Textbook notes on Topology

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The following are notes based on the book *Topology* by Munkres.

Chapter 1

Set Theory and Logic

1.1 Fundamental Concepts

We express that an object a belongs to a set A by the notation:

$$a \in A$$

Similarly,

$$a \notin A$$

We denote the inclusion of a set into another set with:

$$A \subseteq B$$

so that $A = B \iff A \subset B \text{and} B \subset A$. If $A \subset B$ but A is different from A, we say A is a **proper subset** of B, in notation:

$$A \subsetneq B$$

The relation $\subset$ is called **inclusion** and $\subsetneq$ is called proper inclusion.

The Union of Sets and the Meaning of "or"

Given two sets A and B, we can form another set called the union of A and B.

$$A \cup B = \{x | x \in A \text{or } x \in B\}$$

We will use the concept of exclusive or, if the necessity arises.

The Intersection of Sets, the Empty set, and the Meaning of "If...Then"

Another way to form a set from two existing sets is to take the elements in common, that is:

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

The **empty set** is the set with no elements, denoted by $\emptyset$. We say that two elements are disjoint if:

$$A \cap B = \emptyset$$

Some property of this interesting empty set are:

$$A \cap \emptyset = \emptyset \qquad A \cup \emptyset = A$$

Chapter 2

Topological Spaces and Continuous Functions

2.1 Topological Spaces

Definition 1. *A* ***topology*** *on a set X is a collection τ of subsets of X having the sollowing properties:*

- *$\emptyset$ and X are in τ*
- *The union of the elements of any subcollection of τ is in τ*
- *The intersection of the elements of any finite subcollection of τ is in τ.*

A set X for which a topology τ has been specified is called a ***topological space***

Properly, a topological space is an ordered pair (X, τ).

If Z is a topological space with topology τ, we say that a subset U of X is an **open set** of X if U belongs to the collection τ.

Example 1. *If X is any set, the collection of all subsets of X is a topology on X; it is called the* ***discrete topology****. The collection consisting of X and $\emptyset$ only is also a topology on X; we shall call it the* ***indiscrete topology****, or the* ***trivial topology***

Example 2. *Let X be a set; let τ_f be the collection of all subsets of U of X such that $X - U$ is either finite or is all of X. Then τ_f is a topology on X, called the* ***finite complement topology****. Both X and $\varnothing$ are in τ_f, since $X - X$ is finite and $X - \varnothing$ is all of X. If $\{U_\alpha\}$ is an indexed family of nonempty elements of τ_f, to show that $\cup U_\alpha$ is in τ_f, we compute*

$$(\bigcup U_\alpha)^c = \bigcap U_\alpha^c$$

and since each U^c is finite, the union of these set is finite. If $U_1, \ldots, U_n$ are nonempty elements of τ_f, to show that $\cap U_i$ is in τ_f, er compute:

$$\left(\bigcap_{i=1}^{n}\right)^c U_i = \bigcup_{i=1}^{n} U_i^c$$

Observe then that each U_i^c is finite, and finite union of finite set is finite.

Example 3. *Let X be a set; let τ_c be the collection of all subsets U of X such that $X - U$ either is countable or is all of X. Then τ_c is a topology on X.*

Definition 2. *Suppose that τ and τ' are two topologies on a given set X. If $\tau' \supset \tau$, we say that τ' is **finer** than τ; if τ' properly contains τ, we say τ' is **strictly finer** than τ. We also say that τ is **coarser** thanτ', or **strictly coarser**, in these two respective situations. We say τ is **comparable** with τ' if either $\tau' \supset \tau$ or $\tau \supset \tau'$*

Basis for a Topology

Definition 3. *If X is a set, a **basis** for a topology on X is a collection $\mathfrak{B}$ of subsets of X (called **basis elements**) such that:*

- *For each $x \in X$, there is at least one basis element B containing x.*
- *If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.*

*If $\mathfrak{B}$ satisfies these two conditions, then we define the **topology τ generated by** $\mathfrak{B}$ as follows: A subset U of X is said to be open in X (that is , to be an element of τ) if for each $x \in U$, there is a basis element $B \in \mathfrak{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of τ*

Example 4. *If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology on X.*

Lemma 1. *The collection τ generated by the basis $\mathfrak{B}$ is, in fact a topology on X*

Proof. If U is the empty set, it satisfies the defining condition of openness vacuously. Likewise, X is in τ, since for each $x \in X$ there is some basis element B containing x and contained in X. Now let us take an indexed family $\{U_\alpha\}_{\alpha \in J}$ of elements of τ, that is the collection $\subset$ in τ and show that:

$$U = \bigcup_{\alpha \in J} U_\alpha \in \tau$$

Given $x \in U$, there exist an index α such that $x \in U_\alpha$. Since U_α is open, there is a basis element B such that $x \in B \subset U_\alpha$. Then $x \in B$ and $B \subset U$, so that U is open, by definition. Now let us take two elements U_1 and U_2 of τ and show that the intersection belongs to τ. Given $x \in U_1 \cap U_2$, choose a basis element B_1 containing x such that $B_1 \subset U_1$; choose also a basis B_2 containing x such that $B_2 \subset U_2$. The second condition for a basis enables us to choose a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$. Then $x \in B_3$ and $B_3 \subset U_1 \cap U_2$, so $U_1 \cap U_2$ belongs to τ. Finally, we show by induction that any finite intersection $U_! \cap \ldots \cap U_n$ of elements in τ is inτ. The fact is trivial for $n = 1$; we suppose it true for $n-1$ and prove it for n. Now

$$(U_1 \cap \ldots U_n) = (U_1 \cap \ldots U_{n-1}) \cap U_n$$

. By hypothesis, $U_1 \cap \ldots U_{n-1}$ belongs to τ; by the result proven above, the intersection of $U_1 \cap \ldots U_{n-1}$ and U_n also belongs to τ □

Another point of view to view a basis for a topology is summarized in the following lemma:

Lemma 2. *Let X be a set; let $\mathfrak{B}$ be a basis for a topology τ on X. Then τ equals the collection of all unions of elements of $\mathfrak{B}$*

Proof. By double inclusion, Given a collection of elements in $\mathfrak{B}$, they are also elements of τ. Because τ is a topology, their union is in τ. Conversely, given $U \in \tau$, choose for each $x \in U$ an element B_x of $\mathfrak{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathfrak{B}$ □

As an important observation, this says that every open set U in X can be written as a union of basis elements. This expression for U is not unique. In summary, we have just described two ways from going from the basis to the topology it generates, now for the other way around:

Lemma 3. *Let X be a topological space. Suppose that $\mathfrak{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathfrak{C}$ such that $x \in C \subset U$. Then $\mathfrak{C}$ is a basis for the topology of X.*

Proof. We must show that $\mathfrak{C}$ is a basis. The first condition for a basis is easy: Given $x \in X$, since X is itself an open set, there is by hypothesis an element C of $\mathfrak{C}$ such that $x \in C \subset X$. To check the second condition, let x belong to $C_1 \cap C_2$, where C_1 and C_2 are elements of $\mathfrak{C}$. Since $C_!$ and C_2 are open, so is $C_1 \cap C_2$. Therefore, there exists by hypothesis an element C_3in $\mathfrak{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

Let τ be the collection of open sets of X; we must show that the topology τ' generated by $\mathfrak{C}$ equals the topology τ. First, note that if U belongs to τ and if $x \in U$ then there is by hypothesis an element C of $\mathfrak{C}$ such that $x \in C \subset U$. It follows that U belongs to the topology τ', by definition. Conversely, if W belongs to the topology τ', then W equals a union of elements of $\mathfrak{C}$, by the preceding lemma. Since each element of $\mathfrak{C}$ belongs to τ and τ is a topology. W also belongs to τ □

When topologies are given by bases, we want to have a criterion for deciding which topology is finer than other. The following provides a criterion:

Lemma 4. *Let $\mathfrak{B}$ and $\mathfrak{B}'$ be bases for the topologies τ and τ', respectively, on X. Then the following are equivalent:*

- *τ' is finer than τ, i.e., $\tau' \supset \tau$*
- *For each $x \in X$ and each basis element $B \in \mathfrak{B}$ containing x, there is a basis element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$.*

Proof. "$2 \implies 1$". Given an element U of τ, we wish to show that $U \in \tau'$. Let $x \in U$. Since $\mathfrak{B}$ generates τ, there is an element $B \in \mathfrak{B}$ such that $x \in B \subset U$. Condition (2) tell us there exists an element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$. Then $x \in B' \subset U$, so $U \in \tau'$, by definition.

"$1 \implies 2$". We are given $x \in X$ and $B \in \mathfrak{B}$, with $x \in B$. Now B belongs to τ by definition and $\tau \subset \tau'$ by condition (1); therefore, $B \in \tau'$. Since τ' is generated by $\mathfrak{B}'$, there is an element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$ □

2.2 Closed Sets and limit Points

Definition 4. *A subset A of a topological space X is said to be **closed** if the set $X - A$ is open.*

Observe that a set can be open, or closed or both.

Theorem 1. *Let X be atopological space, Then the following conditions holds:*

- *$\varnothing$ and X are closed.*
- *Arbitary intersections of closed sets are closed.*
- *Finite unions of closed sets are closed.*

Proof. Observe that $\varnothing$ and X are closed necause they are the complements of the open set X and $\varnothing$ respictively.

For the second proof, suposse that we are given a collection of closed sets $\{A_\alpha\}_{\alpha \in J}$, and we apply then DeMorgan's Law,

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup (X - A_\alpha)$$

□

Theorem 2. *Let Y be a subspace of X. Then a a set A is closed in Y if and only if it equals the intersection of a closed set of X*

Compactness and Connectedness

Definition A subset K if a topological space X is *compact* if every open cover of K has a finite subcover. A subset is compact if and only if it is compact as a topological space whit it is given its relative topologt, that is, if and only if every cover by relatively open sets has a finite subcover.

Proposition Let X be a topological space, and let $K \subseteq X$.

- If K is a compact subset of X, then K is closed.
- If K is compact and F is a closed set contained in K, then F is compact.
- The continuous image of a compact set is compact.

Corollary IF X is a compact space and $f : X \to \mathbb{R}$ is a continuous function, then there are points a and b in X such that $f(a) \leq f(x) \leq f(b)$ for all x in X.

Proposition If K is a closed subset of a topological space X, then K is compact if and only if every collection of closed subsets of K having the FIP has a nonempty intersection.

Compactness in Metric Spaces

If $\mathfrak{G}$ is a collection of subset of subsets of X and $E \subseteq X$, then $\mathfrak{G}$ is a *cover* of E if $E \subseteqq \cup\{G : G \in \mathfrak{G}\}$