Topics in Algebra

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These are few notes and compilation of exercises of the book *Topics in Algebra.* by I. N. Herstein.

1 Preliminary notions

1.1 Set Theory

Given a set S we shall use the notation throughout $a \in S$ to read "*a is an element of S*". The set A will be said to be a *subset* of S if every element in A is an element of S, We shall write $A \subset S$. Two sets A and B are equal, if both $A \subset B$ and $B \subset A$. A set D will be called *proper subset* of S if $D \subset S$ but $D \neq S$. The null set is the set having no elements; it is a subset of every set. Given a set S we shall use the notation $A \{a \in S | P(a)\}$ to read " A is the set of all elements in S for which the property P holds.

Definition 1. *The **union** of the two sets A and B, written as $A \cup B$, is the set $\{x | x \in A \ or \ \in B\}$*

Remark when we say that x is in A or x is in B, we mean x is in at least one of A or B, and may be in both.

Definition 2. *The **intersection** of the two sets A and B, written as $A \cap B$, is the set $\{x | x \in A \ and \ x \in B\}$*

Two sets are said to be disjoint *if their intersection is empty.*

Proposition 1. *For any three sets, A, B, C we have*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof. We will prove first $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. Observe $B \subset B \cup C$, so that $A \cap B \subset A \cap (B \cup C)$, in the same line of reasoning, $C \subset B \cap C$, so that $A \cap C \subset A \cap (B \cap C)$, and we conclude $(A \cap B) \cup (A \cap C) \subset (A \cap (B \cup C)) \cup (A \cap (B \cup C)) = A \cap (B \cup C)$. Now for the other direction, let $x \in A \cap (B \cup C)$, so that $x \in A$, and $x \in B \cup C$; suppose the former, and we have that $x \in (A \cap B)$. The second possibility, namely, $x \in C$, implies that $x \in A \cap C$. Thus in either case, $x \in (A \cap C) \cup (A \cap B)$, whence $A \cap (B \cup C) \subset (A \cap B)\,(A \cap C)$. Combining the two concluding assertions, they give us the equality of both sets $\square$

Given a set T, we say that T serves as an *index set* for the family $F = \langle A_\alpha \}$ of sets if for every $\alpha \in T$, there exist a set of A_α in the family F. By the iunion of the set A_α, where α is in T, we mean the set $\{x | x \in A_\alpha$ for at least one $\alpha \in T\}$. We denote it by $\cup_{\alpha \in T} A_\alpha$. Similarly, we denote the intersection of the sets A_α by $\cap_{\alpha \in T} A_\alpha$. The sets A_α are mutually disjoint if for $\alpha \neq \beta$, $A_\alpha \cap A_\beta$ is the null set.

Definition 3. *Given the two sets A, B then the* ***difference set****, $A - B$, is the set $\{x \in A | x \notin N\}$*

Proposition 2. *For any set B, the set A satisfies*

$$A = (A \cap B) \cup (A - B).$$

Proof. Again, using the same strategy used before, we will show first $(A \cap B) \cup (A - B) \subset A$, Observe that $A \cap B \subset A$, and $A - B \subset A$ so that $(A \cap B) \cup (A - B) \subset A$ Now, for the converse, we want to see $A \subset (A \cap B)$, first observe that if we suppose that $x \in A$, then either $x \in (A - B)$ or $x \in A \cap B$, so that in either case, eventually $x \in ((A \cap B) \cup (A - B))$, so we conclude $A \subset (A \cap B)$. Finally, combining the concluding assertions we have $A = (A \cap B)$ which proves our proposition. $\square$

Observe $B \cap (A - B)$ is the null set. Observe than when $B \subset A$, we call $A - B$ the complement of B in A.