Preliminary on Topology

David Cardozo

February 8, 2015

Definition 1. *A* ***rule of assignment*** *is a subset r of the cartesian product $C \times D$ of two sets, having the property that each element of C appears as the first coordinate of at most one ordered pair belonging to r*

An equivalent formulation is

$$[(c, d) \in r \text{ and } (c, d') \in r] \implies [d = d']$$

Definition 2. *a* ***function*** *f is a rule of assignment r, together with a set B that contains the image set of r . The domain A of the rule r is also called the* ***domain*** *of the function f; the image set of r is also called the* ***image set*** *of f; and the set b is called the* ***range*** *of f*

Definition 3. *If $f : A \to B$ and if A_0 is a subset og A, we define the* ***restriction*** *of f to A_0 to be the function mapping A_0 into B whose rule is:*

$$\{(a, f(a)) | a \in A_0\}$$

It is denoted by $f \restriction_{A_0}$.

Definition 4. *Given functions $f : A \to B$ and $g : B \to C$, we define the* ***composite*** *$g \circ f$ of f and g as the function $g \circ f : A \to C$ defined by the equation $(g \circ f)(a) = g(f(a))$. Formally, $g \circ f : A \to C$ is the function whose rule is:*

$$\{(a, c) | \textit{For some } b \in B, f(a) = b \textit{ and } g(b) = c\}$$

We take note that $g \circ f$ is defined only when the range of f equals the domain of g

Definition 5. *A function $f : A \to B$ is said to be* ***injective*** *(or* ***one-to-one****) if for each pair of distinct points of A, their images under f are distinct. It is said to be* ***surjective*** *(or f is said to map A* ***onto*** *B) if every element of B is the image of some element of A under the function f, If f is both injective and surjective, it is said to be* ***bijective***

An important remark of facts is that, the composite of two surjective functions is surjective, and the composite of two injective functions is injective.

If f is bijective there exists a function from B to A called the **inverse** of f. It is denoted by f^{-1} and is defined by letting $f^{-1}(b)$ be that unique element a of A for which $f(a) = b$.

Lemma 1. *Let $f : A \to B$. If there are functions $g : B \to A$ and $h : B \to A$ such that $g(f(a)) = a$ for every a in A, and $f(h(b)) = b$ for every b in B, then f is bijective and $g = h = f^{-1}$*

Proof. To prove that f is surjective, observe that the function h that goes from B to A maps every element to $h(b) \in A$ so that, every element of b has an element of A such that $f(h(b)) = b$. To prove that it is injective, suppose for the sake of contradiction that there exists elements of A such that $a \neq a'$ but $f(a) = f(a')$ then, applying g to both sides of this equation we have that $g(f(a)) = g(f(a'))$ and by the conditions of the hypothesis we have that $a = a'$ which is a contradiction. To see that both are equal observe that:

$$\begin{aligned} h(b) &= g(f(h(b))) \\ &= g(b) \end{aligned}$$

and the proof is complete. □

Definition 6. *Let $f : A \to B$. If A_0 is a subset of A, we denote by $f(A_0)$ the set of all images of points of A_0 under the function f; this set is called the* ***image*** *of A_0 under f. Formally,*

$$f(A_0) = \{b | b = f(a) \text{ for at least one } a \in A_0\}$$

On the other hand, if B_0 is a subset of B, we denote by $f^{-1}(B_0)$ the set of all elements of A whose images under f lie in B_0; it is called the ***preimage*** *of B_0 under f (or the "counterimage" or the "inverse image" of B_0). Formally,*

$$f^{-1}(B_0) = \{a | f(a) \in B_0\}$$

Of course, there may be no points aof A whose images lie in B_0; in that case, $f^{-1}(B_0)$ is empty.

We make the following remark, and that is that: If $f : A \to B$ and if $A_0 \subset A$ and $B_0 \subset B$, then:

$$A_0 \subset f^{-1}(f(A_0)) \quad \text{and} \quad f(f^{-1}(B_0)) \subset B_0$$

The first inclusion is an equality if f is injective, and the second inclusion is an equality if f is surjective.

Relations

Definition 7. *A* ***relation*** *on a set A is a subset C of the Cartesian product $A \times A$*

Equuivalence Relations and Partitions An **equivalence relation** on a set A is a relation C on A having the following three properties:

1. (Reflexivity) xCx for every $x \in A$
2. (Symmetry) If xCy, then yCx
3. (Transitivity) If xCy and yCz, then xCz

Given an equivalent realtion $\sim$ on a set A and an element x of A, we define a certain subset E of A, called the **equivalence class** determined by x, via:

$$E = \{y|y \sim x\}$$

Lemma 2. *Two equivalence classes E and E' are either disjoint or equal*

Proof. Let E be the equivalence class determined by x, and let E' be the equivalence class determined by x'. Suppose that $E \cap E'$ is not empty; let y be a point of $E \cap E'$, observe that we have $y\ x$ and $y\ x'$ so that we conclude that $x\ x'$. If now w is any point of E, we have $w\ x$ so that we conclude $E \subset E'$, by symmetry of the argument we see that the equality holds. $\square$

Definition 8. *A* ***partition*** *of a set A is a collection of disjoint subsets of A whose union is all of A*

Order Relations

A relation C in a set A is called an **order relation** (or a **simple order** or a **linear order**) if it has the following properties:

- (Comparability) For every x and y in A for which $x \neq y$, either xCy or yCx.
- (Nonreflexivity) For no x in A does the relation xCx holds.
- (Transitivity) If xCy and yCz, then xCZ.

Definition 9. *If X is a set and , is an order relation X, and if $a < b$, we use the notation (a, b) to denote the set:*

$$\{x|a < x < b\}$$

it is called an ***open interval*** *in X. If this set is empty, we call a the immediate predecessor of b, and we call b the* ***immediate successor*** *of a*

Definition 10. *Suppose that A and B are two sets with the order relations $<_A$ and $<_B$ respectively. We say that A and B have the same* ***order type*** *if there is a bijective correspondence between them that preserves order; that is, if there exists a bijective function $f : A \to B$ such that:*

$$a_1 <_A a_2 \implies f(a_1) <_B f(a_2)$$

The following is very useful in Topology:

Definition 11. *Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. Define an order relation $<$ on $A \times B$ by defining:*

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$, or if $a_1 = a_2$ and $b_1 <_B b_2$. It is called the ***dictionary order relation*** *on $A \times B$*

Some terminology is defined, Supoose that A is a set ordered by the realtion $<$. Let A_0 be a subset of A. We say that the element b is the **largest element** of A_0 if $b \in A_0$ and if $x \leq b$ for every $x \in A_0$. Similarly, we say that a is the **smallest element**, if $a \in A_0$ and if $a \leq x$ for every $x \in A_0$.

We say that the subset A_0 of A is **bounded above** if there is an element b of A such that $x \leq b$ for every $x \in A_0$; the element b is called an **upper bound** fo A_0. If the set of all upper bounds for A_0 has a smallest element, that element is called the **least upper bound**, or the **supremum**, of A_0. It is denoted by $\sup A_0$; it may or may not belong to A_0. If it does, it is the largest element of A_0. Similarly, A_0 is **bounded below** if there is an element a of A such that $a \leq x$ for every $x \in A_0$; the element a is called a **lower bound** for A_0. If the set of all lower bounds for A_0 has a largest element, that element is called the **greatest lower bound**, or the **infimum**, of A_0. It is denoted by $\inf A_0$; it may or may not belong to A_0. If it does, it is the smallest element of A_0

Definition 12. *An ordered set A is said to have the* ***least upper bound property*** *if every nonempty subset A_0 of A that is bounded above has a least upper bound. Analogously, the set A is said to have the* ***greatest lower bound property*** *if every nonempty subset A_0 of A that is bounded below has a greatest lower bound.*

The Integers and the Real Numbers

Definition 13. *A* ***binary operation*** *on a set A is a function f mapping $A \times A$ into A*

Assumption

We assume there exists a set $\mathbb{R}$, called the set of **real numbers**, two binary operations $+$ and $\cdot$ on $\mathbb{R}$ called the addition and multiplication operations, respectively, and an order relation $<$ on $\mathbb{R}$, such that the following properties hold:

Algebraic Properties

1. $(x + y) + z = x + (y + z)$

   $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

2. $x + y = y + x$

   $x \cdot y = y \cdot x$

3. There exist a unique element of $\mathbb{R}$ called **zero**, denoted by 0, such that $x + 0 = x$ for all x in $\mathbb{R}$.

   There exists a unique element of $\mathbb{R}$ called **one**, different from 0 and denoted by 1, such that $x \cdot 1 = x$ for all $x \in \mathbb{R}$

4. For each x in $\mathbb{R}$, there exists a unique y in $\mathbb{R}$ such that $x + y = 0$

   For each x in $\mathbb{R}$ different from 0, there exists a unique y in $\mathbb{R}$ such that $x \cdot y = 1$

5. $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ for all $x, y, z \in \mathbb{R}$

A Mixed Algebraic and Order Property

6. If $x > y$, then $x + z > y + z$.

   If $x > y$ and $z > 0$, then $x \cdot z > y \cdot z$

Order Properties

7. The order relation $<$ has the least upper bound property.

8. If $x < y$, there exists an element z such that $x < z$ and $z < y$

We define a number x to be **positive** if $x > 0$ and to be **negative** if $x < 0$. We denote the positive reals via $\mathbb{R}_+$.

Any set with two binary operations satisfying (1) - (5) is called a **field**; if the field has an order relation satisfying (6) is called an **ordered field**. Any set with an order relation satisfying (7) and (8) is called by topologist a **linear continuum**.