An Introduction to Computational Group Theory (GAP)

Groups, Algorthims, and Programming

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What is GAP?

1. **DIEHTLYAPL**

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GAP

Group, Algorithms, and Programming. Computer algebra system **CAS**

Terminal Tool

Read input $\rightarrow$ Evaluate $\rightarrow$ Print.

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Research Tool

You shape it to your needs.

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Computational Group Theory

The study of algorithms for groups. Produce algorithms to answer questions about concrete groups: combinatorial structures.

Can we calculate the objects we define theoretically ?

Questions Concrete Groups

Complexity Theory

P=NP

Graph Isomorphism [Solved] Travel

Recall GAP is a command line tool

```
~$ mkdir ClassTutorial
~$ cd ClassTutorial/
~/ClassTutorial$ gap
 ┌───────┐   GAP, Version 4.7.5
 │  GAP  │   http://www.gap-system.org
 └───────┘   Architecture: x86_64...
Libs used:   gmp, readline
Loading the library and packages ...
Components: trans 1.0, prim 2.1, ...
Packages:    Alnuth 3.0.0, AtlasRep 1.5.0 ...
Try '?help' for help.
gap>
```

The GAP console

Internal types of data structures: Integers are built in, Boolean Values: true, false. And also Gap doesn't need to declare types of variables

```
$ gap>TeachingMode(true);
#I Teaching Mode is turned ON
$ gap>8=9;
false
gap> 1^13 + 12^3 = 9^3 + 10^3;
true
gap> FirstPerfectNumber := 6;
6
gap> 22+FirstPerfectNumber;
28
```

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Functions

GAP comes with a lot of Built-in functions that are commonly used in Group Theory, but it also provide ways to define our own functions.

Question

Is $2^{13} - 1$ a prime number ?

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Question

Is $2^{13}-1$ a prime number ?

```
gap> IsPrime(2^13 -1);
true
```

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Functions

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Question

Is $2^{13} - 1$ a prime number ?

```
gap> IsPrime(2^13 -1);
true
```

Let us define our own functions:

```
gap> AddOne := function(x) return x+1; end;
function( x ) ... end
```

Shortcut: `AddOne:= (x -> x+1);`

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Lists

Sometimes we will a “container” a list structure to keep data. Built-in functions include: Test Membership, and Long List Constructor:

```
gap> L1 := [4,5,6,7,8,12];
[ 4, 5, 6, 7, 8, 12 ]
gap> Position(L1,7);
gap> List([1..10],x->x);
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 ]
gap> List([5,9..45],x->x);
[ 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45 ]
gap> List([20..29],x->x^2);
[ 400, 441, 484, 529, 576, 625, 676, 729, 784, 841 ]
```

Bad Feature on GAP: **List themselves are pointers to lists**, for copying lists(vector or matrices) `ShallowCopy`

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Vectors and Matrices

```
gap> vec := [-1,2,1];
[ -1, 2, 1 ]
gap> M:=[[1,2,3],[4,5,6],[7,8,9]];
[ [ 1, 2, 3 ], [ 4, 5, 6 ], [ 7, 8, 9 ] ]
gap> Display(M);
[ [  1,  2,  3 ],
[  4,  5,  6 ],
[  7,  8,  9 ] ]
gap> vec*M;
[ 14, 16, 18 ]
gap> M*vec;
[ 6, 12, 18 ]
gap> M[3][2];
8
gap> vec*vec; #The inner product
6
```

While vectors in GAP are usually considered as row vectors, scalar products or matrix/vector products automatically consider the second factor as column vector.

Hill Cipher

Plain text is divided into sets of *n* letters, each of which is replaced by a set of *n* cipher letters is called a **polygraphic system**.

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*	*M*	*N*	*O*	*P*	*Q*	*R*	*S*	*T*	*U*	*V*	*W*	*X*	*Y*	*Z*
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0

Theorem

A square matrix *A with entries in* $\mathcal{Z}_m$ *is invertible modulo m if and only if the residue of* $\det(A)$ *modulo m has a reciprocal modulo m*

Corollary

A *square matrix A* with entires in $\mathcal{Z}_{26}$ is invertible modulo 26 if and only if the residue of $\det(A)$ mod 26 is not divisible by 2 or 13

Algorithm for the Hill Cipher

1. Choose a 2×2 matrix with integers:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

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3. Successively convert each plaintext pair into a column vector:

$$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Form the product $A\vec{p}$ the **ciphertext vector**

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   Form the product $A\vec{p}$ the **ciphertext vector**

4. Convert each ciphertext vector into its alphabetic equivalent

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Message to encode: **I AM HIDING**, using the matrix:

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

I A	M H	I D	I N	G G
9 1	13 8	9 4	9 14	7 7

```
gap> im:=Integers mod 26; # Represent numbers for mod 26
gap> A := [[1,2],[0,3]]*One(im);
gap> p1 := [9,1]*One(im);
gap> p2 := [13,8]*One(im);
gap> p3 := [9,4]*One(im);
gap> p4 := [9,14]*One(im);
gap> p5 := [7,7]*One(im);
```

Hill Cipher

Hell 2-cipher

```
gap> Result := List([A*p1,A*p2,A*p3,A*p4,A*p5],x->x);
gap> Display(Result);
matrix over Integers mod 26:
[ [  11,   3 ],
[   3,  24 ],
[  17,  12 ],
[  11,  16 ],
[  21,  21 ] ]
```

11 3	3 24	17 12	11 16	21 21
K C	C X	Q L	K P	U U

So we will transmit the following:

KCCXQLKPUU

Breaking a Hill Cipher

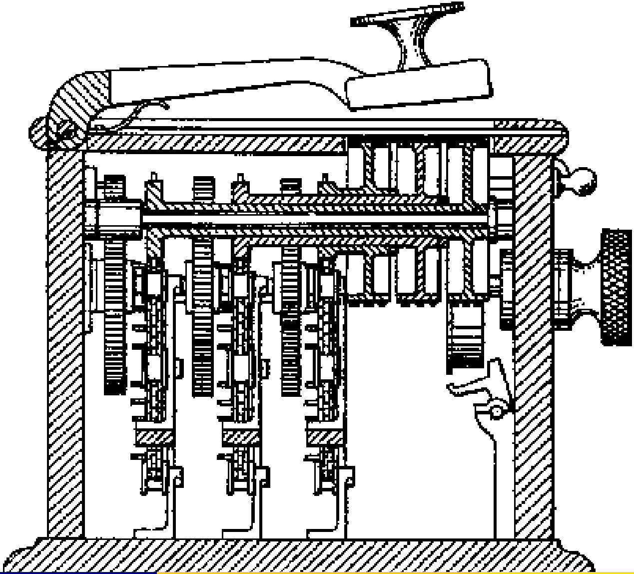

Figure: Polish Card

Creation of the Republic of Poland, was cumbersome at best. In 1934, a card was intercept from Galicia from a Polish Army officer (some sites mention Edward Rydz-Śmigły) to Warsaw.

IOSBTHXESPXHOPDE

It was customary in Poland to star a letter with DEAR

It is a basic result in linear algebra that a linear transformation is completely determined by its values at a basis. Thus suggests that if we have a Hill n-cipher, and if $\vec{p}_1, \ldots \vec{p}_n$ are linearly independent plaintext vectors whose corresponding cipher vectors: $A\vec{p}_1, \ldots, A\vec{p}_n$ are known, then there is enough information available to determine the matrix A and hence $A^{-1} \mod 26$

Theorem

Let $\vec{p}_1, \ldots \vec{p}_n$ be linearly independent plaintext vectors, and let $\vec{c}_1, \ldots, \vec{c}_n$ be the corresponding ciphertext vectors in a Hill n-cipher if:

$$P = \begin{pmatrix} \vec{p}_1^T \\ \vdots \\ \vec{p}_n^T \end{pmatrix} \quad C = \begin{pmatrix} \vec{c}_1^T \\ \vdots \\ \vec{c}_n^T \end{pmatrix}$$

Where both are $n \times n$, then the sequence of elementary row operations that reduces C to I transforms P to $(A^{-1})T$

Solution to Problem using Gap

```
gap> G:= CyclicGroup(6);
<fp group of size 6 on the generators [ a ]>
gap> Elements(G);
[ <identity ...>, a, a^2, a^3, a^4, a^5 ]
gap> List(Elements(G),Order);
[ 1, 6, 3, 2, 3, 6 ]
gap> ShowMultiplicationTable(G);
*     | <id> a    a^2  a^3  a^4  a^5
-----+------------------------------
<id> | <id> a    a^2  a^3  a^4  a^5
a    | a    a^2  a^3  a^4  a^5  <id>
a^2  | a^2  a^3  a^4  a^5  <id> a
a^3  | a^3  a^4  a^5  <id> a    a^2
a^4  | a^4  a^5  <id> a    a^2  a^3
a^5  | a^5  <id> a    a^2  a^3  a^4
```

Abelian Group

The Fundamental Theorem Of Finite Abelian Groups

A finite Abelian group is isomorphic to a direct product of cyclic groups of prime-power order

```
gap> G:=AbelianGroup([2,4,5]);
<fp group of size 40 on the generators [ f1, f2, f3 ]>
gap> gens:= GeneratorsOfGroup(G);
[ f1, f2, f3 ]
gap> List(gens,Order);
[ 2, 4, 5 ]
```

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Working with: $\left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^{\times}$

```
gap> G:= Units(Integers mod 21);
<group of size 12 with 2 generators>
gap> e:=Elements(G);
[ ZmodnZObj( 1, 21 ), ZmodnZObj( 2, 21 ), ZmodnZObj( 4, 21
ZmodnZObj( 5, 21 ), ZmodnZObj( 8, 21 ), ZmodnZObj( 10, 21 )
ZmodnZObj( 11, 21 ), ZmodnZObj( 13, 21 ), ZmodnZObj( 16, 21
ZmodnZObj( 17, 21 ), ZmodnZObj( 19, 21 ), ZmodnZObj( 20, 21
gap> List(e,Order);
[ 1, 6, 3, 6, 2, 6, 6, 2, 3, 6, 6, 2 ]
```

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ZmodnZObj( 11, 21 ), ZmodnZObj( 13, 21 ), ZmodnZObj( 16, 21
ZmodnZObj( 17, 21 ), ZmodnZObj( 19, 21 ), ZmodnZObj( 20, 21
gap> List(e,Order);
[ 1, 6, 3, 6, 2, 6, 6, 2, 3, 6, 6, 2 ]
```

Interpret:

```
List(e,x->Position(e,Inverse(x)));
```

Finally: `gap> ShowGcd(5,9)`

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Dihedral Group and Advanced functions

This will be short, just to show a few advanced commands:

```
gap> D3 := DihedralGroup(6);
<fp group of size 6 on the generators [ r, s ]>
gap> D4:=DihedralGroup(8);
<fp group of size 8 on the generators [ r, s ]>
gap> List([D3,D4],x->IsSolvable(x));
[ true, true ]
gap> List([D3,D4],x->IsNilpotent(x));
[ false, true ]
```

We can then compute the multiplication table:

```
gap> ShowMultiplicationTable(DihedralGroup(6));
*    | <id> r^-1 r    s    r*s  s*r
-----+------------------------------
<id> | <id> r^-1 r    s    r*s  s*r
r^-1 | r^-1 r    <id> s*r  s    r*s
r    | r    <id> r^-1 r*s  s*r  s
s    | s    r*s  s*r  <id> r^-1 r
r*s  | r*s  s*r  s    r    <id> r^-1
s*r  | s*r  s    r*s  r^-1 r    <id>
```

Subgroups

Subgroups in GAP are created and stored by giving generators of the subgroup. The function `Subgroup(group,generators)` constructs a subgroup with given generators. Compared with `Group`, GAP tests whether the generators are actually in the group.

The commands `Normalizer(group,sub)` and `Centralizer(group,sub)` return associated subgroups.

The command `AllSubgroups` should be used with caution:

```
gap> D6:=DihedralGroup(IsPermGroup,6);
Group([ (1,2,3), (2,3) ])
gap> AllSubgroups(D6);
[ Group(()), Group([ (2,3) ]), Group([ (1,2) ]),
  Group([ (1,3) ]),
  Group([ (1,2,3) ]),
  Group([ (1,2,3), (2,3) ]) ]
```

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Subgroup Lattice

GAP provides very general functionality to determine the subgroup structure of a group. To reduce storage this is typically done up to conjugacy. `ConjugacyClassesSubgroups(G)` returns a list of conjugacy classes of subgroups of *G*. For each class *C* in this list `Representative(C)` returns one subgroup in this class. `Stabilizer(C)` returns the normalizer of this Representative in *G*. `Size(C)` returns the number of conjugate subgroups in the class (the index of the normalizer). Last but not least, `NormalSubgroups(G)` returns a list of all normal subgroups.

Subgroup Lattice II

`LatticeSubgroups(G)` determines an object L that represents the lattice of subgroups of G .(The classes of subgroups can be also obtained from this object as `ConjugacyClassesSubgroups(L)`.) For such a lattice object, `MaximalSubgroupsLattice(L)` returns a list *M* that describes maximality inclusion.

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Activity

Let's draw the lattice of S_4 using GAP

```
gap> L:=LatticeSubgroups(G);
DotFileLatticeSubgroups(L,"tester.dot");
```

This will produce a dot file that can be drawn with:

```
dot -Teps tester.dot > output.eps
```

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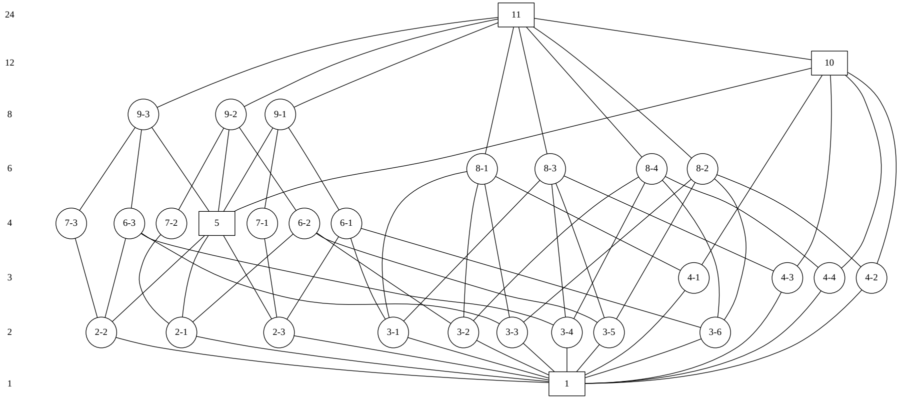
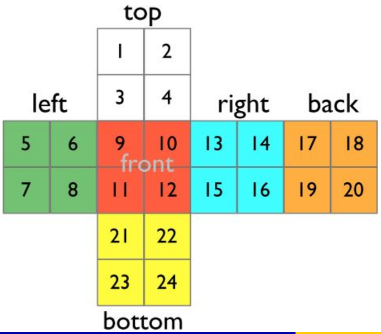


Figure: Lattice of S_4



Many puzzles can be described in this way: Each state of the puzzle corresponds to a permutation, the task of solving the puzzle then corresponds to expressing the permutation as a product of generators.