Textbook notes on Topology

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The following are notes based on the book *Topology* by Munkres.

Chapter 1

Set Theory and Logic

1.1 Fundamental Concepts

We express that an object a belongs to a set A by the notation:

$$a \in A$$

Similarly,

$$a \notin A$$

We denote the inclusion of a set into another set with:

$$A \subseteq B$$

so that $A = B \iff A \subset B \text{and} B \subset A$. If $A \subset B$ but A is different from A, we say A is a **proper subset** of B, in notation:

$$A \subsetneq B$$

The relation $\subset$ is called **inclusion** and $\subsetneq$ is called proper inclusion.

The Union of Sets and the Meaning of "or"

Given two sets A and B, we can form another set called the union of A and B.

$$A \cup B = \{x | x \in A \text{or } x \in B\}$$

We will use the concept of exclusive or, if the necessity arises.

The Intersection of Sets, the Empty set, and the Meaning of "If...Then"

Another way to form a set from two existing sets is to take the elements in common, that is:

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

The **empty set** is the set with no elements, denoted by $\emptyset$. We say that two elements are disjoint if:

$$A \cap B = \emptyset$$

Some property of this interesting empty set are:

$$A \cap \emptyset = \emptyset \qquad A \cup \emptyset = A$$

1.1.1 Exercises

Exercise 1. Prove that a set of n (different) elements has exactly 2^n (different) subsets.

Solution. The following is an application of the principle of mathematical induction. First, we observe that for a set with only one element, there are exactly $2 = 2^1$, now suppose that the statement holds for a set with n elements, now consider the set with $n+1$ elements, say $\{x_1, \ldots, x_{n+1}\}$, observe that we can write this set $\{x_1, \ldots, x_n\} \cup \{x_{n+1}\}$, using out induction hypothesis, the set $\{x_1, \ldots, x_n\}$ has 2^n subsets, and the subsets of $\{x_1, \ldots x_{n+1}\}$ can be obtained as:

$$\overbrace{2^n}^{\text{subsets of n}} \quad + \quad \underbrace{2^n}_{\text{adding the new element}}$$

and we have

$$2^n + 2^n = 2^n(1+1) = 2^{n+1}$$

Exercise 2. Formulate and prove DeMorgan's laws for arbitrary unions and intersections.

Solution.

Proposition 1.1.1. *Let $\mathcal{A}$ a family of sets, then*

$$X - \bigcup_{A \in \mathcal{A}} A = \bigcap_{A \in \mathcal{A}} X - A$$

Proof. Let $x \in X - \bigcup_{A \in \mathcal{A}} A$, so that $x \notin \cup_{A \in \mathcal{A}} A$, or in other words, $\forall A \in \mathcal{A}(x \notin A)$, and since $x \in X$, we can combine the last two statements to give $\forall A \in \mathcal{A}(x \in X$ and $x \notin A$, more suggestively, $\forall A \in \mathcal{A}(x \in X - A)$, and finally this implies $\cap_{A \in \mathcal{A}} X - A$ $\square$

Proposition 1.1.2. *Let $\mathcal{A}$ be a family of sets, then:*

$$X - \bigcap_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} X - A$$

Which again is similar.

1.1.2 Functions

Exercise 3. Let $f : A \to B$. Let $A_0 \subset A$ and $B_0 \subset B$.

- Show that $A_0 \subset f^{-1}(f(A_0))$ and that the equality holds if f is injective.
- Show that $f(f^{-1}(B_0)) \subset B_0$ and that equality holds if f is surjective.

Solution.

- Let $a \in A_0$, so that $f(a) \in f(A_0)$, and this is the condition for that $a \in f^{-1}f(A_0)$. Now for the converse, we add the hypothesis that f is injective, so let $x \in f^{-1}(f(A_0))$, observe that this implies $f(x) \in f(A_0)$, or in other words $f(x) = f(a)$ for at least one $a \in A_0$, since f is injective, then $x = a$ and $x \in A_0$.
- Let $b \in f(f^{-1}(B_0))$, so that $b = f(a)$ for some $a \in f^{-1}(B_0)$, similarly $f(a) \in B_0$ so that $b \in B_0$. Now for the converse we add the hypothesis that f is surjective, now let $b \in B_0$, since f is surjective, there exist $a \in A$ for which $f(a) = b$ so that $a \in f^{-1}(B_0)$, and combining the two previous statements we get $b \in f(f^{-1}(B_0))$

Exercise 4. Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that f^{-1} preserves inclusions, unions, intersections, and differences of sets

- $B_0 \cap B_1 \implies f^{-1}(B_0) \subset f^{-1}(B_1)$.
- $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$

Solution.

- Suppose $B_0 \subset B_1$, let $x \in f^{-1}(B_0)$ that is $f(x) = b$ for some $b \in B_0$, that is, $b \in B_1$, and this last assertion allow us to conclude that $f^{-1}(B_0) \subset f^{-1}(B_1)$
- Let $a \in f^{-1}(B_0 \cup B_1)$, that is $f(a) = b$ for an element b that either belongs to B_0 or B_1, so that $a \in f^{-1}(B_0) \cup f^{-1}(B_1)$. Conversely, ...

Exercise 5. Prove the following:

- $$f^{-1}(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f^{-1}(A_i)$$

Solution. "$\subset$"

$$\begin{aligned} x \in f^{-1}\left(\bigcup_{i \in I} A_i\right) &\implies \exists j \in I(f(x) \in A_j) \\ &\implies x \in \bigcup_{i \in I} f^{-1}(A_i) \end{aligned}$$

"⊃", let $A_i = B_i$ for all $i \in I$

$$\begin{aligned} x \in \bigcup_{i\in I} f^{-1}(B_i) &\implies \exists j \in I(x \in f^{-1}(B_j)) \\ &\implies f(x) \in B_j \implies f(x) \in \bigcup_{i\in I} B_i \\ &\implies x \in f^{-1}\left(\bigcup_{i\in I} B_i\right) \end{aligned}$$

Exercise 6. Prove the following:

$$f^{-1}(\bigcap_{i\in I} B_i) = \bigcap_{i\in I} f^{-1}(B_i)$$

Solution. "⊂"

$$\begin{aligned} x \in f^{-1}\left(\bigcap_{i\in I} B_i\right) &\implies f(x) \in \bigcap_{i\in I} B_i \\ &\implies \forall i \in I(f(x) \in B_i) \implies \forall i \in I(x \in f^{-1}(B_i)) \\ &\implies x \in \bigcap_{i\in I} f^{-1}(B_i) \end{aligned}$$

"⊃"

$$\begin{aligned} x \in \bigcap_{i\in I} f^{-1}(B_i) &\implies \forall i \in I(f(x) \in B_i) \\ &\implies f(x) \in \bigcap_{i\in I} B_i \\ &\implies x \in f^{-1}\left(\bigcap_{i\in I} B_i\right) \end{aligned}$$

Exercise 7. Prove that for an arbitrary class of sets:

$$f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$$

Solution. Let us take $y \in f(\bigcup_{i\in I} A_i)$, so that $y = f(x)$ for some $x \in A_i$, say $x \in A_j$, then $y \in f(A_j)$ so that $y \in \bigcup_{i\in I} f(A_i)$

Exercise 8. Let $f : A \to B$ and $g : B \to C$.

- $C_0 \subset C$, show that $(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$
- If f and g are injective, show that $g \circ f$ is injective.
- If $g \circ f$ is injective, what can you say about the injectivity of f and g?

- If f and g are surjective, show that $g \circ f$ is surjective.
- If $g \circ f$ is surjective, what can you say about surjectivity of f and g

Solution.
- Let $a \in (g \circ f)^{-1}(C_0)$, so that $(g \circ f)(a) \in C_0$, and $g(f(a)) \in C_0$ so that, $f(a) \in g^{-1}(C_0)$ which finally $a \in f^{-1}(g^{-1}(C_0))$
- Let $a, a' \in A$, suppose that $(g \circ f)(a) = (g \circ f)(a')$, by definition $g(f(a)) = g(f(a'))$, since g is injective, we have then $f(a) = f(a')$ again f is injective, then $a = a'$ so that $g \circ f$ is injective.
- If $g \circ f$ is injective, then f is injective.

  Proof. Let $a_1, a_2 \in A$, such that $f(a_1) = f(a_2)$, then $g(f(a_1)) = g(f(a_2))$, so that $(g \circ f)(a_1) = (g \circ f)(a_2)$ and since $g \circ f$ is injective $a_1 = a_2$ ☐
- Let $c \in C$, since the function g is surjective there exist b such that $c = g(b)$, now, since $b \in B$ and f is injective, there exist $a \in A$ for which $c = g(f(a)) = (g \circ f)(a)$, since c was arbitrary we have shown then $g \circ f$ is surjective.
- We propose that g is surjective.

  Proof. Let $c \in C$. Since $g \circ f$ is surjective, there exist a such that $g(f(a)) = c$. By definition for some $b \in B$, $f(a) = b$, and $g(b) = c$, then g is surjective. ☐
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  Theorem 1. *Let $f : A \to B$ and $g : B \to C$ and $g \circ f$ is bijective, then f is injective and g is surjective*

1.1.3 Relations

Definition. A **relation** on a set A is a subset C of the cartesian product $A \times A$

Equivalence Relations and Partitions

An **equivalence relation** on a set A is a relation C on A having the following three properties:

1. (Reflexivity) xCx
2. (Symmetry) xCy, then yCx
3. (Transitivity) xCy and $yCz \implies xCz$

Given an equivalence relation $\sim$ on a set A and an element x of A, we define a certain subset E of A, called the **equivalence class** determined by x, by the equation:

$$E = \{y | y \sim x\}$$

Lemma 2. *Two equivalence classes E and E' are either disjoint or equal.*

Proof. Let E be the equivalence class determined by x, and E' be the equivalence class determined by x', suppose $E \cap E'$ is not empty, so that $y \in E \cap E'$, since y belongs to E and E', $x \sim y$ and $y \sim x'$ so that $x \sim x'$ so that $x' \in E$ and similarly $x \in E'$ and we conclude that $E = E'$ □

Definition 1.1.1. A **partition** of a set A is a collection of disjoint nonempty subsets of A whose union is all of A

Lemma 3. *Given any partition $\mathcal{D}$ of A, there is exactly one equivalence relation on A from which is derived.*

Proof. To show that the partition $\mathcal{D}$ comes from some equivalence relation, let us define the equivalence relation on A as: let $x \in A$ $x \sim y$ if and only if x, y belong to the same element $\mathcal{D}$. Observe that this relation complies with all the rules to be a equivalence relation. Now for the uniqueness suppose $\sim_1$ and $\sim_2$ gives arise to the same partition $\mathcal{D}$ to do that we will show that $x \sim_1 y \iff x \sim_2 y$ but that is simple, consider E as the equivalence class determined by x in $\sim_1$ and consider E' with $\sim_2$, since E_1 is an element of $\mathcal{D}$ then it must equal the unique element D of $\mathcal{D}$, so that $E = D = E'$ □

Order Relations

A relation C on a set A is called an **order relation** (or a **simple order** or a **linear order**) if it has the following properties:

1. (Comparability) For every x and y in A for which $x \neq y$ either xCy or yCx
2. (Non reflexivity) For no x in A does the relation xCx holds.
3. (Transitivity) If xCy and yCz, then xCz

Definition 1.1.2. If X is a set and $<$ is an order relation on X, and if $a < b$ we use the notation (a, b) to denote the set:

$$\{x|a < x < b\}$$

; it is called an **open interval** in X. If this set is empty, we call a the **immediate predecessor** of b and b the **immediate successor** of a

Definition 1.1.3. Suppose that A and B are two sets with order relations $<_A$ and $<_B$respectively. We say that A and B have the same **order type** if there is a bijective function $f : A \to B$ such that:

$$a_1 <_A a_2 \implies f(a_1) <_B f(a_2)$$

Definition 1.1.4. Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. Define an order relation $<$ on $A \times B$ by defining:

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 < a_2$, or if $a_1 = a_2$ and $b_1 <_b b_2$. It is called the **dictionary order relation** on $A \times B$

For the next definitions Let A be a set ordered by the relation$<$

Definition. Let A_0 be a subset of A. We say that the element b is the **largest element** of A_0 if $b \in A_0$ and if $x \leq b$ for every $x \in A_0$

Similarly, we define:

Definition 1.1.5. Let A_0 be a subset of A. We say that the element a is the **smallest element** of A_0 if $a \in A_0$ and if $a \leq x$ for every $x \in A_0$

We make the remark that a set has at most one largest element and at most one smallest element.

Definition. We say the subset A_0 is **bounded above** if there is an element b of A such that $x \leq b$ for every $x \in A_0$; the element b is called an **upper bound** for A_0. If the set of all upper bounds for A_0 has a smallest element, that element is called the **least upper bound**, or the **supremum** of A_0 often denoted as $\sup A_0$

We make the remark that the supremum may not be in A_0 if it is in A_0 then it is the **largest element** of A_0. Similarly

Definition. We say the subset A_0 is **bounded below** if there is an element a of A such that $a \leq x$ for every $x \in A_0$; the element a is called a **lower bound** for A_0. If the set of all lower bounds for A_0 has a largest element, that element is called the **greatest lower bound** or the **infimum** of A_0. It is denoted by $\inf A_0$

We may do the remark that $\inf A_0$ may not belong to A_0, if it does, then it is the **smallest element**

Now, we can define the least upper bound property.

Definition 1.1.6. An ordered set A is said to have the **least upper bound property** if every nonempty subset A_0 of A that is bounded above has a least upper bound. Analogously, the set A is sad to have the **greatest lower bound property** if every nonempty subset A_0 of A that is bounded below has a greatest lower bound

Exercises

Exercise 9. Find the flaw in the following proof: suppose that C is a symmetric and transitive relation, then $xCy \implies yCx$ and by transitive then xCx so that C is reflexive

Solution. This is true if either xCy or yCx if neither of them holds then, this does not prove that xCx

Exercise 10. Show that given any collection of equivalence relations on a set A, their intersection is an equivalence relation on A

Solution. Let S_i be a collection of equivalence relations on A. consider $S = \bigcup_{i \in I} S_i$:

1. Let $x \in A$, since each of the S_i is a reflexive relation xS_ix for each $i \in I$, that is xSx
2. Let $x, y \in S$ so that xS_iy for all $i \in I$, then since S_i is symmetric $yS_ix \forall i \in I$ so that ySx
3. Consider x, y, z such that xSy and ySz, that is: for all $i \in I$ the following holds: xS_iy and yS_iz since each S_i is transitive xS_iz for all i so that xSz

Exercise 11. Show that an element in an ordered set has at most one immediate successor and at most one immediate predecessor. Show that a subset of an ordered set has at most one smallest element and at most one largest element

Solution. Consider an element $a \in A$, and A is an ordered set, consider $a < b$, and $a < b'$ for which:

$$\{x|a < x < b\} = \varnothing$$

and

$$\{x|a < x < b'\} = \varnothing$$

that is, both b and b' are the immediate successor of a, then observe that if $b < b'$, b will be in the first set, similarly b' will be in the second set, we conclude then that $b = b'$.

For the second question, arguments similar as above holds.

Exercise 12. Prove the following:

Theorem 4. *If an ordered set A has the least upper bound property, then it has the greatest lower bound property.*

Solution. Proof. Consider B subset of A, and B is bounded below, consider the set of all lower bounds for B, since B is bounded below the above set is nonempty, and bounded above with every element of B since it is bounded above, it has a supremum, so that we have shown that the set of all lower bounds has a supremum, that is B has an infimum. Therefore A has the greatest lowest bound property. □

Exercise 13. If C is a relation on a set A, define a new relation D on A by letting $(b, a) \in D$ if $(a, b) \in C$

- show that C is symmetric if and only if $C = D$.
- Show that if C is an order relation, D is also an order relation.
- Prove the converse of the theorem in Exercise 13.

Solution.

- Suppose C is symmetric, consider xDy or in other words yCx, since C is symmetric, we have that xCy, so that yDx, and D is symmetric. Similar argument above for the case of D being the symmetric relation is similar.

- Suppose C is an order relation, since C is an order, then either yDx or xDy holds, similar for non-reflexivity, and transitivity.

For the converse of the above theorem.

1.2 The Integers and the Real Numbers

Definition 1.2.1. A **binary opertation** on a set A is a function f mapping $A \times A$ into A

We assume there exists a set $\mathbb{R}$ called the set of **real numbers**, two binary operations $+$ and $\cdot$ on $\mathbb{R}$, called the addition and multiplication operations, respectively, and on order relation $<$ on $\mathbb{R}$, such that the following properties hold:

Algebraic Properties

1. $(x + y) + z = x + (y + z)$

   $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all x, y, z in $\mathbb{R}$

2. $x + y = y + x$,

   $x \cdot y = y \cdot x$ for all x, y in $\mathbb{R}$

3. There exists a unique element of $\mathbb{R}$ called **zero**, denoted by 0, such that $x + 0 = x$ for all $x \in \mathbb{R}$.

   There exists a unique element of $\mathbb{R}$ called **one**, different from 0 and denoted by 1, such that $x \cdot 1 = x$ for all $x \in \mathbb{R}$.

4. For each x in $\mathbb{R}$, there exist a unique y in $\mathbb{R}$ such that $x + y = 0$.

   For each x in $\mathbb{R}$ different from 0, there exists a unique y in $\mathbb{R}$ such that $x \cdot y = 1$.

5. $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ for all $x, y, z \in \mathbb{R}$

A *Mixed algebraic and Order Property.*

6. if $x > y$, then $x + z > y + z$

7. $x > y$ and $z > 0$, then $x \cdot z > y \cdot z$

Order Properties

8. The order relation $<$ has the least upper bound property.

9. If $x < y$, there exists an element z such that $x < z$ and $z < y$

If a set A with operations $+, \cdot$ holds the (1) - (5) properties, then it is a **Field**, if it also has an order relation that holds (6) then it is an ordered field, if this order also gold (7) and (8) hen it is a **linear continuum**.

Definition 1.2.2. A subset A of the real numbers is said to be **inductive** if it contains the number 1, and if for every x in A, the number $x+1$ is also in A. Let $\mathcal{A}$ be the collection of all inductive subsets of $\mathbb{R}$. Then the set $\mathbb{Z}_+$ of **positive integers** is defined by the equation:

$$\mathbb{Z}_+ = \bigcap_{A \in \mathcal{A}} A$$

We observe that the set $\mathbb{R}_+$ is inductive, since it has the element 1 and the property that $x > 0$ then $x + 1 > 0$, therefore $\mathbb{Z}_+ \subset \mathbb{R}_+$ Basic properties of $\mathbb{Z}_+$ are:

- $\mathbb{Z}_+$ is inductive.
- (Principle of Induction). If A is an inductive set of positive integers, then $A = \mathbb{Z}_+$

We define the set $\mathbb{Z}$ of **integers** to be the set consisting of the positive integers and the number 0, and the negatives of the elements $\mathbb{Z}_+$. If n is a positive integer, we use the symbol S_n to denote the set of all positive integers less than n; we call it a **section** of the positive integers. S_1 is empty, and S_{n+1} denotes the set of positive integers between 1 and n, inclusive.

$$\{1, \ldots, n\} = S_{n+1}$$

Theorem 5 (Well-ordering property)**.** *Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.*

Proof. We first prove that, for each $n \in \mathbb{Z}_+$ the following statement holds: *Every nonempty subset of $\{1, \ldots, n\}$ has a smallest element.* Let A be the set of all positive integers n for which this statement holds, Then A contains 1, since if $n = 1$, the only nonempty subset of $\{1, \ldots, n\}$ is the set 1 itself. Then supposing A contains n, we show that contains $n + 1$. So let C be a nonempty subset of $\{1, \ldots, n + 1\}$. If C consists of the single element $n + 1$ then that element is the smallest element of C. Otherwise, consider the set $C \cap \{1, \ldots, n\}$ which is nonempty. Because $n \in A$, this set has a smallest element, which will automatically be the smallest element of C also. Thus, A is inductive, so we conclude that $A = \mathbb{Z}_+$; hence the statement is true for all $n \in \mathbb{Z}_+$.

Now we prove the theorem. Suppose that D is a nonempty subset of $\mathbb{Z}_+$. Choose an element n of D. Then the set $D \cap \{1, \ldots, n\}$ is nonempty, so that A has a smallest element k. Then k is automatically the smallest element of D as well. □

Theorem 6 (Strong induction principle)**.** *Let A be a set of positive integers. Suppose that for each positive integer n, the statement $S_n \cap A$ implies the statement $n \in A$. Then $A = \mathbb{Z}_+$*

Proof. If A does not equal all of $\mathbb{Z}_+$, let n be the smallest positive integer that is not in A. Then every positive integer less than n is in A, so that $S_n \subset A$. Our hypothesis implies that $n \in A$, contrary to assumption. □

Exercise

Exercise 14. Prove the following "laws of algebra" for: $\mathbb{R}$.

- $0 \cdot x = 0$

Solution. • Using hint:

$$x \cdot x = (x + 0) \cdot x$$
$$x \cdot x = x \cdot x + 0 \cdot x$$

by the property that $x + y = x \implies y = 0$ we conclude $0 \cdot x = 0$

Exercise 15. • Show that if $\mathcal{A}$ is a collection of inductive sets, then the intersection of the elements of $\mathcal{A}$ is an inductive set.

- Prove
  - $\mathbb{Z}_+$ is inductive.
  - (Principle of Induction). If A is an inductive set of positive integers, then $A = \mathbb{Z}_+$

Solution. • Suppose that $\mathcal{A}$ is a set of inductive sets, observe that since an inductive set has the element 1, then 1 is in the intersection of these sets, now suppose x belongs to the intersection of the elements of $\mathcal{A}$, so that x is in every of the inductive sets, since all of them are inductive, $x + 1$ is also in all the sets, therefore in the intersection of $\mathcal{A}$.

- In a case by case scenario:
  - By the above point $\mathbb{Z}_+$ is inductive.
  - Suppose that A is an inductive set of positive integers, then $A \subset \mathbb{Z}$, and by being inductive $Z \subset A$

Exercise 16. • Prove by induction that given $n \in \mathbb{Z}_+$, every nonempty subset of $\{1, \ldots, n\}$ has a largest element.

- Explain why you cannot conclude from the above that every nonempty subset of $\mathbb{Z}_+$ has a largest element.

Solution.

Let A be the set of all positive integers for which this statement holds. A contains 1 since the set $\{1\}$ has only one subset itself, and it contains the largest element. Now consider C subset of $\{1, \ldots, n + 1\}$, if $n + 1 \in C$, then C has already its largest element, now consider $n + 1 \notin C$, so that C is a subset of $\{1, \ldots, n\}$ and by induction hypothesis it has a largest element.

We are assuming that the set is contained in a set of the form $\{1, \ldots, n\}$, but there are sets that are not contained in sets of that form.

Exercise 17. • Show that $\mathbb{R}$ has the greatest lower bound property.

Solution. • Consider A subset of $\mathbb{R}$ that is bounded above, consider the set $L = \{x | x \geq a, \forall a \in A\}$

Chapter 2

Topological Spaces and Continuous Functions

2.1 Topological Spaces

Definition. A **topology** on a set X is a collection τ of subsets of X having the sollowing properties:

- $\emptyset$ and X are in τ
- The union of the elements of any subcollection of τ is in τ
- The intersection of the elements of any finite subcollection of τ is in τ.

A set X for which a topology τ has been specified is called a **topological space**

Properly, a topological space is an ordered pair (X, τ).
If Z is a topological space with topology τ, we say that a subset U of X is an **open set** of X if U belongs to the collection τ.

Example 2.1.1. If X is any set, the collection of all subsets of X is a topology on X; it is called the **discrete topology**. The collection consisting of X and $\emptyset$ only is also a topology on X; we shall call it the **indiscrete topology**, or the **trivial topology**

Example 2.1.2. Let X be a set; let τ_f be the collection of all subsets of U of X such that $X - U$ is either finite or is all of X. Then τ_f is a topology on X, called the **finite complement topology**. Both X and $\varnothing$ are in τ_f, since $X - X$ is finite and $X - \varnothing$ is all of X. If $\{U_\alpha\}$ is an indexed family of nonempty elements of τ_f, to show that $\cup U_\alpha$ is in τ_f, we compute

$$(\bigcup U_\alpha)^c = \bigcap U_\alpha^c$$

and since each U^c is finite, the union of these set is finite. If $U_1, \ldots, U_n$ are nonempty elements of τ_f, to show that $\cap U_i$ is in τ_f, er compute:

$$\left(\bigcap_{i=1}^{n}\right)^{c} U_i = \bigcup_{i=1}^{n} U_i^c$$

Observe then that each U_i^c is finite, and finite union of finite set is finite.

Example 2.1.3. Let X be a set; let τ_c be the collection of all subsets U of X such that $X - U$ either is countable or is all of X. Then τ_c is a topology on X.

Definition. Suppose that τ and τ' are two topologies on a given set X. If $\tau' \supset \tau$, we say that τ' is **finer** than τ; if τ' properly contains τ, we say τ' is **strictly finer** than τ. We also say that τ is **coarser** thanτ', or **strictly coarser**, in these two respective situations. We say τ is **comparable** with τ' if either $\tau' \supset \tau$ or $\tau \supset \tau'$

Basis for a Topology

Definition. If X is a set, a **basis** for a topology on X is a collection $\mathfrak{B}$ of subsets of X (called **basis elements**) such that:

- For each $x \in X$, there is at least one basis element B containing x.
- If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathfrak{B}$ satisfies these two conditions, then we define the **topology** τ **generated by** $\mathfrak{B}$ as follows: A subset U of X is said to be open in X (that is , to be an element of τ) if for each $x \in U$, there is a basis element $B \in \mathfrak{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of τ

Example 2.1.4. If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology on X.

Lemma 7. *The collection τ generated by the basis $\mathfrak{B}$ is, in fact a topology on X*

Proof. If U is the empty set, it satisfies the defining condition of openness vacuously. Likewise, X is in τ, since for each $x \in X$ there is some basis element B containing x and contained in X. Now let us take an indexed family $\{U_\alpha\}_{\alpha \in J}$ of elements of τ, that is the collection $\subset$ in τ and show that:

$$U = \bigcup_{\alpha \in J} U_\alpha \in \tau$$

Given $x \in U$, there exist an index α such that $x \in U_\alpha$. Since U_α is open, there is a basis element B such that $x \in B \subset U_\alpha$. Then $x \in B$ and $B \subset U$, so that U is open, by definition. Now let us take two elements U_1 and U_2 of τ and show that the intersection belongs to τ. Given $x \in U_1 \cap U_2$, choose a basis element B_1 containing x such that $B_1 \subset U_1$; choose also a basis B_2 containing x such that $B_2 \subset U_2$. The second condition for a basis enables us to choose a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$. Then $x \in B_3$ and $B_3 \subset U_1 \cap U_2$, so $U_1 \cap U_2$ belongs to τ. Finally, we show by induction that any finite intersection $U_! \cap \ldots \cap U_n$ of elements in τ is inτ. The fact is trivial for $n = 1$; we suppose it true for $n-1$ and prove it for n. Now

$$(U_1 \cap \ldots U_n) = (U_1 \cap \ldots U_{n-1}) \cap U_n$$

. By hypothesis, $U_1 \cap \ldots U_{n-1}$ belongs to τ; by the result proven above, the intersection of $U_1 \cap \ldots U_{n-1}$ and U_n also belongs to τ □

Another point of view to view a basis for a topology is summarized in the following lemma:

Lemma 8. *Let X be a set; let $\mathfrak{B}$ be a basis for a topology τ on X. Then τ equals the collection of all unions of elements of $\mathfrak{B}$*

Proof. By double inclusion, Given a collection of elements in $\mathfrak{B}$, they are also elements of τ. Because τ is a topology, their union is in τ. Conversely, given $U \in \tau$, choose for each $x \in U$ an element B_x of $\mathfrak{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathfrak{B}$ □

As an important observation, this says that every open set U in X can be written as a union of basis elements. This expression for U is not unique. In summary, we have just described two ways from going from the basis to the topology it generates, now for the other way around:

Lemma 9. *Let X be a topological space. Suppose that $\mathfrak{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathfrak{C}$ such that $x \in C \subset U$. Then $\mathfrak{C}$ is a basis for the topology of X.*

Proof. We must show that $\mathfrak{C}$ is a basis. The first condition for a basis is easy: Given $x \in X$, since X is itself an open set, there is by hypothesis an element C of $\mathfrak{C}$ such that $x \in C \subset X$. To check the second condition, let x belong to $C_1 \cap C_2$, where C_1 and C_2 are elements of $\mathfrak{C}$. Since $C_!$ and C_2 are open, so is $C_1 \cap C_2$. Therefore, there exists by hypothesis an element C_3in $\mathfrak{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

Let τ be the collection of open sets of X; we must show that the topology τ' generated by $\mathfrak{C}$ equals the topology τ. First, note that if U belongs to τ and if $x \in U$ then there is by hypothesis an element C of $\mathfrak{C}$ such that $x \in C \subset U$. It follows that U belongs to the topology τ', by definition. Conversely, if W belongs to the topology τ', then W equals a union of elements of $\mathfrak{C}$, by the preceding lemma. Since each element of $\mathfrak{C}$ belongs to τ and τ is a topology. W also belongs to τ □

When topologies are given by bases, we want to have a criterion for deciding which topology is finer than other. The following provides a criterion:

Lemma 10. *Let $\mathfrak{B}$ and $\mathfrak{B}'$ be bases for the topologies τ and τ', respectively, on X. Then the following are equivalent:*

- *τ' is finer than τ, i.e., $\tau' \supset \tau$*
- *For each $x \in X$ and each basis element $B \in \mathfrak{B}$ containing x, there is a basis element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$.*

Proof. "$2 \implies 1$". Given an element U of τ, we wish to show that $U \in \tau'$. Let $x \in U$. Since $\mathfrak{B}$ generates τ, there is an element $B \in \mathfrak{B}$ such that $x \in B \subset U$. Condition (2) tell us there exists an element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$. Then $x \in B' \subset U$, so $U \in \tau'$, by definition.

"$1 \implies 2$". We are given $x \in X$ and $B \in \mathfrak{B}$, with $x \in B$. Now B belongs to τ by definition and $\tau \subset \tau'$ by condition (1); therefore, $B \in \tau'$. Since τ' is generated by $\mathfrak{B}'$, there is an element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$ □

2.2 Closed Sets and limit Points

Definition. A subset A of a topological space X is said to be **closed** if the set $X - A$ is open.

Observe that a set can be open, or closed or both.

Theorem 11. *Let X be atopological space, Then the following conditions holds:*

- *$\varnothing$ and X are closed.*
- *Arbitary intersections of closed sets are closed.*
- *Finite unions of closed sets are closed.*

Proof. Observe that $\varnothing$ and X are closed necause they are the complements of the open set X and $\varnothing$ respictively.

For the second proof, suposse that we are given a collection of closed sets $\{A_\alpha\}_{\alpha \in J}$, and we apply then DeMorgan's Law,

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup (X - A_\alpha)$$

□

Theorem 12. *Let Y be a subspace of X. Then a a set A is closed in Y if and only if it equals the intersection of a closed set of X*

Compactness and Connectedness

Definition A subset K if a topological space X is *compact* if every open cover of K has a finite subcover. A subset is compact if and only if it is compact as a topological space whit it is given its relative topologt, that is, if and only if every cover by relatively open sets has a finite subcover.

Proposition Let X be a topological space, and let $K \subseteq X$.

- If K is a compact subset of X, then K is closed.
- If K is compact and F is a closed set contained in K, then F is compact.
- The continuous image of a compact set is compact.

Corollary IF X is a compact space and $f : X \to \mathbb{R}$ is a continuous function, then there are points a and b in X such that $f(a) \leq f(x) \leq f(b)$ for all x in X.

Proposition If K is a closed subset of a topological space X, then K is compact if and only if every collection of closed subsets of K having the FIP has a nonempty intersection.

Compactness in Metric Spaces

If $\mathfrak{G}$ is a collection of subset of subsets of X and $E \subseteq X$, then $\mathfrak{G}$ is a *cover* of E if $E \subsetneqq \cup\{G : G \in \mathfrak{G}\}$. *subcover* of E