Textbook notes on Topology

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The following are notes based on the book *Topology* by Munkres.

Chapter 1

Set Theory and Logic

1.1 Fundamental Concepts

We express that an object a belongs to a set A by the notation:

$$a \in A$$

Similarly,

$$a \notin A$$

We denote the inclusion of a set into another set with:

$$A \subseteq B$$

so that $A = B \iff A \subset B \text{and} B \subset A$. If $A \subset B$ but A is different from A, we say A is a **proper subset** of B, in notation:

$$A \subsetneq B$$

The relation $\subset$ is called **inclusion** and $\subsetneq$ is called proper inclusion.

The Union of Sets and the Meaning of "or"

Given two sets A and B, we can form another set called the union of A and B.

$$A \cup B = \{x | x \in A \text{or } x \in B\}$$

We will use the concept of exclusive or, if the necessity arises.

The Intersection of Sets, the Empty set, and the Meaning of "If...Then"

Another way to form a set from two existing sets is to take the elements in common, that is:

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

The **empty set** is the set with no elements, denoted by $\emptyset$. We say that two elements are disjoint if:

$$A \cap B = \emptyset$$

Some property of this interesting empty set are:

$$A \cap \emptyset = \emptyset \qquad A \cup \emptyset = A$$

Chapter 2

Topological Spaces and Continuous Functions

2.1 Topological Spaces

Definition 1. *A* ***topology*** *on a set X is a collection τ of subsets of X having the sollowing properties:*

- *$\emptyset$ and X are in τ*
- *The union of the elements of any subcollection of τ is in τ*
- *The intersection of the elements of any finite subcollection of τ is in τ.*

A set X for which a topology τ has been specified is called a ***topological space***

Properly, a topological space is an ordered pair (X, τ).
If Z is a topological space with topology τ, we say that a subset U of X is an **open set** of X if U belongs to the collection τ.

Example 1. *If X is any set, the collection of all subsets of X is a topology on X; it is called the* ***discrete topology****. The collection consisting of X and $\emptyset$ only is also a topology on X; we shall call it the* ***indiscrete topology****, or the* ***trivial topology***

Example 2. *Let X be a set; let τ_f be the collection of all subsets of U of X such that $X - U$ is either finite or is all of X. Then τ_f is a topology on X, called the* ***finite complement topology****. Both X and $\varnothing$ are in τ_f, since $X - X$ is finite and $X - \varnothing$ is all of X. If $\{U_\alpha\}$ is an indexed family of nonempty elements of τ_f, to show that $\cup U_\alpha$ is in τ_f, we compute*

$$(\bigcup U_\alpha)^c = \bigcap U_\alpha^c$$

and since each U^c is finite, the union of these set is finite. If $U_1, \ldots, U_n$ are nonempty elements of τ_f, to show that $\cap U_i$ is in τ_f, er compute:

$$\left(\bigcap_{i=1}^{n}\right)^{c} U_i = \bigcup_{i=1}^{n} U_i^c$$

Observe then that each U_i^c is finite, and finite union of finite set is finite.

Example 3. *Let X be a set; let τ_c be the collection of all subsets U of X such that $X - U$ either is countable or is all of X. Then τ_c is a topology on X.*

Definition 2. *Suppose that τ and τ' are two topologies on a given set X. If $\tau' \supset \tau$, we say that τ' is **finer** than τ; if τ' properly contains τ, we say τ' is **strictly finer** than τ. We also say that τ is **coarser** thanτ', or **strictly coarser**, in these two respective situations. We say τ is **comparable** with τ' if either $\tau' \supset \tau$ or $\tau \supset \tau'$*

Basis for a Topology

Definition 3. *If X is a set, a **basis** for a topology on X is a collection B of subsets of X (called **basis elements**) such that:*

- *For each $x \in X$, there is at least one basis element B containing x.*
- *If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element $B3$ containing x such that $B_3 \subset B_1 \cap B_2$.*

*If B satisfies these two conditions, then we define the **topology** τ **generated by** B as follows: A subset U of X is said to be open in X (that is , to be an element of τ) if for each $x \in U$, there is a basis element $b \in B$ such that $x \in b$ and $b \subset U$. Note that each basis element is itself an element of τ*