

Textbook notes on Topology

David Cardozo

January 25, 2015

The following are notes based on the book *Topology* by Munkres.

Chapter 1

Set Theory and Logic

1.1 Fundamental Concepts

We express that an object a belongs to a set A by the notation:

$$a \in A$$

Similarly,

$$a \notin A$$

We denote the inclusion of a set into another set with:

$$A \subseteq B$$

so that $A = B \iff A \subseteq B \text{ and } B \subseteq A$. If $A \subseteq B$ but A is different from B , we say A is a **proper subset** of B , in notation:

$$A \subsetneq B$$

The relation \subseteq is called **inclusion** and \subsetneq is called proper inclusion.

The Union of Sets and the Meaning of "or"

Given two sets A and B , we can form another set called the union of A and B .

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

We will use the concept of exclusive or, if the necessity arises.

The Intersection of Sets, the Empty set, and the Meaning of "If...Then"

Another way to form a set from two existing sets is to take the elements in common, that is:

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

The **empty set** is the set with no elements, denoted by \emptyset . We say that two elements are disjoint if:

$$A \cap B = \emptyset$$

Some property of this interesting empty set are:

$$A \cap \emptyset = \emptyset \quad A \cup \emptyset = A$$

Chapter 2

Topological Spaces and Continuous Functions

2.1 Topological Spaces

Definition 1. A **topology** on a set X is a collection τ of subsets of X having the following properties:

- \emptyset and X are in τ
- The union of the elements of any subcollection of τ is in τ
- The intersection of the elements of any finite subcollection of τ is in τ .

A set X for which a topology τ has been specified is called a **topological space**

Properly, a topological space is an ordered pair (X, τ) .

If Z is a topological space with topology τ , we say that a subset U of X is an **open set** of X if U belongs to the collection τ .

Example 1. If X is any set, the collection of all subsets of X is a topology on X ; it is called the **discrete topology**. The collection consisting of X and \emptyset only is also a topology on X ; we shall call it the **indiscrete topology**, or the **trivial topology**

Example 2. Let X be a set; let τ_f be the collection of all subsets of U of X such that $X - U$ is either finite or is all of X . Then τ_f is a topology on X , called the **finite complement topology**. Both X and \emptyset are in τ_f , since $X - X$ is finite and $X - \emptyset$ is all of X . If $\{U_\alpha\}$ is an indexed family of nonempty elements of τ_f , to show that $\cup U_\alpha$ is in τ_f , we compute

$$(\cup U_\alpha)^c = \cap U_\alpha^c$$

and since each U^c is finite, the union of these set is finite. If U_1, \dots, U_n are nonempty elements of τ_f , to show that $\cap U_i$ is in τ_f , er compute:

$$\left(\bigcap_{i=1}^n \right)^c U_i = \bigcup_{i=1}^n U_i^c$$

Observe then that each U_i^c is finite, and finite union of finite set is finite.

Example 3. Let X be a set; let τ_c be the collection of all subsets U of X such that $X - U$ either is countable or is all of X . Then τ_c is a topology on X .

Definition 2. Suppose that τ and τ' are two topologies on a given set X . If $\tau' \supset \tau$, we say that τ' is **finer** than τ ; if τ' properly contains τ , we say τ' is **strictly finer** than τ . We also say that τ is **coarser** than τ' , or **strictly coarser**, in these two respective situations. We say τ is **comparable** with τ' if either $\tau' \supset \tau$ or $\tau \supset \tau'$

Basis for a Topology

Definition 3. If X is a set, a **basis** for a topology on X is a collection \mathfrak{B} of subsets of X (called **basis elements**) such that:

- For each $x \in X$, there is at least one basis element B containing x .
- If x belongs to the intersection of two basis elements B_1 and B_2 , then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If \mathfrak{B} satisfies these two conditions, then we define the **topology τ generated by \mathfrak{B}** as follows: A subset U of X is said to be open in X (that is, to be an element of τ) if for each $x \in U$, there is a basis element $B \in \mathfrak{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of τ

Example 4. If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology on X .

Lemma 1. The collection τ generated by the basis \mathfrak{B} is, in fact a topology on X

Proof. If U is the empty set, it satisfies the defining condition of openness vacuously. Likewise, X is in τ , since for each $x \in X$ there is some basis element B containing x and contained in X . Now let us take an indexed family $\{U_\alpha\}_{\alpha \in J}$ of elements of τ , that is the collection \subset in τ and show that:

$$U = \bigcup_{\alpha \in J} U_\alpha \in \tau$$

Given $x \in U$, there exist an index α such that $x \in U_\alpha$. Since U_α is open, there is a basis element B such that $x \in B \subset U_\alpha$. Then $x \in B$ and $B \subset U$, so that U is open, by definition. Now let us take two elements U_1 and U_2 of τ and show

that the intersection belongs to τ . Given $x \in U_1 \cap U_2$, choose a basis element B_1 containing x such that $B_1 \subset U_1$; choose also a basis B_2 containing x such that $B_2 \subset U_2$. The second condition for a basis enables us to choose a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$. Then $x \in B_3$ and $B_3 \subset U_1 \cap U_2$, so $U_1 \cap U_2$ belongs to τ . Finally, we show by induction that any finite intersection $U_1 \cap \dots \cap U_n$ of elements in τ is in τ . The fact is trivial for $n = 1$; we suppose it true for $n - 1$ and prove it for n . Now

$$(U_1 \cap \dots \cap U_n) = (U_1 \cap \dots \cap U_{n-1}) \cap U_n$$

. By hypothesis, $U_1 \cap \dots \cap U_{n-1}$ belongs to τ ; by the result proven above, the intersection of $U_1 \cap \dots \cap U_{n-1}$ and U_n also belongs to τ \square

Another point of view to view a basis for a topology is summarized in the following lemma:

Lemma 2. *Let X be a set; let \mathfrak{B} be a basis for a topology τ on X . Then τ equals the collection of all unions of elements of \mathfrak{B}*

Proof. By double inclusion, Given a collection of elements in \mathfrak{B} , they are also elements of τ . Because τ is a topology, their union is in τ . Conversely, given $U \in \tau$, choose for each $x \in U$ an element B_x of \mathfrak{B} such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of \mathfrak{B} \square

As an important observation, this says that every open set U in X can be written as a union of basis elements. This expression for U is not unique. In summary, we have just described two ways from going from the basis to the topology it generates, now for the other way around:

Lemma 3. *Let X be a topological space. Suppose that \mathfrak{C} is a collection of open sets of X such that for each open set U of X and each x in U , there is an element C of \mathfrak{C} such that $x \in C \subset U$. Then \mathfrak{C} is a basis for the topology of X .*

Proof. We must show that \mathfrak{C} is a basis. The first condition for a basis is easy: Given $x \in X$, since X is itself an open set, there is by hypothesis an element C of \mathfrak{C} such that $x \in C \subset X$. To check the second condition, let x belong to $C_1 \cap C_2$, where C_1 and C_2 are elements of \mathfrak{C} . Since C_1 and C_2 are open, so is $C_1 \cap C_2$. Therefore, there exists by hypothesis an element C_3 in \mathfrak{C} such that $x \in C_3 \subset C_1 \cap C_2$.

Let τ be the collection of open sets of X ; we must show that the topology τ' generated by \mathfrak{C} equals the topology τ . First, note that if U belongs to τ and if $x \in U$ then there is by hypothesis an element C of \mathfrak{C} such that $x \in C \subset U$. It follows that U belongs to the topology τ' , by definition. Conversely, if W belongs to the topology τ' , then W equals a union of elements of \mathfrak{C} , by the preceding lemma. Since each element of \mathfrak{C} belongs to τ and τ is a topology. W also belongs to τ \square

When topologies are given by bases, we want to have a criterion for deciding which topology is finer than other. The following provides a criterion:

Lemma 4. *Let \mathfrak{B} and \mathfrak{B}' be bases for the topologies τ and τ' , respectively, on X . Then the following are equivalent:*

- τ' is finer than τ , i.e., $\tau' \supset \tau$
- For each $x \in X$ and each basis element $B \in \mathfrak{B}$ containing x , there is a basis element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$.

Proof. "2 \implies 1". Given an element U of τ , we wish to show that $U \in \tau'$. Let $x \in U$. Since \mathfrak{B} generates τ , there is an element $B \in \mathfrak{B}$ such that $x \in B \subset U$. Condition (2) tell us there exists an element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$. Then $x \in B' \subset U$, so $U \in \tau'$, by definition.

"1 \implies 2". We are given $x \in X$ and $B \in \mathfrak{B}$, with $x \in B$. Now B belongs to τ by definition and $\tau \subset \tau'$ by condition (1); therefore, $B \in \tau'$. Since τ' is generated by \mathfrak{B}' , there is an element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$ \square