Preliminary on Topology

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1 Set Theory and Logic

Definition 1.1. A **rule of assignment** is a subset r of the cartesian product $C \times D$ of two sets, having the property that each element of C appears as the first coordinate of at most one ordered pair belonging to r

An equivalent formulation is

$$[(c,d) \in r \text{ and } (c,d') \in r] \implies [d = d']$$

Definition 1.2. a **function** f is a rule of assignment r, together with a set B that contains the image set of r . The domain A of the rule r is also called the **domain** of the function f; the image set of r is also called the **image set** of f; and the set b is called the **range** of f

Definition 1.3. If $f : A \to B$ and if A_0 is a subset og A, we define the **restriction** of f to A_0 to be the function mapping A_0 into B whose rule is:

$$\{(a, f(a)) | a \in A_0\}$$

It is denoted by $f \restriction_{A_0}$.

Definition 1.4. Given functions $f : A \to B$ and $g : B \to C$, we define the **composite** $g \circ f$ of f and g as the function $g \circ f : A \to C$ defined by the equation $(g \circ f)(a) = g(f(a))$. Formally, $g \circ f : A \to C$ is the function whose rule is:

$$\{(a, c) | \text{For some } b \in B, f(a) = b \text{ and } g(b) = c\}$$

We take note that $g \circ f$ is defined only when the range of f equals the domain of g

Definition 1.5. A function $f : A \to B$ is said to be **injective** (or **one-to-one**) if for each pair of distinct points of A, their images under f are distinct. It is said to be **surjective** (or f is said to map A **onto** B) if every element of B is the image of some element of A under the function f, If f is both injective and surjective, it is said to be **bijective**

An important remark of facts is that, the composite of two surjective functions is surjective, and the composite of two injective functions is injective.

If f is bijective there exists a function from B to A called the **inverse** of f. It is denoted by f^{-1} and is defined by letting $f^{-1}(b)$ be that unique element a of A for which $f(a) = b$.

Lemma 1. *Let $f : A \to B$. If there are functions $g : B \to A$ and $h : B \to A$ such that $g(f(a)) = a$ for every a in A, and $f(h(b)) = b$ for every b in B, then f is bijective and $g = h = f^{-1}$*

Proof. To prove that f is surjective, observe that the function h that goes from B to A maps every element to $h(b) \in A$ so that, every element of b has an element of A such that $f(h(b)) = b$. To prove that it is injective, suppose for the sake of contradiction that there exists elements of A such that $a \neq a'$ but $f(a) = f(a')$ then, applying g to both sides of this equation we have that $g(f(a)) = g(f(a'))$ and by the conditions of the hypothesis we have that $a = a'$ which is a contradiction. To see that both are equal observe that:

$$\begin{aligned} h(b) &= g(f(h(b))) \\ &= g(b) \end{aligned}$$

and the proof is complete. □

Definition 1.6. Let $f : A \to B$. If A_0 is a subset of A, we denote by $f(A_0)$ the set of all images of points of A_0 under the function f; this set is called the **image** of A_0 under f. Formally,

$$f(A_0) = \{b | b = f(a) \text{ for at least one } a \in A_0\}$$

On the other hand, if B_0 is a subset of B, we denote by $f^{-1}(B_0)$ the set of all elements of A whose images under f lie in B_0; it is called the **preimage** of B_0 under f (or the "counterimage" or the "inverse image" of B_0). Formally,

$$f^{-1}(B_0) = \{a | f(a) \in B_0\}$$

Of course, there may be no points $a$of A whose images lie in B_0; in that case, $f^{-1}(B_0)$ is empty.

We make the following remark, and that is that: If $f : A \to B$ and if $A_0 \subset A$ and $B_0 \subset B$, then:

$$A_0 \subset f^{-1}(f(A_0)) \quad \text{and} \quad f(f^{-1}(B_0)) \subset B_0$$

The first inclusion is an equality if f is injective, and the second inclusion is an equality if f is surjective.

Relations

Definition 1.7. A **relation** on a set A is a subset C of the Cartesian product $A \times A$

Equuivalence Relations and Partitions An **equivalence relation** on a set A is a relation C on A having the following three properties:

1. (Reflexivity) xCx for every $x \in A$
2. (Symmetry) If xCy, then yCx
3. (Transitivity) If xCy and yCz, then xCz

Given an equivalent realtion $\sim$ on a set A and an element x of A, we define a certain subset E of A, called the **equivalence class** determined by x, via:

$$E = \{y | y \sim x\}$$

Lemma 2. *Two equivalence classes E and E' are either disjoint or equal*

Proof. Let E be the equivalence class determined by x, and let E' be the equivalence class determined by x'. Suppose that $E \cap E'$ is not empty; let y be a point of $E \cap E'$, observe that we have y x and y x' so that we conclude that x x'. If now w is any point of E, we have w x so that we conclude $E \subset E'$, by symmetry of the argument we see that the equality holds. $\square$

Definition 1.8. A **partition** of a set A is a collection of disjoint subsets of A whose union is all of A

Order Relations

A relation C in a set A is called an **order relation** (or a **simple order** or a **linear order**) if it has the following properties:

- (Comparability) For every x and y in A for which $x \neq y$, either xCy or yCx.
- (Nonreflexivity) For no x in A does the relation xCx holds.

- (Transitivity) If xCy and yCz, then xCZ.

Definition 1.9. If X is a set and , is an order relation X, and if $a < b$, we use the notation (a, b) to denote the set:

$$\{x|a < x < b\}$$

it is called an **open interval** in X. If this set is empty, we call a the immediate predecessor of b, and we call b the **immediate successor** of a

Definition 1.10. Suppose that A and B are two sets with the order relations $<_A$ and $<_B$ respectively. We say that A and B have the same **order type** if there is a bijective correspondence between them that preserves order; that is, if there exists a bijective function $f : A \to B$ such that:

$$a_1 <_A a_2 \implies f(a_1) <_B f(a_2)$$

The following is very useful in Topology:

Definition 1.11. Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. Define an order relation $<$ on $A \times B$ by defining:

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$, or if $a_1 = a_2$ and $b_1 <_B b_2$. It is called the **dictionary order relation** on $A \times B$

Some terminology is defined, Supoose that A is a set ordered by the realtion $<$. Let A_0 be a subset of A. We say that the element b is the **st element** of A_0 if $b \in A_0$ and if $x \leq b$ for every $x \in A_0$. Similarly, we say that a is the **smallest element**, if $a \in A_0$ and if $a \leq x$ for every $x \in A_0$.

We say that the subset A_0 of A is **bounded above** if there is an element b of A such that $x \leq b$ for every $x \in A_0$; the element b is called an **upper bound** fo A_0. If the set of all upper bounds for A_0 has a smallest element, that element is called the **least upper bound**, or the **supremum**, of A_0. It is denoted by $\sup A_0$; it may or may not belong to A_0. If it does, it is the st element of A_0. Similarly, A_0 is **bounded below** if there is an element a of A such that $a \leq x$ for every $x \in A_0$; the element a is called a **lower bound** for A_0. If the set of all lower bounds for A_0 has a st element, that element is called the **greatest lower bound**, or the **infimum**, of A_0. It is denoted by $\inf A_0$; it may or may not belong to A_0. If it does, it is the smallest element of A_0

Definition 1.12. An ordered set A is said to have the **least upper bound property** if every nonempty subset A_0 of A that is bounded above has a least upper bound. Analogously, the set A is said to have the **greatest lower bound property** if every nonempty subset A_0 of A that is bounded below has a greatest lower bound.

The Integers and the Real Numbers

Definition 1.13. A **binary operation** on a set A is a function f mapping $A \times A$ into A

Assumption

We assume there exists a set $\mathbb{R}$, called the set of **real numbers**, two binary operations $+$ and $\cdot$ on $\mathbb{R}$ called the addition and multiplication operations, respectively, and an order relation $<$ on $\mathbb{R}$, such that the following properties hold:

Algebraic Properties

1. $(x + y) + z = x + (y + z)$

   $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

2. $x + y = y + x$

   $x \cdot y = y \cdot x$

3. There exist a unique element of $\mathbb{R}$ called **zero**, denoted by 0, such that $x + 0 = x$ for all x in $\mathbb{R}$.

   There exists a unique element of $\mathbb{R}$ called **one**, different from 0 and denoted by 1, such that $x \cdot 1 = x$ for all $x \in \mathbb{R}$

4. For each x in $\mathbb{R}$, there exists a unique y in $\mathbb{R}$ such that $x + y = 0$

   For each x in $\mathbb{R}$ different from 0, there exists a unique y in $\mathbb{R}$ such that $x \cdot y = 1$

5. $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ for all $x, y, z \in \mathbb{R}$

A Mixed Algebraic and Order Property

6. If $x > y$, then $x + z > y + z$.

   If $x > y$ and $z > 0$, then $x \cdot z > y \cdot z$

Order Properties

7. The order relation $<$ has the least upper bound property.

8. If $x < y$, there exists an element z such that $x < z$ and $z < y$

We defn a number x to be **positive** if $x > 0$ and to be **negative** if $x < 0$. We denote the positive reals via $\mathbb{R}_+$.

Any set with two binary operations satisfying (1) - (5) is called a **field**; if the field has an order relation satisfying (6) is called an **ordered field**. Any set with an order relation satisfying (7) and (8) is called by topologist a **linear continuum**.

Definition 1.14. A subset A of the real numbers is said to be **inductive** if it contains the number 1, and if for every x in A, the number $x+1$ is also in A. Let $\mathcal{A}$ be the collection of all inductive subsets of $\mathbb{R}$. Then the set $\mathbb{Z}_+$ of **positive integers** is defined by the equation:

$$\mathbb{Z}_+ = \bigcap_{A \in \mathcal{A}} A$$

The following are properties of: $\mathbb{Z}_+$

1. $\mathbb{Z}_+$ is inductive.

2. (Principle of Induction) If A is an inductive set of positive integers then $A = \mathbb{Z}_+$

We define the set $\mathbb{Z}$ of **integers** to be the set consisting of the positive integers $\mathbb{Z}_+$ and the number 0. The set $\mathbb{Q}$ of quotients of integers is called the set of **rational numbers**. We define

$$\{1, \ldots, n\} = S_{n+1}$$

Theorem 3. ***Well-ordering property****. Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.*

Proof. We first prove that, for each $n \in \mathbb{Z}_+$, the following statement holds: Every nonempty subset of $\{1, \ldots, n\}$ has a smallest element. By induction. Let A be the set of all positive integers n for which this statement holds. Then A contains 1, then, supposing A contains n, we show that it contains $n+1$. So let C be a nonempty subset of the set $\{1, \ldots, n\}$. If C consists of only the element $n+1$ we are done, otherwise consider the set $C \cap \{1, \ldots n\}$ which is nonempty. Because $n \in A$, this set has a smallest element, which will automatically be the smallest element of C also. Thus A is inductive, and we conclude $A = \mathbb{Z}_+$. Now we prove the theorem. Suppose that D is a nonempty subset of $\mathbb{Z}_+$. Choose an element n of D. Then the set $A = D \cap \{1, .., D\}$ is nonempty, os that A has a smallest element k. The element k is automatically the smallest element of D as well. $\square$

Cartesian Product

Definition 1.15. Let $\mathcal{A}$ be a nonempty collection of sets. An **indexing function** for $\mathcal{A}$ is a surjective function f from some set J , called the **index set**, to $\mathcal{A}$. The collection $\mathcal{A}$, together with the indexing function f, is called an **indexed family of sets**. Given $\alpha \in J$, we shall denote the set $f(\alpha)$ by the symbol A_α. And we shall denote the indexed family itself by the symbol:

$$\{A_\alpha\}_{\alpha \in J}$$

The indexing function it is not required to be injective. Two especially useful index sets are the set $\{1, \dots, n\}$ of the positive integers to n and the set $\mathbb{Z}_+$.

Definition 1.16. Let m be a positive integer. Given a set X, we define an **m-tuple** of elements of X to be a function.

$$\mathbf{x} : \{1, \dots, m\} \to X$$

. If $\mathbf{x}$ is m-tuple, we often denote the value of $\mathbf{x}$ at i by the symbol x_i and call it the ith **coordinate** of $\mathbf{x}$. And we often denote the function $\mathbf{x}$ itself by the symbol:

$$(x_1, \dots, x_m)$$

Now let $\{A_1, \dots A_m\}$ be a family of sets indexed with the set $\{1, \dots, m\}$. Let $X = A_1 \cup \dots \cup A_m$. We define the **cartesian product** of this indexed family, denoted by:

$$\prod_{i=1}^{m} A_1 \quad \text{or} \quad A_1 \times \dots \times A_m.$$

to be the set of all m-tuples $(x_1, \dots, x_m)$ of elements of X such that $x_i \in A_i$ for each i.

Definition 1.17. Given a set X, we define an **ω-tuple** of elements of X to be a function:

$$\mathbf{x} : \mathbb{Z}_+ \to X$$

we also call such a function a **sequence**, or an **infinite sequence**, of elements of X. If $\mathbf{x}$ is an ω-tuple, we often denote $\mathbf{x}$ by the symbol:

$$(x_1, \dots) \quad \text{or} \quad (x_n))n \in \mathbb{Z}_+$$

Now let $\{A_1, \ldots\}$ be a family of sets, indexed with the positive integers; let X be the union of the sets in this family. The **Cartesian product** of this indexed family of sets, denoted by:

$$\prod_{i \in \mathbb{Z}_+} A_i \quad \text{or} \quad A_1 \times \ldots$$

is defined to be the set of all ω-tuples $(x_1, X_2, \ldots)$ of elements of X such that $x_i \in A_i$ for each i

Examples If $\mathbb{R}$ is the set of real numbers, then R^m denotes the set of all m-tuples of real numbers; it is often called **euclidean m-space**. Analogously, R^ω is sometimes called "infinite-dimensional euclidean space".

Finite sets

Definition 1.18. A set is said to be **finite** if there is a bijective correspondence of A with some section of the positive integers. That is, A is finite if it is empty or if there is a bijection:

$$f : A \to \{1, \ldots, n\}$$

for some positive integer n. In the former case, we say that A has **cardinality** 0; in the latter case, we say that A has **cardinality** n

Lemma Let n be a positive integer. Let A be a set; let a_0 be an element of A. Then there exists a bijective correspondence f of the set A with the set $\{arg1\}, \ldots n+1$ if and only if there exists a bijective correspondence g of the set $A - \{a_0\}$ with the set $\{1, \ldots, n\}$.

From this lemma we derive some important results.

Theorem 4. *LetA be a set; suppose that there exists a bijection $f : A \to \{1, \ldots n\}$ for some $n \in \mathbb{Z}_+$. Let B be a proper subset of A. Then there exists no bijection $g : B \to \{1, \ldots n\}$; but (provided $B \neq \varnothing$) there does exist a bijection $h : B \to \{1, \ldots, m\}$ for some $m < n$*

Corollary If A is finite, there is no bijection of A with a proper subset of itself.

Proof Assume that B is a proper subset of A and that $f : A \to B$ is a bijection. By assumption, there is a bijection $g : A \to \{1, \ldots, n\}$ for some n. The composite $g \circ f^{-1}$ is then a bijection of B with $\{1, \ldots, n\}$. This contradicts the preceding theorem.

Corollary The cardinality of a finite set A is uniquely determined by A.

Proof Let $m < n$. Suppose there are bijections:

$$f : A \to \{1, \dots, n\}$$
$$g : A \to \{1, \dots, m\}$$

Then the composite:

$$g \circ f^{-1} : \{1, \dots, n\} \to \{1, \dots, m\}$$

is a bijection of the finite set $\{1, \dots, n\}$ with a proper subset of itself, contradicting the corollary just proved.

Corollary If B is a subset of the finite set A, then B is finite. IF B is a proper subset of A, then the cardinality of B is less than the cardinality of A.

Corollary Let B be a nonempty set. Then the following are equivalent:

1. B is finite.

2. There is a surjective function from a section of the positive integers onto B.

3. There is an injective function from B into a section of the positive integers.

Proof $1 \implies 2$. Since B is nonempty, there is, for some n, a bijective function $f : \{1, \dots, n\} \to B$

$2 \implies 2$ If $f : \{1, \dots, n\} \to B$ is surjective, define $g : B \to \{1, \dots, n\}$ by the equation:

$$g(b) = \text{ smallest element of } f^{-1}(\{b\})$$

Because f is surjective, the set $f^{-1}\{(b)\}$ is nonempty; then the well-ordering property of $\mathbb{Z}_+$ tell us that $g(b)$ is uniquely defined. The map g is injective, for if $b \neq b'$ then the sets $f^{-1}(\{b\})$ and $f^{-1}(\{b'\})$

$3 \implies 1$ If $g : B \to \{1, \dots, n\}$ is injective, then changing the range of g gives a bijection of B with a subset of $\{1, \dots, n\}$. It follows from the preceding corollary that B is finite. $\square$

Corollary Finite unions and finite cartesian products of finite sets are finite.

Countable and Uncountable Sets

Definition 1.19. A set A is said to be **infinite** if it is not finite. It is said to be **countably infinite** if there is a bijective correspondence:

$$f : A \to \mathbb{Z}_+$$

Definition 1.20. A set is said to be **countable** if it is either finite or countably infinite. A set that is not countable is said to be **uncountable**

a useful criterion for showing that a set is countable is the following:

Theorem 5. *Let B be a nonempty set. Then the following are equivalent:*

1. *B is countable.*
2. *There is a surjective functions $f : \mathbb{Z}_+ \to B$.*
3. *There is an injective function $g : B \to \mathbb{Z}_+$*

Lemma If C is an infinite subset of $\mathbb{Z}_+$, then C is countably infinite.

To avoid logical problems, we define then the following principle.

Principle of recursive Let A be a set. Given a formula that defines $h(1)$ as a unique element of A, and for $i > 1$ defines $h(i)$ uniquely as an element of A in terms of the values of h for positive integers less than i, this formula determines a unique function $h : \mathbb{Z}_+ \to A$

Corollary A subset of a countable set is countable.

Proof Suppose $A \subset B$, where B is countable. There is an injection f of B into $\mathbb{Z}_+$; the restriction of f to A is an injection of A into $\mathbb{Z}_+$

Corollary The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countably infinite

Proof In view of 5, it suffices to construct an injective map $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$. We define f by the equation:

$$f(n,m) = 2^n \cdot 3^m$$

f is injective, for suppose that $2^n 3^m = 2^p 3^q$. If $n < p$, then $3^m = 2^{p-n} 3^q$, contradicting the fact that 3^m is odd for all m. Therefore, $n = p$. as a result, $3^m = 3^q$, then if $m < q$, it follows that $1 = 3^{q-m}$, another contradiction hence $m = q$

Theorem 6. *A countable union of countable sets is countable, also a finite product of countable sets is countable.*

Observe that is very tempting to assert that countable products of countable sets should be countable; but this assertion is in fact not true.

Theorem 7. *Let X denote the two element set $\{0, 1\}$. Then the set X^ω is uncountable.*

Proof We show that, given any function:

$$g : \mathbb{Z}_+ \to X^\omega$$

g is not surjective. For this purpose, let us denote $g(n)$ as

$$g(n) = (x_{n1}, x_{n2}, x_{n3}, \ldots, x_{nm}, \ldots)$$

where each x_{ij} is either 0 or 1. Then we define an element $\mathbf{y} = (y_1, y_2, \ldots)$ of X^ω by letting:

$$y_n = \begin{cases} 0 & \text{if } x_n n = 1 \\ 1 & \text{if } x_n n = 0 \end{cases}$$

Now $\mathbf{y}$ is an element of X^ω, and $\mathbf{y}$ does not lie in the image of g; given n, the point $g(n)$ and the point $\mathbf{y}$ differ in at least one coordinate. $\square$

Theorem 8. *Let A be a set. There is no injective map $f : \mathbb{P}(A) \to A$, and there is no surjective map $g : A \to \mathbb{P}(A)$*

Infinte Sets and the Axiom of Choice

Theorem 9. *Let A be a set. The following statements about A are equivalent:*

1. *There exists an injective function $f : \mathbb{Z}_+ \to A$*
2. *There exists a bijection of A with a proper subset of itself.*
3. *A is infinite.*

The proof of this theorem allows us to discuss an important method of forming sets. **The Axiom of Choice**

Axiom of choice Given a collection $\mathcal{A}$ of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of $\mathcal{A}$; that is, a set C such that C is contained in the union of the elements of $\mathcal{A}$, and for each $A \in \mathcal{A}$, the set $C \cap A$ contains a single element.

Lemma (Existence of a choice function) Given a collection $\mathcal{B}$ of nonempty sets (not necessarily disjoint), there exists a function:

$$c : \mathcal{B} \to \bigcup_{B \in \mathcal{B}} B$$

such that $c(B)$ is an element of B, for each $b \in \mathcal{B}$

Well-Ordered Sets

Definition 1.21. A set A with an order relation $<$ is said to be **well-ordered** if every nonempty subset A has a smallest element.