# Textbook notes on Probability

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The following are notes based on the book *Measures, Integrals and Martingales* by Schilling.

A word of caution:

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

Positive or negative is always understood in non-strict sense $\geq 0$ or $\leq 0$; to exclude 0, the author explicitly says strictly.

# Chapter 1

# Prologue

The least we should expect from a reasinable measure $\mu$ is that it is:

Well-defined, takes values in $[0, \infty]$, and $\mu(\emptyset) = 0$;
additive, i.e $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$.
Is invariant under congruences.

When dealing with advanced stuff the finite additivity is not enough for this and we have to use a property called ($\sigma$ - additivity):

$$\text{area}(\bigcup_{j \in N})A_j = \sum_{j \in N} \text{area}(A_j)$$

Where $A_j$ is the disjoint union of the sets, i.e the sets $A_j$ are pairwise disjoint.

## 1.1 The Pleasures of counting

Throughout this section $X$ and $Y$ denote two arbitrary sets. For any two sets $A, B$ we write:

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ and } B\}$$
$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$
$$A \backslash B = \{x : x \in A \text{ and } x \notin B\};$$

for this book $A \subset B$ means $A$ is contained in $B$ without excluding the possibility that $A = B$; for the latter we write $A \subsetneq B$. If $A \subset X$, we set $A^c := X \backslash A$ for the complement of $A$ (relative to $X$).

Recall the distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

and de Morgan's identities:

$$(A \cap B)^c = A^c \cup B^c$$

$$(A \cup B)^c = A^c \cap B^c$$

Observe that these identities also hold for arbitrarily many sets $A_i \subset X, i \in I$ (I stands for an arbitrary index set),

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$$

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$$

A map $f : X \to Y$ is called:

$$\begin{gathered} \textit{injective} \text{ (or one-one) } \iff f(x) = f(x)' \implies x = x' \\ \textit{Surjective} \text{ (or onto ) } \iff f(X) := f(x) \in Y : x \in X = Y \\ \textit{Bijective} \iff \text{is injective and surjective.} \end{gathered}$$

Set operations and direct images under a map $f$ are not in general necessarily compatible, that is

$$\begin{aligned} f(A \cup B) &= f(A) \cup f(B) \\ f(A \cap B) &\neq f(A) \cap f(B) \\ f(A \backslash B) &\neq f(A) \backslash f(B) \end{aligned}$$

However Inverse images and set operation are always compatible. For $C, C_i, D \subset Y$ one has:

$$\begin{aligned} f^{-1}\left(\bigcup_{i \in I} C_i\right) &= \bigcup_{i \in I} f^{-1}(C_i) \\ f^{-1}\left(\bigcap_{i \in I} C_i\right) &= \bigcap_{i \in I} f^{-1}(C_i) \\ f^{-1}(C \backslash D) &= f^{-1}(C) \backslash f^{-1}(D) \end{aligned}$$