Review of Differential Equations

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These notes contains the main methods for solutions of differential equations which were covered in the course *Ecuaciones Diferenciales* at Universidad de los Andes.

1 First Order Differential Equations

Definition 1 *A first order differential equation of the form:*

$$\frac{dy}{dx} = g(x)h(y) \tag{1}$$

is said to be separable or to have separable variables.

Theorem 1 *The solution for a differential equation that is separable can be found by the following method:*

$$\frac{dy}{dx} = g(x)h(y)$$
$$\frac{dy}{h(y)} = g(x)dx$$

So that the general solution can be found as:

$$\int \frac{dy}{h(y)} = \int g(x)dx \tag{2}$$

Definition 2 *A first order differential equations of the form:*

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(y) \tag{3}$$

and $a_1 \neq 0$ is said to be a linear equation in the variable y.

Remark Observe that the linear equation (3) can be transformed into a more known form of the linear equation:

$$\frac{dy}{dx} + P(x)y = f(x) \tag{4}$$

by dividing (3) by $a_1(x)$ since $a_1(x) \neq 0$

Theorem 2 *A linear differential equation of the form:*

$$\frac{dy}{dx} + P(x)y = f(x)$$

has a solution of the form:

$$y(x) = \frac{\int \mu(x)f(x)dx}{\mu(x)} \tag{5}$$

where $\mu(x)$ is defined as:

$$\mu(x) = e^{\int P(x)dx}$$

Remark In conclusion, observe that for solving a linear first order differential equations we take the following steps:

- Put linear equation into the form (4).
- Identify correctly $P(x)$ and find the integrating factor $\mu(x) = e^{\int P(x)dx}$. For easiness choose the integration constant to be zero.
- Use (5) to find the solution.

So before we define of what it is a exact differential equation, we need the definition of an exact differential from vector calculus.

Definition 3 *A differential expression $M(x,y)dx+N(x,y)dy$ is an **exact differential** if it corresponds to the differential of some function $f(x,y)$, that is, given a function of two variables, say $z = f(x,y)$, it happens that $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x,y)dx + N(x,y)dy$.*

Remark Observe that in the special case that we take a function of the form $f(x,y) = c$ where c is any real number, the differential of $f(x,y)$ is:

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

Remember that the derivative of a number is zero.

Example Let $f(x,y) = 5x^3 + 12xy + 5y^2 = c$, where c is any real number, the differential of f is $(15x^2+12y)dx+(12x+10y) = 0$. Since $\frac{\partial f}{\partial x} = 15x^2+12y$, $\frac{\partial f}{\partial y} = 12x+10y$, and $df = 0$ since $f(x,y) = c$ and the derivative of a number (in this case c) is zero. We say $(15x^2+12y)dx+(12x+10y)$ is an exact differential, since is the differential of $f(x,y) = 5x^3 + 12xy + 5y^2 = c$

Definition 4 *A first order differential equation of the form*

$$M(x,y)dx + N(x,y)dy = 0 \tag{6}$$

*is said to be an **exact equation** if the expression on the left-hand side is an exact differential.*

So the question at this moment is: given an expression of the form $M(x,y)dx+N(x,y)dy$. How we know is an exact differential? The following theorem provides an answer.

Theorem 3 *Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives. Then a necessary and sufficient condition that $M(x,y)dx + N(x,y)dy$ be an exact differential is:*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{7}$$

So that to check if a differential