as:

$$\mu(x) = e^{\int P(x)dx}$$

Remark In conclusion, observe that for solving a linear first order differential equations we take the following steps:

- Put linear equation into the form (4).
- Identify correctly $P(x)$ and find the integrating factor $\mu(x) = e^{\int P(x)dx}$. For easiness choose the integration constant to be zero.
- Use (5) to find the solution.

So before we define of what it is a exact differential equation, we need the definition of an exact differential from vector calculus.

Definition 3 *A differential expression $M(x,y)dx + N(x,y)dy$ is an* ***exact differential*** *if it corresponds to the differential of some function $f(x,y)$, that is, given a function of two variables, say $z = f(x,y)$, it happens that $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x,y)dx + N(x,y)dy$.*

Remark Observe that in the special case that we take a function of the form $f(x,y) = c$ where c is any real number, the differential of $f(x,y)$ is:

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

Remember that the derivative of a number is zero.

Example Let $f(x,y) = 5x^3 + 12xy + 5y^2 = c$, where c is any real number, the differential of f is $(15x^2 + 12y)dx + (12x + 10y) = 0$. Since $\frac{\partial f}{\partial x} = 15x^2 + 12y$, $\frac{\partial f}{\partial y} = 12x + 10y$, and $df = 0$ since $f(x,y) = c$ and the derivative of a number (in this case c) is zero. We say $(15x^2 + 12y)dx + (12x + 10y)$ is an exact differential, since is the differential of $f(x,y) = 5x^3 + 12xy + 5y^2 = c$

Definition 4 *A first order differential equation of the form*

$$M(x,y)dx + N(x,y)dy = 0 \tag{6}$$

is said to be an ***exact equation*** *if the expression on the left-hand side is an exact differential.*

So the question at this moment is: given an expression of the form $M(x,y)dx + N(x,y)dy$. How we know is an exact differential? The following theorem provides an answer.

Theorem 3 *Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives. Then a necessary and sufficient condition that $M(x,y)dx + N(x,y)dy$ be an exact differential is:*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{7}$$

So that to check a first order differential equation is an exact equation, put the differential equation into the form (6) and check if the partial derivatives are equal.

Now if the differential equation is an exact equation the solution is found with the following method:

Method of solution for an exact equation Given an equation in the form:

$$M(x,y)dx + N(x,y)dy = 0$$

First determine if the expression on the left hand side is an exact differential, $i.e$ the left hand side complies with the equality on (7).

If it does, Observe that there exist a function f such that:

$$\frac{\partial f}{\partial x} = M(x,y)$$

So that, we can find f by integrating $M(x,y)$ with respect to x while holding y constant:

$$f(x,y) = \int M(x,y)dx + g(y)$$

Where $g(y)$ appears since it is the constant of integration (Observe that since we are integrating with respect to x the constant of integration can be any function that depends only on y).

Now, if we differentiate the previous equation with respect to y, and assume that $\frac{\partial f}{\partial y} = N(x,y)$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)dx + g'(y) = N(x,y)$$

Isolating $g'(y)$ and integrating we find $g(y)$

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx \tag{8}$$

So that:

$$g(y) = \int (N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx)dy$$

And finally, we have the solution of the equation given by $f(x,y) = c$.

Before going to do some examples, let us remark two things in this method of solution. First, observe that the expression on (8) is independent of x, that because

$$\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx\right] = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\int M(x,y)dx\right) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

So observe that the hypothesis that the differential expression is an exact differential is been used (that is why is so important to check for the exact differential criterion, that is, the equality (7) holds). Second, observe that we could have also started assuming that $\frac{\partial f}{\partial y} = N(x, y)$. After integrating N with respect to y and then differentiating that result, we would have found:

$$f(x, y) = \int N(x, y)dy + h(x) \quad \text{and} \quad h'(x) = M(x, y) - \frac{\partial}{\partial x}\int N(x, y)dy$$

which are analogous to the equations we discussed in the method of solution for exact equations; usually you usd the last two relations when if integrating with respect to x is rather difficult or impossible, whereas integrating to y is much more easy.

It may seem to the reader that the previous discussion is rather complex and complicated, but once she sees the examples, he would find exact equations easy:

Examples Insert examples here

Definition 5 *The differential equation*

$$\frac{dy}{dx} + P(x)y = f(x)y^n \tag{9}$$

where n is any real number, is called ***Bernoulli's equation.***

Observe that for the values $n = 0$ or $n = 1$, (9) is a linear equation for which we know how to solve. The following method of solution is used for when $n \neq 0, 1$.

Method of Solution for Bernoulli Equation

Assume we have the following differential equation:

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Since $y^n \neq 0$ for $n \neq 0, 1$ we will divide y^n and rewrite as:

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = f(x) \tag{10}$$

Observe $\frac{y}{y^n} = y^{1-n}$.

The previous equation suggest that we should do the substitution $z = y^{1-n}$ (Also called the Bernoulli substitution), which implies:

$$\frac{dz}{dx} = (1 - n) \cdot y^{-n}\frac{dy}{dx}$$

or in a more suggestive way

$$\left(\frac{1}{1-n}\right)\frac{dz}{dx} = y^{-n}\frac{dy}{dx}$$

so that (10) transforms into:

$$\left(\frac{1}{1-n}\right)\frac{dz}{dx} + P(x)z = f(x)$$

or

$$\frac{dz}{dx} + (1-n)\,P(x)z = (1-n)\,f(x)$$

Which is a linear equation of the form (4), for which we can find z via (5). Observe that work is not finished there, at the end we need to go our original variables with the relation $z = y^{1-n}$, that is $y = z^{1/1-n}$

2 The Laplace Transform

Many problems encountered in physics and engineering involves forces that are represented by discontinuous functions, observe that many of the methods described above for solving differential equations are rather complicated or awkward to use, in this section, we will define and get acquainted with a tool that will allow us to solve a complicated differential equation, by translating the problem into an algebraic problem, which is more easy to solve.

2.1 Mathematical Background

The following section contains a small discussion on improper integrals and the existence theorems of the Laplace transform, as such, it can be omitted.

Definition 6 *An improper integral over an bounded interval is defined as a limit of integrals over finite intervals; thus*

$$\int_0^\infty f(t)dt = \lim_{A\to\infty} \int_a^A f(t)dt$$

if the limit exist, we say that the improper integral ***converges****. Otherwise we say that the integral* ***diverges****.*

The following definitions will be important for studying the existence of improper integrals.

Definition 7 *A function f is said to be* ***piecewise continuous*** *on an interval $\alpha \le t \le \beta$. If there exist a partition $P = \{\alpha = t_0 < t_1 < \ldots t_n = \beta\}$ so that:*

- *f is continuous in each open subinterval (t_{i-1}, t_i)*
- *f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval*

In different words, we say that f is piecewise continuous on an interval, if it is continuous there except for a finite number of jump discontinuities (not easy to show). Now since we are studying the existence of the improper integral, observe that if the integral has a closed form solution, i.e, you can "evaluate" the integral, it is usually simple to see if the limit exist. Observe then, that we cannot say much of the existence of the limit, if we do not have a closed form solution. For circumventing this problem, we will make use of the following theorem that will allow us to compare and test improper integrals:

Theorem 4 *If f is piecewise continuous for $t \geq a$, if $|f(t)| \leq g(t)$ when $t \geq M$ for some positive constant M, and if $\int_M^\infty g(t)dt$ converges, then $\int_a^\infty f(t)dt$ also converges. On the other hand, if $f(t) \geq g(t) \geq 0$ for $t \geq M$, and if $\int_M^\infty g(t)dt$ diverges, then $\int_a^\infty f(t)dt$ also diverges*

The proof of this theorem can be found in any Calculus book, or it can be thoroughly studied in a real analysis class. However, observe that our geometric intuition can give us an image of why the preceding theorem is true, in the way of comparing areas of the functions. Now, we will see that the Laplace transform is an special transform of a larger set of integral transforms.

Definition 8 *An **integral transform** is a relation of the form:*

$$F(s) = \int_\alpha^\beta K(s,t)f(t)dt$$

*Where K is a given function known as the **kernel** of the transformation. In all cases, α and β are given, and are elements of the extended real line ($\mathbb{R} \cup \{-\infty, \infty\}$)*

Observe that the previous relation is acting on f, that is, we are giving an arbitrary function f and we are getting a different function F.

There are many useful integral transformation in applied mathematics, but for this section we will use the Laplace transform, which is defined by:

Definition 9 *The **Laplace transform** of a function f is given by:*

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)dt$$