

Textbook notes on Topology

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The following are notes based on the book *Topology* by Munkres.

Chapter 1

Set Theory and Logic

1.1 Fundamental Concepts

We express that an object a belongs to a set A by the notation:

$$a \in A$$

Similarly,

$$a \notin A$$

We denote the inclusion of a set into another set with:

$$A \subseteq B$$

so that $A = B \iff A \subseteq B$ and $B \subseteq A$. If $A \subseteq B$ but A is different from B , we say A is a **proper subset** of B , in notation:

$$A \subsetneq B$$

The relation \subseteq is called **inclusion** and \subsetneq is called proper inclusion.

The Union of Sets and the Meaning of "or"

Given two sets A and B , we can form another set called the union of A and B .

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

We will use the concept of exclusive or, if the necessity arises.

The Intersection of Sets, the Empty set, and the Meaning of "If...Then"

Another way to form a set from two existing sets is to take the elements in common, that is:

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

The **empty set** is the set with no elements, denoted by \emptyset . We say that two elements are disjoint if:

$$A \cap B = \emptyset$$

Some property of this interesting empty set are:

$$A \cap \emptyset = \emptyset \quad A \cup \emptyset = A$$

1.1.1 Exercises

Exercise 1. Prove that a set of n (different) elements has exactly 2^n (different) subsets.

Solution. The following is an application of the principle of mathematical induction. First, we observe that for a set with only one element, there are exactly $2 = 2^1$, now suppose that the statement holds for a set with n elements, now consider the set with $n + 1$ elements, say $\{x_1, \dots, x_{n+1}\}$, observe that we can write this set $\{x_1, \dots, x_n\} \cup \{x_{n+1}\}$, using out induction hypothesis, the set $\{x_1, \dots, x_n\}$ has 2^n subsets, and the subsets of $\{x_1, \dots, x_{n+1}\}$ can be obtained as:

$$\overbrace{2^n}^{\text{subsets of } n} + \underbrace{2^n}_{\text{adding the new element}}$$

and we have

$$2^n + 2^n = 2^n(1 + 1) = 2^{n+1}$$

Exercise 2. Formulate and prove DeMorgan's laws for arbitrary unions and intersections.

Solution.

Proposition 1.1.1. Let \mathcal{A} a family of sets, then

$$X - \bigcup_{A \in \mathcal{A}} A = \bigcap_{A \in \mathcal{A}} X - A$$

Proof. Let $x \in X - \bigcup_{A \in \mathcal{A}} A$, so that $x \notin \bigcup_{A \in \mathcal{A}} A$, or in other words, $\forall A \in \mathcal{A}(x \notin A)$, and since $x \in X$, we can combine the last two statements to give $\forall A \in \mathcal{A}(x \in X \text{ and } x \notin A)$, more suggestively, $\forall A \in \mathcal{A}(x \in X - A)$, and finally this implies $\bigcap_{A \in \mathcal{A}} X - A$ \square

Proposition 1.1.2. Let \mathcal{A} be a family of sets, then:

$$X - \bigcap_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} X - A$$

Which again is similar.

1.1.2 Functions

Exercise 3. Let $f : A \rightarrow B$. Let $A_0 \subset A$ and $B_0 \subset B$.

- Show that $A_0 \subset f^{-1}(f(A_0))$ and that the equality holds if f is injective.
- Show that $f(f^{-1}(B_0)) \subset B_0$ and that equality holds if f is surjective.

Solution. • Let $a \in A_0$, so that $f(a) \in f(A_0)$, and this is the condition for that $a \in f^{-1}f(A_0)$. Now for the converse, we add the hypothesis that f is injective, so let $x \in f^{-1}(f(A_0))$, observe that this implies $f(x) \in f(A_0)$, or in other words $f(x) = f(a)$ for at least one $a \in A_0$, since f is injective, then $x = a$ and $x \in A_0$.

- Let $b \in f(f^{-1}(B_0))$, so that $b = f(a)$ for some $a \in f^{-1}(B_0)$, similarly $f(a) \in B_0$ so that $b \in B_0$. Now for the converse we add the hypothesis that f is surjective, now let $b \in B_0$, since f is surjective, there exist $a \in A$ for which $f(a) = b$ so that $a \in f^{-1}(B_0)$, and combining the two previous statements we get $b \in f(f^{-1}(B_0))$

Exercise 4. Let $f : A \rightarrow B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that f^{-1} preserves inclusions, unions, intersections, and differences of sets

- $B_0 \cap B_1 \implies f^{-1}(B_0) \subset f^{-1}(B_1)$.
- $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$

Solution. • Suppose $B_0 \subset B_1$, let $x \in f^{-1}(B_0)$ that is $f(x) = b$ for some $b \in B_0$, that is, $b \in B_1$, and this last assertion allow us to conclude that $f^{-1}(B_0) \subset f^{-1}(B_1)$

- Let $a \in f^{-1}(B_0 \cup B_1)$, that is $f(a) = b$ for an element b that either belongs to B_0 or B_1 , so that $a \in f^{-1}(B_0) \cup f^{-1}(B_1)$. Conversely, ...

Exercise 5. Prove the following:

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$$f^{-1}\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f^{-1}(A_i)$$

Solution. “ \subset ”

$$\begin{aligned} x \in f^{-1}\left(\bigcup_{i \in I} A_i\right) &\implies \exists j \in I (f(x) \in A_j) \\ &\implies x \in \bigcup_{i \in I} f^{-1}(A_i) \end{aligned}$$

“ \supset ”, let $A_i = B_i$ for all $i \in I$

$$\begin{aligned} x \in \bigcup_{i \in I} f^{-1}(B_i) &\implies \exists j \in I (x \in f^{-1}(B_j)) \\ &\implies f(x) \in B_j \implies f(x) \in \bigcup_{i \in I} B_i \\ &\implies x \in f^{-1}\left(\bigcup_{i \in I} B_i\right) \end{aligned}$$

Exercise 6. Prove the following:

$$f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i)$$

Solution. “ \subset ”

$$\begin{aligned} x \in f^{-1}\left(\bigcap_{i \in I} B_i\right) &\implies f(x) \in \bigcap_{i \in I} B_i \\ &\implies \forall i \in I (f(x) \in B_i) \implies \forall i \in I (x \in f^{-1}(B_i)) \\ &\implies x \in \bigcap_{i \in I} f^{-1}(B_i) \end{aligned}$$

“ \supset ”

$$\begin{aligned} x \in \bigcap_{i \in I} f^{-1}(B_i) &\implies \forall i \in I (x \in f^{-1}(B_i)) \\ &\implies \forall i \in I (f(x) \in B_i) \\ &\implies f(x) \in \bigcap_{i \in I} B_i \\ &\implies x \in f^{-1}\left(\bigcap_{i \in I} B_i\right) \end{aligned}$$

Exercise 7. Prove that for an arbitrary class of sets:

$$f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i)$$

Solution. Let us take $y \in f(\bigcup_{i \in I} A_i)$, so that $y = f(x)$ for some $x \in A_i$, say $x \in A_j$, then $y \in f(A_j)$ so that $y \in \bigcup_{i \in I} f(A_i)$

Exercise 8. Let $f : A \rightarrow B$ and $g : B \rightarrow C$.

- $C_0 \subset C$, show that $(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$
- If f and g are injective, show that $g \circ f$ is injective.
- If $g \circ f$ is injective, what can you say about the injectivity of f and g ?

- If f and g are surjective, show that $g \circ f$ is surjective.
- If $g \circ f$ is surjective, what can you say about surjectivity of f and g

Solution. • Let $a \in (g \circ f)^{-1}(C_0)$, so that $(g \circ f)(a) \in C_0$, and $g(f(a)) \in C_0$ so that, $f(a) \in g^{-1}(C_0)$ which finally $a \in f^{-1}(g^{-1}(C_0))$

- Let $a, a' \in A$, suppose that $(g \circ f)(a) = (g \circ f)(a')$, by definition $g(f(a)) = g(f(a'))$, since g is injective, we have then $f(a) = f(a')$ again f is injective, then $a = a'$ so that $g \circ f$ is injective.
- If $g \circ f$ is injective, then f is injective.

Proof. Let $a_1, a_2 \in A$, such that $f(a_1) = f(a_2)$, then $g(f(a_1)) = g(f(a_2))$, so that $(g \circ f)(a_1) = (g \circ f)(a_2)$ and since $g \circ f$ is injective $a_1 = a_2$ \square

- Let $c \in C$, since the function g is surjective there exist b such that $c = g(b)$, now, since $b \in B$ and f is injective, there exist $a \in A$ for which $c = g(f(a)) = (g \circ f)(a)$, since c was arbitrary we have shown then $g \circ f$ is surjective.
- We propose that g is surjective.

Proof. Let $c \in C$. Since $g \circ f$ is surjective, there exist a such that $g(f(a)) = c$. By definition for some $b \in B$, $f(a) = b$, and $g(b) = c$, then g is surjective. \square

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Theorem 1. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ and $g \circ f$ is bijective, then f is injective and g is surjective

1.1.3 Relations

Definition. A **relation** on a set A is a subset C of the cartesian product $A \times A$

Equivalence Relations and Partitions

An **equivalence relation** on a set A is a relation C on A having the following three properties:

1. (Reflexivity) $x C x$
2. (Symmetry) $x C y$, then $y C x$
3. (Transitivity) $x C y$ and $y C z \implies x C z$

Given an equivalence relation \sim on a set A and an element x of A , we define a certain subset E of A , called the **equivalence class** determined by x , by the equation:

$$E = \{y | y \sim x\}$$

Lemma 2. *Two equivalence classes E and E' are either disjoint or equal.*

Proof. Let E be the equivalence class determined by x , and E' be the equivalence class determined by x' , suppose $E \cap E'$ is not empty, so that $y \in E \cap E'$, since y belongs to E and E' , $x \sim y$ and $y \sim x'$ so that $x \sim x'$ so that $x' \in E$ and similarly $x \in E'$ and we conclude that $E = E'$ \square

Definition 1.1.1. A **partition** of a set A is a collection of disjoint nonempty subsets of A whose union is all of A

Lemma 3. *Given any partition \mathcal{D} of A , there is exactly one equivalence relation on A from which is derived.*

Proof. To show that the partition \mathcal{D} comes from some equivalence relation, let us define the equivalence relation on A as: let $x \in A$ $x \sim y$ if and only if x, y belong to the same element \mathcal{D} . Observe that this relation complies with all the rules to be a equivalence relation. Now for the uniqueness suppose \sim_1 and \sim_2 gives arise to the same partition \mathcal{D} to do that we will show that $x \sim_1 y \iff x \sim_2 y$ but that is simple, consider E as the equivalence class determined by x in \sim_1 and consider E' with \sim_2 , since E_1 is an element of \mathcal{D} then it must equal the unique element D of \mathcal{D} , so that $E = D = E'$ \square

Order Relations

A relation C on a set A is called an **order relation** (or a **simple order** or a **linear order**) if it has the following properties:

1. (Comparability) For every x and y in A for which $x \neq y$ either xCy or yCx
2. (Non reflexivity) For no x in A does the relation xCx holds.
3. (Transitivity) If xCy and yCz , then xCz

Definition 1.1.2. If X is a set and $<$ is an order relation on X , and if $a < b$ we use the notation (a, b) to denote the set:

$$\{x | a < x < b\}$$

; it is called an **open interval** in X . If this set is empty, we call a the **immediate predecessor** of b and b the **immediate successor** of a

Definition 1.1.3. Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. We say that A and B have the same **order type** if there is a bijective function $f : A \rightarrow B$ such that:

$$a_1 <_A a_2 \implies f(a_1) <_B f(a_2)$$

Definition 1.1.4. Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. Define an order relation $<$ on $A \times B$ by defining:

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 < a_2$, or if $a_1 = a_2$ and $b_1 <_B b_2$. It is called the **dictionary order relation** on $A \times B$

For the next definitions Let A be a set ordered by the relation $<$

Definition. Let A_0 be a subset of A . We say that the element b is the **largest element** of A_0 if $b \in A_0$ and if $x \leq b$ for every $x \in A_0$

Similarly, we define:

Definition 1.1.5. Let A_0 be a subset of A . We say that the element a is the **smallest element** of A_0 if $a \in A_0$ and if $a \leq x$ for every $x \in A_0$

We make the remark that a set has at most one largest element and at most one smallest element.

Definition. We say the subset A_0 is **bounded above** if there is an element b of A such that $x \leq b$ for every $x \in A_0$; the element b is called an **upper bound** for A_0 . If the set of all upper bounds for A_0 has a smallest element, that element is called the **least upper bound**, or the **supremum** of A_0 often denoted as $\sup A_0$

Exercises

Exercise 9. Find the flaw in the following proof: suppose that C is a symmetric and transitive relation, then $xCy \implies yCx$ and by transitive then xCx so that C is reflexive

Solution. This is true if either xCy or yCx if neither of them holds then, this does not prove that xCx

Exercise 10. Show that given any collection of equivalence relations on a set A , their intersection is an equivalence relation on A

Solution. Let S_i be a collection of equivalence relations on A . consider $S = \bigcup_{i \in I} S_i$:

1. Let $x \in A$, since each of the S_i is a reflexive relation $xS_i x$ for each $i \in I$, that is xSx
2. Let $x, y \in S$ so that $xS_i y$ for all $i \in I$, then since S_i is symmetric $yS_i x \forall i \in I$ so that ySx
3. Consider x, y, z such that xSy and ySz , that is: for all $i \in I$ the following holds: $xS_i y$ and $yS_i z$ since each S_i is transitive $xS_i z$ for all i so that xSz

Chapter 2

Topological Spaces and Continuous Functions

2.1 Topological Spaces

Definition. A **topology** on a set X is a collection τ of subsets of X having the following properties:

- \emptyset and X are in τ
- The union of the elements of any subcollection of τ is in τ
- The intersection of the elements of any finite subcollection of τ is in τ .

A set X for which a topology τ has been specified is called a **topological space**

Properly, a topological space is an ordered pair (X, τ) .

If Z is a topological space with topology τ , we say that a subset U of X is an **open set** of X if U belongs to the collection τ .

Example 2.1.1. If X is any set, the collection of all subsets of X is a topology on X ; it is called the **discrete topology**. The collection consisting of X and \emptyset only is also a topology on X ; we shall call it the **indiscrete topology**, or the **trivial topology**

Example 2.1.2. Let X be a set; let τ_f be the collection of all subsets of U of X such that $X - U$ is either finite or is all of X . Then τ_f is a topology on X , called the **finite complement topology**. Both X and \emptyset are in τ_f , since $X - X$ is finite and $X - \emptyset$ is all of X . If $\{U_\alpha\}$ is an indexed family of nonempty elements of τ_f , to show that $\cup U_\alpha$ is in τ_f , we compute

$$(\cup U_\alpha)^c = \cap U_\alpha^c$$

and since each U^c is finite, the union of these set is finite. If U_1, \dots, U_n are nonempty elements of τ_f , to show that $\cap U_i$ is in τ_f , er compute:

$$\left(\bigcap_{i=1}^n \right)^c U_i = \bigcup_{i=1}^n U_i^c$$

Observe then that each U_i^c is finite, and finite union of finite set is finite.

Example 2.1.3. Let X be a set; let τ_c be the collection of all subsets U of X such that $X - U$ either is countable or is all of X . Then τ_c is a topology on X .

Definition. Suppose that τ and τ' are two topologies on a given set X . If $\tau' \supset \tau$, we say that τ' is **finer** than τ ; if τ' properly contains τ , we say τ' is **strictly finer** than τ . We also say that τ is **coarser** than τ' , or **strictly coarser**, in these two respective situations. We say τ is **comparable** with τ' if either $\tau' \supset \tau$ or $\tau \supset \tau'$

Basis for a Topology

Definition. If X is a set, a **basis** for a topology on X is a collection \mathfrak{B} of subsets of X (called **basis elements**) such that:

- For each $x \in X$, there is at least one basis element B containing x .
- If x belongs to the intersection of two basis elements B_1 and B_2 , then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If \mathfrak{B} satisfies these two conditions, then we define the **topology τ generated by \mathfrak{B}** as follows: A subset U of X is said to be open in X (that is , to be an element of τ) if for each $x \in U$, there is a basis element $B \in \mathfrak{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of τ

Example 2.1.4. If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology on X .

Lemma 4. *The collection τ generated by the basis \mathfrak{B} is, in fact a topology on X*

Proof. If U is the empty set, it satisfies the defining condition of openness vacuously. Likewise, X is in τ , since for each $x \in X$ there is some basis element B containing x and contained in X . Now let us take an indexed family $\{U_\alpha\}_{\alpha \in J}$ of elements of τ , that is the collection \subset in τ and show that:

$$U = \bigcup_{\alpha \in J} U_\alpha \in \tau$$

Given $x \in U$, there exist an index α such that $x \in U_\alpha$. Since U_α is open, there is a basis element B such that $x \in B \subset U_\alpha$. Then $x \in B$ and $B \subset U$, so that U is open, by definition. Now let us take two elements U_1 and U_2 of τ and show

that the intersection belongs to τ . Given $x \in U_1 \cap U_2$, choose a basis element B_1 containing x such that $B_1 \subset U_1$; choose also a basis B_2 containing x such that $B_2 \subset U_2$. The second condition for a basis enables us to choose a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$. Then $x \in B_3$ and $B_3 \subset U_1 \cap U_2$, so $U_1 \cap U_2$ belongs to τ . Finally, we show by induction that any finite intersection $U_1 \cap \dots \cap U_n$ of elements in τ is in τ . The fact is trivial for $n = 1$; we suppose it true for $n - 1$ and prove it for n . Now

$$(U_1 \cap \dots \cap U_n) = (U_1 \cap \dots \cap U_{n-1}) \cap U_n$$

. By hypothesis, $U_1 \cap \dots \cap U_{n-1}$ belongs to τ ; by the result proven above, the intersection of $U_1 \cap \dots \cap U_{n-1}$ and U_n also belongs to τ \square

Another point of view to view a basis for a topology is summarized in the following lemma:

Lemma 5. *Let X be a set; let \mathfrak{B} be a basis for a topology τ on X . Then τ equals the collection of all unions of elements of \mathfrak{B}*

Proof. By double inclusion, Given a collection of elements in \mathfrak{B} , they are also elements of τ . Because τ is a topology, their union is in τ . Conversely, given $U \in \tau$, choose for each $x \in U$ an element B_x of \mathfrak{B} such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of \mathfrak{B} \square

As an important observation, this says that every open set U in X can be written as a union of basis elements. This expression for U is not unique. In summary, we have just described two ways from going from the basis to the topology it generates, now for the other way around:

Lemma 6. *Let X be a topological space. Suppose that \mathfrak{C} is a collection of open sets of X such that for each open set U of X and each x in U , there is an element C of \mathfrak{C} such that $x \in C \subset U$. Then \mathfrak{C} is a basis for the topology of X .*

Proof. We must show that \mathfrak{C} is a basis. The first condition for a basis is easy: Given $x \in X$, since X is itself an open set, there is by hypothesis an element C of \mathfrak{C} such that $x \in C \subset X$. To check the second condition, let x belong to $C_1 \cap C_2$, where C_1 and C_2 are elements of \mathfrak{C} . Since C_1 and C_2 are open, so is $C_1 \cap C_2$. Therefore, there exists by hypothesis an element C_3 in \mathfrak{C} such that $x \in C_3 \subset C_1 \cap C_2$.

Let τ be the collection of open sets of X ; we must show that the topology τ' generated by \mathfrak{C} equals the topology τ . First, note that if U belongs to τ and if $x \in U$ then there is by hypothesis an element C of \mathfrak{C} such that $x \in C \subset U$. It follows that U belongs to the topology τ' , by definition. Conversely, if W belongs to the topology τ' , then W equals a union of elements of \mathfrak{C} , by the preceding lemma. Since each element of \mathfrak{C} belongs to τ and τ is a topology. W also belongs to τ \square

When topologies are given by bases, we want to have a criterion for deciding which topology is finer than other. The following provides a criterion:

Lemma 7. Let \mathfrak{B} and \mathfrak{B}' be bases for the topologies τ and τ' , respectively, on X . Then the following are equivalent:

- τ' is finer than τ , i.e., $\tau' \supset \tau$
- For each $x \in X$ and each basis element $B \in \mathfrak{B}$ containing x , there is a basis element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$.

Proof. "2 \implies 1". Given an element U of τ , we wish to show that $U \in \tau'$. Let $x \in U$. Since \mathfrak{B} generates τ , there is an element $B \in \mathfrak{B}$ such that $x \in B \subset U$. Condition (2) tell us there exists an element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$. Then $x \in B' \subset U$, so $U \in \tau'$, by definition.

"1 \implies 2". We are given $x \in X$ and $B \in \mathfrak{B}$, with $x \in B$. Now B belongs to τ by definition and $\tau \subset \tau'$ by condition (1); therefore, $B \in \tau'$. Since τ' is generated by \mathfrak{B}' , there is an element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$ \square

2.2 Closed Sets and limit Points

Definition. A subset A of a topological space X is said to be **closed** if the set $X - A$ is open.

Observe that a set can be open, or closed or both.

Theorem 8. Let X be a topological space, Then the following conditions holds:

- \emptyset and X are closed.
- Arbitrary intersections of closed sets are closed.
- Finite unions of closed sets are closed.

Proof. Observe that \emptyset and X are closed because they are the complements of the open set X and \emptyset respectively.

For the second proof, suppose that we are given a collection of closed sets $\{A_\alpha\}_{\alpha \in J}$, and we apply then DeMorgan's Law,

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha)$$

\square

Theorem 9. Let Y be a subspace of X . Then a set A is closed in Y if and only if it equals the intersection of a closed set of X

Compactness and Connectedness

Definition A subset K of a topological space X is *compact* if every open cover of K has a finite subcover. A subset is compact if and only if it is compact as a topological space with it given its relative topology, that is, if and only if every cover by relatively open sets has a finite subcover.

Proposition Let X be a topological space, and let $K \subseteq X$.

- If K is a compact subset of X , then K is closed.
- If K is compact and F is a closed set contained in K , then F is compact.
- The continuous image of a compact set is compact.

Corollary If X is a compact space and $f : X \rightarrow \mathbb{R}$ is a continuous function, then there are points a and b in X such that $f(a) \leq f(x) \leq f(b)$ for all x in X .

Proposition If K is a closed subset of a topological space X , then K is compact if and only if every collection of closed subsets of K having the FIP has a nonempty intersection.

Compactness in Metric Spaces

If \mathfrak{G} is a collection of subset of subsets of X and $E \subseteq X$, then \mathfrak{G} is a *cover* of E if $E \subseteq \bigcup \{G : G \in \mathfrak{G}\}$. *subcover* of E